

# Solution to selected problems.

## Chapter 1. Preliminaries

1.  $\forall A \in \mathcal{F}_S, \forall t \geq 0, A \cap \{T \leq t\} = (A \cap \{S \leq t\}) \cap \{T \leq t\}$ , since  $\{T \leq t\} \subset \{S \leq t\}$ . Since  $A \cap \{S \leq t\} \in \mathcal{F}_t$  and  $\{T \leq t\} \in \mathcal{F}_t$ ,  $A \cap \{T \leq t\} \in \mathcal{F}_t$ . Thus  $\mathcal{F}_S \subset \mathcal{F}_T$ .

2. Let  $\Omega = \mathbb{N}$  and  $\mathcal{F} = \mathcal{P}(\mathbb{N})$  be the power set of the natural numbers. Let  $\mathcal{F}_n = \sigma(\{2\}, \{3\}, \dots, \{n+1\})$ ,  $\forall n$ . Then  $(\mathcal{F}_n)_{n \geq 1}$  is a filtration. Let  $S = 3 \cdot 1_3$  and  $T = 4$ . Then  $S \leq T$  and

$$\{\omega : S(\omega) = n\} = \begin{cases} \{3\} & \text{if } n = 3 \\ \emptyset & \text{otherwise} \end{cases}$$

$$\{\omega : T(\omega) = n\} = \begin{cases} \Omega & \text{if } n = 4 \\ \emptyset & \text{otherwise} \end{cases}$$

Hence  $\{S = n\} \in \mathcal{F}_n, \{T = n\} \in \mathcal{F}_n, \forall n$  and  $S, T$  are both stopping time. However  $\{\omega : T - S = 1\} = \{\omega : 1_{\{3\}}(\omega) = 1\} = \{3\} \notin \mathcal{F}_1$ . Therefore  $T - S$  is not a stopping time.

3. Observe that  $\{T_n \leq t\} \in \mathcal{F}_t$  and  $\{T_n < t\} \in \mathcal{F}_t$  for all  $n \in \mathbb{N}, t \in \mathbb{R}_+$ , since  $T_n$  is stopping time and we assume usual hypothesis. Then

- (1)  $\sup_n T_n$  is a stopping time since  $\forall t \geq 0, \{\sup_n T_n \leq t\} = \cap_n \{T_n \leq t\} \in \mathcal{F}_t$ .
- (2)  $\inf_n T_n$  is a stopping time since  $\{\inf_n T_n < t\} = \cup_n \{T_n < t\} \in \mathcal{F}_t$
- (3)  $\limsup_{n \rightarrow \infty} T_n$  is a stopping time since  $\limsup_{n \rightarrow \infty} T_n = \inf_m \sup_{n \geq m} T_n$  (and (1), (2).)
- (4)  $\liminf_{n \rightarrow \infty} T_n$  is a stopping time since  $\liminf_{n \rightarrow \infty} T_n = \sup_m \inf_{n \geq m} T_n$  (and (1), (2).)

4.  $T$  is clearly a stopping time by exercise 3, since  $T = \liminf_n T_n$ .  $\mathcal{F}_T \subset \mathcal{F}_{T_n}, \forall n$  since  $T \leq T_n$ , and  $\mathcal{F}_T \subset \cap_n \mathcal{F}_{T_n}$  by exercise 1. Pick a set  $A \in \cap_n \mathcal{F}_{T_n}$ .  $\forall n \geq 1, A \in \mathcal{F}_{T_n}$  and  $A \cap \{T_n \leq t\} \in \mathcal{F}_t$ . Therefore  $A \cap \{T \leq t\} = A \cap (\cap_n \{T_n \leq t\}) = \cap_n (A \cap \{T_n \leq t\}) \in \mathcal{F}_t$ . This implies  $\cap_n \mathcal{F}_{T_n} \subset \mathcal{F}_T$  and completes the proof.

5. (a) By completeness of  $L^p$  space,  $X \in L^p$ . By Jensen's inequality,  $E|M_t|^p = E|E(X|\mathcal{F}_t)|^p \leq E[E(|X|^p|\mathcal{F}_t)] = E|X|^p < \infty$  for  $p > 1$ .

(b) By (a),  $M_t \in L^p \subset L^1$ . For  $t \geq s \geq 0, E(M_t|\mathcal{F}_s) = E(E(X|\mathcal{F}_t)|\mathcal{F}_s) = E(X|\mathcal{F}_s) = M_s$  a.s.  $\{M_t\}$  is a martingale. Next, we show that  $\{M_t\}$  is continuous. By Jensen's inequality, for  $p > 1$ ,

$$E|M_t^n - M_t|^p = E|E(M_t^n - X|\mathcal{F}_t)|^p \leq E|M_t^n - X|^p, \quad \forall t \geq 0. \quad (1)$$

It follows that  $\sup_t E|M_t^n - M_t|^p \leq E|M_\infty^n - X|^p \rightarrow 0$  as  $n \rightarrow \infty$ . Fix arbitrary  $\varepsilon > 0$ . By Chebychev's and Doob's inequality,

$$P\left(\sup_t |M_t^n - M_t| > \varepsilon\right) \leq \frac{1}{\varepsilon^p} E(\sup_t |M_t^n - M_t|^p) \leq \left(\frac{p}{p-1}\right)^p \frac{\sup_t E|M_t^n - M_t|^p}{\varepsilon^p} \rightarrow 0. \quad (2)$$

Therefore  $M^n$  converges to  $M$  uniformly in probability. There exists a subsequence  $\{n_k\}$  such that  $M_{n_k}$  converges uniformly to  $M$  with probability 1. Then  $M$  is continuous since for almost all  $\omega$ , it is a limit of uniformly convergent continuous paths.

6. Let  $p(n)$  denote a probability mass function of Poisson distribution with parameter  $\lambda t$ . Assume  $\lambda t$  is integer as given.

$$\begin{aligned}
E|N_t - \lambda t| &= E(N_t - \lambda t) + 2E(N_t - \lambda t)^- = 2E(N_t - \lambda t)^- = 2 \sum_{n=0}^{\lambda t} (\lambda t - n)p(n) \\
&= 2e^{-\lambda t} \sum_{n=0}^{\lambda t} (\lambda t - n) \frac{(\lambda t)^n}{n!} = 2\lambda t e^{-\lambda t} \left( \sum_{n=0}^{\lambda t} \frac{(\lambda t)^n}{n!} - \sum_{n=0}^{\lambda t-1} \frac{(\lambda t)^n}{n!} \right) \\
&= 2e^{-\lambda t} \frac{(\lambda t)^{\lambda t}}{(\lambda t - 1)!}
\end{aligned} \tag{3}$$

7. Since  $N$  has stationary increments, for  $t \geq s \geq 0$ ,

$$E(N_t - N_s)^2 = EN_{t-s}^2 = \text{Var}(N_{t-s}) + (EN_{t-s})^2 = \lambda(t-s)[1 + \lambda(t-s)]. \tag{4}$$

As  $t \downarrow s$  (or  $s \uparrow t$ ),  $E(N_t - N_s)^2 \rightarrow 0$ .  $N$  is continuous in  $L^2$  and therefore in probability.

8. Suppose  $\tau_\alpha$  is a stopping time. A 3-dimensional Brownian motion is a strong Markov process we know that  $P(\tau_\alpha < \infty) = 1$ . Let's define  $W_t := B_{\tau_\alpha+t} - B_{\tau_\alpha}$ .  $W$  is also a 3-dimensional Brownian motion and its augmented filtration  $\mathcal{F}_t^W$  is independent of  $\mathcal{F}_{\tau_\alpha+} = \mathcal{F}_{\tau_\alpha}$ . Observe  $\|W_0\| = \|B_{\tau_\alpha}\| = \alpha$ . Let  $S = \inf_t \{t > 0 : \|W_t\| \leq \alpha\}$ . Then  $S$  is a  $F_t^W$  stopping time and  $\{S \leq s\} \in \mathcal{F}_s^W$ . So  $\{S \leq s\}$  has to be independent of any sets in  $\mathcal{F}_{\tau_\alpha}$ . However  $\{\tau_\alpha \leq t\} \cap \{S \leq s\} = \emptyset$ , which implies that  $\{\tau_\alpha \leq t\}$  and  $\{S \leq s\}$  are dependent. Since  $\{\tau_\alpha \leq t\} \in \mathcal{F}_{\tau_\alpha}$ , this contradicts the fact that  $\mathcal{F}_{\tau_\alpha}$  and  $\mathcal{F}_t^W$  are independent. Hence  $\tau_\alpha$  is not a stopping time.

9. (a) Since  $L^2$ -space is complete, it suffices to show that  $S_t^n = \sum_{i=1}^n M_t^i$  is a Cauchy sequence w.r.t.  $n$ . For  $m \geq n$ , by independence of  $\{M^i\}$ ,

$$E(S_t^n - S_t^m)^2 = E\left(\sum_{i=n+1}^m M_t^i\right)^2 = \sum_{i=n+1}^m E(M_t^i)^2 = t \sum_{i=n+1}^m \frac{1}{i^2}. \tag{5}$$

, Since  $\sum_{i=1}^{\infty} 1/i^2 < \infty$ , as  $n, m \rightarrow \infty$ ,  $E(S_t^n - S_t^m)^2 \rightarrow 0$ .  $\{S_t^n\}_n$  is Cauchy and its limit  $M_t$  is well defined for all  $t \geq 0$ .

(b) First, recall Kolmogorov's convergence criterion: *Suppose  $\{X_i\}_{i \geq 1}$  is a sequence of independent random variables. If  $\sum_i \text{Var}(X_i) < \infty$ , then  $\sum_i (X_i - EX_i)$  converges a.s.*

For all  $i$  and  $t$ ,  $\Delta M_t^i = (1/i)\Delta N_t^i$  and  $\Delta M_t^i > 0$ . By Fubini's theorem and monotone convergence theorem,

$$\sum_{s \leq t} \Delta M_s = \sum_{s \leq t} \sum_{i=1}^{\infty} \Delta M_s^i = \sum_{i=1}^{\infty} \sum_{s \leq t} \Delta M_s^i = \sum_{i=1}^{\infty} \sum_{s \leq t} \frac{\Delta N_s^i}{i} = \sum_{i=1}^{\infty} \frac{N_t^i}{i}. \tag{6}$$

Let  $X_i = (N_t^i - t)/i$ . Then  $\{X_i\}_i$  is a sequence of independent random variables such that  $EX_i = 0$ ,  $\text{Var}(X_i) = 1/i^2$  and hence  $\sum_{i=1}^{\infty} \text{Var}(X_i) < \infty$ . Therefore Kolmogorov's criterion implies that  $\sum_{i=1}^{\infty} X_i$  converges almost surely. On the other hand,  $\sum_{i=1}^{\infty} t/i = \infty$ . Therefore,  $\sum_{s \leq t} \Delta M_s = \sum_{i=1}^{\infty} N_t^i/i = \infty$ .

10. (a) Let  $N_t = \sum_i \frac{1}{i}(N_t^i - t)$  and  $L_t = \sum_i \frac{1}{i}(L_t^i - t)$ . As we show in exercise 9(a),  $N, M$  are well defined in  $L^2$  sense. Then by linearity of  $L^2$  space  $M$  is also well defined in  $L^2$  sense since

$$M_t = \sum_i \frac{1}{i} [(N_t^i - t) - (L_t^i - t)] = \sum_i \frac{1}{i}(N_t^i - t) - \sum_i \frac{1}{i}(L_t^i - t) = N_t - L_t. \quad (7)$$

Both terms in right side are martingales change only by jumps as shown in exercise 9(b). Hence  $M_t$  is a martingale which changes only by jumps.

(b) First show that given two independent Poisson processes  $N$  and  $L$ ,  $\sum_{s>0} \Delta N_s \Delta L_s = 0$  a.s., i.e.  $N$  and  $L$  almost surely don't jump simultaneously. Let  $\{T_n\}_{n \geq 1}$  be a sequence of jump times of a process  $L$ . Then  $\sum_{s>0} \Delta N_s \Delta L_s = \sum_n \Delta N_{T_n}$ . We want to show that  $\sum_n \Delta N_{T_n} = 0$  a.s. Since  $\Delta N_{T_n} \geq 0$ , it is enough to show that  $E \Delta N_{T_n} = 0$  for  $\forall n \geq 1$ .

Fix  $n \geq 1$  and let  $\mu_{T_n}$  be a induced probability measure on  $\mathbb{R}_+$  of  $T_n$ . By conditioning,

$$E(\Delta N_{T_n}) = E[E(\Delta N_{T_n} | T_n)] = \int_0^\infty E(\Delta N_t | T_n = t) \mu_{T_n}(dt) = \int_0^\infty E(\Delta N_t) \mu_{T_n}(dt), \quad (8)$$

where last equality is by independence of  $N$  and  $T_n$ . It follows that  $E \Delta N_{T_n} = E \Delta N_t$ . Since  $\Delta N_t \in L^1$  and  $P(\Delta N_t = 0) = 1$  by problem 25,  $E \Delta N_t = 0$ , hence  $E \Delta N_{T_n} = 0$ .

Next we show that the previous claim holds even when there are countably many Poisson processes. assume that there exist countably many independent Poisson processes  $\{N^i\}_{i \geq 1}$ . Let  $A \subset \Omega$  be a set on which more than two processes of  $\{N^i\}_{i \geq 1}$  jump simultaneously. Let  $\Omega_{ij}$  denotes a set on which  $N^i$  and  $N^j$  don't jump simultaneously. Then  $P(\Omega_{ij}) = 1$  for  $i \neq j$  by previous assertion. Since  $A \subset \cup_{i>j} \Omega_{ij}^c$ ,  $P(A) \leq \sum_{i>j} P(\Omega_{ij}^c) = 0$ . Therefore jumps don't happen simultaneously almost surely.

