4. The Homogeneous Case

Das Spiel dauert 90 Minuten.
[The game lasts 90 minutes.]
(Sepp Herberger)

Well, not quite: sometimes a soccer game lasts 93 minutes and 36 seconds. Also the writing of this chapter took some unexpected turns. Initially, our intention was to compose a small collection of the necessary background material for our true goal, namely to write a linchpin chapter about Hilbert functions. Hilbert functions arise in graded situations, and thus we were naturally led to ask the following basic questions.

1) Which gradings on the polynomial ring are useful for Computational Commutative Algebra?

2) How are the graded and the non-graded settings related to each other? Can one pass from one to the other?

3) What, if any, are the computational advantages of being in a graded setting?

4) How can one compute some typical invariants of homogeneous ideals and graded modules, such as the minimal number of generators and the graded Betti numbers?

As we struggled to answer these questions, the chapter grew and grew. Time and again we discovered that the intricacies of some topic put up more resistance than we had expected, and now that everything is finished, Chapter 4 has more than 150 pages. So, what is all the fuss about?

Instead of delving into a detailed, proposition by proposition account of the chapter, let us walk though a typical example originating in algebraic geometry. As we explain in Tutorial 27, the Zariski closure of the set $\{(t, t^3, t^4) \in \mathbb{A}^3_{\mathbb{Q}} \mid t \in \mathbb{Q}\}$ is an affine variety. In Tutorial 39.h we saw that the vanishing ideal of this curve is $I = (x_1 - t,\, x_2 - t^3,\, x_3 - t^4) \cap \mathbb{Q}[x_1, x_2, x_3]$, and using Theorem 3.4.5 we find that $I = (x_1x_2 - x_3,\, x_1^3 - x_2)$. Moreover, a glance at $\mathrm{LT}_{\mathtt{DegRevLex}}(I) = (x_1x_2,\, x_1^3,\, x_1^2x_3,\, x_2^3)$ convinces us that I is not principal, i.e. that the given system of generators is minimal.

However, this is not the end of the story. Usually, algebraic geometers are even more interested in the projective closure of this curve, i.e. the curve $C = \{(u^4 : tu^3 : t^3u : t^4) \in \mathbb{P}^3_{\mathbb{Q}} \mid (t : u) \in \mathbb{P}^1_{\mathbb{Q}}\}$. The homogeneous vanishing ideal of C is the homogenization $I^{\mathrm{hom}} = (f^{\mathrm{hom}} \mid f \in I)$ of I in

$P = \mathbb{Q}[x_0, \ldots, x_3]$ where $f^{\mathrm{hom}} = x_0^{\deg(f)} f(\frac{x_1}{x_0}, \ldots, \frac{x_3}{x_0})$ denotes the homogenization of a polynomial $f \in \mathbb{Q}[x_1, x_2, x_3]$. How can we compute I^{hom}? As we shall see, it is not sufficient to homogenize the generators of I. Instead, Section 4.3 contains three methods for computing I^{hom}. Firstly, Corollary 4.3.8 shows $I^{\mathrm{hom}} = (x_0x_3 - x_1x_2,\, x_0^2x_2 - x_1^3) :_P (x_0)^\infty$. Secondly, Proposition 4.3.21 yields $I^{\mathrm{hom}} = (x_1x_2 - x_0x_3,\, x_1^3 - x_0^2x_2,\, x_1^2x_3 - x_0x_2^2,\, x_2^3 - x_1x_3^2)$, since $G = \{x_1x_2 - x_3,\, x_1^3 - x_2,\, x_1^2x_3 - x_2^2,\, x_2^3 - x_1x_3^2\}$ is a `DegRevLex`-Gröbner basis of I. The third method is based on Tutorial 51 and requires us to compute the elimination ideal $(x_0 - y_0^4,\, x_1 - y_0^3y_1,\, x_2 - y_0y_1^3,\, x_3 - y_1^4) \cap P$.

After determining the homogeneous vanishing ideal I^{hom} of C in the ring P, we would like to calculate some invariants associated to it. For instance, what is the minimal number of generators of I^{hom}? By the Graded Version of Nakayama's Lemma 1.7.15, the irredundant system of generators consisting of the four polynomials given above is minimal. In fact, using the variant of Buchberger's algorithm we present in Theorem 4.6.3, we can discover this property while performing a suitable Gröbner basis computation. Another way of expressing the insight we have just gained is to say that there is a homogeneous surjective P-linear map $\varphi : F \longrightarrow I^{\mathrm{hom}}$, where $F = P(-2) \oplus P(-3)^3$ and $\mathrm{Ker}(\varphi)$ is contained in $(x_0, \ldots, x_3)\, F$.

A finer set of invariants can be attached to I^{hom} if we notice that it is actually a bigraded ideal. This means that if we use a $\mathbb{Z}^2$-grading on P for which $\deg(x_0) = \binom{1}{0}$, $\deg(x_1) = \binom{1}{1}$, $\deg(x_2) = \binom{1}{3}$, and $\deg(x_3) = \binom{1}{4}$, then I^{hom} is a homogeneous ideal with respect to this grading. Hence we can use the results in Sections 4.7 and 4.8 to compute the minimal bigraded free resolution

$$\begin{aligned}0 \longrightarrow P\big(-\tbinom{5}{10}\big) \xrightarrow{\lambda} P\big(-\tbinom{4}{6}\big) \oplus P\big(-\tbinom{4}{7}\big) \oplus P\big(-\tbinom{4}{9}\big) \oplus P\big(-\tbinom{4}{10}\big) \xrightarrow{\psi} \\ \xrightarrow{\psi} P\big(-\tbinom{2}{4}\big) \oplus P\big(-\tbinom{3}{3}\big) \oplus P\big(-\tbinom{3}{6}\big) \oplus P\big(-\tbinom{3}{9}\big) \xrightarrow{\varphi} I^{\mathrm{hom}} \longrightarrow 0\end{aligned}$$

of I^{hom}. The multiplicities of the shifts appearing in this resolution are called the graded Betti numbers of I^{hom}. They encode subtle properties of the geometry of C.

In the light of this example, let us come back to the basic questions we asked above and discuss them. Let $P = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K.

1) The gradings on P have to be over an explicitly given, easily computable monoid. For our purposes, the monoid $\Gamma = \mathbb{Z}^m$ is well-suited. Moreover, each indeterminate should be homogeneous, and non-zero constants should be homogeneous of degree zero. Fixing the degree of each indeterminate gives a uniquely defined grading on P with these properties. Let $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ be the matrix whose columns are the degrees of the indeterminates. We shall say that the grading on P is given by W. Such gradings are studied in Sections 4.1 and 4.2.A. Later we introduce and study gradings of positive type and positive gradings, because they offer additional nice

properties such as finite dimensional homogeneous components of finitely generated graded modules, the applicability of the graded version of Nakayama's lemma, a natural and easily computable well-ordering on the set of degrees of non-zero homogeneous polynomials, or the existence of a degree compatible term ordering.

2) There are several ways of passing from a non-homogeneous polynomial f (or ideal I) to a homogeneous one. The first method is to consider only those terms in the polynomial having maximal degree. They yield the degree form of f. In analogy with the theory developed in Chapter 2, we can now define the degree form ideal of I and study Macaulay bases, i.e. sets of polynomials whose degree forms generate the degree form ideal of I. This is done in Section 4.2.B. Another approach is followed in Section 4.3, where we introduce new indeterminates and use them to homogenize a polynomial by multiplying terms of non-maximal degree with suitable power products of the homogenizing indeterminates. The ideal generated by the homogenizations of the polynomials in I is then called the homogenization of I and denoted by I^{hom}. After a detailed discussion of the algebraic and computational aspects of the passage from I to I^{hom} (and back), we also explain its relationship to Macaulay bases, Gröbner bases, and flat families. Finally, in Tutorial 50, we show how to compute the homogeneous ideal obtained from a non-homogeneous one by simply taking the ideal generated by the homogeneous elements contained in it.

3) In Chapters 2 and 3 we have already encountered situations where the difficulty of computing a particular Gröbner basis depended strongly on the chosen term ordering. In many cases, the degree reverse lexicographic term ordering turned out to be a good choice. So, what is so special about it? This is the topic of Section 4.4, where we use term orderings which are similar to `DegRevLex` for computing saturations, colon modules, addition of an indeterminate, and reduction modulo an indeterminate in an efficient way if the ideals and modules in question are suitably graded. Further advantages of using gradings surface in Section 4.5, where we examine the computation of Gröbner bases in a graded setting. The homogeneous version of Buchberger's algorithm permits substantial optimizations. This is because we can use the fact that it proceeds degree by degree and also that the S-vectors it has to process are homogeneous. Moreover, if we stop the computation after a fixed degree is finished, we have computed a truncated Gröbner basis and, as we shall see in Section 4.5.B, this may be all we need for a particular problem.

4) The first and most obvious invariant of a homogeneous ideal or graded module is its minimal number of generators. Due to the Graded Version of Nakayama's Lemma 1.7.15, every irredundant homogeneous system of generators is minimal, and all of them have the same number of elements. In Section 4.6, we prove that a slight variation of the Homogeneous Buchberger Algorithm 4.5.5 yields a minimal homogeneous system of generators $\mathcal{V}_{\min}$ contained in the original system of generators. The next step is to compute

a minimal homogeneous system of generators of $\mathrm{Syz}_P(\mathcal{V}_{\min})$, i.e. a minimal homogeneous presentation of the ideal or module we started with. This step is taken in Section 4.7, where we not only introduce several strategies for computing such minimal homogeneous presentations, but also prove that the ranks and shifts of the graded free modules involved in such a presentation are invariants of M, i.e. they do not depend on the chosen presentation. Continuing this process of computing minimal homogeneous systems of generators of syzygy modules, we obtain a minimal graded free resolution

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of a finitely generated graded P-module M, where $F_0, F_1, \ldots$ are finitely generated graded free P-modules. After proving in Section 4.8.A that a finite minimal graded free resolution of M exists and is essentially unique (up to isomorphisms of graded free modules), we present three methods for computing it in Section 4.8.B.

After all is said and done, it seems that the advantages of working in a graded setting more than compensate for the additional conceptual and notational toil and trouble it requires. Do you think that this is an obvious conclusion, something French people would call a "lapalissade"? At times *lapalissades* contain profound truths.

Monsieur de La Palisse est mort
il est mort devant Pavie
un quart d'heure avant sa mort
il était encore en vie.
(Bernard de la Monnoye)

4.1 Polynomial Rings Graded by Matrices

Begin at the beginning
and go on till you come to the end;
then stop.
(Lewis Carroll)

Before beginning our journey into the realm of gradings on polynomial rings, let us have a look at where we are. In Volume 1 we devoted all of Section 1.7 to the hidden secrets of general gradings. This generality was motivated by the goal of providing correct proofs in the theory of Gröbner bases of modules. Now, where do we want to go from there? For actual computations, gradings on polynomial rings given by arbitrary commutative monoids are too general. Therefore we need to restrict our attention to a more limited class of gradings. How can we manufacture a suitable notion?

To get some inspiration, let us have another look back at the path we took in Volume 1. In Example 1.7.2 we introduced the most common grading on the polynomial ring, namely the standard grading, and in Section 1.4 we saw that the most useful term orderings on a polynomial ring are those defined by matrices. What is the connection here? In both cases there is a matrix of integers which plays a key role. Comparison of terms is based on taking scalar products of the rows of this matrix with logarithms of terms. For instance, the standard degree of a term $t \in \mathbb{T}^n$ is given by the product $\deg(t) = W \cdot \log(t)^{\mathrm{tr}}$, where W is the matrix $W = (1\ 1\ \cdots\ 1)$.

Thus we are led in a natural way to *connect the dots.* On the polynomial ring $P = K[x_1, \ldots, x_n]$ we define a $\mathbb{Z}^m$-grading by means of a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$. The degrees of the indeterminates are the columns of W, and constant polynomials are homogeneous of degree $(0, \ldots, 0)^{\mathrm{tr}}$. So, what are the applications of such gradings? A first answer comes from Proposition 4.1.8 which yields a characterization of monomial ideals as the *most homogeneous* ones, since they are the only ideals which are homogeneous with respect to every grading. But in order to be able to answer the question more satisfactorily, we need to examine the basic properties of gradings by matrices and to generalize this notion to include graded modules.

Do we really have to take the long and twisty road of modules again? Absolutely yes! As we frequently saw in Volume 1, and as is becoming increasingly clear, Computational Commutative Algebra is really about computations in and with modules. This aspect is taken care of in Subsection B, where we show that the most important operations on graded modules respect the grading. For actual computations, arbitrary gradings by matrices are too general. Therefore in the last subsection we restrict our attention further and turn to the subject of *gradings of positive type.* These are the gradings defined by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ such that some linear combination of its rows with integer coefficients has all entries positive (see Definition 4.1.17).

Why are they called of positive type, and why are they important? Concerning these questions, we offer you two pieces of good news and one piece of bad news. The good news is that, under a grading of positive type, the vector space of all elements of any given degree has finite dimension over K (see Proposition 4.1.19). Moreover, the Graded Version of Nakayama's Lemma 1.7.15 holds for such gradings, so that irredundant systems of generators are minimal (see Proposition 4.1.22). Hence gradings of positive type enjoy two of the most useful properties we can hope for. So, what is the bad news? Do you really want the bad news right now? Let's postpone it to a subsequent section and be happy for a while!

4.1.A Gradings by Matrices

Having introduced the general notions of graded rings and modules in Section 1.7, we are now going to concentrate on certain special kinds of graded object. We are interested in gradings on polynomial rings which have two properties: the indeterminates are homogeneous, and the constants are homogeneous of degree zero.

In what follows, we let K be a field, $n \geq 1$, and $P = K[x_1, \dots, x_n]$ a polynomial ring over K. Furthermore, we let $(\Gamma, +)$ be a monoid whose identity element is denoted by 0. As in Volume 1, we assume that Γ is a commutative monoid.

Recall that in Section 1.7 we defined a Γ-grading on P to be a decomposition $P = \bigoplus_{\gamma \in \Gamma} P_\gamma$ into additive subgroups such that $P_\gamma \cdot P_{\gamma'} \subseteq P_{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. A P-module M will be called a **graded P-module** if it is a Γ-graded P-module in the sense of Definition 1.7.4, i.e. if we have a decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ into additive subgroups such that $P_\gamma \cdot M_{\gamma'} \subseteq M_{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. In other words, a graded P-module is understood to be graded over the same monoid. By Proposition 1.7.10, a finitely generated module is graded if and only if it has a finite system of generators consisting of homogeneous elements. In this case, as in Volume 1, we shall say that M has a finite homogeneous system (or set) of generators.

Proposition 4.1.1. *Let Γ be a monoid, and let $\gamma_1, \dots, \gamma_n \in \Gamma$.*

a) There exists exactly one Γ-grading on P such that the non-zero constant polynomials are homogeneous of degree 0 and, for $i = 1, \dots, n$, the indeterminate x_i is homogeneous of degree γ_i.

b) Under this grading, the set $\{\gamma \in \Gamma \mid P_\gamma \neq 0\}$ is the submonoid of Γ generated by $\{\gamma_1, \dots, \gamma_n\}$.

Proof. To prove a), let us show existence first. For $\alpha_1, \dots, \alpha_n \in \mathbb{N}$ and $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, we let t be homogeneous of degree $\deg(t) = \alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n$. More generally, given $\gamma \in \Gamma$, we say that a polynomial $f \in P$ is homogeneous of degree γ if all terms in its support are homogeneous of degree γ. This means that $P_\gamma = \bigoplus_{\alpha_1\gamma_1+\cdots+\alpha_n\gamma_n=\gamma} K\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for all $\gamma \in \Gamma$. Then we have

$$P = \bigoplus_{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n} K\, x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \bigoplus_{\gamma\in\Gamma}\left(\bigoplus_{\alpha_1\gamma_1+\cdots+\alpha_n\gamma_n=\gamma} K\, x_1^{\alpha_1}\cdots x_n^{\alpha_n}\right) = \bigoplus_{\gamma\in\Gamma} P_\gamma$$

and it is clear that $P_\gamma \cdot P_{\gamma'} \subseteq P_{\gamma+\gamma'}$ for all $\gamma,\gamma' \in \Gamma$. Thus we have defined a Γ-grading on P which has the desired properties.

Now we show uniqueness. Given any Γ-grading on P with the stated properties, the conditions $\deg(x_1) = \gamma_1,\dots,\deg(x_n) = \gamma_n$ imply that $\deg(x_1^{\alpha_1}\cdots x_n^{\alpha_n}) = \alpha_1\gamma_1+\cdots+\alpha_n\gamma_n$ for $\alpha_1,\dots,\alpha_n \in \mathbb{N}$. So, in fact, the above definition is forced upon us by the rules satisfied by a Γ-grading, i.e. we have proved uniqueness.

The proof of b) follows from the observation that, for any $\gamma \in \Gamma$, the vector space P_γ is generated by the terms of degree γ. The degrees of those terms are contained in the submonoid of Γ generated by the set $\{\gamma_1,\dots,\gamma_n\}$. Conversely, every degree in this submonoid is the degree of a term because P is an integral domain. □

Even in the case $\Gamma = \mathbb{Z}$, the gradings on P provided by this proposition are still very general, as the following example shows.

Example 4.1.2. Let the polynomial ring $P = K[x_1,x_2]$ be equipped with the $\mathbb{Z}$-grading defined by $K \subseteq P_0$, $x_1 \in P_{-1}$, and $x_2 \in P_1$. Then we have $x_1x_2 \in P_0$, and hence $K \subset P_0$. In fact, it is easy to see that $\dim_K(P_0) = \infty$.

Our main interest will be in gradings for which all homogeneous components are finite dimensional K-vector spaces. The following definition generalizes Example 1.7.2 and is a case in point.

Definition 4.1.3. A K-algebra R is called a **standard graded K-algebra** if it is $\mathbb{N}$-graded, satisfies $R_0 = K$ and $\dim_K(R_1) < \infty$, and if R is generated by the elements of R_1 as a K-algebra.

Standard graded algebras are characterized by the existence of special presentations as follows.

Remark 4.1.4. Using Corollary 1.1.14 and Remark 1.7.9, we see that a standard graded K-algebra R is an algebra of the form $R \cong P/I$, where $P = K[x_1,\dots,x_n]$ is $\mathbb{N}$-graded such that $K = P_0$, each x_i is homogeneous of degree one, and I is a homogeneous ideal in P. Conversely, every algebra of this form P/I is a standard graded K-algebra.

In this and the following chapters, standard graded K-algebras will play an important role. But not every $\mathbb{N}$-graded, finitely generated K-algebra is standard graded.

Example 4.1.5. Let $P = K[x_1,x_2]$ be equipped with the standard grading. Then the K-subalgebra $S = K[x_1^2, x_1x_2, x_2^2]$ of P is a finitely generated $\mathbb{N}$-graded algebra, but it is not standard graded, since $S_1 = \{0\}$.

To bring all the graded algebras which we want to examine in this chapter under one umbrella requires a concept of gradings whose level of generality is somewhere between the very wide class of gradings considered in Proposition 4.1.1 and the rather limited class of standard graded K-algebras. For our purposes, the following concept will prove most useful.

Definition 4.1.6. Let $m \geq 1$, and let the polynomial ring $P = K[x_1, \dots, x_n]$ be equipped with a $\mathbb{Z}^m$-grading such that $K \subseteq P_0$ and $x_1, \dots, x_n$ are homogeneous elements.

a) For $j = 1, \dots, n$, let $(w_{1j}, \dots, w_{mj}) \in \mathbb{Z}^m$ be the degree of x_j. The matrix $W = (w_{ij}) \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ is called the **degree matrix** of the grading. In other words, the columns of the degree matrix are the degrees of the indeterminates. The rows of the degree matrix are called the **weight vectors** of the indeterminates $x_1, \dots, x_n$.
b) Conversely, given a matrix $W = (w_{ij}) \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, we can consider the $\mathbb{Z}^m$-grading on P for which $K \subseteq P_0$ and the indeterminates are homogeneous elements whose degrees are given by the columns of W. In this case, we say that P is **graded by** W.
c) Let $d \in \mathbb{Z}^m$. The set of homogeneous polynomials of degree d is denoted by $P_{W,d}$, or simply by P_d if it is clear which grading we are considering. A polynomial $f \in P_{W,d}$ is also called **homogeneous of degree** d, and we write $\deg_W(f) = d$.

To aid the reader in understanding our notation, we point out that a degree is a vector in $\mathbb{Z}^m$. If we use it in matrix equations, its representation is a column. Sometimes we also denote it by a row if no confusion arises. If a grading on P is defined by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, the degree of a term $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is given by $\deg_W(t) = W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}}$. So, we have $\{d \in \mathbb{Z}^m \mid P_{W,d} \neq 0\} = \{W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$.

Example 4.1.7. Let $P = K[x_1, x_2, x_3, x_4]$ be graded by the matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

and let $f = x_1x_4 - x_2x_3$. Then f is homogeneous of degree $(2,1,1)$, because $W \cdot \log(x_1x_4)^{\mathrm{tr}} = W \cdot \log(x_2x_3)^{\mathrm{tr}} = (2,1,1)^{\mathrm{tr}}$. The principal ideal generated by f is a homogeneous ideal by Proposition 1.7.10.

A first non-trivial example of a graded object is given by the following characterization of monomial ideals as the "most homogeneous" ideals. Recall that a square matrix is called non-singular if its determinant is non-zero.

Proposition 4.1.8. *Let I be an ideal of P. Then the following conditions are equivalent.*

a) *The ideal I is monomial.*
b) *There is a non-singular matrix $W \in \mathrm{Mat}_n(\mathbb{Z})$ such that I is homogeneous with respect to the grading on P given by W.*
c) *For every $m \geq 1$ and every matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, the ideal I is homogeneous with respect to the grading on P given by W.*

Proof. Since I is generated by terms, and terms are homogeneous with respect to the gradings we are considering, Proposition 1.7.10 shows that a) implies c). Obviously, b) is a special case of c). Therefore it suffices to show that b) implies a).

We take a homogeneous polynomial $f \in I_{W,d}$ and show that there is only one term in its support. Let $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $t' = x_1^{\beta_1} \cdots x_n^{\beta_n}$ be terms in the support of f. Then $d = \deg_W(t) = \deg_W(t')$ implies $W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}} = W \cdot (\beta_1, \dots, \beta_n)^{\mathrm{tr}}$. Since we have $\det(W) \neq 0$, the $\mathbb{Z}$-linear map defined by W is injective. Therefore we obtain $(\alpha_1, \dots, \alpha_n) = (\beta_1, \dots, \beta_n)$, and hence $t = t'$. $\square$

In general, this proposition does not hold for monomial submodules of graded free P-modules (see Exercises 5 and 6). If two matrices in $\mathrm{Mat}_{m,n}(\mathbb{Z})$ can be transformed into each other by elementary row operations, the relation between the corresponding gradings on P is fairly simple to understand.

Proposition 4.1.9. *Let $m \geq 1$, let $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let $V \in \mathrm{Mat}_{\ell,m}(\mathbb{Z})$ for some $\ell \geq 1$.*

a) *The gradings on P given by W and $V \cdot W$ are related by $P_{W,d} \subseteq P_{V\cdot W, V\cdot d}$ for all $d \in \mathbb{Z}^m$. In particular, the map $(\mathrm{id}_P, \psi) : (P, \mathbb{Z}^m) \longrightarrow (P, \mathbb{Z}^\ell)$, where $\psi : \mathbb{Z}^m \longrightarrow \mathbb{Z}^\ell$ is the left multiplication by V, is a homomorphism of graded rings in the sense of Definition 1.7.7.*
b) *If ψ is injective, then we have $P_{W,d} = P_{V\cdot W, V\cdot d}$ for all $d \in \mathbb{Z}^m$.*

Proof. First we prove a). Since P is generated by $\mathbb{T}^n$ as a K-vector space, and since terms are homogeneous, it suffices to prove the claim for a term $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in P_{W,d}$. In this case we have $\deg_W(t) = d$, and therefore $\deg_{V\cdot W}(t) = V \cdot W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}} = V \cdot d$.

Now we prove b). Let $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in P_{V\cdot W, V\cdot d}$ for some $d \in \mathbb{Z}^m$. Then $\deg_{V\cdot W}(t) = V \cdot d$ implies $V \cdot W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}} = V \cdot d$. By hypothesis, ψ is injective, so we get $W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}} = d$, i.e. we have $\deg_W(t) = d$. Thus the inclusion $P_{W,d} \subseteq P_{V\cdot W, V\cdot d}$ is in fact an equality. $\square$

Let us apply this proposition in the setting of Example 4.1.7.

Example 4.1.10. Let $P = K[x_1, x_2, x_3, x_4]$ be graded by $W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. First we want to determine the homogeneous component $P_{W,d}$ of degree $d = (2, 1, 1)$.

By definition, this vector space is generated by the terms $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that $W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}} = d$. The set of solutions in $\mathbb{N}^n$ of this system of equations is $\{(1,0,0,1), (0,1,1,0)\}$. Thus we have $P_{W,d} = K \cdot x_1 x_4 \oplus K \cdot x_2 x_3$.

Now let w_1, w_2, w_3 be the rows of W. We replace w_2 by $w_1 + w_2$ and w_3 by $w_1 + w_3$, i.e. we form the matrix $V \cdot W$, where $V = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Then we get $V \cdot d = (2,3,3)^{\mathrm{tr}}$, and therefore the homogeneous component of P of degree $e = (2,3,3)$ with respect to the grading defined by $V \cdot W$ is given by $P_{V \cdot W, e} = K \cdot x_1 x_4 \oplus K \cdot x_2 x_3$.

