Preface

"Il ne semblait pas que cette importante théorie pût encore être perfectionnée, lorsque les deux géomètres qui ont le plus contribué à la rendre complète, en ont fait de nouveau le sujet de leurs méditations...". By these words, Siméon Denis Poisson announced in 1809 [293] that he had found an improvement in the theory of Lagrangian mechanics, which was being developed by Joseph-Louis Lagrange and Pierre-Simon Laplace. In that pioneering paper, Poisson introduced (we slightly modernize his writing) the notation

$$(a,b) = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial q_i}\frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i}\frac{\partial b}{\partial q_i} \right), \tag{0.1}$$

where a and b are two functions of the coordinates q_i and the conjugate quantities $p_i = \frac{\partial R}{\partial \dot{q}_i}$ for a mechanical system with Lagrangian function R. He proved that, if a and b are first integrals of the system then (a,b) also is. This (a,b) is nowadays denoted by $\{a,b\}$ and called the Poisson bracket of a and b. Mathematicians of the 19th century already recognized the importance of this bracket. In particular, William Hamilton used it extensively to express his equations in an essay in 1835 [168] on what we now call Hamiltonian dynamics. Carl Jacobi in his "Vorlesungen über Dynamik" around 1842 (see [185]) showed that the Poisson bracket satisfies the famous Jacobi identity:

$$\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0. \tag{0.2}$$

This same identity is satisfied by Lie algebras, which are infinitesimal versions of Lie groups, first studied by Sophus Lie and his collaborators in the end of the 19th century [213].

In our modern language, a Poisson structure on a manifold M is a 2-vector field Π (Poisson tensor) on M, such that the corresponding bracket (Poisson bracket) on the space of functions on M, defined by

$$\{f,g\} := \langle \mathrm{d}f \wedge \mathrm{d}g, \Pi \rangle \ , \tag{0.3}$$

satisfies the Jacobi identity. (M,Π) is then called a Poisson manifold. This notion of Poisson manifolds generalizes both symplectic manifolds and Lie algebras. The

Poisson tensor of the original bracket of Poisson is

$$\Pi = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}, \tag{0.4}$$

which is nondegenerate and corresponds to a symplectic 2-form, namely

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i \ . \tag{0.5}$$

On the other hand, each finite-dimensional Lie algebra gives rise to a linear Poisson tensor on its dual space and vice versa.

Poisson manifolds play a fundamental role in Hamiltonian dynamics, where they serve as phase spaces. They also arise naturally in other mathematical problems as well. In particular, they form a bridge from the “commutative world” to the “noncommutative world”. For example, Lie groupoids give rise to noncommutative operator algebras, while their infinitesimal versions, called Lie algebroids, are nothing but “fiber-wise linear” Poisson structures. Poisson geometry, i.e., the geometry of Poisson structures, which began as an outgrowth of symplectic geometry, has seen rapid growth in the last three decades, and has now become a very large theory, with interactions with many other domains of mathematics, including Hamiltonian dynamics, integrable systems, representation theory, quantum groups, noncommutative geometry, singularity theory, and so on.

This book arises from its authors’ efforts to study Poisson structures, and in particular their normal forms. As a result, the book aims to offer a quick introduction to Poisson geometry, and to give an extensive account on known results about the theory of normal forms of Poisson structures and related objects. This theory is relatively young. Though some earlier results may be traced back to V.I. Arnold, it really took off with a fundamental paper of Alan Weinstein in 1983 [346], in which he proved a formal linearization theorem for Poisson structures, a local symplectic realization theorem, and the following splitting theorem: locally any Poisson manifold can be written as the direct product of a symplectic manifold with another Poisson manifold whose Poisson tensor vanishes at a point. Since then, a large number of other results have emerged, many of them very recently.

Here is a brief summary of this book, which only highlights a few important points from each chapter. For a more detailed list of what the book has to offer, the reader may look at the table of contents.

The book consists of eight chapters and some appendices. Chapter 1 is based on lectures given by the authors in Montpellier and Toulouse for graduate students, and is a small self-contained introduction to Poisson geometry. Among other things, we show how Poisson manifolds can be viewed as singular foliations with symplectic leaves, and also as quotients of symplectic manifolds. The reader will also find in this chapter a section about the Schouten bracket of multi-vector fields, which was discovered by Schouten in 1940 [311], and whose importance goes beyond Poisson geometry.

Starting from Chapter 2, the book contains many recent results which have not been previously available in book form. A few results in this book are even original and not published elsewhere.

Chapter 2 is about Poisson cohomology, a natural and important invariant introduced by André Lichnerowicz in 1977 [211]. In particular, we show the role played by this cohomology in normal form problems, and its relations with de Rham cohomology of manifolds and Chevalley–Eilenberg cohomology of Lie algebras. Some known methods for computing Poisson cohomology are briefly discussed, including standard tools from algebraic topology such as the Mayer–Vietoris sequence and spectral sequences, and also tools from singularity theory. Many authors, including Viktor Ginzburg, Johannes Huebschmann, Mikhail Karasev, Jean-Louis Koszul, Izu Vaisman, Ping Xu, etc., contributed to the understanding of Poisson cohomology, and we discuss some of their results in this chapter. However, the computation of Poisson cohomology remains very difficult in general.