Going back to the main proof, by (a) and the fact that  $N$  and  $L$  don't jump simultaneously,  $\forall t > 0$ ,

$$\sum_{s \leq t} |\Delta M_s| = \sum_{s \leq t} |\Delta N_s| + \sum_{s \leq t} |\Delta L_s| = \infty \quad a.s. \quad (9)$$

11. **Continuity:** We use notations adopted in Example 2 in section 4 (P33). Assume  $E|U_1| < \infty$ . By independence of  $U_i$ , elementary inequality, Markov inequality, and the property of Poisson process, we observe

$$\begin{aligned} \lim_{s \rightarrow t} P(|Z_t - Z_s| > \epsilon) &= \lim_{s \rightarrow t} \sum_k P(|Z_t - Z_s| > \epsilon | N_t - N_s = k) P(N_t - N_s = k) \\ &\leq \lim_{s \rightarrow t} \sum_k P\left(\sum_{i=1}^k |U_i| > \epsilon\right) P(N_t - N_s = k) \leq \lim_{s \rightarrow t} \sum_k \left[k P(|U_1| > \frac{\epsilon}{k})\right] P(N_t - N_s = k) \\ &\leq \lim_{s \rightarrow t} \frac{E|U_1|}{\epsilon} \sum_k k^2 P(N_t - N_s = k) = \lim_{s \rightarrow t} \frac{E|U_1|}{\epsilon} \{\lambda(t-s)\} = 0 \end{aligned} \quad (10)$$

**Independent Increment:** Let  $F$  be a distribution function of  $U$ . By using independence of  $\{U_k\}_k$  and strong Markov property of  $N$ , for arbitrary  $t, s : t \geq s \geq 0$ ,

$$\begin{aligned}
E\left(e^{iu(Z_t - Z_s) + ivZ_s}\right) &= E\left(e^{iu \sum_{k=N_s+1}^{N_t} U_k + iv \sum_{k=1}^{N_s} U_k}\right) \\
&= E\left(E\left(e^{iu \sum_{k=N_s+1}^{N_t} U_k + iv \sum_{k=1}^{N_s} U_k} \middle| \mathcal{F}_s\right)\right) \\
&= E\left(e^{iv \sum_{k=1}^{N_s} U_k} E\left(e^{iu \sum_{k=N_s+1}^{N_t} U_k} \middle| \mathcal{F}_s\right)\right) = E\left(e^{iv \sum_{k=1}^{N_s} U_k} E\left(e^{iu \sum_{k=N_s+1}^{N_t} U_k}\right)\right) \\
&= E\left(e^{iv \sum_{k=1}^{N_s} U_k}\right) E\left(e^{iu \sum_{k=N_s+1}^{N_t} U_k}\right) = E\left(e^{ivZ_s}\right) E\left(e^{iu(Z_t - Z_s)}\right).
\end{aligned} \tag{11}$$

This shows that  $Z$  has independent increments.

**Stationary increment:** Since  $\{U_k\}_k$  are i.i.d and independent of  $N_t$ ,

$$\begin{aligned}
E\left(e^{iuZ_t}\right) &= E\left(E\left(e^{iuZ_t} \middle| N_t\right)\right) = E\left[\left(\int e^{iux} F(dx)\right)^{N(t)}\right] = \sum_{n \geq 0} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \left(\int e^{iux} F(dx)\right)^n \\
&= \exp\left\{-t\lambda \left(\int [1 - e^{iux}] F(dx)\right)\right\}.
\end{aligned} \tag{12}$$

By (11) and (12), set  $v = u$ ,

$$\begin{aligned}
E\left(e^{iu(Z_t - Z_s)}\right) &= E\left(e^{iuZ_t}\right) / E\left(e^{iuZ_s}\right) = \exp\left\{-(t-s)\lambda \left(\int [1 - e^{iux}] F(dx)\right)\right\} \\
&= E\left(e^{iu(Z_{t-s})}\right)
\end{aligned} \tag{13}$$

Hence  $Z$  has stationary increments.

**12.** By exercise 12, a compound Poisson process is a Lévy process and has independent stationary increments.

$$\begin{aligned}
E|Z_t - \lambda t E U_1| &\leq E(E(|Z_t| | N_t)) + \lambda t E|U_1| \leq \sum_{n=1}^{\infty} E\left(\sum_{i=1}^n |U_i| \middle| N_t = n\right) P(N_t = n) + \lambda t E|U_1| \\
&= E|U_1| E N_t + \lambda t E|U_1| = 2\lambda t E|U_1| < \infty,
\end{aligned} \tag{14}$$

For  $t \geq s$ ,  $E(Z_t | \mathcal{F}_s) = E(Z_t - Z_s + Z_s | \mathcal{F}_s) = Z_s + E(Z_{t-s})$ . Since

$$EZ_t = E(E(Z_t | N_t)) = \sum_{n=1}^{\infty} E\left(\sum_{i=1}^n U_i \middle| N_t = n\right) P(N_t = n) = \lambda t E U_1 \tag{15}$$

$E(Z_t - E U_1 \lambda t | \mathcal{F}_s) = Z_s - E U_1 \lambda s$  a.s. and  $\{Z_t - E U_1 \lambda t\}_{t \geq 0}$  is a martingale.

**13.** By Lévy decomposition theorem and hypothesis,  $Z_t$  can be decomposed as

$$Z_t = \int_{\mathbb{R}} x N_t(\cdot, dx) + t\left(\alpha - \int_{|x| \geq 1} x \nu(dx)\right) = Z'_t + \beta t \tag{16}$$

where  $Z'_t = \int_{\mathbb{R}} x N_t(\cdot, dx)$ ,  $\beta = \alpha - \int_{|x| \geq 1} x \nu(dx)$ . By theorem 43,  $E(e^{iuZ'_t}) = \int_{\mathbb{R}} (1 - e^{iux}) \nu(dx)$ .  $Z'_t$  is a compound Poisson process (See problem 11). Arrival rate (intensity) is  $\lambda$  since  $E(\int_{\mathbb{R}} N_t(\cdot, dx)) = t \int_{\mathbb{R}} \nu(dx) = \lambda t$ . Since  $Z_t$  is a martingale,  $EZ'_t = -\beta t$ . Since  $EZ'_t = E(\int_{\mathbb{R}} x N_t(\cdot, dx)) = t \int_{\mathbb{R}} x \nu(dx)$ ,  $\beta = -\int_{\mathbb{R}} x \nu(dx)$ . It follows that  $Z_t = Z'_t - \lambda t \int_{\mathbb{R}} x \frac{\nu}{\lambda}(dx)$  is a compensated compound Poisson process. Then problem 12 shows that  $EU_1 = \int_{\mathbb{R}} x \frac{\nu}{\lambda}(dx)$ . It follows that the distribution of jumps  $\mu = (1/\lambda)\nu$ .

**14.** Suppose  $EN_t < \infty$ . At first we show that  $Z_t \in L^1$ .

$$\begin{aligned} E|Z_t| &\leq E\left(\sum_{i=1}^{N_t} |U_i|\right) = \sum_{n=0}^{\infty} E\left(\sum_{i=1}^n |U_i|\right) P(N_t = n) = \sum_{n=0}^{\infty} \sum_{i=1}^n E(|U_i|) P(N_t = n) \\ &\leq \sup_i E|U_i| \sum_{n=0}^{\infty} n P(N_t = n) = EN_t \sup_i E|U_i| < \infty. \end{aligned} \quad (17)$$

Then

$$\begin{aligned} E(Z_t | \mathcal{F}_s) &= E(Z_s + \sum_{i=1}^{\infty} U_i 1_{\{s < T_i \leq t\}} | \mathcal{F}_s) = Z_s + E(\sum_{i=1}^{\infty} U_i 1_{\{s < T_i \leq t\}} | \mathcal{F}_s) \\ &= Z_s + \sum_{i=1}^{\infty} E(U_i E(1_{\{s < T_i \leq t\}} | \mathcal{F}_s \vee \sigma(U_i : i \geq 1)) | \mathcal{F}_s) = Z_s + \sum_{i=1}^{\infty} E(U_i E(1_{\{s < T_i \leq t\}} | \mathcal{F}_s) | \mathcal{F}_s) \quad (18) \\ &= Z_s + \sum_{i=1}^{\infty} E(U_i | \mathcal{F}_s) E(1_{\{T_i \leq t\}} | \mathcal{F}_s) 1_{\{T_i > s\}} = Z_s, \quad a.s. \end{aligned}$$

since  $E(U_i | \mathcal{F}_s) 1_{\{T_i > s\}} = 0$  a.s. Note that if we drop the assumption  $EN_t < \infty$ ,  $Z_t$  is not a martingale in general. (Though it is a local martingale.)

**15.** By Lévy decomposition theorem (theorem 42),

$$Z_t = \int_{|x| < 1} x(N_t(\cdot, dx) - t\nu(dx)) + \alpha t + \sum_{0 < s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}. \quad (19)$$

The first term in right side is a martingale (theorem 41). Then  $\alpha = -\int_{|x| > 1} x \nu(dx)$  since  $Z_t$  is a martingale. Note that  $\alpha$  is well defined by a condition  $\sum \beta_k^2 \alpha_k < \infty$ . Hence  $Z_t = \int_{\mathbb{R}} x(N_t(\cdot, dx) - t\nu(dx))$ . Let  $A = \mathbb{R} \setminus \{\beta_k\}_k$ . Then  $EN_1^A = \int_A \nu(dx) = 0$  by theorem 38. Therefore mass of  $N(\cdot, dx)$  is concentrated on countable set  $\{\beta_k\}_k$ . Fix  $k \geq 1$  and let  $N_t^k$  denote  $N_t(\cdot, \beta_k)$ . Then  $N_t^k$  is a Poisson process and its rate is  $\alpha_k$  since by theorem 38,  $EN_1^k = \int_{\{\beta_k\}} \nu(dx) = \alpha_k$ . Therefore,

$$Z_t = \beta_k(N_t^k - \alpha_k t) \quad (20)$$

To verify that  $Z_t \in L^2$ , we observe that  $E(\sum_{k=m}^n \beta_k(N_t^k - \alpha_k t))^2 = t \sum_{k=m}^n \beta_k^2 \alpha_k \rightarrow 0$  as  $n, m \rightarrow \infty$  since  $\sum_{k=1}^{\infty} \beta_k^2 \alpha_k < \infty$ . Therefore,  $Z_t$  is a Cauchy limit in  $L^2$ . Since  $L^2$ -space is complete,  $Z_t \in L^2$ .

**16.** Let  $\mathcal{F}_t$  be natural filtration of  $B_t$  satisfying usual hypothesis. By stationary increments property of Brownian motion and symmetry,

$$W_t = B_{1-t} - B_1 \stackrel{d}{=} B_1 - B_{1-t} \stackrel{d}{=} B_{1-(1-t)} = B_t \quad (21)$$

This shows  $W_t$  is Gaussian.  $W_t$  has stationary increments because  $W_t - W_s = B_{1-s} - B_{1-t} \stackrel{d}{=} B_{t-s}$ . Let  $\mathcal{G}_t$  be a natural filtration of  $W_t$ . Then  $\mathcal{G}_t = \sigma(-(B_1 - B_{1-s}) : 0 \leq s \leq t)$ . By independent increments property of  $B_t$ ,  $\mathcal{G}_t$  is independent of  $\mathcal{F}_{1-t}$ . For  $s > t$ ,  $W_s - W_t = -(B_{1-t} - B_{1-s}) \in \mathcal{F}_{1-t}$  and hence independent of  $\mathcal{G}_t$ .

**17. a)** Fix  $\varepsilon > 0$  and  $\omega$  such that  $X(\omega)$  has a sample path which is right continuous, with left limits. Suppose there exists infinitely many jumps larger than  $\varepsilon$  at time  $\{s_n\}_{n \geq 1} \in [0, t]$ . (If there are uncountably many such jumps, we can arbitrarily choose countably many of them.) Since  $[0, t]$  is compact, there exists a subsequence  $\{s_{n_k}\}_{k \geq 1}$  converging to a cluster point  $s_* \in [0, 1]$ . Clearly we can take further subsequence converging to  $s^* \in [0, 1]$  monotonically either from above or from below. To simplify notations, Suppose  $\exists \{s_n\}_{n \geq 1} \uparrow s^*$ . (The other case  $\{s_n\}_{n \geq 1} \downarrow s^*$  is similar.) By left continuity of  $X_t$ , there exists  $\delta > 0$  such that  $s \in (s^* - \delta, s^*)$  implies  $|X_s - X_{s^*-}| < \varepsilon/3$  and  $|X_{s-} - X_{s^*-}| < \varepsilon/3$ . However for  $s_n \in (s^* - \delta, s^*)$ ,

$$|X_{s_n} - X_{s^*-}| = |X_{s_n-} - X_{s^*-} + \Delta X_{s_n}| \geq |\Delta X_{s_n}| - |X_{s_n-} - X_{s^*-}| > \frac{2\varepsilon}{3} \quad (22)$$

This is a contradiction and the claim is shown.

**b)** By a), for each  $n$  there is a finitely many jumps of size larger than  $1/n$ . But  $J = \{s \in [0, t] : |\Delta X_s| > 0\} = \cup_{n=1}^{\infty} \{s \in [0, t] : |\Delta X_s| > 1/n\}$ . We see that cardinality of  $J$  is countable.

**18.** By corollary to theorem 36 and theorem 37, we can immediately see that  $J^\varepsilon$  and  $Z - J^\varepsilon$  are Lévy processes. By Lévy -Khinchine formula (theorem 43), we can see that  $\psi_{J^\varepsilon} \psi_{Z-J^\varepsilon} = \psi_Z$ . Thus  $J^\varepsilon$  and  $Z - J^\varepsilon$  are independent. (For an alternative rigorous solution without Lévy -Khinchine formula, see a proof of theorem 36.)

**19.** Let  $T_n = \inf\{t > 0 : |X_t| > n\}$ . Then  $T_n$  is a stopping time. Let  $S_n = T_n 1_{\{X_0 \leq n\}}$ . We have  $\{S_n \leq t\} = \{T_n \leq t, X_0 \leq n\} \cup \{X_0 > n\} = \{T_n \leq t\} \cup \{X_0 > n\} \in \mathcal{F}_t$  and  $S_n$  is a stopping time. Since  $X$  is continuous,  $S_n \rightarrow \infty$  and  $X^{S_n} 1_{\{S_n > 0\}} \leq n, \forall n \geq 1$ . Therefore  $X$  is locally bounded.

**20.** Let  $H$  be an arbitrary unbounded random variable. (e.g. Normal) and let  $T$  be a positive random variable independent of  $H$  such that  $P(T \geq t) > 0, \forall t > 0$  and  $P(T < \infty) = 1$ . (e.g. Exponential). Define a process  $Z_t = H 1_{\{T \geq t\}}$ .  $Z_t$  is a càdlàg process and adapted to its natural filtration with  $Z_0 = 0$ . Suppose there exists a sequence of stopping times  $T_n \uparrow \infty$  such that  $Z^{T_n}$  is bounded by some  $K_n \in \mathbb{R}$ . Observe that

$$Z_t^{T_n} = \begin{cases} H & T_n \geq T > t \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

Since  $Z^{T_n}$  is bounded by  $K_n$ ,  $P(Z^{T_n} > K_n) = P(H > K_n)P(T_n \geq T > t) = 0$ . It follows that  $P(T_n \geq T > t) = 0, \forall n$  and hence  $P(T_n \leq T) = 1$ . Moreover  $P(\cap_n \{T_n \leq T\}) = P(T = \infty) = 1$ . This is a contradiction.