4.1.B Graded Modules

Here we take a look at graded modules over graded polynomial rings. Let a $\mathbb{Z}^m$-grading on the polynomial ring $P = K[x_1, \ldots, x_n]$ be defined by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let $\delta_1, \ldots, \delta_r \in \mathbb{Z}^m$. In Section 1.7 we defined a very general notion of graded P-modules. In the following we consider modules graded over the $\mathbb{Z}^m$-monomodule $\mathbb{Z}^m$, i.e. we use the same monoid that we used for the grading of P. Thus a graded P-module has a decomposition $M = \bigoplus_{d \in \mathbb{Z}^m} M_d$ and we have $P_d \cdot M_{d'} \subseteq M_{d+d'}$ for all $d, d' \in \mathbb{Z}^m$. We shall now see that the usual ideal-theoretic and module-theoretic operations again produce homogeneous ideals and graded modules when they are applied to such objects.

Proposition 4.1.11. *Let P be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let $I \subseteq P$ be a homogeneous ideal, let U be a graded P-module, and let M and N be graded submodules of U.*

- *a) The sum $M + N$ and the intersection $M \cap N$ are graded submodules of U.*
- *b) The colon ideal $N :_P M$ is a homogeneous ideal in P.*
- *c) The ideal $\mathrm{Ann}_P(M)$ is a homogeneous ideal in P.*
- *d) The colon module $N :_M I$ and the saturation $N :_M I^\infty$ are graded submodules of U.*
- *e) The radical ideal $\sqrt{I}$ is a homogeneous ideal in P.*

Proof. The proof of a) is straightforward, so let us proceed to b). Clearly, the claim is true if $N :_P M = (0)$. Let $f \in P$ be a non-zero polynomial such that $f \cdot M \subseteq N$, and let $f = \sum_{d \in \mathbb{Z}^m} f_d$ be its decomposition into homogeneous components. For every homogeneous vector $v \in M$, we have $fv = \bigoplus_{d \in \mathbb{Z}^m} f_d v \in N$. This sum is the decomposition of fv into homogeneous components. Since N is a graded module, we get $f_d v \in N$ for all $d \in \mathbb{Z}^m$. Hence we see that $f_d \in N :_P M$ for all $d \in \mathbb{Z}^m$, as we wanted to show. Claim c) is a special case of b), and d) follows in a similar way.

To prove e), we write $f \in \sqrt{I}$ as the sum of its homogeneous components $f = f_{d_1} + \cdots + f_{d_\ell}$, where $d_1 >_{\mathtt{Lex}} d_2 >_{\mathtt{Lex}} \cdots >_{\mathtt{Lex}} d_\ell$. We shall show that $f - f_{d_1} \in \sqrt{I}$. Then the claim follows by induction on ℓ. Let $i > 0$ be a number such that $f^i \in I$. When we expand this power, we obtain a sum of homogeneous polynomials among which $f_{d_1}^i$ has the largest degree with respect to `Lex`. Hence $f_{d_1}^i$ is the homogeneous component of degree $i \cdot d_1$

of f^i, and the fact that I is a homogeneous ideal implies $f_{d_1}^i \in I$. Thus we get $f_{d_1} \in \sqrt{I}$, as desired. $\square$

Corollary 4.1.12. *Let I be a monomial ideal in P. Then the radical $\sqrt{I}$ is the monomial ideal generated by the squarefree parts of the terms in the minimal monomial system of generators of I.*

Proof. Let $W \in \mathrm{Mat}_n(\mathbb{Z})$ be a non-singular matrix. By Proposition 4.1.8, the ideal I is homogeneous with respect to the grading given by W. Then part e) of the proposition says that $\sqrt{I}$ is homogeneous, too. Now another application of Proposition 4.1.8 shows that $\sqrt{I}$ is a monomial ideal. Clearly, if a term $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is one of the minimal monomial generators of I, then its squarefree part $\prod_{\{i \mid \alpha_i > 0\}} x_i$ is contained in $\sqrt{I}$. On the other hand, every term in $\sqrt{I}$ has a power which is a multiple of one of the minimal monomial generators of I, and therefore the term is a multiple of one of those squarefree parts. $\square$

As usual, we want to relate different graded modules to each other using suitable homomorphisms. Recall that in Definition 1.7.7 we introduced homomorphisms of rings graded by monoids, and of modules graded by monomodules. In the present situation, that definition specializes in the following way.

Let P be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let M, N be two $\mathbb{Z}^m$-graded P-modules. A P-linear map $\varphi : M \longrightarrow N$ is called a **homomorphism of graded modules** or a **homogeneous P-linear map** if $\varphi(M_d) \subseteq N_d$ for all $d \in \mathbb{Z}^m$. Important examples of such maps are constructed as follows.

Remark 4.1.13. Let P be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let M be a graded P-module. Given homogeneous elements $v_1, \ldots, v_r \in M$ with $\deg_W(v_i) = \delta_i \in \mathbb{Z}^m$ for $i = 1, \ldots, r$, the P-linear map $\varphi : F \longrightarrow M$ defined by $e_i \mapsto v_i$ for $i = 1, \ldots, r$ is a homomorphism of graded modules. We shall say that φ is the map **induced by** $(v_1, \ldots, v_r)$.

An easy way to construct graded modules comes from the observation that, given a homogeneous P-linear map $\lambda : M \longrightarrow N$ between $\mathbb{Z}^m$-graded P-modules, the kernel $\mathrm{Ker}(\lambda)$ is a graded submodule of M and the image $\mathrm{Im}(\lambda)$ is a graded submodule of N. Another kind of graded modules are graded submodules of graded free modules. Let us briefly recall the pertinent definitions.

According to Definition 1.7.6, the grading on P induces a $\mathbb{Z}^m$-grading on the graded free P-module $F = \bigoplus_{i=1}^r P(-\delta_i)$: this grading is given by

$$F_d = \bigoplus_{i=1}^r P_{W,\, d-\delta_i}$$

for all $d \in \mathbb{Z}^m$. Thus a term $te_i \in \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$, where $i \in \{1, \ldots, r\}$ and $t \in \mathbb{T}^n$, is a homogeneous element of F of degree $\deg_W(te_i) =$

$\deg_W(t) + \delta_i$. In particular, the module F is the graded free P-module such that $\deg_W(e_i) = \delta_i$ for $i = 1, \dots, r$.

Given a graded P-submodule M of F, we shall briefly say that M is **graded by** W. The homogeneous components of M will be denoted by $M_{W,d}$ for $d \in \mathbb{Z}^m$. Recall, from Definition 1.7.8, that a graded submodule of P is also called a **homogeneous ideal** of P.

Under the change of grading homomorphism considered in Proposition 4.1.9, a graded submodule of a graded free module is transformed into a graded submodule with respect to the new grading, as the following proposition shows.

Proposition 4.1.14. *Let P be graded by $W \in \operatorname{Mat}_{m,n}(\mathbb{Z})$, let $r \geq 1$, let $\delta_1, \dots, \delta_r \in \mathbb{Z}^m$, and let M be a graded submodule of F. Moreover, let $V \in \operatorname{Mat}_{\ell,m}(\mathbb{Z})$ for some $\ell \geq 1$. Then M is also a graded submodule of $\bigoplus_{i=1}^r P(-V \cdot \delta_i)$ with respect to the grading defined by $V \cdot W$.*

Proof. Using Proposition 4.1.9 and the definition of the grading on F, we see that

$$F_{W,d} = \bigoplus_{i=1}^{r} P_{W,\, d-\delta_i} \subseteq \bigoplus_{i=1}^{r} P_{V\cdot W,\, V\cdot d - V\cdot\delta_i} = (\bigoplus_{i=1}^{r} P(-V \cdot \delta_i))_{V\cdot W,\, V\cdot d}$$

for all $d \in \mathbb{Z}^m$. Since a system of generators of M consisting of homogeneous elements with respect to the grading defined by W is also homogeneous with respect to the grading defined by $V \cdot W$, the claim follows. □

This proposition has a number of useful consequences.

Remark 4.1.15. Suppose we are in the setting of the proposition.

a) Let $V = (a_1\ a_2\ \cdots\ a_m) \in \operatorname{Mat}_{1,m}(\mathbb{Z})$, and let us form the linear combination $V \cdot W$ of the rows $w_1, \dots, w_m$ of W. Then M is a graded module with respect to the grading defined by the $1 \times n$-matrix $V \cdot W$.
b) Let $a \in \mathbb{Z} \setminus \{0\}$. Then M is a graded submodule of F with respect to the grading given by W if and only if it is a graded submodule with respect to the grading given by $a\,W$. This follows from the observation that the inclusion $F_{W,d} \subseteq F_{aW,ad}$ given by the corollary is, in fact, an equality.

Corollary 4.1.16. *Let $W \neq 0$ have $\mathbb{Z}$-linearly dependent rows, and let W' be a submatrix of W which consists of a maximal linearly independent set of rows. Denote the number of rows of W' by m', and let $V \in \operatorname{Mat}_{m,m'}(\mathbb{Z})$ be the matrix such that $V \cdot W' = W$. Then we have $P_{W',d'} = P_{W,V\cdot d'}$ for all $d' \in \mathbb{Z}^{m'}$.*

Proof. It suffices to show that the inclusion $P_{W',d'} \subseteq P_{V\cdot W', V\cdot d'}$ given by the proposition is an equality. Let $t \in \mathbb{T}^n$ be a term of degree $V \cdot d' = W \cdot \log(t)^{\mathrm{tr}}$. Now, the linear map defined by V is injective, so $d' = W' \cdot \log(t)^{\mathrm{tr}}$. Hence t is homogeneous of degree d' with respect to the grading given by W', and the above inclusion is indeed an equality. □

In view of this corollary, we shall, from now on, assume tacitly that the matrix W has $\mathbb{Z}$-linearly independent rows, unless we explicitly say something else.

4.1.C Gradings of Positive Type

Our first goal in this subsection is to find hypotheses which force the homogeneous components of P to be finite dimensional K-vector spaces. For instance, this is the case for the grading defined by $W = (1\ 1\ \cdots\ 1)$. Moreover, Proposition 4.1.9.a shows that it is then also the case for every grading defined by a matrix whose first row is $(1\ 1\ \cdots\ 1)$. Inspired by this observation, we introduce the following notions.

Definition 4.1.17. Let $m \geq 1$, let P be graded by a matrix W of rank m in $\mathrm{Mat}_{m,n}(\mathbb{Z})$, and let $w_1, \ldots, w_m$ be the rows of W.

a) The grading on P given by W is called **of non-negative type** if there exist $a_1, \ldots, a_m \in \mathbb{Z}$ such that the entries of $v = a_1 w_1 + \cdots + a_m w_m$ corresponding to the non-zero columns of W are positive. In this case, we shall also say that W is a matrix of non-negative type.
b) We say that the grading on P given by W is **of positive type** if there exist $a_1, \ldots, a_m \in \mathbb{Z}$ such that all entries of $a_1 w_1 + \cdots + a_m w_m$ are positive. In this case, we shall also say that W is a matrix of positive type.

For instance, the matrix $W = (-1\ -1\ \ 0)$ is of non-negative type, the matrix $W' = (-1\ -1)$ is of positive type, but $W'' = (1\ -1)$ is neither. Gradings of non-negative type are intimately connected to $\mathbb{N}$-gradings, as the following proposition shows.

Proposition 4.1.18. *Let P be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ of non-negative type, let $\Gamma = \{d \in \mathbb{Z}^m \mid P_{W,d} \neq 0\}$, and suppose that $V = (a_1\ a_2\ \cdots\ a_m) \in \mathrm{Mat}_{1,m}(\mathbb{Z})$ is such that $V \cdot W = a_1 w_1 + \cdots + a_m w_m$ has all entries non-negative.*

a) The map $(\mathrm{id}_P, \psi) : (P, \mathbb{Z}^m) \longrightarrow (P, \mathbb{Z})$, where $\psi : \mathbb{Z}^m \longrightarrow \mathbb{Z}$ is left multiplication by V, is a homomorphism of graded rings. It satisfies $\psi(\Gamma) \subseteq \mathbb{N}$.
b) There exists a matrix $U \in \mathrm{Mat}_m(\mathbb{Z})$ such that $\det(U) \neq 0$ and the non-zero columns of $U \cdot W$ have all entries positive.
c) If W is of positive type, there exists a matrix $U \in \mathrm{Mat}_m(\mathbb{Z})$ such that $\det(U) \neq 0$ and $U \cdot W$ has positive entries only.

Proof. The first part of claim a) follows from Proposition 4.1.9.a, while the second part is a consequence of the fact that $V \cdot W$ has non-negative entries only.

To prove b), we note that if $a_i \neq 0$ for some $i \in \{1, \dots, m\}$, then we can replace the i^{th} row of W by $a_1 w_1 + \cdots + a_m w_m$ using a suitable transformation given by a matrix $U_1 \in \mathrm{Mat}_m(\mathbb{Z})$. Next we multiply by a permutation matrix $U_2 \in \mathrm{Mat}_m(\mathbb{Z})$ such that this row becomes the first row. Finally, by adding sufficiently high multiples of this new first row to the other rows, we can make all entries in the non-zero columns of W positive. Let $U_3 \in \mathrm{Mat}_m(\mathbb{Z})$ correspond to those row operations. Then $U = U_3 \cdot U_2 \cdot U_1$ is the desired matrix.

Finally we note that c) follows from b), because a matrix of positive type has all columns non-zero. □

Now we are ready to show that polynomial rings with gradings of positive type and finitely generated graded modules over them have finite dimensional homogeneous components.

Proposition 4.1.19. *Let P be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ of positive type, and let M be a finitely generated graded P-module.*

a) We have $P_0 = K$.
b) For all $d \in \mathbb{Z}^m$, we have $\dim_K(M_d) < \infty$.

Proof. First we show a). Let $V = (a_1 \; a_2 \; \cdots \; a_m) \in \mathrm{Mat}_{1,m}(\mathbb{Z})$ be such that $V \cdot W$ has positive entries only. Using Proposition 4.1.9, we see that $P_{W,0} \subseteq P_{V \cdot W,0}$. Now it suffices to note that every term $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \neq 1$ has positive degree $\deg_{V \cdot W}(t) = V \cdot W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}} > 0$.

In order to prove b), we choose a finite homogeneous system of generators of M and consider the corresponding representation $M \cong F/N$ where N is a graded submodule of F.

Clearly, it suffices to prove the claim for F. We do this by showing it is true for each $P(-\delta_i)$. Since $P(-\delta_i)_d = P_{d-\delta_i}$, it suffices to prove that $\dim_K(P_d) < \infty$ for all $d \in \mathbb{Z}^m$. Since W is of positive type, there exists a matrix $V \in \mathrm{Mat}_{1,m}(\mathbb{Z})$ such that $V \cdot W$ has all entries positive. We use Proposition 4.1.9.a to see that $P_{W,d} \subseteq P_{V \cdot W, V \cdot d}$. Hence we only have to show that the K-vector spaces $P_{V \cdot W,i}$ are finite dimensional for all $i \in \mathbb{Z}$. Their vector space bases $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid V \cdot W \cdot (\alpha_1, \dots, \alpha_n)^{\mathrm{tr}} = i\}$ are finite, because $V \cdot W$ has positive entries only. □

A further advantage of considering finitely generated graded P-modules in the case of gradings of positive type is that Nakayama's Lemma applies to them. In Corollary 1.7.16 we used the notion of a minimal system of generators of M, and in Corollary 3.1.12 we mentioned irredundant systems of generators. Let us give the precise definitions.

Definition 4.1.20. Let R be a ring and M a finitely generated R-module.

a) A finite system of generators of M is called a **minimal system of generators** if its number of elements is minimal among all systems of generators of M.

b) A system of generators of M is called an **irredundant system of generators** if no proper subset generates M.

Minimal systems of generators are irredundant. Over arbitrary rings, the two notions do not coincide. For instance, when $R = \mathbb{Z}$ and M is the ideal generated by $\{2\}$, the system of generators $\{4,6\}$ is irredundant, but not minimal. One of the most important consequences of Nakayama's Lemma is that, in the case of gradings of positive type, irredundant systems of homogeneous generators of finitely generated graded modules are minimal. This will be shown in Proposition 4.1.22.

To formulate our next result, we need two additional objects. Let P be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let $M \neq 0$ be a graded P-module. Then the set $\Gamma = \{d \in \mathbb{Z}^m \mid P_d \neq 0\}$ is clearly a submonoid of $\mathbb{Z}^m$, and we can define the Γ-submonomodule Σ of $\mathbb{Z}^m$ generated by $\{d \in \mathbb{Z}^m \mid M_d \neq 0\}$. It is easy to see that $\Sigma = \{d \in \mathbb{Z}^m \mid M_d \neq 0\}$ if M is a submodule of a graded free P-module. If the grading on P is of non-negative type, these monomodules are well-ordered, as the following proposition shows.

Proposition 4.1.21. *Let P be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ of non-negative type.*

a) There exists a monoid ordering τ on $\mathbb{Z}^m$ such that the restriction of τ to Γ is a well-ordering.
b) For every finitely generated, graded P-module M, the restriction of τ to the monomodule Σ is a well-ordering.
c) If W is of positive type, there exists a monoid ordering τ on Γ which is a well-ordering and for which the set $P_+ = \bigoplus_{d >_\tau 0} P_d$ is the ideal generated by $\{x_1, \dots, x_n\}$.

Proof. First we show a). Using Proposition 4.1.18.b, we find $U \in \mathrm{Mat}_m(\mathbb{Z})$ such that $\det(U) \neq 0$ and all non-zero columns of $U \cdot W$ have positive entries only. For vectors $d, d' \in \mathbb{Z}^m$, we define $d \geq_\tau d'$ if and only if $U \cdot d \geq_{\mathtt{Lex}} U \cdot d'$. Clearly, this rule specifies a monoid ordering τ on $\mathbb{Z}^m$. Its restriction to the submonoid Γ is still a monoid ordering. It remains to show that $\tau|_\Gamma$ is a well-ordering on Γ.

Suppose that $d_1 >_\tau d_2 >_\tau \cdots$ is an infinite descending chain of elements of Γ. Then, by definition, we have $U \cdot d_1 >_{\mathtt{Lex}} U \cdot d_2 >_{\mathtt{Lex}} \cdots$. For every $d_i \in \Gamma$, there exists a term $t_i = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$ such that $d_i = W \cdot (\alpha_{i1}, \dots, \alpha_{in})^{\mathrm{tr}}$. Therefore we have $U \cdot d_i = U \cdot W \cdot (\alpha_{i1}, \dots, \alpha_{in})^{\mathrm{tr}}$, and this vector has all entries non-negative. Since $\mathtt{Lex}$ is a term ordering on $\mathbb{N}^n$, Proposition 1.4.18 and Theorem 1.4.19 imply that an infinite chain $U \cdot d_1 >_{\mathtt{Lex}} U \cdot d_2 >_{\mathtt{Lex}} \cdots$ does not exist.

To prove b), we let $\{m_1, \dots, m_s\} \subseteq M \setminus \{0\}$ be a homogeneous system of generators of M and define $\gamma_i = \deg(m_i)$ for $i = 1, \dots, s$. By Corollary 1.7.11, we have $\Sigma \subseteq \cup_{i=1}^s (\Gamma + \gamma_i)$. Let σ be the restriction of τ to the Γ-submonomodule Σ of $\mathbb{Z}^m$. Clearly, σ is a module ordering which is compatible with τ.

Now we show that the restriction of τ to $\cup_{i=1}^{s}(\Gamma+\gamma_i)$ is a well-ordering. Suppose there is an infinite descending chain $d_1 >_\tau d_2 >_\tau \cdots$ in $\cup_{i=1}^{s}(\Gamma+\gamma_i)$. Then one of the sets $\Gamma+\gamma_i$ has to contain infinitely many degrees d_j, i.e. there exist indices $j_1 < j_2 < \cdots$ such that $d_{j_1} >_\tau d_{j_2} >_\tau \cdots$ is a chain in $\Gamma+\gamma_i$. Now the chain $d_{j_1}-\gamma_i >_\tau d_{j_2}-\gamma_i >_\tau \cdots$ contradicts the fact that τ is a well-ordering on Γ.

Finally, we note that c) follows from the observation that the well-ordering τ constructed in the proof of a) satisfies $\deg_W(x_i) >_\tau 0$, since $\deg_{U\cdot W}(x_i) >_{\mathtt{Lex}} 0$ for $i=1,\ldots,n$. $\square$

Using this proposition, we see that the hypotheses of the Graded Version of Nakayama's Lemma 1.7.15 are satisfied for gradings of non-negative type. If the grading is actually of positive type, we obtain the result we strived for.

Proposition 4.1.22. *Let P be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ of positive type, and let $M \neq 0$ be a finitely generated graded P-module.*

a) A set of homogeneous elements $m_1,\ldots,m_s$ generates the P-module M if and only if their residue classes $\overline{m}_1,\ldots,\overline{m}_s$ generate the K-vector space $M/(x_1,\ldots,x_n)M$.

b) Every homogeneous system of generators of M contains a minimal one. All irredundant systems of homogeneous generators of M are minimal and have the same number of elements.

Proof. By Proposition 4.1.21.c, there exists a well-ordering τ on Γ such that $P_+ = \bigoplus_{d>_\tau 0} P_d = (x_1,\ldots,x_n)$, and therefore $P/P_+ \cong K$. Hence a) follows from Corollary 1.7.16.a. Now we prove b). Since $P_0 = K$ is a field, Corollary 1.7.16.b shows that every homogeneous system of generators of M contains a subset which is minimal among the homogeneous systems of generators of M and whose residue classes form a K-basis of $M/(x_1,\ldots,x_n)M$. This subset is also minimal among all systems of generators of M because, for any set of generators of M, their set of residue classes generates $M/(x_1,\ldots,x_n)M$. $\square$

This proposition is not true in general if W is of non-negative type.

Example 4.1.23. Let $P = \mathbb{Q}[x,y]$ be graded by the matrix $W = (0\ 1)$, and let $I = (xy,\, y-xy)$. Then W is of non-negative type, I is a homogeneous ideal, and $\{xy,\, y-xy\}$ is an irredundant homogeneous system of generators of I. However, since $I=(y)$, this system of generators is not minimal. Notice that we have $P_+ = (y)$ and $P/P_+ \cong K[x]$ here.

Exercise 1. Let K be a field, let Γ be a monoid, let R be a Γ-graded K-algebra for which $K \subseteq R_0$, and let $S \subset R$ be a K-subalgebra which is generated, as a K-algebra, by homogeneous elements. Show that if we define $S_\gamma = S \cap R_\gamma$ for all $\gamma \in \Gamma$, then S is a Γ-graded K-subalgebra of R.

Exercise 2. Let K be a field, and let $P = K[x_1, x_2]$ be equipped with the standard grading. In Example 4.1.5 we have seen that the ring $S = K[x_1^2, x_1x_2, x_2^2]$ is a graded K-subalgebra of P but is not a standard graded K-algebra. On the other hand, it is easy to see (by hand or by using the CoCoA procedure `Elim`) that $S \cong K[y_1, y_2, y_3]/(y_1y_3 - y_2^2)$. If we endow $K[y_1, y_2, y_3]$ with the standard grading, the ideal $(y_1y_3 - y_2^2)$ is homogeneous and S is a standard graded K-algebra. Explain the apparent contradiction.

Exercise 3. Let $P = K[x_1, \ldots, x_n]$ be standard graded. For a polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in P$ and for $i \in \{1, \ldots, n\}$, we define

$$\frac{\partial f}{\partial x_i} = \sum_{\alpha \in \mathbb{N}^n} \alpha_i \, c_\alpha \, x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} \, x_i^{\alpha_i - 1} \, x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n}$$

and call it the **partial derivative** of f by x_i.

a) For $f, g \in P$ and $i \in \{1, \ldots, n\}$, show that $\frac{\partial (f+g)}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}$.
b) Let $d \geq 0$, and let $f \in P_d$ be a homogeneous polynomial of degree d. Prove **Euler's formula** $d\,f = \sum_{i=1}^n x_i \cdot \frac{\partial f}{\partial x_i}$.

Exercise 4. Let $P = K[x_1, \ldots, x_n]$ be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let $w_1, \ldots, w_m$ be the rows of W, and let $a_1, \ldots, a_m \in \mathbb{Z}$. Show that the pair (id_P, ψ), where $\psi : \mathbb{Z}^m \longrightarrow \mathbb{Z}$ is defined by $\psi(e_i) = a_i$ for $i = 1, \ldots, m$, is a homomorphism of graded rings.

Exercise 5. Let K be a field, let $P = K[x_1, \ldots, x_n]$ be graded by a non-singular matrix $W \in \mathrm{Mat}_n(\mathbb{Z})$, and let M be a graded submodule of a graded free P-module $F = \oplus_{i=1}^r P(-\delta_i)$, where $\delta_1, \ldots, \delta_r \in \mathbb{Z}^n$. Prove that M is generated by vectors of the form $v = (c_1t_1, \ldots, c_rt_r)$, where $c_1, \ldots, c_r \in K$ and $t_1, \ldots, t_r \in \mathbb{T}^n$ are terms with the property that $\deg_W(t_1) + \delta_1 = \cdots = \deg_W(t_r) + \delta_r$.

Exercise 6. Let $P = K[x_1, \ldots, x_n]$, let $r > 0$, and let M be a P-submodule of P^r. Show that the following conditions are equivalent.

a) The module M is a monomial module.
b) For every $m \geq 1$ and every matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and for all vectors $\delta_1, \ldots, \delta_r \in \mathbb{Z}^m$, the module M is a graded submodule of $\bigoplus_{i=1}^r P(-\delta_i)$ with respect to the grading on P given by W.
c) For some non-singular matrix $W \in \mathrm{Mat}_n(\mathbb{Z})$ and for all vectors $\delta_1, \ldots, \delta_r \in \mathbb{Z}^m$, the module M is a graded submodule of $\bigoplus_{i=1}^r P(-\delta_i)$ with respect to the grading on P given by W.