Chapter 3 is about a kind of normal form for Poisson structures, which are comparable to Poincaré–Birkhoff normal forms for vector fields, and which are called Levi decompositions because they are analogous to Levi–Malcev decompositions for finite-dimensional Lie algebras. The results of this chapter are due mainly to Aissa Wade [342] (the formal case), the second author and Monnier [369, 263] (the analytic and smooth cases). The proof of the formal case is purely algebraic and relatively simple. The analytic and smooth cases make use of the fast convergence methods of Kolmogorov and Nash–Moser.

Chapter 4 is about linearization of Poisson structures. The results of Chapter 3 are used in this chapter. In particular, Conn's linearization results for Poisson structures with a semi-simple linear part [80, 81] may be viewed as special cases of Levi decomposition. Among results discussed at length in this chapter, we will mention here Weinstein's theorem on the smooth degeneracy of real semisimple Lie algebras of real rank greater than or equal to 2 [348], and our result on the formal and analytic nondegeneracy of the Lie algebra $\mathfrak{aff}(n)$ [120].

In Chapter 5 we explain the links among quadratic Poisson structures, r-matrices, and the theory of Poisson–Lie groups introduced by Drinfeld [107]. So far, all quadratic Poisson structures known to us can be obtained from r-matrices, which have their origins in the theory of integrable systems. Some important contributions of Semenov–Tian–Shansky, Lu, Weinstein and other people can be found in this chapter. We then show how the curl vector field (also known as modular vector field) led the first author and other people to a classification of "nonresonant" quadratic Poisson structures, and quadratization results for Poisson structures which begin with a nonresonant quadratic part. Let us mention that Poisson–Lie groups are classical versions of *quantum groups*, a subject which is beyond the scope of this book.

Chapter 6 is devoted to n-ary generalizations of Poisson structures, which go under the name of Nambu structures. Though originally invented by physicists Nambu [275] and Takhtajan [328], these Nambu structures turn out to be dual to integrable differential forms and play an important role in the theory of singular foliations. A linearization theorem for Nambu structures [119] is given in this chapter. Its proof at one point makes use of Malgrange's "Frobenius with singularities" theorem [233, 234]. Malgrange's theorem is also discussed in this chapter, together with many other results on singular foliations and integrable differential forms. In particular, we present generalizations of Kupka's stability theorem [204], which are due to de Medeiros [244, 245], Camacho and Lins Neto [59], and ourselves.

Chapter 7 deals with Lie groupoids. Among other things, it contains a recent slice theorem due to Weinstein [354] and the second author [370]. This slice theorem is a normal form theorem for proper Lie groupoids near an orbit, and generalizes the classical Koszul–Palais slice theorem for proper Lie group actions. We also discuss symplectic groupoids, an important object of Poisson geometry introduced independently by Karasev [189], Weinstein [349], and Zakrzewski [364] in the 1980s. A local normal form theorem for proper symplectic groupoids is also given.

Chapter 8 is about Lie algebroids, introduced by Pradines [294] in 1967 as infinitesimal versions of Lie groupoids. They correspond to fiber-wise linear Poisson structures, and many results about general Poisson structures, including the splitting theorem and the Levi decomposition, apply to them. Our emphasis is again on their local normal forms, though we also discuss cohomology of Lie algebroids, and the problem of integrability of Lie algebroids, including a recent strong theorem of Crainic and Fernandes [86].

Finally, Appendix A is a collection of discussions which help make the book more self-contained or which point to closely related subjects. It contains, among other things, Vorobjev's description of a neighborhood of a symplectic leaf [340], toric characterization of Poincaré–Birkhoff normal forms of vector fields, a brief introduction to deformation quantization, including a famous theorem of Kontsevich [195] on the existence of deformation quantization for an arbitrary Poisson structure, etc.

The book is biased towards what we know best, i.e., local normal forms. May the specialists in Poisson geometry forgive us for not giving more discussions on other topics, due to our lack of competence. Familiarity with symplectic manifolds is not required, though it will be helpful for reading this book. There are many nice books readily available on symplectic geometry. On the other hand, books on Poisson geometry are relatively rare. The only general introductory reference to date is Vaisman [333]. Some other references are Cannas da Silva and Weinstein [60] (a nice book about geometric models for noncommutative algebras, where Poisson geometry plays a key role), Karasev and Maslov [190] (a book on Poisson manifolds with an emphasis on quantization), Mackenzie [228] (a general reference on Lie groupoids and Lie algebroids), Ortega and Ratiu [288] (a comprehensive book on symmetry and reduction in Poisson geometry), and a book in preparation by Xu [362] (with an emphasis on Poisson groupoids). We hope that our book is complementary to the above books, and will be useful for students and researchers interested in the subject.

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