**21. a)** let  $a = (1-t)^{-1}(\int_t^1 Y(s)ds)$  and Let  $M_t = Y(\omega)1_{(0,t)}(\omega) + a1_{[t,1)}(\omega)$ . For arbitrary  $B \in \mathcal{B}([0,1])$ ,  $\{\omega : M_t \in B\} = ((0,t) \cap \{Y \in B\}) \cup (\{a \in B\} \cap (t,1))$ .  $(0,t) \cap \{Y \in B\} \subset (0,t)$  and hence in  $\mathcal{F}_t$ .  $\{a \in B\} \cap (t,1)$  is either  $(t,1)$  or  $\emptyset$  depending on  $B$  and in either case in  $\mathcal{F}_t$ . Therefore  $M_t$  is adapted.

Pick  $A \in \mathcal{F}_t$ . Suppose  $A \subset (0,t)$ . Then clearly  $E(M_t : A) = E(Y : A)$ . Suppose  $A \supset (t,1)$ . Then

$$\begin{aligned} E(M_t : A) &= E(Y : A \cap (0,t)) + E\left(\frac{1}{1-t} \int_t^1 Y(s)ds : (t,1)\right) \\ &= E(Y : A \cap (0,t)) + \int_t^1 Y(s)ds = E(Y : A \cap (0,t)) + E(Y : (t,1)) = E(Y : A). \end{aligned} \quad (24)$$

Therefore for  $\forall A \in \mathcal{F}_t$ ,  $E(M_t : A) = E(Y : A)$  and hence  $M_t = E(Y|\mathcal{F}_t)$  a.s.

**b)** By simple calculation  $EY^2 = 1/(1-2\alpha) < \infty$  and  $Y \in L^2 \subset L^1$ . It is also straightforward to compute  $M_t = (1-t)^{-1} \int_t^1 Y(s)ds = (1-\alpha)^{-1}Y(t)$  for  $\omega \in (t,1)$ .

**c)** From b),  $M_t(\omega) = Y(\omega)1_{(0,t)}(\omega) + 1/(1-\alpha)^{-1}Y(t)1_{(t,1)}(\omega)$ . Fix  $\omega \in (0,1)$ . Since  $0 < \alpha < 1/2$ ,  $Y/(1-\alpha) > Y$  and  $Y$  is a increasing on  $(0,1)$ . Therefore,

$$\sup_{0 < t < 1} M_t = \left( \sup_{0 < t < \omega} \frac{Y(t)}{1-\alpha} \right) \vee \left( \sup_{\omega \leq t < 1} Y(\omega) \right) = \frac{Y(\omega)}{1-\alpha} \vee Y(\omega) = \frac{Y(\omega)}{1-\alpha} \quad (25)$$

For each  $\omega \in (0,1)$ ,  $M_t(\omega) = Y(\omega)$  for all  $t \geq \omega$  and especially  $M_\infty(\omega) = Y(\omega)$ . Therefore

$$\left\| \sup_t M \right\|_{L^2} = \frac{1}{1-\alpha} \|Y\|_{L^2} = \frac{1}{1-\alpha} \|M_\infty\|_{L^2} \quad (26)$$

**22. a)** By simple computation,  $\frac{1}{1-t} \int_t^1 s^{-1/2}ds = 2/(1+\sqrt{t})$  and claim clearly holds.

**b)** Since  $T$  is a stopping time,  $\{T > \varepsilon\} \in \mathcal{F}_\varepsilon$  and hence  $\{T > \varepsilon\} \subset (0, \varepsilon]$  or  $\{T > \varepsilon\} \supset (\varepsilon, 1)$  for any  $\varepsilon > 0$ . Assume  $\{T > \varepsilon\} \subset (0, \varepsilon]$  for all  $\varepsilon > 0$ . Then  $T(\omega) \leq \varepsilon$  on  $(\varepsilon, 1)$  for all  $\varepsilon > 0$  and it follows that  $T \equiv 0$ . This is contradiction. Therefore there exists  $\varepsilon_0$  such that  $\{T > \varepsilon_0\} \supset (\varepsilon_0, 1)$ . Fix  $\varepsilon \in (0, \varepsilon_0)$ . Then  $\{T > \varepsilon\} \supset \{T > \varepsilon_0\} \supset (\varepsilon_0, 1)$ . On the other hand, since  $T$  is a stopping time,  $\{T > \varepsilon\} \subset (0, \varepsilon]$  or  $\{T > \varepsilon\} \supset (\varepsilon, 1)$ . Combining these observations, we know that  $\{T > \varepsilon\} \supset (\varepsilon, 1)$  for all  $\varepsilon \in (0, \varepsilon_0)$ .  $\forall \varepsilon \in (0, \varepsilon_0)$ , there exists  $\delta > 0$  such that  $\varepsilon - \delta > 0$  and  $\{T > \varepsilon - \delta\} \supset (\varepsilon - \delta, 1)$ . Especially  $T(\varepsilon) > \varepsilon - \delta$ . Taking a limit of  $\delta \downarrow 0$ , we observe  $T(\varepsilon) \geq \varepsilon$  on  $(0, \varepsilon_0)$ .

**c)** Using the notation in b),  $T(\omega) \geq \omega$  on  $(0, \varepsilon_0)$  and hence  $M_T(\omega) = \omega^{-1/2}$  on  $(0, \varepsilon_0)$ . Therefore,  $EM_T^2 \geq E(1/\omega : (0, \varepsilon_0)) = \int_0^{\varepsilon_0} \omega^{-1}d\omega = \infty$ . Since this is true for all stopping times not identically equal to 0,  $M$  cannot be locally a  $L^2$  martingale.

**d)** By a),  $|M_t(\omega)| \leq \omega^{-1/2} \vee 2$  and  $M$  has bounded path for each  $\omega$ . If  $M_T 1_{\{T > 0\}}$  were a bounded random variable, then  $M_T$  would be bounded as well since  $M_0 = 2$ . However, from c)  $M_T \notin L^2$  unless  $T \equiv 0$  a.s. and hence  $M_T 1_{\{T > 0\}}$  is unbounded.

**23.** Let  $M$  be a positive local martingale and  $\{T_n\}$  be its fundamental sequence. Then for  $t \geq s \geq 0$ ,  $E(M_t^{T_n} 1_{\{T_n > 0\}} | \mathcal{F}_s) = M_s^{T_n} 1_{\{T_n > 0\}}$ . By applying Fatou's lemma,  $E(M_t | \mathcal{F}_s) \leq M_s$  a.s. Therefore

a positive local martingale is a supermartingale. By Doob's supermartingale convergence theorem, positive supermartingale converges almost surely to  $X_\infty \in L^1$  and closable. Then by Doob's optional stopping theorem  $E(M_T | \mathcal{F}_S) \leq M_S$  a.s. for all stopping times  $S \leq T < \infty$ . If equality holds in the last inequality for all  $S \leq T$ , clearly  $M$  is a martingale since deterministic times  $0 \leq s \leq t$  are also stopping time. Therefore any positive *honest* local martingale makes a desired example. For a concrete example, see example at the beginning of section 5, chapter 1. (p. 37)

**24. a)** To simplify a notation, set  $\mathcal{F}^1 = \sigma(Z^{T-}, T) \vee \mathcal{N}$ . At first, we show  $\mathcal{G}_{T-} \supset \mathcal{F}^1$ . Since  $\mathcal{N} \subset \mathcal{G}_0$ ,  $\mathcal{N} \subset \mathcal{G}_{T-}$ . Clearly  $T \in \mathcal{G}_{T-}$  since  $\{T \leq t\} = (\Omega \cap \{t < T\})^c \in \mathcal{G}_{T-}$ .

$\forall t > 0$ ,  $\{Z_t^{T-} \in B\} = (\{Z_t \in B\} \cap \{t < T\}) \cup (\{Z_{T-} \in B\} \cap \{t \geq T\} \cap \{T < \infty\})$ .  $\{Z_t \in B\} \cap \{t < T\} \in \mathcal{G}_{T-}$  by definition of  $\mathcal{G}_{T-}$ . Define a mapping  $f : \{T < \infty\} \rightarrow \{T < \infty\} \times \mathbb{R}_+$  by  $f(\omega) = (\omega, T(\omega))$ . Define a  $\sigma$ -algebra  $\mathcal{P}^1$  consists of subsets of  $\Omega \times \mathbb{R}_+$  that makes all left continuous processes with right limits (càglàd processes) measurable. Especially  $\{Z_{t-}\}_{t \geq 0}$  is  $\mathcal{P}$ -measurable i.e.,  $\{(\omega, t) : Z_t(\omega) \in B\} \in \mathcal{P}$ . Without proof, we cite a fact about  $\mathcal{P}$ . Namely,  $f^{-1}(\mathcal{P}) = \mathcal{G}_{T-} \cap \{T < \infty\}$ . Since  $Z_{T(\omega)-}(\omega) = Z_- \circ f(\omega)$ ,  $\{Z_{T-} \in B\} \cap \{T < \infty\} = f^{-1}(\{(\omega, t) : Z_t(\omega) \in B\}) \cap \{T < \infty\} \in \mathcal{G}_{T-} \cap \{T < \infty\}$ . Note that  $\{T < \infty\} = \cap_n \{n < T\} \in \mathcal{G}_{T-}$ . Then  $Z^{T-} \in \mathcal{G}_{T-}$  and  $\mathcal{G}_{T-} \supset \mathcal{F}^1$ .

Next We show  $\mathcal{G}_{T-} \subset \mathcal{F}^1$ .  $\mathcal{G}_0 \subset \mathcal{F}^1$ , since  $Z_0^{T-} = Z_0$ . Fix  $t > 0$ . We want to show that for all  $A \in \mathcal{G}_t$ ,  $A \cap \{t < T\} \in \mathcal{F}^1$ . Let  $\Lambda = \{A : A \cap \{t < T\} \in \mathcal{F}^1\}$ . Let

$$\Pi = \{\cap_{i=1}^n \{Z_{s_i} \leq x_i\} : n \in \mathbb{N}_+, 0 \leq s_i \leq t, x_i \in \mathbb{R}\} \cup \mathcal{N} \quad (27)$$

Then  $\Pi$  is a  $\pi$ -system and  $\sigma(\Pi) = \mathcal{G}_t$ . Observe  $\mathcal{N} \subset \mathcal{F}^1$  and

$$(\cap_{i=1}^n \{Z_{s_i} \leq x_i\}) \cap \{t < T\} = (\cap_{i=1}^n \{Z_{s_i}^{T-} \leq x_i\}) \cap \{t < T\} \in \mathcal{F}^1, \quad (28)$$

$\Pi \subset \Lambda$ . By Dynkin's theorem ( $\pi - \lambda$  theorem),  $\mathcal{G}_t = \sigma(\Pi) \subset \Lambda$  hence the claim is shown.

**b)** To simplify the notation, let  $\mathcal{H} \triangleq \sigma(T, Z^T) \wedge \mathcal{N}$ . Then clearly  $\mathcal{H} \subset \mathcal{G}_T$  since  $\mathcal{N} \subset \mathcal{G}_T$ ,  $T \in \mathcal{G}_T$ , and  $Z_t^T \in \mathcal{G}_T$  for all  $t$ . It suffices to show that  $\mathcal{G}_T \subset \mathcal{H}$ . Let

$$\mathcal{L} = \left\{ A \in \mathcal{G}_\infty : E[1_A | \mathcal{G}_T] = E[1_A | \mathcal{H}] \right\} \quad (29)$$

Observe that  $\mathcal{L}$  is a  $\lambda$ -system (Dynkin's system) and contains all the null set since so does  $\mathcal{G}_\infty$ . Let

$$\mathcal{C} = \left\{ \bigcap_{j=1}^n \{Z_{t_j} \in B_j\} : n \in \mathbb{N}, t_j \in [0, \infty), B_j \in \mathcal{B}(\mathbb{R}) \right\}. \quad (30)$$

Then  $\mathcal{C}$  is a  $\pi$ -system such that  $\sigma(\mathcal{C}) \vee \mathcal{N} = \mathcal{G}_\infty$ . Therefore by Dynkin's theorem, provided that  $\mathcal{C} \subset \mathcal{L}$ ,  $\sigma(\mathcal{C}) \subset \mathcal{L}$  and thus  $\mathcal{G} \subset \mathcal{L}$ . For arbitrary  $A \in \mathcal{G}_T \subset \mathcal{G}_\infty$ ,  $1_A = E[1_A | \mathcal{G}_T] = E[1_A | \mathcal{H}] \in \mathcal{H}$  and hence  $A \in \mathcal{H}$ .

It remains to show that  $\mathcal{C} \subset \mathcal{L}$ . Fix  $n \in \mathbb{N}$ . Since  $\mathcal{H} \subset \mathcal{G}_T$ , it suffices to show that

$$E \left[ \prod_{j=1}^n 1_{\{Z_{t_j} \in B_j\}} \middle| \mathcal{G}_T \right] \in \mathcal{H}. \quad (31)$$

---

<sup>1</sup> $\mathcal{P}$  is called a predictable  $\sigma$ -algebra. Its definition and properties will be discussed in chapter 3.



For this, let  $t_0 = 0$  and  $t_{n+1} = \infty$  and write

$$E \left[ \prod_{j=1}^n 1_{\{Z_{t_j} \in B_j\}} \middle| \mathcal{G}_T \right] = \sum_{k=1}^{n+1} 1_{\{T \in [t_{k-1}, t_k)\}} \prod_{j < k} 1_{\{Z_{t_j \wedge T} \in B_j\}} E \left[ \prod_{j \geq k} 1_{\{Z_{t_j \vee T} \in B_j\}} \middle| \mathcal{G}_T \right].$$

Let  $\xi \triangleq \prod_{j \geq k} 1_{\{Z_{t_j \vee T} \in B_j\}} \in \mathcal{G}_\infty$ . Then by the strong Markov property of  $Z$ ,

$$E \left[ \prod_{j \geq k} 1_{\{Z_{t_j \vee T} \in B_j\}} \middle| \mathcal{G}_T \right] = E \left[ \xi \circ \theta_T \middle| \mathcal{G}_T \right] = E_{Z_T}[\xi]. \quad (32)$$

This verifies (31) and completes the proof. (This solution is by Jason Swanson).