Furthermore, give an example which shows that a) is not equivalent to the following condition.

d) For every $m \geq 1$ and every matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and for some vectors $\delta_1, \ldots, \delta_r \in \mathbb{Z}^m$, the module M is a graded submodule of $\bigoplus_{i=1}^r P(-\delta_i)$ with respect to the grading on P given by W.

Exercise 7. Let K be a field, and let $P = K[x_1, \ldots, x_n]$ be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$.

a) Given an invertible matrix $V \in \mathrm{Mat}_m(\mathbb{Z})$, consider the homomorphism of graded rings $(\mathrm{id}_P, \varphi) : (P, \mathbb{Z}^m) \longrightarrow (P, \mathbb{Z}^m)$, where the ring on the right-hand side is graded by $V \cdot W$ and $\varphi : \mathbb{Z}^m \longrightarrow \mathbb{Z}^m$ is left multiplication by V. Show that (id_P, φ) is an isomorphism of graded rings, i.e. that there exists a homomorphism of graded rings $(\mathrm{id}_P, \psi) : (P, \mathbb{Z}^m) \longrightarrow (P, \mathbb{Z}^m)$ such that $\varphi \circ \psi = \psi \circ \varphi = \mathrm{id}_{\mathbb{Z}^m}$.
b) Find an example which shows that it is not sufficient in a) to assume that V is non-singular.
c) Can you find necessary and sufficient conditions for two matrices $W, W' \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ to give isomorphic gradings in the sense that there is an isomorphism of graded rings $(\mathrm{id}_P, \psi) : (P, \mathbb{Z}^m) \longrightarrow (P, \mathbb{Z}^m)$?

Exercise 8. Show that $W = \left(\begin{smallmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \end{smallmatrix}\right)$ is of positive type and that $W = \left(\begin{smallmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{smallmatrix}\right)$ is not of non-negative type.

Exercise 9. Let $K[x_1, \ldots, x_n]$ be graded by $W = (w_{ij}) \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let $f \in P$ be a homogeneous polynomial, and let $(a_1, \ldots, a_n) \in K^n$ such that $f(a_1, \ldots, a_n) = 0$.

a) For $m = 1$, show that $f(a_1 b^{w_{11}}, \ldots, a_n b^{w_{1n}}) = 0$ for all $b \in K$.
b) For $m = 2$, show that $f(a_1 b_1^{w_{11}} b_2^{w_{21}}, \ldots, a_n b_1^{w_{1n}} b_2^{w_{2n}}) = 0$ for all $b_1, b_2 \in K$.
c) Find and prove a generalization of a) and b) to arbitrary $m \geq 1$.

Exercise 10. Let $P = K[x]$ be graded by $W \in \mathrm{Mat}_{m,1}(\mathbb{Z}) \setminus \{0\}$. Show that $\dim_K(P_d) \leq 1$ for all $d \in \mathbb{Z}^m$.

Exercise 11. Let $P = K[x_1, \ldots, x_n]$ be graded by $W \in \mathrm{GL}_n(\mathbb{Z})$.

a) Show that $\dim_K(P_d) \leq 1$ for all $d \in \mathbb{Z}^m$.
b) Give an example where $P_d = 0$ for some $d \in \mathbb{Z}^m$.

Exercise 12. Let K be a field, and let $P = K[x_1, \ldots, x_n]$ be graded by a matrix $W \in \mathrm{Mat}_{n-1,n}(\mathbb{Z})$ of rank $n-1$. For each degree $d \in \mathbb{N}^{n-1}$, prove that there exists a line ℓ_d in $\mathbb{Q}^n$ for which $\ell_d \cap \mathbb{N}^n$ coincides with the set $\{\log(t) \mid t \in \mathbb{T}^n,\ \deg_W(t) = d\}$.

Exercise 13. Let $K[x_1, \ldots, x_n]$ be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and assume that $\dim_K(P_0) < \infty$. Prove that $P_0 = K$.

Exercise 14. Let K be a field, let $P = K[x_1, \ldots, x_n]$ be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ of positive type, let I be a homogeneous ideal in P, and let $A = P/I$. Moreover, let $\Sigma = \{d \in \mathbb{Z}^m \mid A_d \neq 0\}$, and let τ be a monoid ordering on $\mathbb{Z}^m$ whose restriction to Σ is a well-ordering (see Proposition 4.1.21). Finally, let $a_1, \ldots, a_s \in A$ be homogeneous elements such that $\deg_W(a_i) >_\tau 0$ for $i = 1, \ldots, s$, and let $A_+ = \bigoplus_{d >_\tau 0} A_d$. Prove that the following conditions are equivalent.

a) $A = K[a_1, \ldots, a_s]$
b) $A_+ = (a_1, \ldots, a_s)$

Exercise 15. Let K be a field, let $P = K[x_1, \ldots, x_n]$ be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ of positive type, let M be a finitely generated graded P-module, and let $\varphi : M \longrightarrow M$ be a homogeneous P-linear map. Prove that the following conditions are equivalent.

a) The map φ is an isomorphism.
b) The map φ is injective.
c) The map φ is surjective.

Moreover, show that φ is an isomorphism if $v - \varphi(v) \in (x_1, \dots, x_n)M$ for all $v \in M$.

Tutorial 45: Homogeneous Maps and Toric Ideals

One of the most important tools for computing kernels and images of K-algebra homomorphisms is the **diagonal ideal**. Given two polynomial rings $P = K[x_1, \dots, x_n]$ and $P' = K[y_1, \dots, y_m]$ over a field K and two ideals $I \subset P$ and $I' \subset P'$, a K-algebra homomorphism $\varphi : P/I \longrightarrow P'/I'$ is determined by polynomials $f_1, \dots, f_n \in P'$ such that $\varphi(x_i + I) = f_i + I'$ for $i = 1, \dots, n$. Then the **diagonal ideal** corresponding to φ is defined to be the ideal

$$J = I'\,Q + (x_1 - f_1, \dots, x_n - f_n)$$

in the polynomial ring $Q = K[x_1, \dots, x_n, y_1, \dots, y_m]$. Its usefulness for studying φ was amply exhibited in Section 3.6. What is the correct analog of these constructions in the graded case? In this tutorial, we shall try to answer this question, and to apply our results to toric ideals.

Let $\ell \geq 1$, let P be graded by $W \in \mathrm{Mat}_{\ell,n}(\mathbb{Z})$, and let P' be graded by $W' \in \mathrm{Mat}_{\ell,m}(\mathbb{Z})$. Suppose that I and I' are homogeneous ideals with respect to these gradings.

a) Show that φ is a homomorphism of graded rings if and only if each polynomial f_i is homogeneous and $\deg_{W'}(f_i) = \deg_W(x_i)$.
b) Now suppose that $I = (0)$ and that $f_1, \dots, f_n \in P'$ are homogeneous polynomials. Prove that there is a unique grading on P by a matrix $W \in \mathrm{Mat}_{\ell,n}(\mathbb{Z})$ such that φ is a homomorphism of graded algebras.
c) Assume that φ is a homomorphism of graded rings. Prove that if W' is of positive type, then W is of positive type, too. (*Hint:* Observe that $W = W'S$ for a suitable matrix $S \in \mathrm{Mat}_{m,n}(\mathbb{N})$.)
d) Next we equip the polynomial ring $Q = K[x_1, \dots, x_n, y_1, \dots, y_m]$ with the grading given by the matrix $W'' = (W|W') \in \mathrm{Mat}_{\ell,m+n}(\mathbb{Z})$. Show that the diagonal ideal J is homogeneous if and only if φ is a homomorphism of graded rings.

For the remainder of this tutorial, we assume that φ is a homomorphism of graded rings and that W, W', W'' are chosen as above.

e) Prove that if W' is of positive type, then W'' is of positive type.
f) Given any homogeneous ideal $\widetilde{J} \subseteq Q$, show that $\widetilde{J} \cap P$ is a homogeneous ideal in P. Use this result and Proposition 3.6.3 to give two proofs for the fact that $J \cap P$ is a homogeneous ideal in P.

Now we specialize the above to the situation where $I' = (0)$ and $f_1, \ldots, f_n$ are terms. Let $\mathcal{A} = (a_{ij}) \in \mathrm{Mat}_{m,n}(\mathbb{N})$ be such that $f_i = y_1^{a_{1i}} \cdots y_m^{a_{mi}}$ for $i = 1, \ldots, n$, and let $I = J \cap P$ be the **toric ideal** associated to $\mathcal{A}$ (see Tutorial 38).

g) Prove that I is homogeneous with respect to the grading given by $\mathcal{A}$. (*Hint:* Use the grading on P' given by the identity matrix $W' = \mathcal{I}_n$.)

h) Let $I_{\mathcal{L}}$ be the **lattice ideal** associated to the lattice of integer solutions of the homogeneous system of Diophantine equations defined by $\mathcal{A}$ (see Tutorial 38.h). Use the identity $I = I_{\mathcal{L}} :_P (x_1 \cdots x_n)^\infty$ to give an alternative proof of g).

Tutorial 46: Projective Varieties

In algebraic geometry, graded rings occur in a variety of ways. The most important way they appear is undoubtedly as the homogeneous coordinate rings of projective varieties. In order to define and study projective varieties, we need to assume that you have mastered Section 2.6, and that you have a good working knowledge both of Tutorial 27 on affine varieties and of the first part of Tutorial 35 on projective spaces.

Let K be a field, let $P = K[x_1, \ldots, x_n]$ and $\overline{P} = K[x_0, \ldots, x_n]$ be standard graded, and let $\mathbb{A}^n_K$ (resp. $\mathbb{P}^n_K$) be the n-dimensional affine (resp. projective) space over K. Moreover, let L be an extension field of K, and let $\overline{K}$ be the algebraic closure of K. For every homogeneous ideal $J \subseteq \overline{P}$, we let

$$\mathcal{Z}_L^+(J) = \{(p_0 : \ldots : p_n) \in \mathbb{P}^n_L \mid f(p_0, \ldots, p_n) = 0 \text{ for all homogeneous } f \in J\}$$

A subset $V \subseteq \mathbb{P}^n_L$ is called a **projective zero-set** defined over K if it is of the form $V = \mathcal{Z}_L^+(J)$ for some homogeneous ideal $J \subseteq \overline{P}$. Projective zero-sets in $\mathbb{P}^n_{\overline{K}}$ are also called **projective varieties**.

a) Show that projective zero-sets have the following properties.
   1) The empty set $\emptyset$ and $\mathbb{P}^n_L$ are projective zero-sets.
   2) If $V_1, \ldots, V_s \subseteq \mathbb{P}^n_L$ are projective zero-sets, then $\cup_{i=1}^s V_i$ is a projective zero-set.
   3) If I is a set of indices and $\{V_i\}_{i \in I}$ a set of projective zero-sets indexed by I, then $\cap_{i \in I} V_i$ is a projective zero-set.

   Conclude that projective zero-sets can be taken as the closed sets of a topology on $\mathbb{P}^n_L$. When $K = L$, this topology is called the **Zariski topology** on $\mathbb{P}^n_K$.

b) Recall that, in Tutorial 36, we defined $\mathbb{P}^n_K$ as a set of equivalence classes $(K^{n+1} \setminus \{0\})/\sim$. Every point $p = (p_0 : \ldots : p_n)$ in $\mathbb{P}^n_K$ is the equivalence class of a punctured line $L(p) = \{(\lambda p_0, \ldots, \lambda p_n) \mid \lambda \in K \setminus \{0\}\}$ in $\mathbb{A}^{n+1}_K$. Show that, for every homogeneous ideal $J \subseteq \overline{P}$, the projective zero-set

$\mathcal{Z}_L^+(J) \subseteq \mathbb{P}_L^n$ is the set of equivalence classes of punctured lines contained in $\mathcal{Z}_L(J) \subseteq \mathbb{A}_L^{n+1}$. In other words, we have $p \in \mathcal{Z}_L^+(J)$ if and only if $L(p) \subseteq \mathcal{Z}_L(J)$.

The affine zero-set $\mathcal{Z}_L(J) \subseteq \mathbb{A}_L^{n+1}$ is called the **affine cone over** $\mathcal{Z}_L^+(J)$.

c) Prove the following projective version of the Weak Nullstellensatz:
For a homogeneous ideal $J \subseteq \overline{P}$*, we have* $\mathcal{Z}_{\overline{K}}^+(J) = \emptyset$ *if and only if* $\sqrt{J} = (x_0, \ldots, x_n)$.
Hint: If there exists a point $q \in \mathcal{Z}_{\overline{K}}(J) \backslash \{0\}$, then the punctured line $L(q)$ is contained in $\mathcal{Z}_{\overline{K}}(J)$. Now use b) and Proposition 3.7.1.

d) Using c), show that, for a homogeneous ideal $J \subseteq \overline{P}$, the following conditions are equivalent.
   1) $\mathcal{Z}_{\overline{K}}^+(J) = \emptyset$
   2) There exists a number $s \geq 1$ such that $(x_0, \ldots, x_n)^s \subseteq J$.
   3) For each $i \in \{0, \ldots, n\}$, there exists a number $\alpha_i \geq 1$ such that $x_i^{\alpha_i} \in J$.

Next we introduce the projective analog of Definition 2.6.15. For a set $S \subseteq \mathbb{P}_L^n$, we let $\mathcal{I}^+(S)$ be the ideal in $\overline{P}$ generated by all homogeneous polynomials $F \in \overline{P}$ such that $F(p_0, \ldots, p_n) = 0$ for all $(p_0 : \ldots : p_n) \in S$ (see Tutorial 16.e). The ideal $\mathcal{I}^+(S)$ is called the **homogeneous vanishing ideal** of S. If we have $L = \overline{K}$ and if $S \subseteq \mathbb{P}_{\overline{K}}^n$ is a projective variety defined over K, then the standard graded K-algebra $\overline{P}/\mathcal{I}^+(S)$ is called the **homogeneous coordinate ring** of S.

e) Show that $\mathcal{I}^+(S)$ is a well-defined homogeneous ideal in $\overline{P}$ which satisfies $\mathcal{Z}_L^+(\mathcal{I}^+(S)) \supseteq S$.

f) Prove the following projective version of the Strong Nullstellensatz:
For every homogeneous ideal $J \subseteq \overline{P}$ *such that* $\sqrt{J} \subset (x_0, \ldots, x_n)$*, we have* $\mathcal{I}^+(\mathcal{Z}_{\overline{K}}^+(J)) = \sqrt{J}$.
Hint: Show that $\mathcal{I}(\mathcal{Z}_{\overline{K}}(J)) = \mathcal{I}^+(\mathcal{Z}_{\overline{K}}^+(J))$ and apply Hilbert's Nullstellensatz 2.6.16.

g) Construct a 1-1 correspondence between non-empty projective varieties $V \subseteq \mathbb{P}_{\overline{K}}^n$ defined over K and reduced standard graded K-algebras $\overline{P}/J$, where the ideal J is a homogeneous radical ideal which is strictly contained in $(x_0, \ldots, x_n)$.

4.2 Degree Forms and Macaulay Bases

First things first,
but not necessarily in that order.
(Dr. Who)

The introduction to the preceding section ended with the preannouncement of some problems related to gradings of positive type. Is it time to face up to reality? Well, maybe not yet. The aim of the introduction to this section is to highlight the most important points, but we do not necessarily have to do it in the order in which they appear in the section. Let us instead start by confronting the following fundamental question.

What is the degree of a polynomial?

To explain the problem, assume that the polynomial ring $K[x_1, \ldots, x_n]$ over a field K is graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$. For the standard grading defined by $W = (1\ 1\ \cdots\ 1)$, we gave the standard answer right away in Section 1.1: the standard degrees of terms are natural numbers, and for a polynomial f, we defined $\deg(f)$ to be the maximum of the degrees of the terms in its support. In the case of the grading defined by an arbitrary matrix W, we have also seen that terms are homogeneous and that their degrees can be computed easily. Furthermore, given two terms, we can decide whether their degrees are equal or not. But which degree is larger?

Let us have a look at a concrete example. Let $K[x_1, x_2]$ be graded by $W = \left(\begin{smallmatrix} 2 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The polynomial $f = x_1 + x_2$ is the sum of two terms of degrees $\deg_W(x_1) = (2, -1)$ and $\deg_W(x_2) = (1, 0)$. So, what is $\deg_W(f)$? There is no canonical answer. Since degrees are elements of $\mathbb{Z}^m$, we have to choose an ordering on $\mathbb{Z}^m$ to decide which homogeneous component of a given polynomial has the largest degree. In the example at hand, we may observe that W also represents a term ordering $\sigma = \mathtt{Ord}(W)$ on $\mathbb{T}^2$. Using σ, we get $x_1 >_\sigma x_2$, since $W \cdot \log(x_1)^{\mathrm{tr}} >_{\mathtt{Lex}} W \cdot \log(x_2)^{\mathrm{tr}}$. As we saw in Section 1.4, the comparison of two terms with respect to the monoid ordering given by a matrix V is effected by lexicographically comparing the vectors obtained from multiplying V by their logarithms. Thus we get a first clue for solving our problem.

Now consider the following example. Let $K[x_1, x_2, x_3]$ be graded by $W = \left(\begin{smallmatrix} 2 & 1 & 3 \\ -1 & 0 & 1 \end{smallmatrix}\right)$. The polynomial $f = x_1x_3 + x_2^5 + x_1^2x_3$ is the sum of two homogeneous polynomials, namely $f_1 = x_1x_3 + x_2^5$ of degree $\deg_W(f_1) = (5, 0)$ and $f_2 = x_1^2x_3$ of degree $\deg_W(f_2) = (7, -1)$. Again we ask ourselves: what is the degree of f? Suppose we complete the matrix W to a non-singular matrix V by appending the row $(1\ 0\ 0)$. The vectors obtained by taking the scalar products of the logarithms of the three terms in the support of f with the rows of V are $(5, 0, 1)$, $(5, 0, 0)$, and $(7, -1, 2)$. Their lexicographical comparison yields $(7, -1, 2) >_{\mathtt{Lex}} (5, 0, 1) >_{\mathtt{Lex}} (5, 0, 0)$. By construction, the degrees of the three terms are the first two components of those vectors.

Therefore, if we want to let the grading interact with the ordering in a natural way, we should compare degrees lexicographically. In other words, there is a natural *law* for ordering degrees, namely `Lex`.

Having decided to apply the principle of `Lex` *and order*, we can venture further. A grading induces a *partial* ordering on the set of terms. Using this partial ordering, we mimic the developments in Volume 1. The homogeneous component of lexicographically largest degree of a vector of polynomials v will be called the *degree form* of v, and plays a role analogous to its leading term. Then we define degree form ideals and, more generally, degree form modules of submodules of graded free modules. A set of vectors is called a *Macaulay basis* of such a module M if their degree forms generate the degree form module of M. In Subsection B we explore this idea and develop part of the theory of Macaulay bases in analogy with, and as a generalization of, the theory of Gröbner bases.

But something is still missing, both theoretically and practically. Theoretically, we would like Macaulay bases to be systems of generators, and practically we would like to compute them. To fulfil these desires, gradings of positive type are not good enough. Finally the bad news has hit! We need the stronger notion of positive gradings. This corresponds to needing term orderings in Gröbner basis theory instead of arbitrary monoid orderings. Positive gradings are characterized by a degree matrix which has the look and feel of the upper part of a matrix defining a term ordering. The subtleties of positive gradings are investigated in Subsection A. If you want to get a good grasp of this topic, we suggest that you compare Propositions 4.1.21 and 4.2.3 carefully.

Finally, *dulcis in fundo*, we use a mix of positive gradings, term orderings and Gröbner basis theory to compute Macaulay bases (see Proposition 4.2.15 and Corollary 4.2.16). At this point you should be quite satisfied, unless you interpret *dulcis in fundo* as the technical latin term for the melted ice cream at the bottom of a tall thin glass which the spoon is unable to reach. However, if you are developing a taste for Macaulay bases, and have an appetite for further sweet delights, skip ahead to Section 4.3.B where two more treats await you. And if it is all a piece of cake for you, try getting your teeth into Tutorials 47 and 48 where several additional generalizations and characterizations of Macaulay bases are on the menu.

4.2.A Positive Gradings

Let K be a field, let $m \geq 1$, and let $P = K[x_1, \dots, x_n]$ be $\mathbb{Z}^m$-graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$. Recall that we always assume that the rows of W are $\mathbb{Z}$-linearly independent. Moreover, let $\delta_1, \dots, \delta_r \in \mathbb{Z}^m$, and let F be the graded free P-module $F = \bigoplus_{i=1}^{r} P(-\delta_i)$. The following definition provides us with a first link between gradings and term orderings.

Definition 4.2.1. Let τ be a monoid ordering on $\mathbb{T}^n$ and σ a module ordering on $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$.

a) The ordering τ is called **compatible with** $\deg_W$, or simply **degree compatible**, if $\deg_W(t) >_{\texttt{Lex}} \deg_W(t')$ implies $t >_\tau t'$ for all $t,t' \in \mathbb{T}^n$.
b) The ordering σ is called **compatible with** $\deg_W$, or simply **degree compatible**, if $\deg_W(te_i) >_{\texttt{Lex}} \deg_W(t'e_j)$ implies $te_i >_\sigma t'e_j$ for all $t,t' \in \mathbb{T}^n$ and all $i,j \in \{1,\dots,r\}$.

In the case of the standard grading, i.e. for $W = (1\ 1\ \cdots\ 1)$, this definition agrees with Definition 1.4.9. The typical situation in which we have a degree compatible monoid ordering is described in the following example.

Example 4.2.2. Suppose that $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ has rank m. If we choose a matrix $W' \in \mathrm{Mat}_{n-m,n}(\mathbb{Z})$ such that $V = \binom{W}{W'}$ is non-singular, then the monoid ordering $\sigma = \texttt{Ord}(V)$ is compatible with $\deg_W$.

Proposition 4.2.3. *Let P be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ of rank m, and let $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$ be the set of terms in F. The following conditions are equivalent.*

a) The first non-zero element in each non-zero column of W is positive.
b) For $i = 1,\dots,n$, we have $\deg_W(x_i) \geq_{\texttt{Lex}} 0$.
c) The restriction of Lex *to the monoid $\Gamma = \{d \in \mathbb{Z}^m \mid P_{W,d} \neq 0\}$ is a well-ordering.*
d) The restriction of Lex *to the monoid $\Gamma = \{d \in \mathbb{Z}^m \mid P_{W,d} \neq 0\}$ is a term ordering.*
e) There exists a term ordering τ on $\mathbb{T}^n$ which is compatible with $\deg_W$.
f) There exists a module term ordering σ on $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$ which is compatible with $\deg_W$.

Proof. Conditions a) and b) are clearly equivalent. Next we prove "a)$\Rightarrow$c)". Since the rank of W is m, there are standard basis vectors $e_{i_1},\dots,e_{i_{n-m}}$ which we can add as new rows to W such that the resulting matrix V is non-singular. Hence every column of V has a non-zero entry, and the first non-zero entry is positive. By Proposition 1.4.12, the monoid ordering $\sigma = \texttt{Ord}(V)$ is a term ordering, and by construction it is compatible with $\deg_W$. It follows that there is no infinite decreasing sequence of terms $t_1 >_\sigma t_2 >_\sigma \cdots$. Since every element of Γ is the degree of a term and since σ is degree compatible, every decreasing chain in Γ is eventually stationary. Now the claim follows from Proposition 1.4.18.

The implication "c)$\Rightarrow$d)" follows from the last part of that proposition, and the implication "d)$\Rightarrow$b)" follows from the definition of a term ordering. Now we prove "a)$\Rightarrow$e)". As above, we choose standard basis vectors $e_{i_1},\dots,e_{i_{n-m}} \in \mathbb{Z}^n$ and append them as new rows to W such that the resulting matrix $V \in \mathrm{Mat}_n(\mathbb{Z})$ is non-singular. Then the term ordering $\sigma = \texttt{Ord}(V)$ is compatible with $\deg_W$.

In order to show "e)$\Rightarrow$f)", we define an ordering σ on $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$ in the following way. Given $te_i, t'e_j \in \mathbb{T}^n\langle e_1,\dots,e_r\rangle$, we let $te_i \ge_\sigma t'e_j$ if $\deg_W(te_i) >_{\mathtt{Lex}} \deg_W(t'e_j)$, or if $\deg_W(te_i) = \deg_W(t'e_j)$ and $i < j$, or if $\deg_W(te_i) = \deg_W(t'e_j)$ and $i = j$ and $t \ge_\tau t'$. It is easy to check that σ is a module ordering. We note that $te_i \ge_\sigma t'e_i$ if and only if $t \ge_\tau t'$, i.e. inside every component of F the ordering σ is induced by τ. Thus we see that $te_i \ge_\sigma e_i$ for all $t \in \mathbb{T}^n$ and all $i \in \{1,\dots,r\}$. Hence σ is a module term ordering. By construction, σ is degree compatible.

Finally we prove "f)$\Rightarrow$b)". For $i = 1,\dots,n$, we have $x_ie_1 >_\sigma e_1$, and therefore $\deg_W(x_ie_1) = \deg_W(x_i) + \deg_W(e_1) \ge_{\mathtt{Lex}} \deg_W(e_1)$. Consequently, we have $\deg_W(x_i) \ge_{\mathtt{Lex}} 0$ for $i = 1,\dots,n$. $\square$

As we shall see, gradings by matrices which satisfy the conditions of this proposition are very useful in a number of ways. For this reason we introduce the following notions.

Definition 4.2.4. Let $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ be a matrix of rank m.

a) The grading on P defined by W is called **non-negative** if the first non-zero element in each non-zero column of W is positive. In this case, we shall also say that W is a non-negative matrix.
b) The grading on P defined by W is called **positive** if no column of W is zero and the first non-zero element in each column is positive. In this case, we shall also say that W is a positive matrix.