**c)** Since  $\mathcal{G}_{T-} \subset \mathcal{G}_T$  and  $Z_T 1_{\{T < \infty\}} \in \mathcal{G}_T$ ,  $\sigma\{\mathcal{G}_{T-}, Z_T\} \subset \mathcal{G}_T$  if  $T < \infty$  a.s. To show the converse, observe that  $Z_t^T = Z_t^{T-} + \Delta Z_T 1_{\{t \geq T\}}$ . Since  $Z_t^{T-}$ ,  $Z_{T-}$ ,  $1_{\{t \geq T\}} \in \mathcal{G}_{T-}$  for all  $t \geq 0$ ,  $Z_t^T \in \sigma(\mathcal{G}_{T-}, Z_T)$  for all  $t \geq 0$ . Therefore  $\mathcal{G}_T = \sigma(\mathcal{G}_{T-}, Z_T)$ .

**d)** Since  $\mathcal{N} \subset \mathcal{G}_{T-}$ ,  $Z_T = Z_{T-}$  a.s. for  $T < \infty$  implies  $\mathcal{G}_T = \sigma(\mathcal{G}_{T-}, Z_T) = \sigma(\mathcal{G}_{T-}) = \mathcal{G}_{T-}$ .

**25.** Let  $Z$  be a Lévy process. By definition Lévy process is continuous in probability, i.e.  $\forall t$ ,  $\lim_n P(|Z_t - Z_{T-1/n}| > \varepsilon) = 0$ .  $\forall \varepsilon > 0$ ,  $\forall t > 0$ ,  $\{|\Delta Z_t| > \varepsilon\} = \cup_n \cap_{n \geq m} \{|Z_t - Z_{T-1/n}| > \varepsilon\}$ . Therefore,

$$P(|\Delta Z_t| > \varepsilon) \leq \liminf_{n \rightarrow \infty} P(|Z_t - Z_{T-1/n}| > \varepsilon) = 0 \quad (33)$$

Since this is true for all  $\varepsilon > 0$ ,  $P(|\Delta Z_t| > 0) = 0$ ,  $\forall t$ .

**26.** To apply results of exercise 24 and 25, we first show following almost trivial lemma.

**Lemma** Let  $T$  be a stopping time and  $t \in \mathbb{R}_+$ . If  $T \equiv t$ , then  $\mathcal{G}_T = \mathcal{G}_t$  and  $\mathcal{G}_{T-} = \mathcal{G}_{t-}$ .

*Proof.*  $\mathcal{G}_{T-} = \{A \cap \{T > s\} : A \in \mathcal{G}_s\} = \{A \cap \{t > s\} : A \in \mathcal{G}_s\} = \{A : A \in \mathcal{G}_s, s < t\} = \mathcal{G}_{t-}$ . Fix  $A \in \mathcal{G}_t$ .  $A \cap \{t \leq s\} = A \in \mathcal{G}_t \subset \mathcal{G}_s$  ( $t \leq s$ ), or  $\emptyset \in \mathcal{G}_s$  ( $t > s$ ) and hence  $\mathcal{G}_t \subset \mathcal{G}_T$ . Fix  $A \in \mathcal{G}_T$ ,  $A \cap \{t \leq s\} \in \mathcal{G}_s$ ,  $\forall s > 0$ . Especially,  $A \cap \{t \leq t\} = A \in \mathcal{G}_t$  and  $\mathcal{G}_T \subset \mathcal{G}_t$ . Therefore  $\mathcal{G}_T = \mathcal{G}_t$ .  $\square$

Fix  $t > 0$ . By exercise 25,  $Z_t = Z_{t-}$  a.s.. By exercise 24 (d),  $\mathcal{G}_t = \mathcal{G}_{t-}$  since  $t$  is a bounded stopping time.

**27.** Let  $A \in \mathcal{F}_t$ . Then  $A \cap \{t < S\} = (A \cap \{t < S\}) \cap \{t < T\} \in \mathcal{F}_{T-}$  since  $A \cap \{t < S\} \in \mathcal{F}_t$ . Then  $\mathcal{F}_{S-} \subset \mathcal{F}_{T-}$ .

Since  $T_n \leq T$ ,  $\mathcal{F}_{T_n-} \subset \mathcal{F}_{T-}$  for all  $n$  as shown above. Therefore,  $\vee_n \mathcal{F}_{T_n-} \subset \mathcal{F}_{T-}$ . Let  $A \in \mathcal{F}_t$ .  $A \cap \{t < T\} = \cup_n (A \cap \{t < T_n\}) \in \vee_n \mathcal{F}_{T_n-}$  and  $\mathcal{F}_{T-} \subset \vee_n \mathcal{F}_{T_n-}$ . Therefore,  $\mathcal{F}_{T-} = \vee_n \mathcal{F}_{T_n-}$ .

**28.** Observe that the equation in theorem 38 depends only on the existence of a sequence of simple functions approximation  $f 1_\Lambda \geq 0$  and a convergence of both sides in  $E\{\sum_j a_j N_j^{\Lambda_j}\} = t \sum_j a_j \nu(\Lambda_j)$ . For this,  $f 1_\Lambda \in L^1$  is enough. (Note that we need  $f 1_\Lambda \in L^2$  to show the second equation.)

**29.** Let  $M_t$  be a Lévy process and local martingale.  $M_t$  has a representation of the form

$$M_t = B_t + \int_{\{|x| \leq 1\}} x [N_t(\cdot, dx) - t\nu(dx)] + \alpha t + \int_{\{|x| > 1\}} x N_t(\cdot, dx) \quad (34)$$

First two terms are martingales. Therefore WLOG, we can assume that

$$M_t = \alpha t + \int_{\{|x| > 1\}} x N_t(\cdot, dx) \quad (35)$$

$M_t$  has only finitely many jumps on each interval  $[0, t]$  by exercise 17 (a). Let  $\{T_n\}_{n \geq 1}$  be a sequence of jump times of  $M_t$ . Then  $P(T_n < \infty) = 1, \forall n$  and  $T_n \uparrow \infty$ . We can express  $M_t$  by a sum of compound Poisson process and a drift term (See Example in P.33):

$$M_t = \sum_{i=1}^{\infty} U_i 1_{\{t \geq T_i\}} - \alpha t. \quad (36)$$

Since  $M_t$  is local martingale by hypothesis,  $M$  is a martingale if and only if  $M_t \in L^1$  and  $E[M_t] = 0, \forall t$ . There exists a fundamental sequence  $\{S_n\}$  such that  $M^{S_n}$  is a martingale,  $\forall n$ . WLOG, we can assume that  $M^{S_n}$  is uniformly integrable. If  $U_1 \in L^1, M_t \in L^1$  for every  $\alpha$  and  $M_t$  becomes martingale with  $\alpha_* = EN_t EU_1/t$ . Furthermore, for any other  $\alpha, M_t$  can't be local martingale since for any stopping time  $T, M_t^T = L_t^T + (\alpha_* - \alpha)(t \wedge T)$  where  $L_t$  is a martingale given by  $\alpha = \alpha_*$  and  $M_t^T$  can't be a martingale if  $\alpha \neq \alpha_*$ . It follows that if  $M_t$  is only a local martingale,  $U_1 \notin L^1$ .

By exercise 24,

$$\mathcal{F}_{T_1-} = \sigma(M^{T_1-}, T_1) \vee \mathcal{N} = \sigma(T_1) \vee \mathcal{N} \quad (37)$$

since  $M_t^{T_1-} = M_t 1_{\{t < T_1\}} + M_{T_1-} 1_{\{t \geq T_1\}} = -\alpha(t \wedge T_1)$  and  $M_{\infty}^{T_1-} = -\alpha T_1$ . Then

$$\{S_n < T_1\} = \cup_{r \in \mathbb{Q}_+} \{S_n \leq r\} \cap \{r < T_1\} \in \mathcal{F}_{T_1-} = \sigma(T_1) \vee \mathcal{N} \quad (38)$$

Therefore,

$$\begin{aligned} E|M_{S_n \wedge T_1}| &= E|U_1 1_{\{S_n \geq T_1\}} - \alpha(S_n \wedge T_1)| \geq E|U_1 1_{\{S_n \geq T_1\}}| - \alpha E|S_n \wedge T_1| \\ &= E[E(|U_1| 1_{\{S_n \geq T_1\}} | \sigma(T_1) \vee \mathcal{N})] - \alpha E|S_n \wedge T_1| \\ &= E[1_{\{S_n \geq T_1\}} E(|U_1| | \sigma(T_1) \vee \mathcal{N})] - \alpha E|S_n \wedge T_1| \\ &\geq E|U_1| P(S_n \geq T_1) - \alpha E T_1 = \infty \end{aligned} \quad (39)$$

This is a contradiction. Therefore,  $M$  is a martingale.

**30.** Let  $T_z = \inf\{s > 0 : Z_t \geq z\}$ . Then  $T_z$  is a stopping time and  $P(T_z < \infty) = 1$ . Let's define a coordinate map  $\omega(t) = Z_t(\omega)$ . Let  $R = \inf\{s < t : Z_s \geq z\}$ . We let

$$Y_s(\omega) = \begin{cases} 1 & s \leq t, \omega(t-s) < z-y \\ 0 & \text{otherwise} \end{cases}, \quad Y'_s(\omega) = \begin{cases} 1 & s \leq t, \omega(t-s) > z+y \\ 0 & \text{otherwise} \end{cases} \quad (40)$$

So that

$$Y_R \circ \theta_R(\omega) = \begin{cases} 1 & R \leq t, Z_t < z - y \\ 0 & \text{otherwise} \end{cases}, \quad Y'_R \circ \theta_R(\omega) = \begin{cases} 1 & R \leq t, Z_t > z + y \\ 0 & \text{otherwise} \end{cases} \quad (41)$$

Strong Markov property implies that on  $\{R < \infty\} = \{T_z \leq t\} = \{S_t \geq z\}$ ,

$$E_0(Y_R \circ \theta_R | \mathcal{F}_R) = E_{Z_R} Y_R, \quad E_0(Y'_R \circ \theta_R | \mathcal{F}_R) = E_{Z_R} Y'_R \quad (42)$$

Since  $Z_R \geq z$  and  $Z$  is symmetric,

$$E_a Y_s = E_a 1_{\{Z_{t-s} < z-y\}} < E_a 1_{\{Z_{t-s} > z+y\}} = E_a Y'_s, \quad \forall a \geq z, s < t. \quad (43)$$

By taking expectation,

$$\begin{aligned} P_0(T_z \leq t, Z_t < z - y) &= E(E_0(Y_R \circ \theta_R | \mathcal{F}_R) : R < \infty) = E(E_{Z_R} Y_R : R < \infty) \\ &\leq E(E_{Z_R} Y'_R : R < \infty) = E(E_0(Y'_R \circ \theta_R | \mathcal{F}_R) : R < \infty) = P_0(T_z \leq t, Z_t > z + y) \\ &= P(Z_t > z + y) \end{aligned} \quad (44)$$

**31.** We define  $T_z, R, \omega(t)$  as in exercise 30. We let

$$Y_s(\omega) = \begin{cases} 1 & s \leq t, \omega(t-s) \geq z \\ 0 & \text{otherwise} \end{cases} \quad (45)$$

Since  $Z_R \geq z$  and  $Z$  is symmetric,

$$E_a Y_s = E_a 1_{\{Z_{t-s} \geq z\}} \geq \frac{1}{2} \quad \forall a \geq z, s < t. \quad (46)$$

By the same reasoning as in exercise 30, taking expectation yields

$$\begin{aligned} P_0(Z_t \geq z) &= P_0(T_z \leq t, Z_t \geq z) = E(E_0(Y_R \circ \theta_R | \mathcal{F}_R) : R < \infty) = E(E_{Z_R} Y_R : R < \infty) \\ &\geq E\left(\frac{1}{2} : R < \infty\right) = \frac{1}{2} P(R < \infty) = \frac{1}{2} P(S_t \geq z) \end{aligned} \quad (47)$$

## Chapter 2. Semimartingales and Stochastic Integrals

1. Let  $x_0 \in \mathbb{R}$  be a discontinuous point of  $f$ . Wlog, we can assume that  $f$  is a right continuous function with left limit and  $\Delta f(x_0) = d > 0$ . Since  $\inf_t \{B_t = x_0\} < \infty$  and due to Strong Markov property of  $B_t$ , we can assume  $x_0 = 0$ . Almost every Brownian path does not have point of decrease (or increase) and it is a continuous process. So  $B$  visit  $x_0 = 0$  and changes its sign infinitely many times on  $[0, \epsilon]$  for any  $\epsilon > 0$ . Therefore,  $Y_t = f(B_t)$  has infinitely many jumps on any compact interval almost surely. Therefore,

$$\sum_{s \leq t} (\Delta Y_s)^2 = \infty \quad a.s.$$

2. By dominated convergence theorem,

$$\lim_{n \rightarrow \infty} E_Q[|X_n - X| \wedge 1] = \lim_{n \rightarrow \infty} E_P \left[ |X_n - X| \wedge 1 \cdot \frac{dQ}{dP} \right] = 0$$

$|X_n - X| \wedge 1 \rightarrow 0$  in  $L^1(Q)$  implies that  $X_n \rightarrow X$  in  $Q$ -probability.

3.  $B_t$  is a continuous local martingale by construction and by independence between  $X$  and  $Y$ ,

$$[B, B]_t = \alpha^2 [X, X]_t + (1 - \alpha^2) [Y, Y]_t = t$$

So by Lévy theorem,  $B$  is a standard 1-dimensional Brownian motion.

$$[X, B]_t = \alpha t, \quad [Y, B]_t = \sqrt{1 - \alpha^2} t$$

4. Assume that  $f(0) = 0$ . Let  $M_t = B_{f(t)}$  and  $\mathcal{G}_t = \mathcal{F}_{f(t)}$  where  $B$  is a one-dimensional standard Brownian motion and  $\mathcal{F}_t$  is a corresponding filtration. Then

$$E[M_t | \mathcal{G}_s] = E[B_{f(t)} | \mathcal{F}_{f(s)}] = B_{f(s)} = M_s$$

Therefore  $M_t$  is a martingale w.r.t.  $\mathcal{G}_t$ . Since  $f$  is continuous and  $B_t$  is a continuous process,  $M_t$  is clearly continuous process. Finally,

$$[M, M]_t = [B, B]_{f(t)} = f(t)$$

If  $f(0) > 0$ , then we only need to add a constant process  $A_t$  to  $B_{f(t)}$  such that  $2A_t M_0 + A_t^2 = -B_{f(0)}^2$  for each  $\omega$  to get a desired result.

5. Since  $B$  is a continuous process with  $\Delta B_0 = 0$ ,  $M$  is also continuous.  $M$  is local martingale since  $B$  is a locally square integrable local martingale.

$$[M, M]_t = \int_0^t H_s^2 ds = \int_0^t 1 ds = t$$

So by Lévy's characterization theorem,  $M$  is also a Brownian motion.

6. Pick arbitrary  $t_0 > 1$ . Let  $X_t^n = 1_{(t_0 - \frac{1}{n}, \infty)}(t)$  for  $n \geq 1$ ,  $X_t = 1_{[t_0, \infty)}$ ,  $Y_t = 1_{[t_0, \infty)}$ . Then  $X^n, X, Y$  are finite variation processes and Semimartingales.  $\lim_n X_t^n = X_t$  almost surely. But

$$\lim_n [X^n, Y]_{t_0} = 0 \neq 1 = [X, Y]_{t_0}$$

7. Observe that

$$[X^n, Z] = [H^n, Y \cdot Z] = H^n \cdot [Y, Z], \quad [X, Z] = [H, Y \cdot Z] = H \cdot [Y, Z]$$

and  $[Y, Z]$  is a semimartingale. Then by the continuity of stochastic integral,  $H^n \rightarrow H$  in ucp implies,  $H^n \cdot [Y, Z] \rightarrow H \cdot [Y, Z]$  and hence  $[X^n, Z] \rightarrow [X, Z]$  in ucp.

8. By applying Ito's formula,

$$\begin{aligned} [f_n(X) - f(X), Y]_t &= [f_n(X_0) - f(X_0), Y]_t + \int_0^t (f'_n(X_s) - f'(X_s)) d[X, Y]_s \\ &\quad + \frac{1}{2} \int_0^t (f''_n(X_s) - f''(X_s)) d[[X, X], Y]_s \\ &= (f_n(X_0) - f(X_0))Y_0 + \int_0^t (f'_n(X_s) - f'(X_s)) d[X, Y]_s \end{aligned}$$

Note that  $[[X, X], Y] \equiv 0$  since  $X$  and  $Y$  are continuous semimartingales and especially  $[X, X]$  is a finite variation process. As  $n \rightarrow \infty$ ,  $(f_n(X_0) - f(X_0))Y_0 \rightarrow 0$  for arbitrary  $\omega \in \Omega$ . Since  $[X, Y]$  has a path of finite variation on compacts, we can treat  $f_n(X) \cdot [X, Y]$ ,  $f(X) \cdot [X, Y]$  as Lebesgue-Stieltjes integral computed path by path. So fix  $\omega \in \Omega$ . Then as  $n \rightarrow \infty$ ,

$$\sup_{0 \leq s \leq t} |f_n(B_s) - f(B_s)| = \sup_{\inf_{s \leq t} B_s \leq x \leq \sup_{s \leq t} B_s} |f_n(x) - f(x)| \rightarrow 0$$

since  $f'_n \rightarrow f$  uniformly on compacts. Therefore

$$\left| \int_0^t (f'_n(X_s) - f'(X_s)) d[X, Y]_s \right| \leq \sup_{0 \leq s \leq t} |f_n(B_s) - f(B_s)| \int_0^t d|[X, Y]|_s \rightarrow 0$$

9. Let  $\sigma_n$  be a sequence of random partition tending to identity. By theorem 22,

$$\begin{aligned} [M, A] &= M_0 A_0 + \lim_{n \rightarrow \infty} \sum_i \left( M^{T_{i+1}^n} - M^{T_i^n} \right) \left( A^{T_{i+1}^n} - A^{T_i^n} \right) \\ &\leq 0 + \lim_{n \rightarrow \infty} \sup_i \left| M^{T_{i+1}^n} - M^{T_i^n} \right| \sum_i \left| A^{T_{i+1}^n} - A^{T_i^n} \right| = 0 \end{aligned}$$

since  $M$  is a continuous process and  $\sum_i \left| A^{T_{i+1}^n} - A^{T_i^n} \right| < \infty$  by hypothesis. Similarly

$$\begin{aligned} [A, A] &= M_0^2 + \lim_{n \rightarrow \infty} \sum_i \left( A^{T_{i+1}^n} - A^{T_i^n} \right) \left( A^{T_{i+1}^n} - A^{T_i^n} \right) \\ &\leq 0 + \lim_{n \rightarrow \infty} \sup_i \left| A^{T_{i+1}^n} - A^{T_i^n} \right| \sum_i \left| A^{T_{i+1}^n} - A^{T_i^n} \right| = 0 \end{aligned}$$

Therefore,

$$[X, X] = [M, M] + 2[M, A] + [A, A] = [M, M]$$

**10.**  $X^2$  is  $P$ -semimartingale and hence  $Q$ -semimartingale by Theorem 2. By Corollary of theorem 15,  $(X_- \cdot X)^Q$  is indistinguishable from  $(X_- \cdot X)^P$ . Then by definition of quadric variation,

$$[X, X]^P = X^2 - (X_- \cdot X)^P = X^2 - (X_- \cdot X)^Q = [X, X]^Q$$

up to evanescent set.

**11. a)**  $\Lambda = [-2, 1]$  is a closed set and  $B$  has a continuous path, by Theorem 4 in chapter 1,  $T(\omega) = \inf t : B_t \notin \Lambda$  is a stopping time.

**b)**  $M_t$  is a uniformly integrable martingale since  $M_t = E[B_T | \mathcal{F}_t]$ . Clearly  $M$  is continuous. Clearly  $N$  is a continuous martingale as well. By Theorem 23,  $[M, M]_t = [B, B]_t^T = t \wedge T$  and  $[N, N]_t = [-B, -B]_t^T = t \wedge T$ . Thus  $[M, M] = [N, N]$ . However  $P(M_t > 1) = 0 \neq P(N_t > 1)$ .  $M$  and  $N$  does not have the same law.

**12.** Fix  $t \in \mathbb{R}_+$  and  $\omega \in \Omega$ . WLOG we can assume that  $X(\omega)$  has a càdlàg path and  $\sum_{s \leq t} |\Delta X_s(\omega)| < \infty$ . Then on  $[0, t]$ , continuous part of  $X$  is bounded by continuity and jump part of  $X$  is bounded by hypothesis. So  $\{X_s\}_{s \leq t}$  is bounded. Let  $K \subset \mathbb{R}$  be  $K = [\inf_{s \leq t} X_s(\omega), \sup_{s \leq t} X_s(\omega)]$ . Then  $f, f', f''$  is bounded on  $K$ . Since  $\sum_{s \leq t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}$  is an absolute convergent series, (see proof of Ito's formula (Theorem 32)), it suffices to show that  $\sum_{s \leq t} \{f'(X_{s-})\Delta X_s\} < \infty$ . By hypothesis,

$$\sum_{s \leq t} |f'(X_{s-})\Delta X_s| \leq \sup_{x \in K} |f'(x)| \sum_{s \leq t} |\Delta X_s| < \infty$$

Since this is true for all  $t \in \mathbb{R}_+$  and almost all  $\omega \in \Omega$ , the claim is shown.

**13.** By definition, stochastic integral is a continuous linear mapping  $J_X: \mathbf{S}_{ucp} \rightarrow \mathbb{D}_{ucp}$ . (Section 4). By continuity,  $H^n \rightarrow H$  under ucp topology implies  $H^n \cdot X \rightarrow H \cdot X$  under ucp topology.

**14.** Let  $\hat{A} = 1 + A$  to simplify notations. Fix  $\omega \in \Omega$  and let  $\hat{A}^{-1} \cdot X = Z$ . Then  $Z_\infty(\omega) < \infty$  by hypothesis and  $\hat{A} \cdot Z = X$ . Applying integration by parts to  $X$  and then divide both sides by  $\hat{A}$  yields

$$\frac{X_t}{\hat{A}} = Z_t - \frac{1}{\hat{A}} \int_0^t Z_s d\hat{A}_s$$

Since  $Z_\infty < \infty$  exists, for any  $\epsilon > 0$ , there exists  $\tau$  such that  $|Z_t - Z_\infty| < \epsilon$  for all  $t \geq \tau$ . Then for  $t > \tau$ ,

$$\frac{1}{\hat{A}_t} \int_0^t Z_s d\hat{A}_s = \frac{1}{\hat{A}_t} \int_0^\tau Z_s d\hat{A}_s + \frac{1}{\hat{A}_t} \int_\tau^t (Z_s - Z_\infty) d\hat{A}_s + Z_\infty \frac{\hat{A}_t - \hat{A}_\tau}{\hat{A}_t}$$

Let's evaluate right hand side. As  $t \rightarrow \infty$ , the first term goes to 0 while the last term converges to  $Z_\infty$  by hypothesis. By construction of  $\tau$ , the second term is smaller than  $\epsilon(\hat{A}_t - \hat{A}_\tau)/\hat{A}_t$ , which converges to  $\epsilon$ . Since this is true for all  $\epsilon > 0$ ,

$$\lim_{t \rightarrow \infty} \frac{1}{\hat{A}} \int_0^t Z_s d\hat{A}_s = Z_\infty$$

Thus  $\lim_{t \rightarrow \infty} X_t/\hat{A}_t = 0$ .

**15.** If  $M_0 = 0$  then by Theorem 42, there exists a Brownian motion  $B$  such that  $M_t = B_{[M,M]_t}$ . By the law of iterated logarithm,

$$\lim_{t \rightarrow \infty} \frac{M_t}{[M, M]_t} = \lim_{t \rightarrow \infty} \frac{B_{[M,M]_t}}{[M, M]_t} = \lim_{\tau \rightarrow \infty} \frac{B_\tau}{\tau} = 0$$

If  $M_0 \neq 0$ , Let  $X_t = M_t - M_0$ . Then  $[X, X]_t = [M, M]_t - M_0^2$ . So  $[X, X]_t \rightarrow \infty$  and

$$\frac{M_t}{[M, M]_t} = \frac{X_t + M_0}{[X, X]_t + M_0^2} \rightarrow 0$$

**17.** Since integral is taken over  $[t - 1/n, t]$  and  $Y$  is adapted,  $X_t^n$  is also an adapted process. For  $t > s \geq 1/n$  such that  $|t - s| \leq 1/n$ ,

$$|X_t^n - X_s^n| = n \left| \int_s^t Y_\tau d\tau - \int_{s-1/n}^{t-1/n} Y_\tau d\tau \right| \leq 2n(t-s) \sup_{s-\frac{1}{n} \leq \tau \leq t} |Y_\tau|$$

Since  $X$  is constant on  $[0, t]$  and  $[1, \infty)$ , let  $\pi$  be a partition of  $[t, 1]$  and  $M = \sup_{s-\frac{1}{n} \leq \tau \leq t} |Y_\tau| < \infty$ . Then for each  $n$ , total variation of  $X^n$  is finite since

$$\sup_{\pi} |X_{t_{i+1}}^n - X_{t_i}^n| \leq 2nM \sum_{\pi} (t_{i+1} - t_i) = 2nM(1-t)$$

Therefore  $X_n$  is a finite variation process and in particular a semimartingale. By the fundamental theorem of calculus and continuity of  $Y$ ,  $Y_s = \lim_{n \rightarrow \infty} n \int_{s-1/n}^s Y_\tau d\tau = \lim_{n \rightarrow \infty} X_t^n$  for each  $t > 0$ . At  $t = 0$ ,  $\lim_n X_n = 0 = Y_0$ . So  $X_n \rightarrow Y$  for all  $t \geq 0$ . However  $Y$  need not be a semimartingale since  $Y_t = (B_t)^{1/3}$  where  $B_t$  is a standard Brownian motion satisfies all requirements but not a semimartingale.

**18.**  $B_t$  has a continuous path almost surely. Fix  $\omega$  such that  $B_t(\omega)$  is continuous. Then  $\lim_{n \rightarrow \infty} X_t^n(\omega) = B_t(\omega)$  by Exercise 17. By Theorem 37, the solution of  $dZ_t^n = Z_{s-}^n dX_s^n$ ,  $Z_0 = 1$  is

$$Z_t^n = \exp \left( X_t^n - \frac{1}{2} [X^n, X^n]_t \right) = \exp(X_t^n),$$

since  $X^n$  is a finite variation process. On the other hand,  $Z_t = \exp(B_t - t/2)$  and

$$\lim_{n \rightarrow \infty} Z_t^n = \exp(B_t) \neq Z_t$$

**19.**  $A^n$  is a clearly càdlàg and adapted. By periodicity and symmetry of triangular function,

$$\int_0^{\frac{\pi}{2}} |dA_s^n| = \frac{1}{n} \int_0^{\frac{\pi}{2}} |d \sin(ns)| = \frac{1}{n} \cdot n \int_0^{\frac{\pi}{2n}} d(\sin(ns)) = \int_0^{\frac{\pi}{2}} d(\sin(s)) = 1$$

Since total variation is finite,  $A_t^n$  is a semimartingale. On the other hand,

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq \frac{\pi}{2}} |A_t^n| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

**20.** Applying Ito's formula to  $u \in C^2(\mathbb{R}^3 - \{0\})$  and  $B_t \in \mathbb{R}^3 \setminus \{0\}$ ,  $\forall t$ , a.s.

$$u(B_t) = u(x) + \int_0^t \nabla u(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \Delta u(B_s) ds = \frac{1}{\|x\|} + \sum_{i=1}^3 \int_0^t \partial_i u(B_s) dB_s^i$$

and  $M_t = u(B_t)$  is a local martingale. This solves **(a)**. Fix  $1 \leq \alpha \leq 3$ . Observe that  $E(u(B_0)^\alpha) < \infty$ . Let  $p, q$  be a positive number such that  $1/p + 1/q = 1$ . Then

$$\begin{aligned} E^x(u(B_t)^\alpha) &= \int_{\mathbb{R}^3} \frac{1}{\|y\|^\alpha} \frac{1}{(2\pi t)^{3/2}} e^{-\frac{\|x-y\|^2}{2t}} dy \\ &= \int_{\{B(0;1) \cap B^c(x;\delta)\}} \frac{1}{\|y\|^\alpha} \frac{1}{(2\pi t)^{3/2}} e^{-\frac{\|x-y\|^2}{2t}} dy + \int_{\mathbb{R}^3 \setminus \{B(0;1) \cap B^c(x;\delta)\}} \frac{1}{\|y\|^\alpha} \frac{1}{(2\pi t)^{3/2}} e^{-\frac{\|x-y\|^2}{2t}} dy \\ &\leq \sup_{y \in \{B(0;1) \cap B^c(x;\delta)\}} \frac{1}{(2\pi t)^{\frac{3}{2}}} e^{-\frac{\|x-y\|^2}{2t}} \cdot \int_{B(0;1)} \frac{1}{\|y\|^\alpha} dy \\ &\quad + \left( \int_{\mathbb{R}^3 \setminus \{B(0;1) \cap B^c(x;\delta)\}} \frac{1}{\|y\|^{\alpha p}} dy \right)^{1/p} \left( \int_{\mathbb{R}^3 \setminus \{B(0;1) \cap B^c(x;\delta)\}} \left( \frac{1}{(2\pi t)^{3/2}} e^{-\frac{\|x-y\|^2}{2t}} \right)^q dy \right)^{1/q} \end{aligned}$$

Pick  $p > 3/\alpha > 1$ . Then the first term goes to 0 as  $t \rightarrow \infty$ . In particular it is finite for all  $t \geq 0$ . The first factor in the second term is finite while the second factor goes to 0 as  $t \rightarrow \infty$  since

$$\int \frac{1}{(2\pi t)^{3/2}} \frac{1}{(2\pi t)^{3(q-1)/2}} e^{-\frac{\|x-y\|^2}{2t/q}} dy = \frac{1}{(2\pi t)^{3(q-1)/2}} \int \frac{1}{q^{3/2}} \frac{1}{(2\pi t/q)^{3/2}} e^{-\frac{\|x-y\|^2}{2t/q}} dy = \frac{1}{(2\pi t)^{3(q-1)/2}} \frac{1}{q^{3/2}}$$

and the second factor is finite for all  $t \geq 0$ . **(b)** is shown with  $\alpha = 1$ . **(c)** is shown with  $\alpha = 2$ . It also shows that

$$\lim_{t \rightarrow \infty} E^x(u((B_t)^2)) = 0$$

which in turn shows **(d)**.