Thus the above proposition implies that, if W defines a non-negative grading defined by W, there exists a term ordering on $\mathbb{T}^n$ which is compatible with $\deg_W$. Notice also that if W is positive, then we have $\deg_W(x_i) >_{\mathtt{Lex}} 0$ for $i = 1,\dots,n$, and hence $P_+ = \bigoplus_{d >_{\mathtt{Lex}} 0} P_{W,d} = (x_1,\dots,x_n)$. The following corollary shows that in a finitely generated graded P-module there exists no infinite decreasing chain $\deg_W(v_1) >_{\mathtt{Lex}} \deg_W(v_2) >_{\mathtt{Lex}} \cdots$ if the grading defined by W is non-negative.

Corollary 4.2.5. *Assume that $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ defines a non-negative grading on P. Let M be a finitely generated, graded P-module, and let Σ be the set $\{d \in \mathbb{Z}^m \mid M_{W,d} \ne 0\}$. Then the relation $\mathtt{Lex}|_\Sigma$ is a well-ordering.*

Proof. By applying Remark 4.1.13 to a finite homogeneous system of generators of M, we construct a surjective homomorphism of graded P-modules of the form $\bigoplus_{i=1}^r P(-\delta_i) \longrightarrow M$. Now the claim follows from Proposition 4.2.3.c, because we have $\Sigma \subseteq \cup_{i=1}^r (\Gamma + \delta_i)$. $\square$

Our next corollary shows that non-negative gradings are of non-negative type and positive gradings are of positive type, as their names suggests.

Corollary 4.2.6. *Let P be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$.*

a) If the grading defined by W is non-negative, it is of non-negative type.

b) *If the grading defined by W is positive, it is of positive type.*

Proof. To prove claim a), we let C_i denote for $i = 1, \ldots, m$ the set of all indices $j \in \{1, \ldots, n\}$ such that the j^{th} column of W has its first non-zero entry in the i^{th} row. If we add high enough multiples of the first row to the rows below, the resulting matrix has strictly positive entries in the columns indexed by C_1. In particular, the second row of this matrix has strictly positive entries in the columns indexed by $C_1 \cup C_2$, and if we add high enough multiples of that row to the rows below, rows $2, 3, \ldots, m$ of the resulting matrix have strictly positive entries in the columns indexed by $C_1 \cup C_2$. Continuing this way, we finally arrive at a matrix whose last row has strictly positive entries in columns $C_1 \cup \cdots \cup C_m$, i.e. in all non-zero columns. Claim b) follows in the same way, except that there are no zero columns in W, so that the last row of the final matrix has positive entries everywhere. $\square$

The converse implications are not true, since for instance the grading on $K[x]$ given by $W = (-1)$ is of positive type, but not positive. However, the following remark says that there exists a change of grading which transforms a grading of non-negative (resp. positive) type into a non-negative (resp. positive) one.

Remark 4.2.7. Let P be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$.

a) If the grading defined by W is of non-negative type, then there exists a non-singular matrix $V \in \mathrm{Mat}_m(\mathbb{Z})$ such that the grading defined by the matrix $W' = V \cdot W$ is non-negative and $P_{W', V\cdot d} = P_{W,d}$ for all $d \in \mathbb{Z}^m$.
b) Similarly, if the grading defined by W is of positive type, then there exists a non-singular matrix $V \in \mathrm{Mat}_m(\mathbb{Z})$ such that the grading defined by the matrix $W' = V \cdot W$ is positive and $P_{W', V\cdot d} = P_{W,d}$ for all $d \in \mathbb{Z}^m$.

These observations follow from Propositions 4.1.18 and 4.1.9.b. Let us order the degrees of homogeneous polynomials with respect to the grading given by W' using the ordering $d \geq_\tau d' \iff W' \cdot d \geq_{\mathtt{Lex}} W' \cdot d'$ introduced in the proof of Proposition 4.1.21.a. Then the isomorphism of graded rings $(\mathrm{id}_P, \psi) : (P, \mathbb{Z}^m) \longrightarrow (P, \mathbb{Z}^m)$, where ψ is the map defined by W', is compatible with the ordering of the degrees.

4.2.B Macaulay Bases

As in the first subsection, we let $P = K[x_1, \ldots, x_n]$ be graded by the matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and we let M be a non-zero P-submodule of the graded free P-module $F = \bigoplus_{i=1}^r P(-\delta_i)$. But, unless stated otherwise, we do not assume anymore that M is a *graded* submodule of F. In Definition 4.1.6, we introduced the concept of degree of a homogeneous element of P. Now we extend this notion as follows.

Definition 4.2.8. Let v be a non-zero vector in M and $v = v_{d_1} + \cdots + v_{d_s}$ its decomposition into homogeneous components, where v_{d_i} has degree $d_i \in \mathbb{Z}^m$ for $i = 1, \dots, s$. Without loss of generality, we may assume that we have $d_1 >_{\mathtt{Lex}} \cdots >_{\mathtt{Lex}} d_s$.

a) The vector $\deg_W(v) = d_1 \in \mathbb{Z}^m$ is called the **degree** of v with respect to the grading given by W. If it is clear which grading we are considering, we shall also write $\deg(v)$.
b) The vector $\mathrm{DF}_W(v) = v_{d_1}$ is called the **degree form** of v with respect to the grading given by W. For the zero vector, we define $\mathrm{DF}_W(0) = 0$.

In the case $r = 1$, $\delta_1 = 0$, and $W = (1\ 1\ \dots\ 1)$, i.e. if we equip P with the standard grading, this definition of the degree of a polynomial and of its degree form agrees with the usual one. Moreover, degree forms generalize leading terms in the following sense.

Remark 4.2.9. Let $W \in \mathrm{Mat}_n(\mathbb{Z})$ be a non-singular matrix which defines a positive grading. Then $\sigma = \mathtt{Ord}(W)$ is a term ordering on $\mathbb{T}^n$ by Proposition 1.4.12. Given a non-zero polynomial $f \in P$, we have $\mathrm{DF}_W(f) = \mathrm{LT}_\sigma(f)$.

It is also instructive to compare the current situation with the definition of σ-degrees and σ-leading forms in Section 2.3.

Remark 4.2.10. In the situation of Section 2.3, consider the case of an ideal $M \subseteq P$. In this case we defined a $\mathbb{T}^n$-grading on the module P^s which was induced by a system of generators $\{g_1, \dots, g_s\}$ of M (see Proposition 2.3.3). By using the injective map $\log : \mathbb{T}^n \longrightarrow \mathbb{Z}^n$ and by letting $\gamma_i = \log(\mathrm{LT}_\sigma(g_i))$ for $i = 1, \dots, s$, we can see that this $\mathbb{T}^n$-grading corresponds exactly to the $\mathbb{Z}^n$-grading on $\bigoplus_{i=1}^s P(-\gamma_i)$ defined by the identity matrix.

Let us illustrate the concepts of degrees and degree forms with some examples.

Example 4.2.11. Let $P = K[x_1, x_2, x_3]$.

a) If we equip P with the grading defined by $W = (1\ 2\ 1)$, then the polynomial $f = x_1^3 - x_1x_2 + x_1$ has degree $\deg_W(f) = 3$. More precisely, it has two homogeneous components $f = f_3 + f_1$, where $\mathrm{DF}_W(f) = f_3 = x_1^3 - x_1x_2$ is the degree form of f and $f_1 = x_1$ has degree one.
b) Now we equip P with the grading defined by $W = (-1\ -1\ -1)$. In this case the same polynomial $f = x_1^3 - x_1x_2 + x_1$ has degree -1 and three homogeneous components f_{-1}, f_{-2}, f_{-3}, where $\mathrm{DF}_W(f) = f_{-1} = x_1$, where $f_{-2} = -x_1x_2 \in P_{-2}$, and where $f_{-3} = x_1^3 \in P_{-3}$.
c) Finally, we equip P with the grading defined by $W = \left(\begin{smallmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \end{smallmatrix}\right)$ and consider the polynomial $g = x_1^2x_2^5 - x_3^3 + x_1^3x_3^2$. It has two homogeneous components $g = g_{(12,9)} + g_{(11,12)}$, where $g_{(12,9)} = x_1^2x_2^5 - x_3^3 \in P_{(12,9)}$ and $g_{(11,12)} = x_1^3x_3^2 \in P_{(11,12)}$. Notice that $(12,9) >_{\mathtt{Lex}} (11,12)$. Thus we have $\mathrm{DF}_W(g) = x_1^2x_2^5 - x_3^3$ and $\deg_W(g) = (12,9)$.

Our next remark collects some useful rules for computing with degree forms. Their proofs are straightforward and proceed exactly as the proofs of the corresponding rules for leading terms (see Proposition 1.5.3).

Remark 4.2.12. Let $v_1, v_2 \in M \setminus \{0\}$, and let $f \in P$.

a) If $\deg_W(v_1) = \deg_W(v_2)$ and $\mathrm{DF}_W(v_1) \neq -\mathrm{DF}_W(v_2)$, then we have $\mathrm{DF}_W(v_1 + v_2) = \mathrm{DF}_W(v_1) + \mathrm{DF}_W(v_2)$.
b) If $\deg_W(v_1) >_{\mathtt{Lex}} \deg_W(v_2)$, then we have $\mathrm{DF}_W(v_1 + v_2) = \mathrm{DF}_W(v_1)$.
c) We have $\mathrm{DF}_W(f v_1) = \mathrm{DF}_W(f) \cdot \mathrm{DF}_W(v_1)$.

To develop the theory of degree forms further, we now introduce the analogs of leading term ideals and Gröbner bases.

Definition 4.2.13. Let M be a submodule of the graded free P-module F.

a) The graded P-submodule $\mathrm{DF}_W(M) = \langle \mathrm{DF}_W(v) \mid v \in M \rangle$ of F is called the **degree form module** of M with respect to the grading given by W.

b) A set $\{v_1, \ldots, v_s\} \subseteq M$ is called a **Macaulay basis** of M with respect to the grading given by W if we have $\mathrm{DF}_W(M) = \langle \mathrm{DF}_W(v_1), \ldots, \mathrm{DF}_W(v_s) \rangle$.

In the literature, Macaulay bases are also called **H-bases**. Under the hypotheses of Remark 4.2.9, a Macaulay basis with respect to the grading given by W is nothing but a Gröbner basis with respect to $\mathtt{Ord}(W)$. More generally, given a suitable partial ordering on the set of degrees of the elements of M, one can define Macaulay bases with respect to this partial ordering. We invite you to explore this notion further in Tutorial 47. Continuing the analogy between Macaulay bases and Gröbner bases, we now imitate Proposition 2.4.3.a and show that a Macaulay basis is additionally a system of generators of M under suitable hypotheses.

Proposition 4.2.14. *Let P be non-negatively graded by the matrix W, and let $\{v_1, \ldots, v_s\}$ be a Macaulay basis of M. Then the set $\{v_1, \ldots, v_s\}$ is a system of generators of M.*

Proof. Without loss of generality, we may assume $v_i \neq 0$ for $i = 1, \ldots, s$. Suppose that the claim is false, i.e. that $\langle v_1, \ldots, v_s \rangle \subset M$. Since the grading is non-negative, the restriction of $\mathtt{Lex}$ to $\Sigma = \{d \in \mathbb{Z}^m \mid F_{W,d} \neq 0\}$ is a well-ordering by Corollary 4.2.5. So, there is an element $v \in M \setminus \langle v_1, \ldots, v_s \rangle$ such that $d = \deg_W(v)$ is minimal. Using Corollary 1.7.11, we can write $\mathrm{DF}_W(v) = f_1 \,\mathrm{DF}_W(v_1) + \cdots + f_s \,\mathrm{DF}_W(v_s)$, where f_i is a homogeneous polynomial of degree $\deg_W(f_i) = d - \deg_W(v_i)$ for $i = 1, \ldots, s$. Now the rules given in Remark 4.2.12 show that $\deg_W(v - f_1 v_1 - \cdots - f_s v_s) <_{\mathtt{Lex}} d$. Because of the minimality of d, this implies that there are polynomials $g_1, \ldots, g_s$ such that $v - f_1 v_1 - \cdots - f_s v_s = g_1 v_1 + \cdots + g_s v_s$. This representation contradicts the supposition that $v \notin M$ and finishes the proof. $\square$

The conclusion of this proposition obviously holds if W is positive, but it need not hold if W is of positive type (see Exercise 5). Degree compatible term orderings allow us to connect the two notions of Gröbner basis and Macaulay basis. Recall that, by Proposition 4.2.3, a degree compatible term ordering exists if and only if P is non-negatively graded by W.

Proposition 4.2.15. *Let the polynomial ring P be non-negatively graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let σ be a term ordering which is compatible with $\deg_W$, and let $G = \{g_1, \dots, g_s\}$ be a σ-Gröbner basis of M.*

a) The set $\{\mathrm{DF}_W(g_1), \dots, \mathrm{DF}_W(g_s)\}$ is a σ-Gröbner basis of $\mathrm{DF}_W(M)$.
b) The set G is a Macaulay basis of M with respect to W.

Proof. First we prove a). Let $v \in M \setminus \{0\}$. Since σ is degree compatible, the term $\mathrm{LT}_\sigma(v)$ has to be one of the terms in $\mathrm{Supp}(\mathrm{DF}_W(v))$. Therefore we have $\mathrm{LT}_\sigma(v) = \mathrm{LT}_\sigma(\mathrm{DF}_W(v))$. By assumption, $\mathrm{LT}_\sigma(v)$ is a multiple of $\mathrm{LT}_\sigma(g_j)$ for some $j \in \{1, \dots, s\}$. This shows a), and b) is an immediate consequence of a) and Proposition 2.4.3.a. $\square$

The converse of part b) of this proposition is not true in general (see Exercise 6). A common situation in which we can apply b) is given by Example 4.2.2. In fact, we can use this idea to compute Macaulay bases effectively as follows.

Corollary 4.2.16. (Computation of Macaulay Bases)
Let P be non-negatively graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let M be a P-submodule of F. Consider the following sequence of instructions.

1) Choose a matrix $W' \in \mathrm{Mat}_{n-m,n}(\mathbb{N})$ such that $V = \binom{W}{W'}$ is non-singular and positive.
2) Using Buchberger's Algorithm 2.5.5, compute a Gröbner basis G of M with respect to the term ordering $\mathtt{Ord}(V)$.
3) Return the result G.

This is an algorithm which computes a Macaulay basis G of M with respect to the grading given by W.

Proof. Clearly, we may assume that $M \neq 0$. By Proposition 4.2.3, a matrix W' as required in step 1) exists. By Proposition 1.4.12, the monoid ordering $\sigma = \mathtt{Ord}(V)$ is then a term ordering. Hence Buchberger's Algorithm can be used to compute a σ-Gröbner basis of M. By the proposition, this Gröbner basis is a Macaulay basis of M with respect to the grading given by W. $\square$

There are several ways to perform step 1) of this algorithm. One possibility is to choose suitable standard basis vectors (see Exercise 9).

Exercise 1. Let K be a field, let $P = K[x_1, \ldots, x_n]$ be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let $\delta_1, \ldots, \delta_r \in \mathbb{Z}^m$, and let $F = \bigoplus_{i=1}^r P(-\delta_i)$. Show that the grading given by W is non-negative if and only if the restriction of `Lex` to the set $\Sigma = \{d \in \mathbb{Z}^m \mid F_{W,d} \neq 0\}$ is a well-ordering.

Exercise 2. Let K be a field, and let $P = K[x_1, \ldots, x_n]$ be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$. Give an alternative proof of Corollary 4.2.6.a using induction on m.

Exercise 3. Let K be a field, let $P = K[x_1, \ldots, x_n]$ be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let I be a non-zero principal ideal in P. Show that $\mathrm{DF}_W(I)$ is generated by the degree form of a generator of I.

Exercise 4. In the following cases, try to find a non-zero matrix $W \in \mathrm{Mat}_{1,3}(\mathbb{Z})$ such that the support of the degree form of the following polynomials contains as many terms as possible.

a) $f_1 = x_1^3 - x_2x_3 + x_1$
b) $f_2 = x_1^2x_2^5 - x_3^3 + x_1^3x_3^2$
c) $f_3 = x_1^3 + x_2^3 + x_3^3 - x_1^2x_2^2x_3^2$

Exercise 5. Let us equip the polynomial ring $P = K[x]$ over a field K with the $\mathbb{Z}^2$-grading defined by $W = \binom{-1}{1}$. Prove that $x - x^2$ is a Macaulay basis of the ideal (x) which does not generate that ideal.

Exercise 6. Let us equip the polynomial ring $P = K[x, y]$ over a field K with the standard grading. Show that $\{x^2 - y^2,\, xy\}$ is a Macaulay basis of the ideal $(x^2 - y^2,\, xy)$, but not a Gröbner basis with respect to any term ordering.

Exercise 7. Let K be a field, let $P = K[x_1, \ldots, x_n]$ be positively graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let $f \in P \setminus \{0\}$.

a) Show that if $\mathrm{DF}_W(f)$ is irreducible then f is irreducible.
b) Show that the converse is not true in general.
c) Let $f = x_1^2 + g(x_2, \ldots, x_n)$. Show that, up to units, f has at most two irreducible factors.

Exercise 8. Let K be a field, and let $P = K[x_1, \ldots, x_n]$ be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ of positive type.

a) Show that there exists a non-singular matrix $U \in \mathrm{Mat}_m(\mathbb{Z})$ such that $U \cdot W$ defines a positive grading on P and such that (id_P, φ), where $\varphi : \mathbb{Z}^m \longrightarrow \mathbb{Z}^m$ is defined by U, is an isomorphism of graded rings.
b) Find an example where such an isomorphism does not preserve Macaulay bases.

Exercise 9. Let $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ be a matrix of rank m. Show that one can choose $n - m$ standard basis vectors $e_{i_1}, \ldots, e_{i_{n-m}}$ such that the matrix

$$V = \begin{pmatrix} W \\ e_{i_1} \\ \vdots \\ e_{i_{n-m}} \end{pmatrix}$$

is non-singular. Moreover, show that the matrix V is positive if W is non-negative.

Tutorial 47: Computation of Macaulay Bases

Computers are useless.
They can only give you answers.
(Pablo Picasso)

In this tutorial we invite you to explore the notion of a Macaulay basis further. Be prepared to work by doing both some thinking and some programming. Let us start with a little programming.

a) Let $P = \mathbb{Q}[x_1, \ldots, x_n]$ be non-negatively graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$. Write a CoCoA function `CompatibleTO`(...) which takes W and computes a matrix $V \in \mathrm{Mat}_n(\mathbb{Z})$ of the kind required by step 1) of the algorithm in Corollary 4.2.16. (*Hint:* You may want to solve Exercise 9 first.)
b) In the setting of a), let I be an ideal of P which is homogeneous with respect to the grading given by W. Implement a CoCoA function `MacaulayBasis`(...) which computes a Macaulay basis of I with respect to this grading.

Switching to the thinking part, we shall now generalize Macaulay bases by using partial monoid orderings on $\mathbb{T}^n$. They are defined as follows. A relation π on a monoid $(\Gamma, \circ)$ is called a **partial monoid ordering** if the following conditions are satisfied for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$.

1) $\gamma_1 \geq_\pi \gamma_1$ (reflexivity)
2) $\gamma_1 \geq_\pi \gamma_2$ and $\gamma_2 \geq_\pi \gamma_3$ imply $\gamma_1 \geq_\pi \gamma_3$ (transitivity)
3) $\gamma_1 \geq_\pi \gamma_2$ implies $\gamma_1 \circ \gamma_3 \geq_\pi \gamma_2 \circ \gamma_3$

Now it is time for you to start working again.

c) Compare this definition to the definition of a monoid ordering in 1.4.1. For each of the additional properties in Definition 1.4.1, find a partial monoid ordering which has this property but not the others.
Hint: Use $\Gamma = \mathbb{T}^1$ and $\Gamma = \mathbb{T}^1 \cup \{\infty\}$.

Next we choose $m \in \{1, \ldots, n\}$, let $v_1, \ldots, v_m \in \mathbb{R}^n$ be $\mathbb{R}$-linearly independent vectors, and define $V \in \mathrm{Mat}_{m,n}(\mathbb{R})$ to be the matrix with rows $v_1, \ldots, v_m$. For two terms $t_1, t_2 \in \mathbb{T}^n$, we set $t_1 \geq_{\mathtt{Ord}(V)} t_2$ if and only if $V \cdot (\log(t_1) - \log(t_2)) \geq_{\mathtt{Lex}} 0$. Now back to work!

d) Prove that $\mathtt{Ord}(V)$ is a partial monoid ordering on $\mathbb{T}^n$. It is called the **partial ordering represented by** V.
e) Consider the matrix $U = (1\ \sqrt{2}\ \sqrt{2}) \in \mathrm{Mat}_{1,3}(\mathbb{R})$. Write a CoCoA function `CompareOrdU`(...) which takes two terms $t_1, t_2 \in \mathbb{T}^3$ and returns the Boolean value corresponding to $t_1 \geq_{\mathtt{Ord}(U)} t_2$.
f) Show that there do not exist $m \geq 1$ and $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ such that $\mathtt{Ord}(U)$ is the lexicographic ordering on the degrees of the terms in $\mathbb{T}^3$ with respect to the grading given by W.
g) Find a number $m \geq 1$ and a matrix $W \in \mathrm{Mat}_{m,3}(\mathbb{R})$ such that $\mathtt{Ord}(W)$ is a term ordering which refines the partial ordering represented by U.

h) Implement a CoCoA program `CompareOrdW(...)` which takes two terms $t_1, t_2 \in \mathbb{T}^3$ and returns the Boolean value corresponding to $t_1 \geq_{\mathtt{Ord}(W)} t_2$.

At this point we can continue to generalize the definitions given in the text. Let K be a field, let $P = K[x_1, \ldots, x_n]$, and let $V \in \mathrm{Mat}_{m,n}(\mathbb{R})$ be a matrix as above. Given a non-zero polynomial $f \in P$, we write $f = c_1 t_1 + \cdots + c_s t_s$ with $c_1, \ldots, c_s \in K \setminus \{0\}$ and $t_1, \ldots, t_s \in \mathbb{T}^n$. Without loss of generality we may assume that $t_1 \geq_{\mathtt{Ord}(V)} t_2 \geq_{\mathtt{Ord}(V)} \cdots \geq_{\mathtt{Ord}(V)} t_s$. Then the polynomial

$$\mathrm{DF}_V(f) = c_1 t_1 + \cdots + c_j t_j \qquad \text{where} \qquad j = \max\{i \mid t_i \geq_{\mathtt{Ord}(V)} t_1\}$$

is called the **degree form** of f with respect to the partial ordering $\mathtt{Ord}(V)$. Furthermore, we define the degree form ideal $\mathrm{DF}_V(I)$ and a Macaulay basis with respect to $\mathtt{Ord}(V)$ of an ideal $I \subseteq P$ as in the text.

i) Prove that we have $t \geq_{\mathtt{Ord}(V)} 1$ for every term $t \in \mathbb{T}^n$ if and only if the first non-zero entry in each non-zero column of V is positive. In this case, we shall say that $\mathtt{Ord}(V)$ is a **non-negative partial ordering**.
j) Generalize Proposition 4.2.14 in the following way. Suppose that $\mathtt{Ord}(V)$ is a non-negative partial ordering and $\{f_1, \ldots, f_s\}$ is a Macaulay basis of an ideal $I \subseteq P$ with respect to $\mathtt{Ord}(V)$. Then $\{f_1, \ldots, f_s\}$ generates I.
k) Let $\mathtt{Ord}(V)$ be a non-negative partial ordering on $\mathbb{T}^n$. Generalize the algorithm of Corollary 4.2.16 to an algorithm which computes a Macaulay basis of a given ideal I in P with respect to $\mathtt{Ord}(V)$.
Hint: Complete V to a non-singular matrix in $\mathrm{Mat}_n(\mathbb{R})$ in order to find a term ordering which is compatible with $\mathtt{Ord}(V)$.
l) Write a CoCoA program `GenMacBasis(...)` which implements this generalized algorithm in the case of the non-negative partial ordering given by $U = (1\ \sqrt{2}\ \sqrt{2})$.
Hint: Use the function written in part h) above. You will need to reimplement Buchberger's Algorithm 2.5.5 for the term ordering represented by this function.

Tutorial 48: Characterizations of Macaulay Bases

As we shall see in this tutorial, it is possible to define Macaulay bases in a very general setting, namely for submodules of Γ-graded free R-modules, at the level of generality we introduced in Section 1.7. We shall show that such general Macaulay bases enjoy properties analogous to the ones we used to characterize Gröbner bases (see Theorem 2.4.1). In fact, these Macaulay bases can be considered as generalizations of Gröbner bases.

Let Γ be a commutative monoid obeying the cancellation law and for which there exists a term ordering σ on Γ. Let R be a Γ-graded ring, let $\gamma_1, \ldots, \gamma_r \in \Gamma$, let $F = \bigoplus_{i=1}^r R(\gamma_i)$ be a Γ-graded free R-module, and let M be an R-submodule of F. Every element $v \in M \setminus \{0\}$ has a unique decomposition $v = v_1 + \cdots + v_s$, where $v_1, \ldots, v_s$ are homogeneous and $\deg(v_1) >_\sigma \deg(v_2) >_\sigma \cdots >_\sigma \deg(v_s)$.