**21.** Applying integration by parts to  $(A^\infty - A_*)(C^\infty - C_*)$ ,

$$(A_\infty - A_*)(C_\infty - C_*) = A_\infty C_\infty - \int_0^\infty (A_\infty - A_{s-}) dC_s - \int_0^\infty (C_\infty - C_{s-}) dA_s + [A, C].$$

Since  $A, C$  are processes with finite variation,

$$\begin{aligned} [A, C]_t &= \sum_{0 < s < t} \Delta A_s \Delta C_s \\ \int_0^t (A_\infty - A_{s-}) dC_s &= \int_0^t (A_\infty - A_s) dC_s + \sum_{0 < s < t} \Delta A_s \Delta C_s \end{aligned}$$

Substituting these equations,

$$(A_\infty - A_t)(C_\infty - C_t) = A_\infty C_\infty - \int_0^t (A_\infty - A_s) dC_s - \int_0^t (C_\infty - C_{s-}) dA_s$$

Letting  $t \rightarrow 0$  and we obtain **(a)**. **(b)** is immediate from this result.



**22.** Claim that for all integer  $k \geq 1$  and prove by induction.

$$(A_\infty - A_s)^k = k! \int_s^\infty dA_{s_1} \int_{s_1}^\infty dA_{s_2} \dots \int_{s_{p-1}}^\infty dA_{s_p} \quad (48)$$

For  $k = 1$ , equation (48) clearly holds. Assume that it holds for  $k = n$ . Then

$$(k+1)! \int_s^\infty dA_{s_1} \int_{s_1}^\infty dA_{s_2} \dots \int_{s_{p-1}}^\infty dA_{s_{k+1}} = (k+1) \int_s^\infty (A_\infty - A_{s_1}) dA_{s_1} = (A_\infty - A_s)^{k+1}, \quad (49)$$

since  $A$  is non-decreasing continuous process,  $(A_\infty - A) \cdot A$  is indistinguishable from Lebesgue-Stieltjes integral calculated on path by path. Thus equation (48) holds for  $n = k + 1$  and hence for all integer  $n \geq 1$ . Then setting  $s = 0$  yields a desired result since  $A_0 = 0$ .

**24. a)**

$$\begin{aligned} \int_0^t \frac{1}{1-s} d\beta_s &= \int_0^t \frac{1}{1-s} dB_s - \int_0^t \frac{1}{1-s} \frac{B_1 - B_s}{1-s} ds \\ &= \int_0^t \frac{1}{1-s} dB_s - B_1 \int_0^t \frac{1}{(1-s)^2} ds + \int_0^t \frac{B_s}{(1-s)^2} ds \\ &= \int_0^t \frac{1}{1-s} dB_s - B_1 \int_0^t \frac{1}{(1-s)^2} ds + \int_0^t B_s d\left(\frac{1}{1-s}\right) \\ &= \frac{B_t}{1-t} - \left[\frac{1}{1-s}, B\right]_t - B_1 \left(\frac{1}{1-t} - \frac{1}{1}\right) \\ &= \frac{B_t - B_1}{1-t} + B_1 \end{aligned}$$

Arranging terms and we have desired result.

**b)** Using integration by parts, since  $[1-s, \int_0^s (1-u)^{-1} d\beta_u]_t = 0$ ,

$$\begin{aligned} X_t &= (1-t) \int_0^t \frac{1}{1-s} d\beta_s = \int_0^t (1-s) \frac{1}{1-s} d\beta_s + \int_0^t \int_0^s \frac{1}{1-u} d\beta_u (-1) ds \\ &= \beta_t + \int_0^t \left(-\frac{X_s}{1-s}\right) ds, \end{aligned}$$

as required. The last equality is by result of (a).

**27.** By Ito's formula and given SDE,

$$d(e^{\alpha t} X_t) = \alpha e^{\alpha t} X_t dt + e^{\alpha t} dX_t = \alpha e^{\alpha t} X_t dt + e^{\alpha t} (-\alpha X_t dt + \sigma dB_t) = \sigma e^{\alpha t} dX_t$$

Then integrating both sides and arranging terms yields

$$X_t = e^{-\alpha t} \left( X_0 + \sigma \int_0^t e^{\alpha s} dB_s \right)$$

**28.** By the law of iterated logarithm,  $\limsup_{t \rightarrow \infty} \frac{B_t}{t} = 0$  a.s. In particular, for almost all  $\omega$  there exists  $t_0(\omega)$  such that  $t > t_0(\omega)$  implies  $B_t(\omega)/t < 1/2 - \epsilon$  for any  $\epsilon \in (0, 1/2)$ . Then

$$\lim_{t \rightarrow \infty} \mathcal{E}(B_t) = \lim_{t \rightarrow \infty} \exp \left\{ t \left( \frac{B_t}{t} - \frac{1}{2} \right) \right\} \leq \lim_{t \rightarrow \infty} e^{-\epsilon t} = 0, \quad \text{a.s.}$$

**29.**  $\mathcal{E}(X)^{-1} = \mathcal{E}(-X + [X, X])$  by Corollary of Theorem 38. This implies that  $\mathcal{E}(X)^{-1}$  is the solution to a stochastic differential equation,

$$\mathcal{E}(X)_t^{-1} = 1 + \int_0^t \mathcal{E}(X)_{s-}^{-1} d(-X_s + [X, X]_s),$$

which is the desired result. Note that continuity assumed in the corollary is not necessary if we assume  $\Delta X_s \neq -1$  instead so that  $\mathcal{E}(X)^{-1}$  is well defined.

**30. a)** By Ito's formula and continuity of  $M$ ,  $M_t = 1 + \int_0^t M_s dB_s$ .  $B_t$  is a locally square integrable local martingale and  $M \in \mathbb{L}$ . So by Theorem 20,  $M_t$  is also a locally square integrable local martingale.

**b)**

$$[M, M]_t = \int_0^t M_s^2 ds = \int_0^t e^{2B_s - s} ds$$

and for all  $t \geq 0$ ,

$$E([M, M]_t) = \int_0^t E[e^{2B_s}] e^{-s} ds = \int_0^t e^s ds < \infty$$

Then by Corollary 3 of Theorem 27,  $M$  is a martingale.

**c)**  $Ee^{B_t}$  is calculated above using density function. Alternatively, using the result of b),

$$Ee^{B_t} = E(M_t e^{\frac{t}{2}}) = e^{\frac{t}{2}} EM_0 = e^{\frac{t}{2}}$$

**31.** Pick  $A \in \mathcal{F}$  such that  $P(A) = 0$  and fix  $t \geq 0$ . Then  $A \cap \{R_t \leq s\} \in \mathcal{F}_s$  since  $\mathcal{F}_s$  contains all  $P$ -null sets. Then  $A \in \mathcal{G}_t = \mathcal{F}_{R_t}$ . If  $t_n \downarrow t$ , then by right continuity of  $R$ ,  $R_{t_n} \downarrow R_t$ . Then

$$\mathcal{G}_t = \mathcal{F}_{R_t} = \cap_n \mathcal{F}_{R_{t_n}} = \cap_n \mathcal{G}_{t_n} = \cap_{s \geq t} \mathcal{G}_s$$

by the right continuity of  $\{\mathcal{F}_t\}_t$  and Exercise 4, chapter 1. Thus  $\{\mathcal{G}_t\}_t$  satisfies the usual hypothesis.

**32.** If  $M$  has a càdlàg path and  $R_t$  is right continuous,  $\bar{M}_t$  has a càdlàg path.  $\bar{M}_t = M_{R_t} \in \mathcal{F}_{R_t} = \mathcal{G}_t$ . So  $\bar{M}_t$  is adapted to  $\{\mathcal{G}_t\}$ . For all  $0 \leq s \leq t$ ,  $R_s \leq R_t < \infty$ . Since  $M$  is uniformly integrable martingale, by optional sampling theorem,

$$\bar{M}_s = M_{R_s} = E(M_{R_t} | \mathcal{F}_{R_s}) = E(\bar{M}_t | \mathcal{G}_s), \quad \text{a.s.}$$

So  $\bar{M}$  is  $\mathcal{G}$ -martingale.

### Chapter 3. Semimartingales and Decomposable Processes

1. Let  $\{T_i\}_{i=1}^n$  be a set of predictable stopping time. For each  $i$ ,  $T_i$  has an announcing sequence  $\{T_{i,j}\}_{j=1}^\infty$ . Let  $S_k := \max_i T_{i,k}$  and  $R_k := \min_i T_{i,k}$ . Then  $S_k, R_k$  are stopping time.  $\{S_k\}$  and  $\{R_k\}$  make announcing sequences of maximum and minimum of  $\{T_i\}_i$ .

2. Let  $T_n = S + (1 - 1/n)$  for each  $n$ . Then  $T_n$  is a stopping time and  $T_n \uparrow T$  as  $n \uparrow \infty$ . Therefore  $T$  is a predictable stopping time.

3. Let  $S, T$  be two predictable stopping time. Then  $S \wedge T, S \vee T$  are predictable stopping time as shown in exercise 1. In addition,  $\Lambda = \{S \vee T = S \wedge T\}$ . Therefore without loss of generality, we can assume that  $S \leq T$ . Let  $\{T_n\}$  be an announcing sequence of  $T$ . Let  $R_n = T_n + n1_{\{T_n \geq S\}}$ . Then  $R_n$  is a stopping time since  $\{R_n \leq t\} = \{T_n \leq t\} \cap (\{t - T_n \leq n\} \cup \{T_n < S\}) \in \mathcal{F}_t$ .  $R_n$  is increasing and  $\lim R_n = T_\Lambda = S_\Lambda$ .

4. For each  $X \in \mathbb{L}$ , define a new process  $X_n$  by  $X_n = \sum_{k \in \mathbb{N}} X_{k/2^n} 1_{[k/2^n, (k+1)/2^n)}$ . Since each summand is an optional set (See exercise 6)  $X^n$  is an optional process. As a mapping on the product space,  $X$  is a pints limit of  $X^n$ . Therefore  $X$  is optional. Then by the definition  $\mathcal{P} \subset \mathcal{O}$ .

5. Suffice to show that all càdlàg processes are progressively measurable. (Then we can apply monotone class argument.) For a càdlàg process  $X$  on  $[0, t]$ , define a new process  $X^n$  by putting  $X_u^n = X_{k/2^n}$  for  $u \in [(k-1)t/2^n, kt/2^n)$ ,  $k = \{1, \dots, 2^n\}$ . Then on  $\Omega \times [0, t]$ ,

$$\{X^n \in B\} = \cup_{k \in \mathbb{N}_+} \left\{ \omega : X_{k/2^n}(\omega) \in B \times \left[ \frac{k-1}{2^n}, \frac{k}{2^n} \right) \right\} \in \mathcal{F}_t \otimes \mathcal{B}([0, t]) \quad (50)$$

since  $X$  is adapted and  $X^n$  is progressively measurable process. Since  $X$  is càdlàg,  $\{X^n\}$  converges pints to  $X$ , which therefore is also  $\mathcal{F}_t \otimes \mathcal{B}([0, t])$  measurable.

6. (1)  $(S, T]$ : Since  $1_{(S, T]} \in \mathbb{L}$ ,  $(S, T] = \{(s, \omega) : 1_{(S, T]} = 1\} \in \mathcal{P}$  by definition.

(2)  $[S, T)$ : Since  $1_{[S, T)} \in \mathbb{D}$ ,  $[S, T) = \{(s, \omega) : 1_{[S, T)} = 1\} \in \mathcal{O}$  by definition.

(3)  $(S, T)$ : Since  $[S, T] = \cup_n [S, T + 1/n)$ ,  $[S, T]$  is an optional set. Then  $(S, T) = [0, T) \cap [0, S]^c$  and  $(S, T)$  is optional as well.

(4)  $[S, T)$  **when  $S, T$  is predictable**: Let  $\{S_n\}$  and  $\{T_n\}$  be announcing sequences of  $S$  and  $T$ . Then  $[S, T) = \cap_m \cup_n (S_m, T_n]$ . Since  $(S_m, T_n]$  is predictable by (1) for all  $m, n$ ,  $[S, T)$  is predictable.