In this situation, we call $\mathrm{DF}_\Gamma(v) = v_1$ the **degree form** of v with respect to the Γ-grading on F. For $v = 0$, we set $\mathrm{DF}_\Gamma(v) = 0$. Then the Γ-graded submodule $\mathrm{DF}_\Gamma(M) = \langle \mathrm{DF}_\Gamma(m) \mid m \in M \rangle$ of F is called the **degree form module** of M. A set of elements $\{m_1, \ldots, m_t\} \subseteq M$ is called a **Macaulay basis** of M with respect to the Γ-grading on F if $\mathrm{DF}_\Gamma(M) = \langle \mathrm{DF}_\Gamma(m_1), \ldots, \mathrm{DF}_\Gamma(m_t) \rangle$.

a) Explain how the situation described at the beginning of the section and Definitions 4.2.8 and 4.2.13 are special cases of the above assumptions and definitions.
b) Imitate the proof of Proposition 4.2.14 to show that every Macaulay basis of M is a system of generators of M.
c) Prove that $\{m_1, \ldots, m_t\} \subseteq M$ is a Macaulay basis of M if and only if every element $v \in M \setminus \{0\}$ has a representation $v = \sum_{i=1}^t f_i m_i$ such that $f_1, \ldots, f_t \in R$ and $\deg(v) = \max_\sigma \{\deg(f_i m_i) \mid i \in \{1, \ldots, t\},\ f_i m_i \neq 0\}$.
Hint: Proceed as in the proof of Proposition 2.1.3.
d) Let $\mathcal{M}$ be the tuple $(m_1, \ldots, m_t)$, and let $v, w \in M$. We write $v \xrightarrow{\mathcal{M}} w$ if there exist $f_1, \ldots, f_t \in R$ such that $v = w + \sum_{i=1}^t f_i m_i$, and such that $\deg(w) <_\sigma \deg(v)$ or $w = 0$. The reflexive, transitive closure of $\xrightarrow{\mathcal{M}}$ is denoted again by $\xrightarrow{\mathcal{M}}$. Show that the following conditions are equivalent.

    1) The elements of $\mathcal{M}$ are a Macaulay basis of M.
    2) For an element $v \in F$, we have $v \xrightarrow{\mathcal{M}} 0$ if and only if $v \in M$.
    3) If $v \in M$ is irreducible with respect to $\xrightarrow{\mathcal{M}}$, then we have $v = 0$.
    4) For every $v \in F$, there exists a unique $w \in M$ such that $v \xrightarrow{\mathcal{M}} w$ and such that w is irreducible with respect to $\xrightarrow{\mathcal{M}}$.
    5) The relation $\xrightarrow{\mathcal{M}}$ on F is confluent.

    Hint: Imitate the proofs of Propositions 2.2.5 and 2.2.8.
e) Prove that the conditions of d) are also equivalent to $\sum_{i=1}^t f_i m_i \xrightarrow{\mathcal{M}} 0$ for all $(f_1, \ldots, f_t) \in \mathrm{Syz}_R(\mathrm{DF}_\Gamma(\mathcal{M}))$. (*Hint:* Use the method of the proof of Proposition 2.3.12.)
f) Show that it suffices to check the condition of e) for a system of generators of $\mathrm{Syz}_R(\mathrm{DF}_\Gamma(\mathcal{M}))$.
g) Explain how one can use the preceding results to characterize Macaulay bases by a suitable criterion for the lifting of syzygies.
h) Show that Gröbner bases are a special case of the general kind of Macaulay bases considered here. Then use the results you proved above to generalize Theorem 2.4.1 accordingly.

4.3 Homogenization

The real world is a special case.
(Anonymous)

Throughout this chapter, we examine gradings on polynomial rings and study some of their applications. For instance, we shall see that many algorithms can be improved if we know beforehand that the input polynomials are homogeneous. But unfortunately not every polynomial occurring in the real world is homogeneous. Therefore we would like somehow to approximate a non-homogeneous polynomial by a homogeneous one. How can we view an arbitrary polynomial as a special case of something homogeneous?

First, let us search for a solution in an easy case. Let $P = \mathbb{Q}[x_1, x_2, x_3]$ be standard graded and $f = x_1^2 + x_2^5 x_3$. The first term in the support of f has degree 2 and the second term has degree 6. In order to view f as a special case of a homogeneous polynomial, we introduce a new indeterminate y_1 and form $F = y_1^4 x_1^2 + x_2^5 x_3$. Then F is a homogeneous polynomial in the standard graded ring $\overline{P} = \mathbb{Q}[y_1, x_1, x_2, x_3]$ and we can recreate f by plugging $y_1 = 1$ into F. In this sense, we can say that this process of *homogenization* allows us to regard arbitrary real-world polynomials as special cases of homogeneous ones. Notice that we would not have had to do anything if f had been homogeneous to start with.

A natural question is whether we can extend everything to the $\mathbb{Z}^m$-graded situation. For the moment, let us study the general case by means of an example. Let $P = \mathbb{Q}[x_1, x_2, x_3]$ be graded by $W = \binom{2\;1\;1}{1\;0\;1}$. Again the polynomial $f = x_1^2 + x_2^5 x_3$ is not homogeneous, since $\deg_W(x_1^2) = (4, 2)$, while $\deg_W(x_2^5 x_3) = (6, 1)$. The degree form of f is $\mathrm{DF}_W(f) = x_2^5 x_3$. First we consider the upper row of weights. By introducing a new indeterminate y_1 of degree one, we may homogenize f with respect to this row by forming $y_1^2 x_1^2 + x_2^5 x_3$. But then there is no way to endow y_1 with a second integer degree such that $y_1^2 x_1^2 + x_2^5 x_3$ becomes homogeneous with respect to the lower row of weights. What we have to do is introduce another indeterminate y_2 and then set $\deg_W(y_1) = (1, 0)$ and $\deg_W(y_2) = (0, 1)$. Then the polynomial $y_1^2 x_1^2 + y_2 x_2^5 x_3$ is the *homogenization* of f.

This approach works quite generally, and the usual homogenization for standard gradings (occurring most frequently in real life) is a special case. Thus the section starts by providing all the necessary rules for dealing with the processes of homogenizing and dehomogenizing polynomials. Then we really get going and play the same game with ideals. Again we provide the good rules for the interaction between homogenization and ideal-theoretic operations (see Propositions 4.3.5, 4.3.10 and 4.3.12) and point out a few bad ones (see Examples 4.3.9 and 4.3.11). As an application, we see that we can compute homogenizations using saturation with respect to the product of the homogenizing indeterminates, an operation we have mastered in Section 3.5. A further application is discussed in Tutorial 51, where we suggest an efficient method for computing the homogenization of implicitization ideals.

In the last part of the first subsection we examine the behaviour of Gröbner bases of ideals under dehomogenization. For this purpose we have to relate a term ordering on the monoid $\mathbb{T}(x_1,\dots,x_n)$ to a term ordering on the monoid $\mathbb{T}(y_1,\dots,y_m,x_1,\dots,x_n)$. After doing this in the correct way in Definition 4.3.13, we can then show that Gröbner bases dehomogenize in a natural way (see Proposition 4.3.18).

What about homogenizing Macaulay bases and Gröbner bases? Since there are some problems in the general case, we assume in Subsection 4.3.B that $P = K[x_1,\dots,x_n]$ is positively $\mathbb{Z}$-graded. Then we continue our investigation by studying the effect of homogenization on Macaulay bases. We characterize Macaulay bases as those systems of generators of an ideal I whose homogenizations generate the homogenization of I. This is not generally true for $m \geq 2$ (see Exercises 8 and 9), but for $m = 1$ it yields an alternative way of computing the homogenization of an ideal (see Corollary 4.3.20). Moreover, we show that Gröbner bases homogenize in the natural way (see Proposition 4.3.21).

The final topic in this section has a more geometric flavour. Given an ideal I in P, the rings P/I and $P/\operatorname{DF}_W(I)$ are shown to be members of a family of rings which is *flat*. Although we do not define this concept here, we show how this family is constructed using the homogenization of I, and that it has good properties. For instance, it turns out that Macaulay bases are characterized by the property that they generate the defining ideal of each fiber of the family (see Proposition 4.3.23). Again these results require a positive grading with $m = 1$ (see Exercise 11).

Should we shed more light on these intricacies? On a fortune cookie we read that "a good example is the best gift we can bestow on others". Therefore we conclude the section with some elementary examples which cast a few rays of illumination into the vast areas of flat families, deformations, and Hilbert functions. In due course, we shall revisit some of them.

What about homogenization of modules? Haven't we been stressing the need to concentrate on modules all along? Well, yes, but... in this section we prefer to leave the job to you, in Tutorial 49. Finally, we mention that there is another nice geometric interpretation of the operation of homogenizing a polynomial ideal. It is related to the process of forming the projective closure of an affine variety. In Tutorial 52 we invite you to explore this interpretation by following our guided tour.

4.3.A Homogenization of Polynomials and Ideals

Let K be a field and $P = K[x_1,\dots,x_n]$ a polynomial ring over K which is graded by a matrix $W \in \operatorname{Mat}_{m,n}(\mathbb{Z})$ of rank $m \geq 1$. In this situation we choose new indeterminates $y_1,\dots,y_m$ and call them **homogenizing indeterminates**. Then we equip the ring $\overline{P} = K[y_1,\dots,y_m,x_1,\dots,x_n]$ with the grading defined by the matrix $\overline{W} = (\mathcal{I}_m \mid W)$, where $\mathcal{I}_m$ denotes the identity matrix of size m.

Clearly, the map $(\imath, \mathrm{id}_{\mathbb{Z}^m}) : (P, \mathbb{Z}^m) \hookrightarrow (\overline{P}, \mathbb{Z}^m)$, where $\imath$ is the inclusion and $\mathrm{id}_{\mathbb{Z}^m}$ is the identity, is a homomorphism of $\mathbb{Z}^m$-graded rings. The processes of homogenizing and dehomogenizing a polynomial are now defined as follows.

Definition 4.3.1. Let $f \in P \setminus \{0\}$ and $F \in \overline{P}$.

a) Write $f = c_1 t_1 + \cdots + c_s t_s$ with $c_1, \ldots, c_s \in K \setminus \{0\}$ and distinct terms $t_1, \ldots, t_s \in \mathbb{T}^n$. For $j = 1, \ldots, s$, let $\deg_W(t_j) = (\tau_{1j}, \ldots, \tau_{mj}) \in \mathbb{Z}^m$. Moreover, for $i = 1, \ldots, m$, let $\mu_i = \max\{\tau_{ij} \mid j = 1, \ldots, s\}$ be the maximum of the i^{th} components of the degrees of the terms $t_1, \ldots, t_s$. Then $(\mu_1, \ldots, \mu_m)$ is called the **top degree** of f with respect to the grading given by W and is denoted by $\mathrm{topdeg}_W(f)$.

b) In the context of a), the **homogenization** of f with respect to the grading given by W is the polynomial

$$f^{\mathrm{hom}} = \sum_{j=1}^{s} c_j \, t_j \, y_1^{\mu_1 - \tau_{1j}} \cdots y_m^{\mu_m - \tau_{mj}} \ \in \overline{P}$$

For the zero polynomial, we set $0^{\mathrm{hom}} = 0$.

c) The polynomial $F^{\mathrm{deh}} = F(1, \ldots, 1, x_1, \ldots, x_n) \in P$ is called the **dehomogenization** of F with respect to $y_1, \ldots, y_m$.

Our first observation is that the process of homogenizing a polynomial can be described informally as follows. Multiply each term in the support of f by the appropriate power product of $y_1, \ldots, y_m$ in order to make it homogeneous of degree $\mathrm{topdeg}_W(f)$. Hence the polynomial f^{hom} is indeed a homogeneous polynomial in $\overline{P}$ of degree $\deg_{\overline{W}}(f^{\mathrm{hom}}) = \mathrm{topdeg}_W(f)$.

Secondly, we note that the operation of dehomogenization has been defined for all polynomials in $\overline{P}$, but usually we shall apply it only to homogeneous polynomials $F \in \overline{P}$. Homogenization and dehomogenization of polynomials obey the following rules.

Proposition 4.3.2. (Rules for Homogenizing Polynomials)
Let polynomials $f, g \in P$ and $F, G \in \overline{P}$ be given.

a) Let $f \neq 0$, let $\mu = \mathrm{topdeg}_W(f)$, and let $f = f_1 + \cdots + f_r$ be the decomposition of f into its homogeneous components, where each $f_i \in P$ is homogeneous of degree $d_i \in \mathbb{Z}^m$. Then we have

$$f^{\mathrm{hom}} = y^{\mu - d_1} f_1 + \cdots + y^{\mu - d_r} f_r$$

where $y^{\mu - d_i} = y_1^{\alpha_{i1}} \cdots y_m^{\alpha_{im}}$ for $\mu - d_i = (\alpha_{i1}, \ldots, \alpha_{im}) \in \mathbb{Z}^m$.

b) Let $(\mu_1, \ldots, \mu_m) = \mathrm{topdeg}_W(f)$. Then we have

$$f^{\mathrm{hom}} = y_1^{\mu_1} \cdots y_m^{\mu_m} \cdot f\Big(\frac{x_1}{y_1^{w_{11}} \cdots y_m^{w_{m1}}}, \ldots, \frac{x_n}{y_1^{w_{1n}} \cdots y_m^{w_{mn}}}\Big)$$

c) We have $(f^{\mathrm{hom}})^{\mathrm{deh}} = f$.

d) If $fg \neq 0$, *we have* $\mathrm{topdeg}_W(fg) = \mathrm{topdeg}_W(f) + \mathrm{topdeg}_W(g)$.

e) We have $(fg)^{\mathrm{hom}} = (f^{\mathrm{hom}})(g^{\mathrm{hom}})$.

f) Let $f, g, f+g \in P \setminus \{0\}$, *and let* d *be the componentwise maximum of* $\mathrm{topdeg}_W(f)$ *and* $\mathrm{topdeg}_W(g)$. *Then we have*

$$y^{d-\mathrm{topdeg}_W(f+g)} \cdot (f+g)^{\mathrm{hom}} = y^{d-\mathrm{topdeg}_W(f)} \cdot f^{\mathrm{hom}} + y^{d-\mathrm{topdeg}_W(g)} \cdot g^{\mathrm{hom}}$$

g) We have $(FG)^{\mathrm{deh}} = F^{\mathrm{deh}}\, G^{\mathrm{deh}}$ *and* $(F+G)^{\mathrm{deh}} = F^{\mathrm{deh}} + G^{\mathrm{deh}}$.

h) Suppose that F *is non-zero and homogeneous. For* $j = 1, \dots, m$, *let* $s_j = \max\{i \geq 0 \mid y_j^i \text{ divides } F\}$. *Then* $y_1^{s_1} \cdots y_m^{s_m} \cdot (F^{\mathrm{deh}})^{\mathrm{hom}} = F$. *In particular, we have* $F^{\mathrm{deh}} \neq 0$.

i) Given two terms $T, T' \in \mathbb{T}(y_1, \dots, y_m, x_1, \dots, x_n)$ *with* $\deg_{\overline{W}}(T) = \deg_{\overline{W}}(T')$ *and* $T^{\mathrm{deh}} = (T')^{\mathrm{deh}}$, *we have* $T = T'$.

Proof. Claim a) follows by combining the terms $t_1, \dots, t_s$ in Definition 4.3.1.a into groups according to their degrees.

To prove b), we decompose $f = c_1 t_1 + \cdots + c_s t_s$ as in the definition, and we write $t_j = x_1^{\alpha_{1j}} \cdots x_n^{\alpha_{nj}}$ for $j = 1, \dots, s$. Then we have the equality $\deg_W(t_j) = W \cdot (\alpha_{1j}, \dots, \alpha_{nj})^{\mathrm{tr}} = (\tau_{1j}, \dots, \tau_{mj})^{\mathrm{tr}}$, and therefore

$$\begin{aligned}
f^{\mathrm{hom}} &= \sum_{j=1}^{s} c_j\, t_j\, y_1^{\mu_1 - \tau_{1j}} \cdots y_m^{\mu_m - \tau_{mj}} \\
&= y_1^{\mu_1} \cdots y_m^{\mu_m} \cdot \sum_{j=1}^{s} c_j\, \frac{x_1^{\alpha_{1j}} \cdots x_n^{\alpha_{nj}}}{y_1^{\tau_{1j}} \cdots y_m^{\tau_{mj}}} \\
&= y_1^{\mu_1} \cdots y_m^{\mu_m} \cdot \sum_{j=1}^{s} c_j \left(\frac{x_1}{y_1^{w_{11}} \cdots y_m^{w_{m1}}}\right)^{\alpha_{1j}} \cdots \left(\frac{x_n}{y_1^{w_{1n}} \cdots y_m^{w_{mn}}}\right)^{\alpha_{nj}} \\
&= y_1^{\mu_1} \cdots y_m^{\mu_m} \cdot f\left(\frac{x_1}{y_1^{w_{11}} \cdots y_m^{w_{m1}}}, \dots, \frac{x_n}{y_1^{w_{1n}} \cdots y_m^{w_{mn}}}\right)
\end{aligned}$$

Claim c) follows easily from the definition. To prove d), we proceed component by component. For $i \in \{1, \dots, m\}$, consider the i^{th} component. Let $(\mu_1, \dots, \mu_m) = \mathrm{topdeg}_W(f)$ and $(\nu_1, \dots, \nu_m) = \mathrm{topdeg}_W(g)$. Then we write $f = f' + f''$ where f' contains the terms of f whose degree has i^{th} component equal to μ_i and f'' contains the remaining terms of f. Similarly, we decompose $g = g' + g''$. Then $fg = f'g' + (f'g'' + f''g' + f''g'')$ where $f'g'$ consists of terms whose degree has i^{th} component equal to $\mu_i + \nu_i$ and the second summand consists of terms whose degree has a smaller i^{th} component. Hence the i^{th} component of $\mathrm{topdeg}_W(fg)$ is $\mu_i + \nu_i$, as we wanted to show.

For the proof of e), it suffices to use b) and d). Claim f) follows easily from a), and claim g) from the fact that dehomogenization is nothing but the substitution homomorphism $y_i \mapsto 1$ for $i = 1, \dots, m$.

Finally we prove h) and i). We write $F = y_1^{s_1} \cdots y_m^{s_m}\, \widetilde{F}$, where $\widetilde{F}$ is not divisible by any y_i. We need to show that $(\widetilde{F}^{\mathrm{deh}})^{\mathrm{hom}} = \widetilde{F}$. To this end, we note that no two terms in the support of $\widetilde{F}$ dehomogenize to the same term, since the dehomogenization of $y_1^{\alpha_1} \cdots y_m^{\alpha_m} x_1^{\beta_1} \cdots x_n^{\beta_n}$ is $x_1^{\beta_1} \cdots x_n^{\beta_n}$ and the exponents $\alpha_1, \ldots, \alpha_m$ are determined by the equation $(\alpha_1, \ldots, \alpha_m)^{\mathrm{tr}} = \deg_{\overline{W}}(\widetilde{F}) - W \cdot (\beta_1, \ldots, \beta_n)^{\mathrm{tr}}$. Moreover, the degree of $(\widetilde{F}^{\mathrm{deh}})^{\mathrm{hom}}$ equals $\deg_{\overline{W}}(\widetilde{F})$, because for every index $j \in \{1, \ldots, m\}$ there exists one term in the support of $\widetilde{F}$ which is not divisible by y_j. Altogether, the homogenization of $\widetilde{F}^{\mathrm{deh}}$ is $\widetilde{F}$. Claim i) is a special case of h). □

Notice also that $\deg_W(f) \leq_{\mathtt{Lex}} \deg_{\overline{W}}(f^{\mathrm{hom}}) = \mathrm{topdeg}_W(f)$. For $m = 1$, the first inequality is an equality. The following example shows that it may be a strict inequality when $m \geq 2$.

Example 4.3.3. Let $P = \mathbb{Q}[x_1, x_2]$ be graded by the matrix $W = \binom{1\ 0}{0\ 1}$, and let $f = x_1 + x_2$. Then we have $\mathrm{topdeg}_W(f) = (1,1)$ and $f^{\mathrm{hom}} = y_2 x_1 + y_1 x_2$. Hence we see that $\deg_W(f) = (1,0) <_{\mathtt{Lex}} \mathrm{topdeg}_W(f)$.

Based on the notions of homogenization and dehomogenization of polynomials, we can now introduce the same processes for ideals.

Definition 4.3.4. Let I be an ideal in P and J an ideal in $\overline{P}$.

a) The ideal $I^{\mathrm{hom}} = (f^{\mathrm{hom}} \mid f \in I)$ in $\overline{P}$ is called the **homogenization** of I with respect to the grading given by W.
b) The set $J^{\mathrm{deh}} = \{F^{\mathrm{deh}} \mid F \in J\}$ in P is called the **dehomogenization** of J with respect to $y_1, \ldots, y_m$.

Obviously, the set J^{deh} is an ideal in P, since it is the image of J under the surjective ring homomorphism $\overline{P} \longrightarrow P$ defined by $y_j \mapsto 1$ for $j = 1, \ldots, m$ and $x_i \mapsto x_i$ for $i = 1, \ldots, n$. In particular, we see that $J^{\mathrm{deh}} = (J : (y_1 \cdots y_m)^\infty)^{\mathrm{deh}}$. Although we have defined the dehomogenization of an arbitrary ideal in $\overline{P}$, we shall mainly be interested in the case of homogeneous ideals, thereby justifying the choice of the name. Homogenization and dehomogenization of ideals are governed by the following rules.

Proposition 4.3.5. (Rules for Homogenizing Ideals)
Let I be an ideal in P and J a homogeneous ideal in $\overline{P}$.

a) We have $(I^{\mathrm{hom}})^{\mathrm{deh}} = I$. In particular, the operation which associates to an ideal in P its homogenization is injective, and the operation which associates to a homogeneous ideal in $\overline{P}$ its dehomogenization is surjective.
b) For a homogeneous polynomial $F \in \overline{P}$, we have

$$\begin{aligned} F \in I^{\mathrm{hom}} \quad &\Longleftrightarrow \quad F^{\mathrm{deh}} \in I \\ &\Longleftrightarrow \quad F = y_1^{s_1} \cdots y_m^{s_m} f^{\mathrm{hom}} \text{ for some } f \in I \text{ and } s_1, \ldots, s_m \in \mathbb{N} \end{aligned}$$

c) For a homogeneous polynomial $F \in \overline{P}$, we have $F^{\mathrm{deh}} \in J^{\mathrm{deh}}$ if and only if $(y_1 \cdots y_m)^s F \in J$ for some $s \geq 0$.

d) We have $J \subseteq (J^{\mathrm{deh}})^{\mathrm{hom}} = J :_{\overline{P}} (y_1 \cdots y_m)^\infty$.
e) If $I \neq P$, *then* $y_1 \cdots y_m$ *is a non-zerodivisor for* $\overline{P}/I^{\mathrm{hom}}$.
f) If I *is a proper homogeneous ideal in* P, *then we have* $I^{\mathrm{hom}} = I \cdot \overline{P}$. *In particular, in this case* $y_1 \cdots y_m$ *is a non-zerodivisor for* $\overline{P}/I \cdot \overline{P}$.

Proof. First we prove a). By Proposition 4.3.2.c, we have $I \subseteq (I^{\mathrm{hom}})^{\mathrm{deh}}$. Now let $f \in (I^{\mathrm{hom}})^{\mathrm{deh}}$. By definition, the polynomial f is of the form $f = (\sum_{i=1}^r G_i g_i^{\mathrm{hom}})^{\mathrm{deh}}$, where $G_1, \ldots, G_r \in \overline{P}$ and $g_1, \ldots, g_r \in I$. Using Proposition 4.3.2.g, it follows that $f = \sum_{i=1}^r G_i^{\mathrm{deh}} g_i \in I$.

Next we show b). For a homogeneous polynomial $F \in I^{\mathrm{hom}}$, we have $F^{\mathrm{deh}} \in (I^{\mathrm{hom}})^{\mathrm{deh}} = I$ by a). Given $F \in \overline{P}$ such that $F^{\mathrm{deh}} \in I$, we have $F = y_1^{s_1} \cdots y_m^{s_m} f^{\mathrm{hom}}$ for $f = F^{\mathrm{deh}}$ and some $s_1, \ldots, s_m \in \mathbb{N}$ by Proposition 4.3.2.h. And if $F = y_1^{s_1} \cdots y_m^{s_m} f^{\mathrm{hom}}$, then we clearly have $F \in I^{\mathrm{hom}}$.

For the proof of c), we note that the implication "$\Leftarrow$" follows from the fact that $(y_1 \cdots y_m)^s F \in J$ implies $F^{\mathrm{deh}} = ((y_1 \cdots y_m)^s F)^{\mathrm{deh}} \in J^{\mathrm{deh}}$. Conversely, let a homogeneous polynomial $F \in \overline{P}$ be given such that $F^{\mathrm{deh}} \in J^{\mathrm{deh}}$. Since J^{deh} is a homomorphic image of J, there exists a polynomial $G \in J$ such that $G^{\mathrm{deh}} = F^{\mathrm{deh}}$. The homogeneous components of G are also in J, because J is a homogeneous ideal. By multiplying them with the appropriate terms in $\mathbb{T}(y_1, \ldots, y_m)$, we may assume that G is a homogeneous polynomial in J which satisfies $G^{\mathrm{deh}} = F^{\mathrm{deh}}$. Thus Proposition 4.3.2.h yields numbers $s_1, \ldots, s_m, s_1', \ldots, s_m' \in \mathbb{N}$ such that $F = y_1^{s_1} \cdots y_m^{s_m} (F^{\mathrm{deh}})^{\mathrm{hom}} = y_1^{s_1} \cdots y_m^{s_m} (G^{\mathrm{deh}})^{\mathrm{hom}}$ and $G = y_1^{s_1'} \cdots y_m^{s_m'} (G^{\mathrm{deh}})^{\mathrm{hom}}$. Altogether, this shows $(y_1 \cdots y_m)^s F \in (G) \subseteq J$ for $s = \max\{0, \max\{s_i' - s_i \mid i = 1, \ldots, m\}\}$.

The inclusion $J \subseteq (J^{\mathrm{deh}})^{\mathrm{hom}}$ in d) follows from Proposition 4.3.2.h, and $(J^{\mathrm{deh}})^{\mathrm{hom}} \subseteq J :_{\overline{P}} (y_1 \cdots y_m)^\infty$ is a consequence of c). To prove the remaining containment, we note that $J :_{\overline{P}} (y_1 \cdots y_m)^\infty$ is clearly a homogeneous ideal in $\overline{P}$. Let a homogeneous polynomial $F \in J :_{\overline{P}} (y_1 \cdots y_m)^\infty$ be given. Then there exists an $r \geq 0$ such that $(y_1 \cdots y_m)^r F \in J$. Therefore we have $F^{\mathrm{deh}} \in J^{\mathrm{deh}}$. Using Proposition 4.3.2.h, we get numbers $s_1, \ldots, s_m \in \mathbb{N}$ such that $F = y_1^{s_1} \cdots y_m^{s_m} (F^{\mathrm{deh}})^{\mathrm{hom}} \in (J^{\mathrm{deh}})^{\mathrm{hom}}$, and this was to be shown.

To prove e), we suppose that $y_1 \cdots y_m F \in I^{\mathrm{hom}}$ for some homogeneous polynomial $F \in \overline{P}$. Part a) implies $F^{\mathrm{deh}} = (y_1 \cdots y_m F)^{\mathrm{deh}} \in (I^{\mathrm{hom}})^{\mathrm{deh}} = I$, and then b) yields $F \in I^{\mathrm{hom}}$. Finally, we observe that a homogeneous polynomial in P is its own homogenization because of Proposition 4.3.2.a. This proves claim f). □

The following example shows that the inclusion in part d) of the preceding proposition can be strict.

Example 4.3.6. Let $P = K[x_1]$ be standard graded and $\overline{P} = K[y_1, x_1]$. Then we have $J \subset (J^{\mathrm{deh}})^{\mathrm{hom}}$ for the principal ideal $J = (y_1 x_1 - y_1^2)$ in $\overline{P}$, since Proposition 4.3.5.d shows $(J^{\mathrm{deh}})^{\mathrm{hom}} = J :_{\overline{P}} (y_1)^\infty = (x_1 - y_1)$ and $x_1 - y_1 \notin J$.

As a consequence of these rules, we can characterize the homogenization of an ideal as follows.

Corollary 4.3.7. (Characterization of the Homogenization)
Let I be an ideal in P. For an ideal J in $\overline{P}$, the following conditions are equivalent.

a) $J = I^{\mathrm{hom}}$
b) The ideal J is homogeneous, has dehomogenization I, and satisfies the equality $J = J :_{\overline{P}} (y_1 \cdots y_m)^\infty$.

Proof. Since a) implies b) by parts a) and d) of the proposition, it remains to prove the converse. Using the hypothesis and part d) of the proposition, we calculate $J = J :_{\overline{P}} (y_1 \cdots y_m)^\infty = (J^{\mathrm{deh}})^{\mathrm{hom}} = I^{\mathrm{hom}}$. □

Another application of the proposition is a way of actually computing homogenizations and dehomogenizations of ideals. Later we will see alternative ways to do this (see Corollary 4.3.20 and Corollary 4.4.15).

Corollary 4.3.8. (Computation of the Homogenization)
Let I be an ideal in P which is generated by polynomials $f_1, \ldots, f_r \in P$, and let J be a homogeneous ideal in $\overline{P}$ which is generated by polynomials $F_1, \ldots, F_s \in \overline{P}$.

a) The homogenization of I can be computed via the formula

$$I^{\mathrm{hom}} = (f_1^{\mathrm{hom}}, \ldots, f_r^{\mathrm{hom}}) :_{\overline{P}} (y_1 \cdots y_m)^\infty$$

b) The dehomogenization of J can be computed via the formula

$$J^{\mathrm{deh}} = (F_1^{\mathrm{deh}}, \ldots, F_s^{\mathrm{deh}})$$

Proof. Claim a) follows from part d) of the proposition, since I is the dehomogenization of $(f_1^{\mathrm{hom}}, \ldots, f_r^{\mathrm{hom}})$, and b) is a consequence of the fact that J^{deh} is an image of J under a surjective ring homomorphism. □

Is the saturation needed in the above corollary? The following example answers this affirmatively.

Example 4.3.9. Let $P = K[x_1, x_2, x_3]$ be equipped with the standard grading given by $W = (1\ 1\ 1)$, let $f_1 = x_1^2 + x_2$, $f_2 = x_1^2 + x_3$, let $I = (f_1, f_2)$, and let $\overline{P} = K[y_1, x_1, x_2, x_3]$. Then we have $(f_1^{\mathrm{hom}}, f_2^{\mathrm{hom}}) \subset I^{\mathrm{hom}}$, since $(f_1 - f_2)^{\mathrm{hom}} = (x_2 - x_3)^{\mathrm{hom}} = x_2 - x_3$ is contained in I^{hom}, but not in the ideal $(f_1^{\mathrm{hom}}, f_2^{\mathrm{hom}}) = (x_1^2 + y_1 x_2,\ x_1^2 + y_1 x_3)$ which does not contain any non-zero homogeneous element of degree one.

Our next two propositions deal with the next level of generality, namely with the behaviour of homogenization and dehomogenization under important ideal-theoretic operations.

Proposition 4.3.10. *Let I, I_1, and I_2 be ideals in P.*

a) If $I_1 \subseteq I_2$, then we have $I_1^{\mathrm{hom}} \subseteq I_2^{\mathrm{hom}}$.
b) We have $(I_1 \cap I_2)^{\mathrm{hom}} = I_1^{\mathrm{hom}} \cap I_2^{\mathrm{hom}}$.
c) We have $(\sqrt{I})^{\mathrm{hom}} = \sqrt{I^{\mathrm{hom}}}$.
d) The ideal I is prime if and only if I^{hom} is prime.

Proof. Claim a) follows immediately from Definition 4.3.4.a. In b), the inclusion "$\subseteq$" follows from a). Now let $F \in I_1^{\mathrm{hom}} \cap I_2^{\mathrm{hom}}$ be a homogeneous polynomial. Then Proposition 4.3.5 shows $F^{\mathrm{deh}} \in I_1 \cap I_2$, and Proposition 4.3.2.h yields $F = (y_1 \cdots y_m)^s (F^{\mathrm{deh}})^{\mathrm{hom}} \in (I_1 \cap I_2)^{\mathrm{hom}}$ for some $s \geq 1$. Since $I_1^{\mathrm{hom}} \cap I_2^{\mathrm{hom}}$ is a homogeneous ideal, this proves the claim.

Next we prove claim c). By Proposition 4.3.5.b, a homogeneous polynomial $F \in \overline{P}$ satisfies $F \in (\sqrt{I})^{\mathrm{hom}}$ if and only if $F^{\mathrm{deh}} \in \sqrt{I}$, and this means that there exists $s \geq 0$ such that $(F^{\mathrm{deh}})^s \in I$. By Proposition 4.3.2.g, the latter condition is equivalent to $(F^s)^{\mathrm{deh}} \in I$ for some $s \geq 0$, and thus, by Proposition 4.3.5.b, equivalent to $F^s \in I^{\mathrm{hom}}$ for some $s \geq 0$, i.e. to $F \in \sqrt{I^{\mathrm{hom}}}$.

For the proof of d) we use Proposition 1.7.12 which says that I^{hom} is a prime ideal if and only if $FG \in I^{\mathrm{hom}}$ implies $F \in I^{\mathrm{hom}}$ or $G \in I^{\mathrm{hom}}$ for homogeneous polynomials $F, G \in \overline{P}$. Notice that this proposition is applicable, because `Lex` defines a monoid ordering on $\mathbb{Z}^m$. In order to prove "$\Rightarrow$", let $F, G \in \overline{P}$ be homogeneous polynomials such that $FG \in I^{\mathrm{hom}}$. Using Proposition 4.3.2.h, we find $r, s \geq 0$ such that $F = (y_1 \cdots y_m)^r (F^{\mathrm{deh}})^{\mathrm{hom}}$ and $G = (y_1 \cdots y_m)^s (G^{\mathrm{deh}})^{\mathrm{hom}}$. Then we use $F^{\mathrm{deh}} G^{\mathrm{deh}} \in I$ to conclude that $F^{\mathrm{deh}} \in I$ or $G^{\mathrm{deh}} \in I$. Hence we have $F = (y_1 \cdots y_m)^r (F^{\mathrm{deh}})^{\mathrm{hom}} \in I^{\mathrm{hom}}$ or $G = (y_1 \cdots y_m)^s (G^{\mathrm{deh}})^{\mathrm{hom}} \in I^{\mathrm{hom}}$. Finally, the implication "$\Leftarrow$" follows from the observation that every element $f \in I$ is the dehomogenization of $f^{\mathrm{hom}} \in I^{\mathrm{hom}}$. $\square$

Although, as we have seen, homogenization is compatible with many ideal-theoretic operations, this is not true for sums and products of ideals. Our next example shows that the corresponding claims in [ZS60], Ch. VII, Thm. 17, do not hold (see also Exercise 4).

Example 4.3.11. Using the same assumptions as in Example 4.3.9, we let $I_1 = (f_1)$ and $I_2 = (f_2)$.

a) We have $I_1^{\mathrm{hom}} = (f_1^{\mathrm{hom}}) = (x_1^2 + y_1 x_2)$ and $I_2^{\mathrm{hom}} = (f_2^{\mathrm{hom}}) = (x_1^2 + y_1 x_3)$. Since the element $f_1 - f_2 = x_2 - x_3 \in I_1 + I_2$ satisfies $(f_1 - f_2)^{\mathrm{hom}} = x_2 - x_3 \in (I_1 + I_2)^{\mathrm{hom}}$ and $(f_1 - f_2)^{\mathrm{hom}} \notin I_1^{\mathrm{hom}} + I_2^{\mathrm{hom}} = (f_1^{\mathrm{hom}}, f_2^{\mathrm{hom}}) = (x_1^2 + y_1 x_2,\, x_1^2 + y_1 x_3)$, we see that the inclusion $I_1^{\mathrm{hom}} + I_2^{\mathrm{hom}} \subseteq (I_1 + I_2)^{\mathrm{hom}}$ can be strict.

b) Consider the ideals $I_3 = (f_1, x_2^2)$ and $I_4 = (f_2, x_2^2)$. Then $f_1 x_2^2 - x_2^2 f_2 = x_2^3 - x_2^2 x_3 \in I_3 \cdot I_4$ implies that we have $x_2^3 - x_2^2 x_3 \in (I_3 \cdot I_4)^{\mathrm{hom}}$. Using Corollary 4.3.8.a, we calculate $I_3^{\mathrm{hom}} = (f_1^{\mathrm{hom}},\, x_2^2) :_{\overline{P}} y_1^\infty = (x_1^2 + y_1 x_2,\, x_2^2)$ and, similarly, $I_4^{\mathrm{hom}} = (f_2^{\mathrm{hom}}, x_2^2) :_{\overline{P}} y_1^\infty = (x_1^2 + y_1 x_3, x_2^2)$. It follows that the ideal $I_3^{\mathrm{hom}} \cdot I_4^{\mathrm{hom}}$ contains no non-zero elements of degree three. Hence the inclusion $I_3^{\mathrm{hom}} \cdot I_4^{\mathrm{hom}} \subseteq (I_3 \cdot I_4)^{\mathrm{hom}}$ can be strict.

Since dehomogenization is nothing but the application of a surjective ring homomorphism, it is well-behaved with respect to many ideal-theoretic operations. The following proposition summarizes this good behaviour.

Proposition 4.3.12. *Let J, J_1, and J_2 be homogeneous ideals in $\overline{P}$.*

a) If $J_1 \subseteq J_2$, then we have $J_1^{\mathrm{deh}} \subseteq J_2^{\mathrm{deh}}$.
b) We have $(J_1 \cap J_2)^{\mathrm{deh}} = J_1^{\mathrm{deh}} \cap J_2^{\mathrm{deh}}$ and $(J_1 + J_2)^{\mathrm{deh}} = J_1^{\mathrm{deh}} + J_2^{\mathrm{deh}}$.
c) We have $(\sqrt{J})^{\mathrm{deh}} = \sqrt{J^{\mathrm{deh}}}$.
d) If J is a prime ideal of $\overline{P}$ which does not contain $y_1 \cdots y_m$, then J^{deh} is a prime ideal of P. Conversely, if J^{deh} is a prime ideal of P and $y_1 \cdots y_m$ is not a zerodivisor for $\overline{P}/J$, then J is a prime ideal of $\overline{P}$.

Proof. It is clear that a) holds, and in b) the only non-trivial part is the containment "$\supseteq$" in the first formula. Let $f \in J_1^{\mathrm{deh}} \cap J_2^{\mathrm{deh}}$. By Proposition 4.3.5.b, there exists a number $s \geq 0$ such that $(y_1 \cdots y_m)^s f^{\mathrm{hom}} \in J_1$ and $(y_1 \cdots y_m)^s f^{\mathrm{hom}} \in J_2$. Hence we get $f = ((y_1 \cdots y_m)^s f^{\mathrm{hom}})^{\mathrm{deh}} \in (J_1 \cap J_2)^{\mathrm{deh}}$.

For the proof of c), we note that, by Proposition 4.3.5.b, a polynomial $f \in P$ satisfies $f = (f^{\mathrm{hom}})^{\mathrm{deh}} \in (\sqrt{J})^{\mathrm{deh}}$ if and only if $(y_1 \cdots y_m)^s f^{\mathrm{hom}} \in \sqrt{J}$ for some $s \geq 0$. This is equivalent to $((y_1 \cdots y_m)^s f^{\mathrm{hom}})^t \in J$ for some $t \geq 0$, and therefore to $f^t = ((y_1 \cdots y_m\, f^{\mathrm{hom}})^t)^{\mathrm{deh}} \in J^{\mathrm{deh}}$ for some $t \geq 0$, i.e. to $f \in \sqrt{J^{\mathrm{deh}}}$.

It remains to prove d). First we assume that J is a prime ideal which does not contain $y_1 \cdots y_m$. Then $J^{\mathrm{deh}} \subset P$ is a proper ideal of P, since $1 \in J^{\mathrm{deh}}$ would imply $(y_1 \cdots y_m)^s \cdot 1 \in J$ for some $s \geq 0$, contradicting our assumption. Now let $f, g \in P$ such that $fg \in J^{\mathrm{deh}}$. By Propositon 4.3.5.b, there exists a number $s \geq 0$ such that $(y_1 \cdots y_m)^s (fg)^{\mathrm{hom}} = (y_1 \cdots y_m)^s f^{\mathrm{hom}} g^{\mathrm{hom}} \in J$. Since $y_1 \cdots y_m \notin J$, this yields $f^{\mathrm{hom}} \in J$ or $g^{\mathrm{hom}} \in J$. Therefore we have $f = (f^{\mathrm{hom}})^{\mathrm{deh}} \in J^{\mathrm{deh}}$ or $g = (g^{\mathrm{hom}})^{\mathrm{deh}} \in J^{\mathrm{deh}}$.

Conversely, let $F, G \in \overline{P}$ be homogeneous polynomials such that $FG \in J$. Then $(FG)^{\mathrm{deh}} = F^{\mathrm{deh}} G^{\mathrm{deh}} \in J^{\mathrm{deh}}$ implies $F^{\mathrm{deh}} \in J^{\mathrm{deh}}$ or $G^{\mathrm{deh}} \in J^{\mathrm{deh}}$. Using Proposition 4.3.5.c, we get $(y_1 \cdots y_m)^s F \in J$ for some $s \geq 0$ or $(y_1 \cdots y_m)^t G \in J$ for some $t \geq 0$. Since $y_1 \cdots y_m$ is a non-zerodivisor for $\overline{P}/J$, this implies $F \in J$ or $G \in J$. Now Propositon 1.7.12 shows that J is a prime ideal of $\overline{P}$. □

The last part of this subsection deals with the behaviour of Gröbner bases under dehomogenization. To this end, we first explain how to relate term orderings on P to term orderings on $\overline{P}$. Recall that $\overline{P}$ is graded by $\overline{W} = (\mathcal{I}_m \mid W)$, and observe that $\overline{W}$ is non-negative (resp. positive) if and only if W is non-negative (resp. positive).

Definition 4.3.13. Let σ be a monoid ordering on $\mathbb{T}^n = \mathbb{T}(x_1, \ldots, x_n)$ and let us consider the relation $\overline{\sigma}^W$ on $\mathbb{T}^{m+n} = \mathbb{T}(y_1, \ldots, y_m, x_1, \ldots, x_n)$

which is defined by the following rule. Given two terms $t_1, t_2 \in \mathbb{T}^{m+n}$, we say that $t_1 \geq_{\overline{\sigma}^W} t_2$ if either we have $\deg_{\overline{W}}(t_1) >_{\mathtt{Lex}} \deg_{\overline{W}}(t_2)$ or we have $\deg_{\overline{W}}(t_1) = \deg_{\overline{W}}(t_2)$ and $t_1^{\mathrm{deh}} \geq_\sigma t_2^{\mathrm{deh}}$. We call $\overline{\sigma}^W$ the **extension** of σ by W. If it is clear which grading we are considering, we shall simply denote it by $\overline{\sigma}$.

Notice that $\overline{\sigma}$ is not necessarily a true extension of σ in the sense that its restriction to $\mathbb{T}^n$ need not coincide with σ. However, this is the case if σ is compatible with $\deg_W$. The extension of a monoid ordering satisfies the following properties.

Proposition 4.3.14. *Let σ be a monoid ordering on $\mathbb{T}^n$, and let $\overline{\sigma}$ be its extension by W.*

a) The relation $\overline{\sigma}$ is a monoid ordering on $\mathbb{T}^{m+n}$ which is compatible with $\deg_{\overline{W}}$.
b) If W is non-negative and the restriction of σ to the terms in P_0 is a term ordering, then $\overline{\sigma}$ is a term ordering on $\mathbb{T}^{m+n}$.
c) If W is positive, then $\overline{\sigma}$ is a term ordering on $\mathbb{T}^{m+n}$.
d) If σ is of the form $\sigma = \mathtt{Ord}(V)$ for a non-singular matrix $V \in \mathrm{Mat}_n(\mathbb{Z})$, then we have $\overline{\sigma} = \mathtt{Ord}\begin{pmatrix} \mathcal{I}_m & W \\ 0 & V \end{pmatrix}$.

Proof. First we prove a). In order to check antisymmetry, we assume that $t = y_1^{\alpha_1} \cdots y_m^{\alpha_m} x_1^{\beta_1} \cdots x_n^{\beta_n}$ and $t' = y_1^{\alpha_1'} \cdots y_m^{\alpha_m'} x_1^{\beta_1'} \cdots x_n^{\beta_n'}$ satisfy $t \geq_{\overline{\sigma}} t'$ and $t' \geq_{\overline{\sigma}} t$. Then we have $\deg_{\overline{W}}(t) = \deg_{\overline{W}}(t')$ and $t^{\mathrm{deh}} = (t')^{\mathrm{deh}}$, and therefore $\overline{W} \cdot (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)^{\mathrm{tr}} = \overline{W} \cdot (\alpha_1', \ldots, \alpha_m', \beta_1', \ldots, \beta_n')^{\mathrm{tr}}$ and $(\beta_1, \ldots, \beta_n) = (\beta_1', \ldots, \beta_n')$. By plugging the second equation into the first one, we find $\mathcal{I}_m \cdot (\alpha_1, \ldots, \alpha_m)^{\mathrm{tr}} = \mathcal{I}_m \cdot (\alpha_1', \ldots, \alpha_m')^{\mathrm{tr}}$, and hence $t = t'$. The other axioms for monoid orderings are easy to check, and the compatibility of $\overline{\sigma}$ with $\deg_{\overline{W}}$ is an immediate consequence of the definition.

Now we show b). Since $\overline{\sigma}$ is compatible with $\deg_{\overline{W}}$, we have $y_i >_{\overline{\sigma}} 1$ for $i = 1, \ldots, m$. For the same reason, if $i \in \{1, \ldots, n\}$ is such that $\deg_{\overline{W}}(x_i) = \deg_W(x_i) >_{\mathtt{Lex}} 0$, then $x_i >_{\overline{\sigma}} 1$. Finally, if $i \in \{1, \ldots, n\}$ is such that $\deg_W(x_i) = 0$, then the hypothesis implies $x_i >_\sigma 1$. By definition of $\overline{\sigma}$, we obtain $x_i >_{\overline{\sigma}} 1$.

Claim c) follows from b), because if W is positive, we have $P_0 = K$. Finally, claim d) follows from the definition of $\overline{\sigma}$ in a straightforward manner. $\square$

Let us clarify the details of this proposition using an example.

Example 4.3.15. Let $P = \mathbb{Q}[x_1, x_2, x_3]$ be graded by $W = (1\ 1\ 0)$, and let $\sigma = \mathtt{Ord}(V)$ be the monoid ordering on $\mathbb{T}^3$ given by $V = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Notice that the grading given by W is not positive, and that σ is not a term ordering. However, the extension $\overline{\sigma}$ of σ by W is a term ordering by part b) of the proposition, and we have $\overline{\sigma} = \mathtt{Ord}\begin{pmatrix} 1 & W \\ 0 & V \end{pmatrix}$ by d).

The following lemma will prove to be a useful tool.

Lemma 4.3.16. *Let $G \in \overline{P}$ be a non-zero homogeneous polynomial, and let $f \in P$ be a non-zero polynomial. Then*

a) There exist $s_1, \dots, s_m \in \mathbb{N}$ such that $\mathrm{LT}_{\overline{\sigma}}(G) = y_1^{s_1} \cdots y_m^{s_m} \, \mathrm{LT}_\sigma(G^{\mathrm{deh}})$. Hence we have $(\mathrm{LT}_{\overline{\sigma}}(G))^{\mathrm{deh}} = \mathrm{LT}_\sigma(G^{\mathrm{deh}})$.

b) If σ is compatible with $\deg_W$, then

$$\mathrm{LT}_{\overline{\sigma}}(f^{\mathrm{hom}}) = y^{\mu - d_1} \, \mathrm{LT}_\sigma(\mathrm{DF}_W(f)) = y^{\mu - d_1} \, \mathrm{LT}_\sigma(f)$$

where $\mu = \mathrm{topdeg}(f)$ and $d_1 = \deg_W(\mathrm{DF}_W(f))$.

c) If σ is compatible with $\deg_W$ and $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ is a one-row matrix, then

$$\mathrm{LT}_{\overline{\sigma}}(f^{\mathrm{hom}}) = \mathrm{LT}_\sigma(\mathrm{DF}_W(f)) = \mathrm{LT}_\sigma(f)$$

Proof. The proof of a) follows from the definition of $\overline{\sigma}$. The proof of b) follows from $f = (f^{\mathrm{hom}})^{\mathrm{deh}}$ and Proposition 4.3.2.a. The proof of c) follows from b) and the fact that $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ implies $\mu = d_1$. □

The assumption that σ is degree compatible cannot be dropped in part b) of this lemma, as the following example shows.

Example 4.3.17. Let $P = K[x_1, x_2]$ graded by $W = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, let $f = x_1 + x_2$, and let $G = f^{\mathrm{hom}} = y_2 x_1 + y_1 x_2$. The term ordering $\sigma = \mathtt{Ord}(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ is clearly not compatible with $\deg_W$. Nevertheless, we have $\mathrm{LT}_{\overline{\sigma}}(G) = y_1 x_2 = y_1 \, \mathrm{LT}_\sigma(G^{\mathrm{deh}})$ in accord with a). But $\mathrm{LT}_{\overline{\sigma}}(f^{\mathrm{hom}}) = y_1 x_2$ does not agree with the right-hand side, since $\mu = (1,1)$ and $d_1 = (1,0)$ and $\mathrm{DF}_W(f) = x_1$ yield $y^{\mu - d_1} \, \mathrm{LT}_\sigma(\mathrm{DF}_W(f)) = y_2 x_1$.

If we use the lexicographic term ordering instead, it is clearly compatible with $\deg_W$, and we obtain $\mathrm{LT}_{\overline{\mathtt{Lex}}}(f^{\mathrm{hom}}) = y_2 x_1 = y^{\mu - d_1} \, \mathrm{LT}_{\mathtt{Lex}}(\mathrm{DF}_W(f))$ in agreement with claim b).

Our next proposition shows how Gröbner bases behave under dehomogenization. Notice that under the hypotheses of the following proposition, the ordering $\overline{\sigma}$ is a term ordering by Proposition 4.3.14.b.

Proposition 4.3.18. (Dehomogenization of Gröbner Bases)
Let P be non-negatively graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let σ be a term ordering on $\mathbb{T}^n$, let $J \subseteq \overline{P}$ be a non-zero homogeneous ideal, and let $\{G_1, \dots, G_s\}$ be a homogeneous $\overline{\sigma}$-Gröbner basis of J. Then $\{G_1^{\mathrm{deh}}, \dots, G_s^{\mathrm{deh}}\}$ is a σ-Gröbner basis of J^{deh}.

Proof. It suffices to show for every non-zero polynomial $f \in J^{\mathrm{deh}}$ that there exists an index $i \in \{1, \dots, s\}$ such that $\mathrm{LT}_\sigma(f)$ is a multiple of $\mathrm{LT}_\sigma(G_i^{\mathrm{deh}})$. Since $f^{\mathrm{hom}} \in (J^{\mathrm{deh}})^{\mathrm{hom}}$, Proposition 4.3.5.c yields a number $\ell \in \mathbb{N}$ such that $(y_1 \cdots y_m)^\ell f^{\mathrm{hom}} \in J$. By assumption, there exist an index $i \in \{1, \dots, s\}$ and a term $t \in \mathbb{T}^n$ such that $(y_1 \cdots y_m)^\ell \, \mathrm{LT}_{\overline{\sigma}}(f^{\mathrm{hom}}) = y_1^{\alpha_1} \cdots y_m^{\alpha_m} \, t \, \mathrm{LT}_{\overline{\sigma}}(G_i)$.