7. Pick a set  $A \in \mathcal{F}_{S_n}$ . Then  $A = A \cap \{S_n < T\} \in \mathcal{F}_{T-}$  by theorem 7 and the definition of  $\{S_n\}_n$ . (Note: The proof of theorem 7 does not require theorem 6). Since this is true for all  $n$ ,  $\vee_n \mathcal{F}_{S_n} \subset \mathcal{F}_{T-}$ . To show the converse let  $\Pi = \{B \cap \{t < T\} : B \in \mathcal{F}_t\}$ . Then  $\Pi$  is closed with respect to finite intersection.  $B \cap \{t < T\} = \cup_n (B \cap \{t < S_n\})$ . Since  $(B \cap \{t < S_n\}) \cap \{S_n \leq t\} = \emptyset \in \mathcal{F}_t$ ,  $B \cap \{t < S_n\} \in \mathcal{F}_{S_n}$ . Therefore  $B \cap \{t < T\} \in \vee_n \mathcal{F}_{S_n}$  and  $\Pi \subset \vee_n \mathcal{F}_{S_n}$ . Then by Dynkin's theorem,  $\mathcal{F}_{T-} \subset \vee_n \mathcal{F}_{S_n}$ .

8. Let  $S, T$  be stopping times such that  $S \leq T$ . Then  $\mathcal{F}_{S_n} \subset \mathcal{F}_T$  and  $\bigvee_n \mathcal{F}_{S_n} \subset \mathcal{F}_T$ . By the same argument as in exercise 7,  $\mathcal{F}_{T-} \subset \bigvee_n \mathcal{F}_{S_n}$ . Since  $\mathcal{F}_{T-} = \mathcal{F}_T$  by hypothesis, we have desired result. (Note:  $\{S_n\}$  is not necessarily an announcing sequence since  $S_n = T$  is possible. Therefore  $\bigvee \mathcal{F}_{S_n} \neq \mathcal{F}_{T-}$  in general.)

9. Let  $X$  be a Lévy process,  $\mathcal{G}$  be its natural filtration and  $T$  be stopping time. Then by exercises 24(c) in chapter 1,  $\mathcal{G}_T = \sigma(\mathcal{G}_{T-}, X_T)$ . Since  $X$  jumps only at totally inaccessible time (a consequence of theorem 4),  $X_T = X_{T-}$  for all predictable stopping time  $T$ . Therefore if  $T$  is a predictable time,  $\mathcal{G}_T = \sigma(\mathcal{G}_{T-}, X_T) = \sigma(\mathcal{G}_{T-}, X_{T-}) = \mathcal{G}_{T-}$  since  $X_{T-} \in \mathcal{G}_{T-}$ . Therefore a completed natural filtration of a Lévy process is quasi left continuous.

10. As given in hint,  $[M, A]$  is a local martingale. Let  $\{T_n\}$  be its localizing sequence, that is  $[M, A]^{T_n}$  is a martingale for all  $n$ . Then  $E([X, X]_t^{T_n}) = E([M, M]_t^{T_n}) + E([A, A]_t^{T_n})$ . Since quadratic variation is non-decreasing non-negative process, first letting  $n \rightarrow \infty$  and then letting  $t \rightarrow \infty$ , we obtain desired result with monotone convergence theorem.

11. By the Kunita-Watanabe inequality for square bracket processes,  $([X + Y, X + Y])^{1/2} \leq ([X, X])^{1/2} + ([Y, Y])^{1/2}$ . It follows that  $[X + Y, X + Y] \leq 2([X, X] + [Y, Y])$ . This implies that  $[X + Y, X + Y]$  is locally integrable and the sharp bracket process  $\langle X + Y, X + Y \rangle$  exists. Since sharp bracket process is a predictable projection (compensator) of square bracket process, we obtain polarization identity of sharp bracket process from one of squared bracket process. Namely,

$$\langle X, Y \rangle = \frac{1}{2}(\langle X + Y, X + Y \rangle - \langle X, X \rangle - \langle Y, Y \rangle) \quad (51)$$

Then the rest of the proof is exactly the same as the one of theorem 25, chapter 2, except that we replace square bracket processes by sharp bracket processes.

12. Since a continuous finite process with finite variation always has a integrable variation, without loss of generality we can assume that the value of  $A$  changes only by jumps. Thus  $A$  can be represented as  $A_t = \sum_{s \leq t} \Delta A_s$ . Assume that  $C = \int_0^\cdot |dA_s|$  is predictable. Then  $T_n = \inf\{t > 0 : C_t \geq n\}$  is a predictable time since it is a debut of right closed predictable set. Let  $\{T_{n,m}\}_m$  be an announcing sequence of  $T_n$  for each  $n$  and define  $S_n$  by  $S_n = \sup_{1 \leq k \leq n} T_{k,n}$ . Then  $S_n$  is a sequence of stopping time increasing to  $\infty$ ,  $S_n < T_n$  and hence  $C_{S_n} \leq n$ . Thus  $EC_{S_n} < n$  and  $C$  is locally integrable. To prove that  $C$  is predictable, we introduce two standard results.

**lemma** Suppose that  $A$  is the union of graphs of a sequence of predictable times. Then there exists a sequence of predictable times  $\{T_n\}$  such that  $A \subset \bigcup_n [T_n]$  and  $[T_n] \cap [T_m] = \emptyset$  for  $n \neq m$ .

Let  $\{S_n\}_n$  be a sequence of predictable stopping times such that  $A \subset \bigcup_n [S_n]$ . Put  $T_1 = S_1$  and for  $n \geq 2$ ,  $B_n = \bigcap_{k=1}^{n-1} [S_k \neq S_n]$ ,  $T_n = (S_n)_{B_n}$ . Then  $B_n \in \mathcal{F}_{S_n-}$ ,  $T_n$  is predictable,  $[T_n] \cap [T_m] = \emptyset$  when  $n \neq m$ , and  $A = \bigcup_{n \geq 1} [T_n]$ . (Note: By the definition of the graph,  $[T_n] \cap [T_m] = \emptyset$  even if  $P(T_n = T_m = \infty) > 0$  as long as  $T_n$  and  $T_m$  are disjoint on  $\Omega \times \mathbb{R}_+$ )  $\square$

**lemma** *Let  $X_t$  be a càdlàg adapted process and predictable. Then there exists a sequence of strictly positive predictable times  $\{T_n\}$  such that  $[\Delta X \neq 0] \subset \cup_n [T_n]$ .*

*Proof.* Let  $S_{n+1}^{1/k} = \inf\{t : t > T_n^{1/k}(\omega), |X_{S_n^{1/k}} - X_t| > 1/k \text{ or } |X_{S_n^{1/k}} - X_{t-}| > 1/k\}$ . Then since  $X$  is predictable, we can show by induction that  $\{S_n\}_{n \geq 1}$  is predictable stopping time. In addition  $[\Delta X \neq 0] \subset \cup_{n,k \geq 1} [S_n^{1/k}]$ . Then by previous lemma, there exists a sequence of predictable stopping times  $\{T_n\}$  such that  $\cup_{n,k \geq 1} [S_n^{1/k}] \subset \cup_n [T_n]$ . Then  $[\Delta X \neq 0] \subset \cup_n [T_n]$   $\square$

**Proof of the main claim:** Combining two lemmas, we see that  $\{\Delta A \neq 0\}$  is the union of a sequence of disjoint graphs of predictable stopping times. Since  $A_t = \sum_{s \leq t} \Delta A_s$  is absolute convergent for each  $\omega$ , it is invariant with respect to the change of the order of summation. Therefore  $A_t = \sum_n \Delta A_{S_n} 1_{\{S_n \leq t\}}$  where  $S_n$  is a predictable time.  $\Delta A_{S_n} 1_{\{S_n \leq t\}}$  is a predictable process since  $S_n$  is predictable and  $A_{S_n} \in \mathcal{F}_{S_n-}$ . This clearly implies that  $|\Delta A_{S_n}| 1_{\{S_n \leq t\}}$  is predictable. Then  $C$  is predictable as well.

**Note:** As for the second lemma, following more general claim holds.

**lemma** *Let  $X_t$  be a càdlàg adapted process. Then  $X$  is predictable if and only if  $X$  satisfies the following conditions (1). There exists a sequence of strictly positive predictable times  $\{T_n\}$  such that  $[\Delta X \neq 0] \subset \cup_n [T_n]$ . (2). For each predictable time  $T$ ,  $X_T 1_{\{T < \infty\}} \in \mathcal{F}_{T-}$ .*

**13.** Let  $\{T_i\}_i$  be a jump time of a counting process and define  $S_i = T_i - T_{i-1}$ . Then by corollary to theorem 23, a compensator  $A$  is given by

$$A_t = \sum_{i \geq 1} \left[ \sum_{j=1}^i \phi_j(S_j) + \phi_{i+1}(t - T_i) \right] 1_{\{T_i \leq t < T_{i+1}\}}, \quad \phi_i(s) = \int_0^s \frac{-1}{F_i(u-)} dF_i(u), \quad (52)$$

where  $F_i(u) = P(S_i > u)$ . For each  $\omega$ , it is clear that If  $F_i$  has a density such that  $dF_i(u) = f_i(u)$  then  $A_t$  is absolutely continuous. Conversely if  $F_k(u)$  does not admit density then on  $[T_{k-1}, T_k)$ ,  $A_t$  is not absolutely continuous.

**14.**  $N_t - 1_{\{t \geq T\}} = 0$  is a trivial martingale. Since Doob-Meyer decomposition is unique, it suffices to show that  $1_{\{t \geq T\}}$  is a predictable process,  $1_{\{t \geq T\}}$  is a predictable process if and only if  $T$  is a predictable time. Let  $S_n = T(1 - 1/n)$ . Then  $T \in \mathcal{F}_0$  implies  $S_n \in \mathcal{F}_0$ . In particular  $\{S_n \leq t\} \in \mathcal{F}_0 \subset \mathcal{F}_t$  and  $S_n$  is a stopping time such that  $S_n < T$  and  $S_n \uparrow T$ . This makes  $\{S_n\}$  an announcing sequence of  $T$ . Thus  $T$  is predictable.

**15.** By the uniqueness of Doob-Meyer decomposition, it suffices to show that  $N_t - \mu\lambda t$  is a martingale. since  $\mu\lambda t$  is clearly a predictable process with finite variation. Let  $C_t$  be a Poisson process associated with  $N_t$ . Then by the independent and stationary increment property of compound

Poisson process,

$$\begin{aligned} E[N_t - N_s | \mathcal{F}_s] &= E \left[ \sum_{i=1}^{C_{t-s}} U_i \right] = \sum_{k=1}^{\infty} E \left[ \sum_{i=1}^{C_{t-s}} U_i | C_{t-s} = k \right] P(C_{t-s} = k) \\ &= \mu \sum_{k=1}^{\infty} k P(C_{t-s} = k) = \mu \lambda (t - s) \end{aligned} \quad (53)$$

Therefore  $N_t - \mu \lambda t$  is a martingale and the claim is shown.

**16.** By direct computation,

$$\lambda(t) = \lim_{h \rightarrow 0} \frac{1}{h} P(t \leq T < t + h | T \geq t) = \lim_{h \rightarrow 0} \frac{1 - e^{-\lambda h}}{h} = \lambda. \quad (54)$$

A case that  $T$  is Weibull is similar and omitted.

**17.** Observe that  $P(U > s) = P(T > 0, U > s) = \exp\{-\mu s\}$ .  $P(T \leq t, U > s) = P(U > s) - P(T > t, U > s)$ . Then

$$\begin{aligned} P(t \leq T < t + h | T \geq t, U \geq t) &= \frac{P(t \leq T < t + h, t \leq U)}{P(t \leq T, t \leq U)} = \frac{\int_t^{t+h} e^{-\mu t} (\lambda + \theta t) e^{-(\lambda + \theta t)u} du}{\exp\{-\lambda t - \mu t - \theta t^2\}} \\ &= \frac{-1[\exp\{-\lambda(t+h) - \theta t(t+h)\} - \exp\{-\lambda t - \theta t^2\}]}{\exp\{-\lambda t - \theta t^2\}} = 1 - \exp\{-(\lambda + \theta t)h\} \end{aligned} \quad (55)$$

and

$$\lambda^\#(t) = \lim_{h \rightarrow 0} \frac{1 - \exp\{-(\lambda + \theta t)h\}}{h} = \lambda + \theta t \quad (56)$$

**18.** There exists a disjoint sequence of predictable times  $\{T_n\}$  such that  $A = \{t > 0 : \Delta \langle M, M \rangle_t \neq 0\} \subset \cup_n [T_n]$ . (See the discussion in the solution of exercise 12 for details.) In addition, all the jump times of  $\langle M, M \rangle$  is a predictable time. Let  $T$  be a predictable time such that  $\langle M, M \rangle_T \neq 0$ . Let  $N = [M, M] - \langle M, M \rangle$ . Then  $N$  is a martingale with finite variation since  $\langle M, M \rangle$  is a compensator of  $[M, M]$ . Since  $T$  is predictable,  $E[N_T | \mathcal{F}_{T-}] = N_{T-}$ . On the other hand, since  $\{\mathcal{F}_t\}$  is a quasi-left-continuous filtration,  $N_{T-} = E[N_T | \mathcal{F}_{T-}] = E[N_T | \mathcal{F}_T] = N_T$ . This implies that  $\Delta \langle M, M \rangle_T = \Delta [M, M]_T = (\Delta M_T)^2$ . Recall that  $M$  itself is a martingale. So  $M_T = E[M_T | \mathcal{F}_T] = E[M_T | \mathcal{F}_{T-}] = M_{T-}$  and  $\Delta M_T = 0$ . Therefore  $\Delta \langle M, M \rangle_T = 0$  and  $\langle M, M \rangle$  is continuous.

**19.** By theorem 36,  $X$  is a special semimartingale. Then by theorem 34, it has a unique decomposition  $X = M + A$  such that  $M$  is local martingale and  $A$  is a predictable finite variation process. Let  $X = N + C$  be an arbitrary decomposition of  $X$ . Then  $M - N = C - A$ . This implies that  $A$  is a compensator of  $C$ . It suffices to show that a local martingale with finite variation is locally integrable. Set  $Y = M - N$  and  $Z = \int_0^t |dY_s|$ . Let  $S_n$  be a fundamental sequence of  $Y$  and set  $T_n = S_n \wedge n \wedge \inf\{t : Z_t > n\}$ . Then  $Y_{T_n} \in L^1$  (See the proof of theorem 38) and  $Z_{T_n} \leq n + |Y_{T_n}| \in L^1$ . Thus  $Y = M - N$  has a locally integrable variation. Then  $C$  is a sum of two process with locally integrable variation and the claim holds.