By part a) of the lemma, there exist numbers $s_1, \ldots, s_m \in \mathbb{N}$ such that $\mathrm{LT}_{\overline{\sigma}}(f^{\mathrm{hom}}) = y_1^{s_1} \cdots y_m^{s_m} \, \mathrm{LT}_\sigma(f)$. If we dehomogenize the two expressions for $\mathrm{LT}_{\overline{\sigma}}(f^{\mathrm{hom}})$ and use part a) of the lemma again, we obtain the equality $\mathrm{LT}_\sigma(f) = t\,(\mathrm{LT}_{\overline{\sigma}}(G_i))^{\mathrm{deh}} = t\,\mathrm{LT}_\sigma(G_i^{\mathrm{deh}})$, as claimed. □

4.3.B Macaulay Bases and Homogenization

In this subsection we consider different aspects of the process of homogenizing an ideal, specifically its relation to Macaulay bases and its geometric interpretation. Since it turns out that the results we are going to discuss do not hold in general, we restrict to the case when $m = 1$ and the polynomial ring $P = K[x_1, \ldots, x_n]$ is graded by a row of positive integers $W = (w_1 \ \cdots \ w_n) \in \mathrm{Mat}_{1,n}(\mathbb{Z})$. Moreover, since we now have only one homogenizing indeterminate, it is customary to denote it by x_0 rather than by y_1. Hence we let $\overline{P} = K[x_0, \ldots, x_n]$ be graded by the matrix $\overline{W} = (1 \ w_1 \ \cdots \ w_n)$. Notice that we have $\deg_W(f) = \deg_{\overline{W}}(f^{\mathrm{hom}})$ and $f^{\mathrm{hom}}(0, x_1, \ldots, x_n) = \mathrm{DF}_W(f)$ for every non-zero polynomial $f \in P$.

In Example 4.3.9 we saw that, to compute the homogenization of an ideal (see Corollary 4.3.8.a), it is usually not sufficient to homogenize a system of generators. Sometimes it is essential to perform the saturation with respect to x_0. This suggests the following natural question. Which systems of generators of I have the property that their homogenizations generate I^{hom}? Our next theorem says that the answer to this question is "Macaulay bases".

Theorem 4.3.19. (Macaulay Bases and Homogenization)
Let I be a proper ideal of P, and let $\{f_1, \ldots, f_r\} \subset P \setminus \{0\}$ be a system of generators of I. Then the following conditions are equivalent.

- *a) We have $I^{\mathrm{hom}} = (f_1^{\mathrm{hom}}, \ldots, f_r^{\mathrm{hom}})$.*
- *b) The set $\{f_1, \ldots, f_r\}$ is a Macaulay basis of I with respect to the grading given by W.*

Proof. First we show that a) implies b). Given a polynomial $f \in I \setminus \{0\}$, we use the hypothesis to find a representation $f^{\mathrm{hom}} = \sum_{i=1}^r G_i f_i^{\mathrm{hom}}$, where G_i is a homogeneous polynomial in $\overline{P}$ of degree $\deg_W(f) - \deg_W(f_i)$. If we substitute $x_0 \mapsto 0$ in this equation and use Proposition 4.3.2, we obtain $\mathrm{DF}_W(f) = \sum_{i=1}^r G_i(0, x_1, \ldots, x_n)\, \mathrm{DF}_W(f_i)$, as we had to show.

Now we prove that b) implies a). The ideal $J = (f_1^{\mathrm{hom}}, \ldots, f_r^{\mathrm{hom}})$ is a homogeneous ideal in $\overline{P}$ which is contained in I^{hom}. Using induction on the degree d, we shall prove $J_d = I_d^{\mathrm{hom}}$ for all $d \geq 0$. For $d = 0$, we have $I_0^{\mathrm{hom}} = J_0 = (0)$, since $I \subset P$ means that I contains no non-zero constant polynomial. For $d > 0$, we let $G \in I_d^{\mathrm{hom}}$. By Proposition 4.3.2.h, the polynomial G is of the form $G = x_0^s (G^{\mathrm{deh}})^{\mathrm{hom}}$ for some $s \geq 0$, and Proposition 4.3.5.e implies $(G^{\mathrm{deh}})^{\mathrm{hom}} \in I_{d-s}^{\mathrm{hom}}$. Thus the claim follows from the induction hypothesis $I_{d-s}^{\mathrm{hom}} = J_{d-s}$ if $s > 0$. Hence we may assume that $s = 0$, i.e. that $G = (G^{\mathrm{deh}})^{\mathrm{hom}}$.

By assumption and Corollary 1.7.11, there are homogeneous polynomials $g_1, \ldots, g_r \in P$ of degree $\deg_W(g_i) = \deg_W(G^{\mathrm{deh}}) - \deg_W(f_i)$ such that $\mathrm{DF}_W(G^{\mathrm{deh}}) = \sum_{i=1}^r g_i \, \mathrm{DF}_W(f_i)$. Hence the polynomial $G^{\mathrm{deh}} \in P$ is of the form $G^{\mathrm{deh}} = \sum_{i=1}^r g_i f_i + h$ with a polynomial $h \in P$ whose degree is $\deg_W(h) < \deg_W(G^{\mathrm{deh}}) = d$. Using Proposition 4.3.2.c, we obtain $G = (G^{\mathrm{deh}})^{\mathrm{hom}} = \sum_{i=1}^r g_i f_i^{\mathrm{hom}} + x_0^t H$ with $t > 0$ and a homogeneous polynomial $H \in \overline{P}$ which is not divisible by x_0. Consequently, we have $H = (H^{\mathrm{deh}})^{\mathrm{hom}}$ and $x_0^t H \in I_d^{\mathrm{hom}}$. Therefore Proposition 4.3.5.e and the induction hypothesis show $H \in I_{d-t}^{\mathrm{hom}} = J_{d-t}$. Altogether, we find $G \in J_d$ as claimed. □

For instance, for a principal ideal $I = (f)$ in P, this theorem shows $I^{\mathrm{hom}} = (f^{\mathrm{hom}})$, since $\{f\}$ is a Macaulay basis of I. Exercises 8 and 9 show that both implications of this theorem can fail in the case $m \geq 2$.

Recall that Corollary 4.3.8 provides a method for computing the homogenization of an ideal using saturation. Theorem 4.3.19 can be used to compute homogenizations in another way if $m = 1$ and P is positively graded.

Corollary 4.3.20. *Given an ideal I in P, consider the following sequence of instructions.*

1) *Choose a non-singular matrix $V \in \mathrm{Mat}_n(\mathbb{Z})$ of the form $\binom{W}{W'}$, where $W' \in \mathrm{Mat}_{n-1,n}(\mathbb{Z})$.*
2) *Compute a Gröbner basis $\{g_1, \ldots, g_s\}$ of I with respect to $\mathtt{Ord}(V)$.*
3) *Return the ideal $(g_1^{\mathrm{hom}}, \ldots, g_s^{\mathrm{hom}})$ and stop.*

This is an algorithm which computes $I^{\mathrm{hom}} = (g_1^{\mathrm{hom}}, \ldots, g_s^{\mathrm{hom}})$.

Proof. It suffices to combine Corollary 4.2.16 and the theorem. □

Proposition 4.3.21. (Homogenization of Gröbner Bases)
Let σ be a term ordering on $\mathbb{T}^n$ which is compatible with $\deg_W$, let I be an ideal in P, and let $\{g_1, \ldots, g_s\}$ be a σ-Gröbner basis of I. Then $\{g_1^{\mathrm{hom}}, \ldots, g_s^{\mathrm{hom}}\}$ is a $\overline{\sigma}$-Gröbner basis of I^{hom}.

Proof. It suffices to show that, for every non-zero homogeneous polynomial $G \in I^{\mathrm{hom}}$, there exists an index $i \in \{1, \ldots, s\}$ such that $\mathrm{LT}_{\overline{\sigma}}(G)$ is a multiple of $\mathrm{LT}_{\overline{\sigma}}(g_i^{\mathrm{hom}})$. Since $G^{\mathrm{deh}} \in I \setminus \{0\}$ by Proposition 4.3.5, and since $\{g_1, \ldots, g_s\}$ is a σ-Gröbner basis of I, we know that $\mathrm{LT}_\sigma(G^{\mathrm{deh}})$ is a multiple of $\mathrm{LT}_\sigma(g_i)$ for some $i \in \{1, \ldots, s\}$. By Lemma 4.3.16.c, it follows that $\mathrm{LT}_{\overline{\sigma}}((G^{\mathrm{deh}})^{\mathrm{hom}})$ is a multiple of $\mathrm{LT}_{\overline{\sigma}}(g_i^{\mathrm{hom}})$. Now the claim is a consequence of Proposition 4.3.2.h. □

By this proposition, the polynomials $g_1^{\mathrm{hom}}, \ldots, g_s^{\mathrm{hom}}$ computed by the algorithm of Corollary 4.3.20 are actually a Gröbner basis of I^{hom} with respect to $\overline{\sigma}$, where $\sigma = \mathtt{Ord}\binom{W}{W'}$. On the other hand, Exercise 9 shows that the proposition can fail in the case $m \geq 2$.

Our next topic is another interpretation of the homogenization process which has a decidedly more geometric flavor. The basic result is part b) of the following theorem. Notice that there exists a natural homomorphism of K-algebras $K[x_0] \longrightarrow \overline{P}/I^{\rm hom}$ which allows us to consider $\overline{P}/I^{\rm hom}$ as a $K[x_0]$-module.

Theorem 4.3.22. (Homogenization as a Free Module)
Let I be a proper ideal in P.

a) We have isomorphisms of K-algebras $\overline{P}/(I^{\rm hom} + (x_0)) \cong P/\operatorname{DF}_W(I)$ and $\overline{P}/(I^{\rm hom} + (x_0 - c)) \cong P/I$ for every $c \in K \setminus \{0\}$.

b) The ring $\overline{P}/I^{\rm hom}$ is a free $K[x_0]$-module.

Proof. First we construct the two isomorphisms in a). By Proposition 4.3.2.a, we have $f^{\rm hom}(0, x_1, \dots, x_n) = \operatorname{DF}_W(f)$ for all $f \in P \setminus \{0\}$. Thus the preimage of $\operatorname{DF}_W(I)$ under the homomorphism $\overline{P} \longrightarrow \overline{P}$ defined by $x_0 \mapsto 0$ and $x_i \mapsto x_i$ for $i = 1, \dots, n$ is the ideal $I^{\rm hom} + (x_0)$. This proves the first claim in a). The second isomorphism is induced by the homomorphism $\varphi : \overline{P} \longrightarrow P$ defined by $\varphi(x_0) = c$ and $\varphi(x_i) = c^{w_i} x_i$ for $i = 1, \dots, n$. Using Proposition 4.3.2.a, we see that φ maps $f^{\rm hom} \in I^{\rm hom}$ to $c^d f$ for $f \in I \setminus \{0\}$ and $d = \deg_W(f)$. Hence φ maps $I^{\rm hom} + (x_0 - c)$ onto I.

It remains to prove that $\varphi^{-1}(I)$ is contained in $I^{\rm hom} + (x_0 - c)$. Let $f \in \overline{P}$ be a non-zero polynomial such that $g = \varphi(f) \in I$. Then the polynomial $h = c^{\deg_W(g)} f - g^{\rm hom}$ satisfies $\varphi(h) = c^{\deg_W(g)} g - c^{\deg_W(g)} g = 0$, and therefore $h = c^{\deg_W(g)} f - g^{\rm hom} \in \operatorname{Ker}(\varphi) = (x_0 - c)$. Since $c \neq 0$, we obtain $f \in I^{\rm hom} + (x_0 - c)$, as we wanted to show. Thus φ induces the desired isomorphism.

Now we prove b). We choose a term ordering σ on $\mathbb{T}^n$ which is compatible with $\deg_W$ and let $\{f_1, f_2, \dots, f_s\}$ be a σ-Gröbner basis of I. By Macaulay's Basis Theorem 1.5.7, the residue classes of the terms in $B = \mathbb{T}^n \setminus \operatorname{LT}_\sigma\{I\}$ form a K-basis of P/I. Our goal is to show that the residue classes of the elements of B in $\overline{P}/I^{\rm hom}$ form a $K[x_0]$-basis. By Proposition 4.3.21, the set $\{f_1^{\rm hom}, \dots, f_s^{\rm hom}\}$ is a $\overline{\sigma}$-Gröbner basis of $I^{\rm hom}$. By Lemma 4.3.16.b, we have $\operatorname{LT}_\sigma(f_i) = \operatorname{LT}_{\overline{\sigma}}(f_i^{\rm hom})$ for $i = 1, \dots, s$. Therefore it follows that the set $\mathbb{T}(x_0, \dots, x_n) \setminus \operatorname{LT}_{\overline{\sigma}}\{I^{\rm hom}\}$ equals $\cup_{i=0}^{\infty} x_0^i B$. Hence the residue classes of the elements of this set form a K-basis of $\overline{P}/I^{\rm hom}$. Clearly, this implies that B is a basis of $\overline{P}/I^{\rm hom}$ as a $K[x_0]$-module. $\square$

In the language of algebraic geometry, the preceding theorem says that, for any ideal $I \subset P$, the algebra homomorphism $K[x_0] \longrightarrow \overline{P}/I^{\rm hom}$ defines a **flat family** whose **special fiber** is $P/\operatorname{DF}_W(I)$ and whose **general fiber** is P/I. Since the ring $K[x_0]$ is a principal ideal domain, flatness (a concept not defined here) is equivalent to freeness (as in part b) of the theorem). Theorem 4.3.19 allows us to characterize Macaulay bases by their behaviour with respect to this flat family.

Proposition 4.3.23. (Macaulay Bases and the Flat Family) *Let I be a proper ideal in P, and let $\{f_1, \ldots, f_s\} \subseteq P$ be a system of generators of I. Then the following conditions are equivalent.*

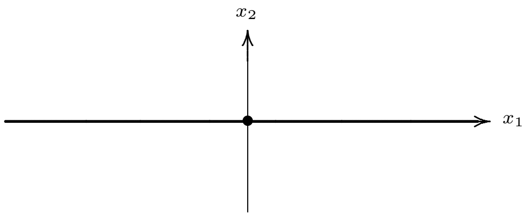
a) The set $\{f_1, \ldots, f_s\}$ is a Macaulay basis of I.
b) For every $c \in K$, the residue classes of $\{f_1^{\mathrm{hom}}, \ldots, f_s^{\mathrm{hom}}\}$ in the ring $\overline{P}/(x_0 - c)$ generate the ideal $(I^{\mathrm{hom}} + (x_0 - c))/(x_0 - c)$.

Proof. By Theorem 4.3.19, condition a) implies $I^{\mathrm{hom}} = (f_1^{\mathrm{hom}}, \ldots, f_s^{\mathrm{hom}})$, and hence b). Conversely, if we assume b) and apply it to the case $c = 0$, we see that $\{\mathrm{DF}_W(f_1), \ldots, \mathrm{DF}_W(f_s)\}$ generates $\mathrm{DF}_W(I)$. Thus $\{f_1, \ldots, f_s\}$ is a Macaulay basis of I. □

Another way to phrase the above statement is that if these conditions hold, then the residue classes of $\{f_1^{\mathrm{hom}}, \ldots, f_s^{\mathrm{hom}}\}$ generate the defining ideal of every fiber of the flat family $K[x_0] \longrightarrow \overline{P}/I^{\mathrm{hom}}$. Let us explain the geometric content of this proposition with a few examples.

Example 4.3.24. Let $P = \mathbb{Q}[x_1, x_2]$ be graded by the matrix $W = (1\ 2)$, let $f_1 = x_1^2 + x_2^2$, let $f_2 = x_1 x_2$, and let $I = (f_1, f_2)$. Then we obtain $f_1^{\mathrm{hom}} = x_0^2 x_1^2 + x_2^2$ and $f_2^{\mathrm{hom}} = x_1 x_2$. The ideal I corresponds geometrically to the origin $\mathcal{Z}(I) = \{(0,0)\} \subseteq \mathbb{A}^2_{\mathbb{Q}}$.

Now we look at the ideal $\tilde{I} = (f_1^{\mathrm{hom}}, f_2^{\mathrm{hom}})$ in $\overline{P}$ and at its residue class ideals $(\tilde{I} + (x_0 - c))/(x_0 - c) \subseteq \overline{P}/(x_0 - c) \cong P$ for various $c \in \mathbb{Q}$. If $c \neq 0$, then that residue class ideal is isomorphic to the ideal $I_c = (c^2 x_1^2 + x_2^2,\ x_1 x_2) \subseteq P$ which corresponds to $\mathcal{Z}(I_c) = \{(0,0)\}$. But if $c = 0$, then the residue class ideal is $I_0 = (x_2^2,\ x_1 x_2)$ and corresponds to the line $\mathcal{Z}(I_0) = \{(\lambda, 0) \mid \lambda \in \mathbb{Q}\}$.

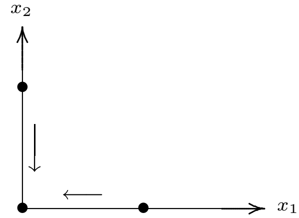


Thus the family $K[x_0] \longrightarrow \overline{P}/\tilde{I}$ has general fiber $\{(0,0)\}$, i.e. a point, and a special fiber which is a line. This means that the family is not flat. The reason is that $\{f_1, f_2\}$ is not a Macaulay basis of I. If we add the third polynomial $f_3 = x_1^3$, then $\{f_1, f_2, f_3\}$ is a Macaulay basis of I and the family $K[x_0] \longrightarrow \overline{P}/(f_1^{\mathrm{hom}}, f_2^{\mathrm{hom}}, f_3^{\mathrm{hom}})$ is flat. Its special fiber $\overline{P}/(I^{\mathrm{hom}} + (x_0)) \cong P/(x_1^3,\ x_1 x_2,\ x_2^2)$ corresponds again to the origin only, because $\mathcal{Z}((x_1^3,\ x_1 x_2,\ x_2^2)) = \{(0,0)\}$.

Example 4.3.25. Let $P = \mathbb{Q}[x_1, x_2]$ be standard graded, let $f_1 = x_1 x_2$, let $f_2 = x_1^2 - x_1$, let $f_3 = x_2^2 - x_2$, and let $I = (f_1, f_2, f_3)$. Geometrically, the ideal I corresponds to three points $\mathcal{Z}(I) = \{(0,0),\ (1,0),\ (0,1)\} \subseteq \mathbb{A}^2_{\mathbb{Q}}$.

It is easy to check that $\{f_1, f_2, f_3\}$ is a Macaulay basis of I. Thus we have $I^{\mathrm{hom}} = (f_1^{\mathrm{hom}}, f_2^{\mathrm{hom}}, f_3^{\mathrm{hom}})$.

Let us look at the family $K[x_0] \longrightarrow \overline{P}/I^{\mathrm{hom}}$. For $c \in \mathbb{Q} \setminus \{0\}$, the fiber $\overline{P}/(I^{\mathrm{hom}} + (x_0 - c))$ corresponds to $\mathcal{Z}((x_1x_2, x_1^2 - cx_1, x_2^2 - cx_2)) = \{(0,0), (c,0), (0,c)\}$. For $c = 0$, the fiber $\overline{P}/(I^{\mathrm{hom}} + (x_0))$ corresponds to $\{(0,0)\}$.



Thus we can interpret the process of letting $c \longrightarrow 0$ as the process of moving the two points $(c,0)$, $(0,c)$ towards the origin. In the special fiber, the three points come together at the origin. This suggests that we should consider the special fiber as a point of **multiplicity** three, a topic which we shall examine more closely in Chapter 5.

Exercise 1. Let K be a field and $P = K[x_1, \ldots, x_n]$. Find a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ such that the grading defined by W is of positive type, but the grading on $K[y_1, \ldots, y_m, x_1, \ldots, x_n]$ defined by $\overline{W} = (\mathcal{I}_m | W)$ is not of positive type.

Exercise 2. Let K be an algebraically closed field, let $\overline{P} = K[x_0, x_1]$ be standard graded, let $d \geq 1$, and let $F \in \overline{P}$ be a non-zero homogeneous polynomial of degree d. Show that F admits a factorization $F = F_1 \cdots F_d$ with homogeneous polynomials $F_1, \ldots, F_d \in P_1$.

Exercise 3. In the context of Subsection 4.3.A, let $f_1, \ldots, f_s \in P$ and $I = (f_1, \ldots, f_s)$. Prove that $I^{\mathrm{hom}} = (f_1^{\mathrm{hom}}, \ldots, f_s^{\mathrm{hom}}) \cdot \overline{P}_{y_1 \cdots y_m} \cap \overline{P}$ where $\overline{P}_{y_1 \cdots y_m}$ is the localization of $\overline{P}$ in the element $y_1 \cdots y_m$.

Exercise 4. Let K be a field, let $P = K[x_1, \ldots, x_n]$ be graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let I_1 and I_2 be two ideals in P, and let $\overline{P} = K[y_1, \ldots, y_m, x_1, \ldots, x_n]$.

a) Show that $(I_1 + I_2)^{\mathrm{hom}} = (I_1^{\mathrm{hom}} + I_2^{\mathrm{hom}}) :_{\overline{P}} (y_1 \cdots y_m)^{\infty}$.
b) Show that $I_1^{\mathrm{hom}} \cdot I_2^{\mathrm{hom}} = (I_1 \cdot I_2)^{\mathrm{hom}}$ if I_1 and I_2 are principal ideals.

Exercise 5. Let K be a field, let $P = K[x_1, \ldots, x_n]$ be standard graded, let $I \subseteq P$ be an ideal, let $f \in P$ be a non-zerodivisor for P/I, and let $\overline{P} = K[x_0, \ldots, x_n]$.

a) Show that f^{hom} is a non-zerodivisor for $\overline{P}/I^{\mathrm{hom}}$.
b) Find an example in which the inclusion $I^{\mathrm{hom}} + (f^{\mathrm{hom}}) \subseteq (I + (f))^{\mathrm{hom}}$ is strict. Under which additional hypothesis on f is this an equality?

Exercise 6. Let K be a field, and let J be a homogeneous ideal in $K[y_1,\dots,y_m,x_1,\dots,x_n]$. Show that J^{deh} can be a prime ideal without J having this property.

Exercise 7. In the situation of Subsection 4.3.B, let $I \subset P$ be a proper ideal in P, and let $\varphi : K[x_0] \longrightarrow \overline{P}/I^{\mathrm{hom}}$ be the canonical ring homomorphism.

a) Show that φ is injective.
b) Prove that, for every non-zero polynomial $f \in K[x_0]$, the image $\varphi(f)$ is a non-zerodivisor in $\overline{P}/I^{\mathrm{hom}}$.

Exercise 8. Let K be a field, and let $P = K[x_1,x_2,x_3,x_4]$ be graded by $W = \left(\begin{smallmatrix} 1&1&1&0\\ 1&0&0&1 \end{smallmatrix}\right)$.

a) Using CoCoA, show that the homogenizations of the polynomials $f_1 = x_1x_2 - x_3^2$ and $f_2 = x_2^2 - x_4^2$ generate the ideal $(f_1,f_2)^{\mathrm{hom}}$.
b) Use the polynomial $f_3 = x_1x_4^2 - x_2x_3^2$ to show that $\{f_1,f_2\}$ is not a Macaulay basis of (f_1,f_2) with respect to the grading given by W.

Exercise 9. Let K be a field, and let $P = K[x_1,x_2,x_3,x_4]$ be graded by $W = \left(\begin{smallmatrix} 1&1&0&0\\ 0&0&1&1 \end{smallmatrix}\right)$.

a) Show that the polynomials $f_1 = x_1 - x_3$ and $f_2 = x_2 - x_4$ form a Macaulay basis of the ideal $I = (f_1,f_2)$ with respect to the grading given by W.
b) Use the polynomial $f_3 = x_1x_4 - x_2x_3$ to prove that $(f_1^{\mathrm{hom}}, f_2^{\mathrm{hom}})$ is strictly contained in I^{hom}.

Exercise 10. In the setting of Proposition 4.3.21, let G be the reduced σ-Gröbner basis of I. Prove that $G^{\mathrm{hom}} = \{g_1^{\mathrm{hom}},\dots,g_s^{\mathrm{hom}}\}$ is the reduced $\overline{\sigma}$-Gröbner basis of I^{hom}.

Exercise 11. Let $P = \mathbb{Q}[x_1,x_2]$ be graded by the identity matrix, let $I = (x_1+x_2,\, x_1^2,\, x_2^2)$, and let $\overline{P} = \mathbb{Q}[y_1,y_2,x_1,x_2]$. Consider $\overline{P}/I^{\mathrm{hom}}$ as a $K[y_1,y_2]$-module via the natural homomorphism $K[y_1,y_2] \longrightarrow \overline{P}/I^{\mathrm{hom}}$ and show that is not free.

Tutorial 49: Homogenization of Modules

I'm against a homogenized society,
because I want the cream to rise.
(Robert Frost)

The definitions in this section require only slight modification and extension in order to generalize the process of homogenization to vectors of polynomials contained in submodules of graded free modules over a graded polynomial ring. If you work out the following problems, you attain a theory of homogenization which has the same level of generality as the remainder of this chapter.