**20.** Let  $\{T_n\}$  be an increasing sequence of stopping times such that  $X^{T_n}$  is a special semimartingale as shown in the statement. Then by theorem 37,  $X_{t \wedge T_n}^* = \sup_{s \leq t} |X_s^{T_n}|$  is locally integrable. Namely, there exists an increasing sequence of stopping times  $\{R_n\}$  such that  $(X_{t \wedge T_n}^*)^{R_n}$  is integrable. Let  $S_n = T_n \wedge R_n$ . Then  $S_n$  is an increasing sequence of stopping times such that  $(X_t^*)^{S_n}$  is integrable. Then  $X_t^*$  is locally integrable and by theorem 37,  $X$  is a special semimartingale.

**21.** Since  $\mathbb{Q} \sim \mathbb{P}$ ,  $d\mathbb{Q}/d\mathbb{P} > 0$  and  $Z > 0$ . Clearly  $M \in L^1(\mathbb{Q})$  if and only if  $MZ \in L^1(\mathbb{P})$ . By generalized Bayes' formula.

$$E_{\mathbb{Q}}[M_t|\mathcal{F}_s] = \frac{E_{\mathbb{P}}[M_t Z_t|\mathcal{F}_s]}{E_{\mathbb{P}}[Z_t|\mathcal{F}_s]} = \frac{E_{\mathbb{P}}[M_t Z_t|\mathcal{F}_s]}{Z_s}, \quad t \geq s \quad (57)$$

Thus  $E_{\mathbb{P}}[M_t Z_t|\mathcal{F}_s] = M_s Z_s$  if and only if  $E_{\mathbb{Q}}[M_t|\mathcal{F}_s] = M_s$ .

**22.** By Rao's theorem.  $X$  has a unique decomposition  $X = M + A$  where  $M$  is  $(\mathcal{G}, \mathbb{P})$ -local martingale and  $A$  is a predictable process with path of locally integrable variation.  $E[X_{t_{i+1}} - X_{t_i}|\mathcal{G}_{t_i}] = E[A_{t_{i+1}} - A_{t_i}|\mathcal{G}_{t_i}]$ . So

$$\sup_{\tau} E \left[ \sum_{i=0}^n |E[A_{t_i} - A_{t_{i+1}}|\mathcal{G}_{t_i}]| \right] < \infty \quad (58)$$

$Y_t = E[X_t|\mathcal{F}_t] = E[M_t|\mathcal{F}_t] + E[A_t|\mathcal{F}_t]$ . Since  $E[E[M_t|\mathcal{F}_t]|\mathcal{F}_s] = E[M_t|\mathcal{F}_s] = E[E[M_t|\mathcal{G}_s]|\mathcal{F}_s] = E[M_s|\mathcal{F}_s]$ ,  $E[M_t|\mathcal{F}_t]$  is a martingale. Therefore

$$\text{Var}_{\tau}(Y) = E \left[ \sum_{i=1}^n |E[E[A_{t_i}|\mathcal{F}_{t_i}] - E[A_{t_{i+1}}|\mathcal{F}_{t_{i+1}}]|\mathcal{F}_{t_i}]| \right] = \sum_{i=1}^n E[|E[A_{t_i} - A_{t_{i+1}}|\mathcal{F}_{t_i}]|] \quad (59)$$

For arbitrary  $\sigma$ -algebra  $\mathcal{F}, \mathcal{G}$  such that  $\mathcal{F} \subset \mathcal{G}$  and  $X \in L_1$ ,

$$E(|E[X|\mathcal{F}]|) = E(|E(E[X|\mathcal{G}]|\mathcal{F})|) \leq E(E(|E[X|\mathcal{G}]|)|\mathcal{F}) = E(|E[X|\mathcal{G}]|) \quad (60)$$

Thus for every  $\tau$  and  $t_i$ ,  $E[|E[A_{t_i} - A_{t_{i+1}}|\mathcal{F}_{t_i}]|] \leq E[|E[A_{t_i} - A_{t_{i+1}}|\mathcal{G}_{t_i}]|]$ . Therefore  $\text{Var}(X) < \infty$  (w.r.t.  $\{\mathcal{G}_t\}$ ) implies  $\text{Var}(Y) < \infty$  (w.r.t  $\{\mathcal{F}_t\}$ ) and  $Y$  is  $(\{\mathcal{F}_t\}, \mathbb{P})$ -quasi-martingale.

**23.** We introduce a following standard result without proof.

**Lemma.** Let  $N$  be a local martingale. If  $E([N, N]_{\infty}^{1/2}) < \infty$  or alternatively,  $N \in \mathcal{H}^1$  then  $N$  is uniformly integrable martingale.

This is a direct consequence of Fefferman's inequality. (Theorem 52 in chapter 4. See chapter 4 section 4 for the definition of  $\mathcal{H}^1$  and related topics.) We also take a liberty to assume Burkholder-Davis-Gundy inequality (theorem 48 in chapter 4) in the following discussion.

Once we accept this lemma, it suffices to show that a local martingale  $[A, M] \in \mathcal{H}^1$ . By Kunita-Watanabe inequality,  $[A, M]_{\infty}^{1/2} \leq [A, A]_{\infty}^{1/4} [M, M]_{\infty}^{1/4}$ . Then by Hölder inequality,

$$E([A, M]_{\infty}^{1/2}) \leq E([A, A]_{\infty}^{1/2})^{\frac{1}{2}} E([M, M]_{\infty}^{1/2}). \quad (61)$$

By hypothesis  $E\left([A, A]_\infty^{1/2}\right)^{\frac{1}{2}} < \infty$ . By BDG inequality and the fact that  $M$  is a bounded martingale,  $E([M, M]^{1/2}) \leq c_1 E[M_\infty^*] < \infty$  for some positive constant  $c_1$ . This complete the proof.

**24.** Since we assume that the usual hypothesis holds throughout this book (see page 3), let  $\mathcal{F}_t^0 = \sigma\{T \wedge s : s \leq t\}$  and redefine  $\mathcal{F}_t$  by  $\mathcal{F}_t = \cap_\epsilon \mathcal{F}_{t+\epsilon}^0 \vee \mathcal{N}$ . Since  $\{T < t\} = \{T \wedge t < t\} \in \mathcal{F}_t$ ,  $T$  is  $\mathcal{F}$ -stopping time. Let  $\mathbb{G} = \{\mathcal{G}_t\}$  be a smallest filtration such that  $T$  is a stopping time, that is a natural filtration of the process  $X_t = 1_{\{T \leq t\}}$ . Then  $\mathbb{G} \subset \mathbb{F}$ .

For the converse, observe that  $\{T \wedge s \in B\} = (\{T \leq s\} \cap \{T \in B\}) \cup (\{T > s\} \cap \{s \in B\}) \in \mathcal{F}_s$  since  $\{s \in B\}$  is  $\emptyset$  or  $\Omega$ ,  $\{T \in B\} \in \mathcal{F}_T$  and in particular  $\{T \leq s\} \cap \{T \in B\} \in \mathcal{F}_s$  and  $\{T > s\} \in \mathcal{F}_s$ . Therefore for all  $t$ ,  $T \wedge s$ ,  $(\forall s \leq t)$  is  $\mathcal{G}_t$  measurable. Hence  $\mathcal{F}_t^0 \subset \mathcal{G}_t$ . This shows that  $\mathbb{G} \subset \mathbb{F}$ . (Note: we assume that  $\mathbb{G}$  satisfies usual hypothesis as well. )

**25.** Recall that the  $\{(\omega, t) : \Delta A_t(\omega) \neq 0\}$  is a subset of a union of disjoint predictable times and in particular we can assume that a jump time of predictable process is a predictable time. (See the discussion in the solution of exercise 12). For any predictable time  $T$  such that  $E[\Delta Z_T | \mathcal{F}_{T-}] = 0$ ,

$$\Delta A_T = E[\Delta A_T | \mathcal{F}_{T-}] = E[\Delta M_T | \mathcal{F}_{T-}] = 0 \quad a.s. \quad (62)$$

**26.** Without loss of generality, we can assume that  $A_0 = 0$  since  $E[\Delta A_0 | \mathcal{F}_{0-}] = E[\Delta A_0 | \mathcal{F}_0] = \Delta A_0 = 0$ . For any finite valued stopping time  $S$ ,  $E[A_S] \leq E[A_\infty]$  since  $A$  is an increasing process. Observe that  $A_\infty \in L_1$  because  $A$  is a process with integrable variation. Therefore  $A \{A_S\}_S$  is uniformly integrable and  $A$  is in class (D). Applying Theorem 11 (Doob-Meyer decomposition) to  $-A$ , we see that  $M = A - \tilde{A}$  is a uniformly integrable martingale. Then

$$0 = E[\Delta M_T | \mathcal{F}_{T-}] = E[\Delta A_T | \mathcal{F}_{T-}] - E[\Delta \tilde{A}_T | \mathcal{F}_{T-}] = 0 - E[\Delta \tilde{A}_T | \mathcal{F}_{T-}]. \quad a.s. \quad (63)$$

Therefore  $\tilde{A}$  is continuous at time  $T$ .

**27.** Assume  $A$  is continuous. Consider an arbitrary increasing sequence of stopping time  $\{T_n\} \uparrow T$  where  $T$  is a finite stopping time.  $M$  is a uniformly integrable martingale by theorem 11 and the hypothesis that  $Z$  is a supermartingale of class D. Then  $\infty > EZ_T = -EA_T$  and in particular  $A_T \in L^1$ . Since  $A_T \geq A_{T_n}$  for each  $n$ . Therefore by Doob's optional sampling theorem, Lebesgue's dominated convergence theorem and continuity of  $A$  yields,

$$\lim_n E[Z_T - Z_{T_n}] = \lim_n E[M_T - N_{T_n}] - \lim_n E[A_T - A_{T_n}] = -E[\lim_n (A_T - A_{T_n})] = 0. \quad (64)$$

Therefore  $Z$  is regular.

Conversely suppose that  $Z$  is regular and assume that  $A$  is not continuous at time  $T$ . Since  $A$  is predictable, so is  $A_-$  and  $\Delta A$ . In particular,  $T$  is a predictable time. Then there exists an announcing sequence  $\{T_n\} \uparrow T$ . Since  $Z$  is regular,

$$0 = \lim_n E[Z_T - Z_{T_n}] = \lim_n E[M_T - M_{T_n}] - \lim_n E[A_T - A_{T_n}] = E[\Delta A_T]. \quad (65)$$

Since  $A$  is an increasing process and  $\Delta A_T \geq 0$ . Therefore  $\Delta A_T = 0$  a.s. This is a contradiction. Thus  $A$  is continuous.



**31.** Let  $T$  be an arbitrary  $\mathbb{F}^\mu$  stopping time and  $\Lambda = \{\omega : X_T(\omega) \neq X_{T-}(\omega)\}$ . Then by Meyer's theorem,  $T = T_\Lambda \wedge T_{\Lambda^c}$  where  $T_\Lambda$  is totally inaccessible time and  $T_{\Lambda^c}$  is predictable time. By continuity of  $X$ ,  $\Lambda = \emptyset$  and  $T_\Lambda = \infty$ . Therefore  $T = T_{\Lambda^c}$ . It follows that all stopping times are predictable and there is no totally inaccessible stopping time.

**32.** By exercise 31, the standard Brownian space supports only predictable time since Brownian motion is clearly a strong Markov Feller process. Since  $\mathcal{O} = \sigma([S, T[: S, T \text{ are stopping time and } S \leq T)$  and  $\mathcal{P} = \sigma([S, T[: S, T \text{ are predictable times and } S \leq T)$ , if all stopping times are predictable  $\mathcal{O} = \mathcal{P}$ .

**35.**  $\mathcal{E}(M_t) \geq 0$  and  $\mathcal{E}(M_t) = \exp[B_t^\tau - 1/2(t \wedge \tau)] \leq e$ . So it is a bounded local martingale and hence martingale. If  $\mathcal{E}(-M)$  is a uniformly integrable martingale, there exists  $\mathcal{E}(-M_\infty)$  such that  $E[\mathcal{E}(-M_\infty)] = 1$ . By the law of iterated logarithm,  $\exp(B_\tau - 1/2\tau)1_{\{\tau=\infty\}} = 0$  a.s. Then

$$E[\mathcal{E}(-M_\infty)] = E\left[\exp\left(-1 - \frac{\tau}{2}\right)1_{\{\tau<\infty\}}\right] \leq e^{-1} < 1. \quad (66)$$

This implies that  $\mathcal{E}(-M)$  is not a uniformly integrable martingale.

**36.** Clearly  $\mathcal{F}_{T-} \subset \mathcal{F}_T$  and  $\sigma\{\Delta M_T : M \text{ a martingale}\} \subset \mathcal{F}_T$ . Therefore  $\mathcal{F}_{T-} \vee \sigma\{\Delta M_T : M \text{ a martingale}\} \subset \mathcal{F}_T$ . For the converse, recall Theorem 6 in chapter 1 and Theorem 5 in chapter 3.

$$\begin{aligned} \mathcal{F}_T &= \sigma\{X_T; X \text{ all adapted càdlàg processes}\} \\ \mathcal{F}_{T-} &= \sigma\{H_T; H \text{ predictable}\} \end{aligned}$$

Pick an  $X_T$ . Assume first that  $X_T$  is bounded. Let  $M_t = E[X_T | \mathcal{F}_{t \wedge T}]$ . Note that  $X_T$  is bounded and in particular in  $L_1$ . So this process is well defined. Then  $M_t$  is a martingale such that  $M_T = X_T$ . Then  $X_T = M_T = M_{T-} + \Delta M_T$  where  $M_{T-}$  is a left continuous process  $M_{t-}$  evaluated at  $T$ .  $M_T \in \sigma\{\Delta M_T : M \text{ a martingale}\}$ . Since  $\{M_{t-}\}$  is a predictable process,  $M_{T-} \in \mathcal{F}_{T-}$ . Thus  $X_T = M_T \in \mathcal{F}_{T-} \vee \sigma\{\Delta M_T : M \text{ a martingale}\}$ . For unbounded  $X_T$ , set  $X_T^n = X_T 1_{\{|X_T| < n\}}$ . Then  $X_T^n \rightarrow X_T$  a.s. while  $X_T^n \in \mathcal{F}_{T-} \vee \sigma\{\Delta M_T : M \text{ a martingale}\}$  for each  $n$ . Then  $X_T \in \mathcal{F}_{T-} \vee \sigma\{\Delta M_T : M \text{ a martingale}\}$  and  $\mathcal{F}_T \subset \mathcal{F}_{T-} \vee \sigma\{\Delta M_T : M \text{ a martingale}\}$ .



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