Let K be a field, let $P = K[x_1, \dots, x_n]$ be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let $\delta_1, \dots, \delta_r \in \mathbb{Z}^m$, and let F be the graded free P-module $F = \bigoplus_{i=1}^r P(-\delta_i)$. We introduce homogenizing indeterminates $y_1, \dots, y_m$ and equip the polynomial ring $\overline{P} = K[y_1, \dots, y_m, x_1, \dots, x_n]$ with the grading defined by $\overline{W} = (\mathcal{I}_m \mid W)$. Then $\overline{F} = \bigoplus_{i=1}^r \overline{P}(-\delta_i)$ is a graded free $\overline{P}$-module in the natural way.

Given a vector $v \in F \setminus \{0\}$, we decompose it into its homogeneous components $v = v_1 + \cdots + v_s$, where v_i is homogeneous of degree $d_i \in \mathbb{Z}^m$. Then we let $d \in \mathbb{Z}^m$ be the componentwise maximum of $\{d_1, \dots, d_s\}$, call it the **top degree** of v, and denote it by $\mathrm{topdeg}_W(v)$. The vector $v^{\mathrm{hom}} = y^{d-d_1} v_1 + \cdots + y^{d-d_s} v_s \in \overline{F}$ is called the **homogenization** of v with respect to the grading given by W. For $v = 0$, we set $v^{\mathrm{hom}} = 0$.

Similarly, given a vector $\overline{v} \in \overline{F}$, we write $\overline{v} = (\bar{f}_1, \dots, \bar{f}_r)$ with polynomials $\bar{f}_1, \dots, \bar{f}_r \in \overline{P}$. Then we call $\overline{v}^{\,\mathrm{deh}} = (\bar{f}_1^{\,\mathrm{deh}}, \dots, \bar{f}_r^{\,\mathrm{deh}}) \in F$ the **dehomogenization** of $\overline{v}$.

a) Discuss how this definition generalizes Definition 4.3.1. In particular, show that it behaves as if the canonical basis elements e_i were *true* indeterminates of degrees δ_i for $i = 1, \dots, r$.
b) Write a CoCoA function `IsWHomog(...)` which takes a vector in F, checks whether it is homogeneous, and returns the corresponding Boolean value.
c) Implement a CoCoA function `WDegree(...)` which takes a homogeneous vector in F or in $\overline{F}$ and returns its degree.
d) Prove the following rules for homogenizing vectors of polynomials.
   1) For all $v \in F$, we have $(v^{\mathrm{hom}})^{\mathrm{deh}} = v$.
   2) For $f \in P \setminus \{0\}$ and $v \in F \setminus \{0\}$, we have $\mathrm{topdeg}_W(fv) = \mathrm{topdeg}_W(f) + \mathrm{topdeg}_W(v)$ as well as $(fv)^{\mathrm{hom}} = (f^{\mathrm{hom}})(v^{\mathrm{hom}})$.
   3) Let $v, w, v+w \in F \setminus \{0\}$, and let d be the componentwise maximum of $d' = \mathrm{topdeg}_W(v)$ and $d'' = \mathrm{topdeg}_W(w)$. Then we have
   $$y^{d-\mathrm{topdeg}_W(v+w)} \cdot (v+w)^{\mathrm{hom}} = y^{d-d'} \cdot v^{\mathrm{hom}} + y^{d-d''} \cdot w^{\mathrm{hom}}$$
   4) For $\bar{f} \in \overline{P}$ and $\overline{v}, \overline{w} \in \overline{F}$, we have $(\bar{f}\,\overline{v})^{\mathrm{deh}} = (\bar{f}^{\mathrm{deh}})(\overline{v}^{\,\mathrm{deh}})$ and $(\overline{v} + \overline{w})^{\mathrm{deh}} = \overline{v}^{\mathrm{deh}} + \overline{w}^{\mathrm{deh}}$.
   5) Let $\overline{v} \in \overline{F} \setminus \{0\}$ be homogeneous and $s_j = \max\{i \geq 0 \mid y_j^i \text{ divides } \overline{v}\}$ for $j = 1, \dots, m$. Then we have
   $$y_1^{s_1} \cdots y_m^{s_m} \cdot (\overline{v}^{\,\mathrm{deh}})^{\mathrm{hom}} = \overline{v}$$
   In particular, we have $\overline{v}^{\,\mathrm{deh}} \neq 0$.
e) Write CoCoA functions `WHomogenize(...)` and `WDehomogenize(...)` which take vectors in F and $\overline{F}$, respectively, and perform the appropriate operation. Use your functions to compute the following homogenizations and dehomogenizations, where we assume that $P = \mathbb{Q}[x_1, \dots, x_n]$ is graded by W.

1) v^{hom} for $v = (x_1^2 + x_2^2,\, x_2^3) \in \mathbb{Q}[x_1,x_2] \oplus \mathbb{Q}[x_1,x_2]((-1,0))$ graded by the matrix $W = (\binom{1\ 1}{2\ 3})$
2) w^{hom} for $w = (x_1^2x_2^2,\, x_2^3x_3^3,\, x_1x_3) \in \mathbb{Q}[x_1,x_2,x_3]^3$ graded by the matrix $W = \binom{1\ 1\ 1}{0\ 1\ 2}$
3) $\overline{v}^{\,\mathrm{deh}}$ for $\overline{v} = (x_1x_2 + x_1y_2,\, y_1^2y_2^4x_1x_2 + y_1^3y_2^4x_2) \in \mathbb{Q}[x_1,x_2]((0,-1)) \oplus \mathbb{Q}[x_1,x_2]((2,3))$ graded by the matrix $W = \binom{\ 1\ 0}{-2\ 1}$

f) Let M be a submodule of F and $\overline{M}$ a graded submodule of $\overline{F}$. Imitate Definition 4.3.4 and introduce the homogenization M^{hom} of M and the dehomogenization $\overline{M}^{\,\mathrm{deh}}$ of $\overline{M}$. Then prove the following rules governing these processes.

1) We have $(M^{\mathrm{hom}})^{\mathrm{deh}} = M$.
2) For a homogeneous vector $\overline{v} \in \overline{P}$, we have $\overline{v} \in M^{\mathrm{hom}}$ if and only if there exist a vector $v \in M$ and numbers $s_1, \dots, s_m \in \mathbb{N}$ such that $\overline{v} = y_1^{s_1} \cdots y_m^{s_m}\, v^{\mathrm{hom}}$.
3) For a homogeneous vector $\overline{v} \in \overline{P}$, we have $\overline{v}^{\,\mathrm{deh}} \in \overline{M}^{\,\mathrm{deh}}$ if and only if $(y_1 \cdots y_m)^s \cdot \overline{v} \in \overline{M}$ for some $s \geq 0$.
4) We have $\overline{M} \subseteq (\overline{M}^{\,\mathrm{deh}})^{\mathrm{hom}} = \overline{M} :_{\overline{F}} (y_1 \cdots y_m)^\infty$.
5) If $M \neq 0$, then $y_1 \cdots y_m$ is a non-zerodivisor for $\overline{F}/M^{\mathrm{hom}}$.
6) If M is a proper graded submodule of F, then $M^{\mathrm{hom}} = \overline{P} \cdot M$.

g) Use these rules to derive algorithms for computing the homogenization and dehomogenization of submodules of F and $\overline{F}$, respectively. Implement these algorithms in two CoCoA functions `HomogModule`(...) and `DehomogModule`(...).

h) Using your function `HomogModule`(...), compute the homogenization of the following submodules.

1) $M_1 = \langle (x_1x_2{+}1,\, x_1{+}x_2), (x_1^3,\, x_2^3), (x_2,\, x_1{+}1) \rangle \subseteq \mathbb{Q}[x_1,x_2]((-1,0)) \oplus \mathbb{Q}[x_1,x_2]((0,-2))$ graded by $W = \binom{1\ 2}{2\ 1}$
2) $M_2 = \langle (x_1^2 + 1, 0, 0), (0, x_2^2 + 1, 0), (0, 0, x_3^2 + 1) \rangle \subseteq \mathbb{Q}[x_1,x_2,x_3]^3$ graded by $W = \binom{1\ 1\ \ \ 1}{1\ 2\ -1}$

i) Let M_1, M_2 be submodules of F and $\overline{M}_1, \overline{M}_2$ graded submodules of $\overline{F}$. Prove the following rules.

1) If $M_1 \subseteq M_2$, then we have $M_1^{\mathrm{hom}} \subseteq M_2^{\mathrm{hom}}$.
2) We have $(M_1 \cap M_2)^{\mathrm{hom}} = M_1^{\mathrm{hom}} \cap M_2^{\mathrm{hom}}$.
3) If $\overline{M}_1 \subseteq \overline{M}_2$, then we have $\overline{M}_1^{\mathrm{deh}} \subseteq \overline{M}_2^{\mathrm{deh}}$.
4) We have $(\overline{M}_1 \cap \overline{M}_2)^{\mathrm{deh}} = \overline{M}_1^{\mathrm{deh}} \cap \overline{M}_2^{\mathrm{deh}}$.
5) We have $(\overline{M}_1 + \overline{M}_2)^{\mathrm{deh}} = \overline{M}_1^{\mathrm{deh}} + \overline{M}_2^{\mathrm{deh}}$.

He who asks is a fool for five minutes,
but he who does not ask remains a fool forever.
(Chinese Proverb)

Tutorial 50: The Homogeneous Part of an Ideal

The idea of homogenization is to approximate an arbitrary polynomial ideal by a homogeneous ideal. The price we have to pay is the introduction of a number of additional indeterminates. Another way to achieve the same goal is to look at the ideal generated by the homogeneous elements in the given ideal. Using this approach we avoid extra indeterminates, but the disadvantage is that the approximation might be very bad, e.g. that the only homogeneous element inside the ideal might be zero. Nevertheless, let us see what we can do, and let us do it effectively.

Let K be a field, let $P = K[x_1, \ldots, x_n]$ be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, and let I be an ideal in P. Then the ideal I_W generated by the homogeneous polynomials in I is called the **W-homogeneous part** of I, or simply the **homogeneous part** of I if it is clear which grading we are considering.

a) Prove that $I_W = I^{\mathrm{hom}} \cap P$.
   Hint: Show that both ideals are homogeneous and have the same homogeneous elements.
b) Using a), show that if I is a prime ideal, then I_W too is a prime ideal.
c) Let σ be a term ordering on $\mathbb{T}^n$. Generalize Tutorial 22.c by showing that I is a homogeneous ideal if and only if its reduced σ-Gröbner basis is homogeneous with respect to the grading given by W.
d) Let $P = K[x_1, \ldots, x_n]$ be standard graded. Find an ideal $I \subseteq P$ such that I_W strictly contains the ideal generated by the homogeneous elements in some reduced Gröbner basis of I.
e) Combine the results obtained so far and develop an algorithm for computing I_W. Implement this algorithm in a CoCoA function `HomogPart(...)` which computes the homogeneous part of an ideal.
   Hint: Use the reduced Gröbner basis of I^{hom} with respect to an elimination ordering for $\{y_1, \ldots, y_m\}$.
f) Apply your function `HomogPart(...)` to compute the homogeneous parts of the following ideals with respect to the gradings given by the stated matrices.
   1) $I_1 = (x_1^2 - x_1,\ x_2^2 - x_1) \subseteq \mathbb{Q}[x_1, x_2]$ graded by $W = (1\ 1)$
   2) $I_2 = (x_1x_2 - 1,\ 2x_1^3x_2 + x_2^3 - x_1^2) \subseteq \mathbb{Q}[x_1, x_2]$ graded by $W = \binom{3\ 2}{1\ 0}$
   3) $I_3 = (x_1x_2x_3 + x_1 + x_2,\ x_1x_2 + x_2^2 + x_2x_3,\ x_1^2x_2^2x_3 + 2x_1 + 2x_2) \subseteq \mathbb{Q}[x_1, x_2, x_3]$ graded by $W = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 3 & -1 \end{pmatrix}$
g) The ideal generated by the terms in I is called the **monomial part** of I and will be denoted by I_{mon}. Explain how one can use the preceding results in order to compute I_{mon}. Write a CoCoA function `MonomialPart(...)` which does this.
h) Compute the monomial parts of the following ideals.
   1) $J_1 = (x_1x_2 + x_1,\ x_1^2x_2^2 + x_1^2x_2 + x_1^2) \subseteq \mathbb{Q}[x_1, x_2]$
   2) $J_2 = (x_1^2 + 2x_2^2,\ 2x_1^2 + 2x_1x_2 + x_2^2,\ x_1x_2 + 2x_2^2) \subseteq \mathbb{Q}[x_1, x_2]$

3) $J_3 = (x_1x_2^2x_3 + x_1x_2x_3,\ x_1x_2^2x_3 + x_2^2x_3,\ x_1^2x_2^3x_3 + x_1^2x_2 + x_1x_2x_3,\ x_1^5x_2^5x_3) \subseteq \mathbb{Q}[x_1, x_2, x_3]$

i) Let $w_1, \ldots, w_m$ be the the rows of W. We can consider them as matrices in $\mathrm{Mat}_{1,n}(\mathbb{Z})$. Show that we have $I_W = (\cdots((I_{w_1})_{w_2})\cdots)_{w_m}$.

j) Prove that $I_W \subseteq \bigcap_{i=1}^m I_{w_i}$ and that this inclusion may be strict. (*Hint:* Consider the ideal $(x^2 + y,\ x + y^2)$ in $\mathbb{Q}[x, y]$.)

k) Using i), implement a second function `HomogPart2(...)`, apply it in the cases of f), and compare the results.

l) Compute the determinant of the matrix

$$W = \begin{pmatrix} 2 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 2 \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{Z})$$

Use this matrix together with i) to write a function `MonomialPart2(...)` which computes I_{mon} in an alternative way. Apply your function in the cases of h) and compare the results.

Tutorial 51: Implicitization and Homogenization

In the current chapter it has been one of our dicta that many computations can be performed more efficiently in a homogeneous setting. To make this principle come alive, we now look at the computation of implicitizations using the technique of homogenizing, performing a homogeneous computation, and dehomogenizing again. In Section 3.6 we defined the implicitization of a tuple of polynomials $(f_1, \ldots, f_n) \in K[y_1, \ldots, y_m]^n$ as the ideal of all algebraic relations among them. Corollary 3.6.3 gives us a concrete method for computing this ideal: form the diagonal ideal $J = (x_1 - f_1, \ldots, x_n - f_n)$ in $Q = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ and use elimination to get the implicitization $J \cap K[x_1, \ldots, x_n]$.

Now we propose another way of computing such implicitizations. We introduce a homogenizing indeterminate x_0 and equip the polynomial ring $\overline{Q} = K[x_0, \ldots, x_n, y_1, \ldots, y_m]$ with a suitable grading which makes the ideal $\overline{J} = (x_1 - f_1^{\mathrm{hom}}, \ldots, x_n - f_n^{\mathrm{hom}})$ homogeneous. The we compute a homogeneous Gröbner basis of $\overline{J} \cap K[x_0, \ldots, x_n]$ and find $J \cap K[x_1, \ldots, x_n]$ by dehomogenizing it. We will guide you through a proof of the correctness of this technique, ask you to implement it and to compare it to the former method. In the second part of the tutorial, we introduce a variation of this algorithm which computes the homogenization of the implicitization with respect to the standard grading. An optimization of this algorithm is contained in Tutorial 69.

Let K be a field, let $P = K[x_1, \ldots, x_n]$, let $P' = K[y_1, \ldots, y_m]$, and let $f_1, \ldots, f_n \in P' \setminus \{0\}$. We equip P with the grading given by the matrix

$W = (d_1 \; \cdots \; d_n)$, where d_i is the degree of f_i with respect to the standard grading. Our goal is to show that we can compute the implicitization of $f_1, \ldots, f_n$ by performing the following steps.

1) Form the polynomial ring $Q = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ and the ideal $J = (x_1 - f_1, \ldots, x_n - f_n)$ in Q.
2) Form the polynomial ring $\overline{P} = K[x_0, \ldots, x_n]$ and equip it with the grading given by $\overline{W} = (1 \; d_1 \; \cdots \; d_n)$.
3) Form the polynomial ring $\overline{Q} = K[x_0, \ldots, x_n, y_1, \ldots, y_m]$ and equip it with the grading given by $\widetilde{W} = (1 \; d_1 \; d_2 \; \cdots \; d_n \; 1 \; \cdots 1)$.
4) For $i = 1, \ldots, n$, compute the homogenization f_i^{hom} of f_i with respect to x_0. Then form the ideal $\overline{J} = (x_1 - f_1^{\mathrm{hom}}, \ldots, x_n - f_n^{\mathrm{hom}})$ in $\overline{Q}$.
5) Compute the elimination ideal $\overline{I} = \overline{J} \cap \overline{P}$.
6) Return the ideal $I = \overline{I}^{\mathrm{deh}}$.

In the following, you will prove that $I = J \cap P$, i.e. that I is the implicitization of J. Moreover, we shall see that $\overline{I}$ is the homogenization of I with respect to the grading given by W.

a) Show that $\overline{J} \cap \overline{P}$ is a homogeneous ideal in $\overline{P}$.
b) Using Proposition 3.6.1.a, prove that $\overline{J}$ is a prime ideal.
c) Using b), show that the ideal $\overline{J} \cap \overline{P}$ is saturated with respect to x_0.
d) Prove that the homogenization of J with respect to x_0 is given by $\overline{J}$. (*Hint:* Use Corollary 4.3.8.)
e) Finally, use Corollary 4.3.7 to prove that $(J \cap P)^{\mathrm{hom}} = \overline{J} \cap \overline{P}$, where P is graded by W and x_0 is the homogenizing indeterminate.
f) Write a CoCoA function `Implicit(...)` which takes $(f_1, \ldots, f_n)$ and computes the implicitization $J \cap P$ using the above method.
g) Apply your function `Implicit(...)` to compute the implicitizations of the following tuples of polynomials. Compare the timings of your function to the timings of the built-in CoCoA command `Elim`.
   1) $(y_1^5 + y_1 y_2 + y_2,\; y_1^4 + y_1 y_2^2 - y_1 - 1,\; y_1^3 + y_2)$ in $\mathbb{Z}/(101)[y_1, y_2]$
   2) $(y_1^7 + y_1 y_2 + y_2,\; y_1^4 + y_1 y_2^2 - y_1 - 1,\; y_1^3 + y_2)$ in $\mathbb{Z}/(101)[y_1, y_2]$
   3) $(y_1^5 + y_1 y_2 + y_2,\; y_1^4 + y_1 y_2^2 - y_1 - 1,\; y_1^3 + y_2)$ in $\mathbb{Q}[y_1, y_2]$
   4) $(y_1^7 + y_1 y_2 + y_2,\; y_1^4 + y_1 y_2^2 - y_1 - 1,\; y_1^3 + y_2)$ in $\mathbb{Q}[y_1, y_2]$

In the second part of this tutorial, we start with the same setting, but we want to compute the homogenization of the implicitization $J \cap P$ with respect to the standard grading on P. In this case, we can modify the above steps as follows.

1') Form the polynomial ring $Q = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ and the ideal $J = (x_1 - f_1, \ldots, x_n - f_n)$ in Q.
2') Form the polynomial ring $\overline{P} = K[x_0, \ldots, x_n]$ and equip it with the standard grading.
3') Form the polynomial ring $\overline{Q} = K[x_0, \ldots, x_n, y_0, \ldots, y_m]$ and equip it with the grading given by the matrix $\widetilde{W} = (d \; d \; \cdots \; d \; 1 \; \cdots 1)$, where $d = \max\{d_1, \ldots, d_n\}$ is repeated $n+1$ times.

4') For $i = 1, \ldots, n$, compute the homogenization f_i^{hom} of f_i with respect to y_0. Form the ideal $\overline{J} = (x_0 - y_0^d, x_1 - y_0^{d-d_1} f_1^{\mathrm{hom}}, \ldots, x_n - y_0^{d-d_n} f_n^{\mathrm{hom}})$ in $\overline{Q}$.

5') Compute the elimination ideal $\overline{I} = \overline{J} \cap \overline{P}$.

In a manner which is similar to the method above we want to show the formula $\overline{I} = (J \cap P)^{\mathrm{hom}}$. This gives us an efficient way of computing the homogenization with respect to the standard grading of the implicitization $J \cap P$, since we have to compute only one Gröbner basis in a homogeneous setting.

h) Show that $\overline{J} \cap \overline{P}$ is homogeneous with respect to the standard grading.

i) Using b), show that the ideal $\overline{J} \cap \overline{P}$ is saturated with respect to x_0.

j) Prove that the homogenization of J with respect to y_0 is the ideal $(x_1 - y_0^{d-d_1} f_1^{\mathrm{hom}}, \ldots, x_n - y_0^{d-d_n} f_n^{\mathrm{hom}})$. (*Hint:* Use Corollary 4.3.8.)

k) Now show that $J \cap P = (\overline{J} \cap \overline{P})^{\mathrm{deh}}$, where the dehomogenization is formed with respect to x_0.
Hint: Prove the two inclusions. For the proof of "$\subseteq$", equip P with the grading given by $(d \;\cdots\; d)$, homogenize $f \in J \cap P$ with respect to y_0 and substitute $y_0^d \mapsto x_0$ in f^{hom}. For the proof of "$\supseteq$", use the substitution $x_0 \mapsto 1,\ y_0 \mapsto 1$.

l) Finally, use Corollary 4.3.7 to prove that $(J \cap P)^{\mathrm{hom}} = \overline{J} \cap \overline{P}$, where P is standard graded and x_0 is the homogenizing indeterminate.

m) Write a CoCoA function `HomImplicit(...)` which takes $(f_1, \ldots, f_n)$ and computes the homogenization of the implicitization $J \cap P$ with respect to the standard grading using steps 1') – 5').

n) Apply your function `HomImplicit(...)` to the examples in g) and measure the timings.

Tutorial 52: Projective Closure

The geometric interpretation of the process of homogenizing a polynomial ideal under the standard grading is the process of forming the projective closure of an affine variety. Furthermore, the geometric interpretation of the process of dehomogenizing a polynomial ideal is the process of passing to the affine part of a projective variety. In this tutorial, we want to study these processes from the geometric point of view.

Let K be a field, let $P = K[x_1, \ldots, x_n]$ and $\overline{P} = K[x_0, \ldots, x_n]$ be standard graded, let $\overline{K}$ be the algebraic closure of K, and let $\mathbb{A}^n$ (resp. $\mathbb{P}^n$) be the n-dimensional affine (resp. projective) space over $\overline{K}$.

a) Show that, for $i = 0, \ldots, n$, there exists an injective map $\imath_i : \mathbb{A}^n \longrightarrow \mathbb{P}^n$ defined by $(p_1, \ldots, p_n) \mapsto (p_1 : \ldots : p_i : 1 : p_{i+1} : \ldots : p_n)$, where the "1" occurs in the $(i+1)^{\mathrm{st}}$ position.

b) Prove that $\mathbb{P}^n$ is covered by the images of $\imath_0, \ldots, \imath_n$.

In the following, we shall identify $\mathbb{A}^n$ with its image U under the map $\imath_0$. Thus, for a projective variety $W \subseteq \mathbb{P}^n$, we can consider $W \cap U$ as a subset of $\mathbb{A}^n$. We call $W \cap U$ the **affine part** of W.

c) Let $J \subseteq \overline{P}$ be a homogeneous ideal which defines a non-empty projective variety $W = \mathcal{Z}^+(J) \subseteq \mathbb{P}^n$. Prove that the affine part of W is the affine variety defined by J^{deh}.

Now let I be an ideal in P, and let $V = \mathcal{Z}(I)$ be the affine variety in $\mathbb{A}^n$ defined by I. Then the projective variety $\overline{V} = \mathcal{Z}^+(I^{\mathrm{hom}}) \subseteq \mathbb{P}^n$ is called the **projective closure** of V.

d) Show that $\overline{V} \cap U = V$ and that $\overline{V}$ is the smallest projective variety whose affine part contains V.
e) Prove that the homogeneous vanishing ideal of $\overline{V}$ is $\mathcal{I}(V)^{\mathrm{hom}}$.
f) Write a CoCoA function `ProjClosure(...)` which takes I and computes the ideal I^{hom} which defines the projective closure of $\mathcal{Z}(I)$. Use your function to compute the projective closure of the following affine varieties. (In each case, you will have to find the vanishing ideal first.)
   1) $V_1 = \{(t^3, t^4, t^5) \mid t \in \overline{K}\} \subseteq \mathbb{A}^3$
   2) $V_2 = \{(t, u, t^2, tu, u^2) \mid t, u \in \overline{K}\} \subseteq \mathbb{A}^5$ (This is the **Veronese surface** which we first met in Tutorial 39.g.)
   3) $V_3 = \{(t, u, v, tu, tv) \mid t, u, v \in \overline{K}\} \subseteq \mathbb{A}^5$

The variety $H^{\mathrm{inf}} = \mathcal{Z}^+(x_0)$ in $\mathbb{P}^n$ is called the **hyperplane at infinity**. Let $W \subseteq \mathbb{P}^n$ be a non-empty projective variety defined by a homogeneous ideal $J \subseteq \overline{P}$. The points in $W^{\mathrm{inf}} = W \cap H^{\mathrm{inf}}$ are called the **points at infinity** of W.

g) Show that H^{inf} is the complement of U in $\mathbb{P}^n$, and that it can be identified with $\mathbb{P}^{n-1}$.
h) Prove that W^{inf} is the projective variety in $\mathbb{P}^{n-1}$ defined by the ideal obtained by setting $x_0 = 0$ in the polynomials in J.
i) Compute the points at infinity of the projective closure of the **twisted cubic curve** $C = \mathcal{Z}((x_1^2 - x_2, x_1^3 - x_3)) \subseteq \mathbb{A}^3$. Then determine the set of points at infinity of the projective variety $\mathcal{Z}^+((x_1^2 - x_0x_2, x_1^3 - x_0^2x_3))$ in $\mathbb{P}^3$ obtained by homogenizing the generators of the defining ideal of C. Compare and explain your findings.

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