CHAPTER II

BOUNDARY VALUE PROBLEMS OF GENERALIZED REGULAR FUNCTIONS AND HYPERBOLIC HARMONIC FUNCTIONS IN REAL CLIFFORD ANALYSIS

This chapter deals with boundary value problems of some functions in real Clifford analysis. In the first three sections, the problems of regular and generalized regular functions are considered, and in the last section, the Dirichlet problem of hyperbolic harmonic functions is discussed. Most results in this chapter have been obtained by us in recent years.

1 The Dirichlet Problem of Regular Functions for a ball in Real Clifford Analysis

In this section, we discuss two boundary value problems of regular functions for a ball in real Clifford analysis, which are obtained from the papers of Luogeng Hua[26]1) and Sha Huang[29]4),5).

Firstly, we give definitions of some differential operators

$$\overline{\partial}_{i-1,i} = e_{i-1}\frac{\partial}{\partial x_{i-1}} + e_i\frac{\partial}{\partial x_i},\ i = 2, 3, ..., n,$$
$$\partial_{i-1,i} = e_{i-1}\frac{\partial}{\partial x_{i-1}} - e_i\frac{\partial}{\partial x_i},\ i = 2, 3, ..., n,$$

and then we have

$$\overline{\partial} = \frac{1}{2}[\overline{\partial}_{12} + \partial_{12} + \sum_{i=2}^{n}(\overline{\partial}_{i-1,i} - \partial_{i-1,i})],$$
$$\partial = \frac{1}{2}[\overline{\partial}_{12} + \partial_{12} - \sum_{i=2}^{n}(\overline{\partial}_{i-1,i} - \partial_{i-1,i})],\quad \frac{\partial}{\partial x_1} = \frac{1}{2}(\overline{\partial}_{12} + \partial_{12}),$$

$$\frac{\partial}{\partial x_i} = \frac{-e_i}{2}(\overline{\partial}_{i-1,i} - \partial_{i-1,i}),\ i = 2, 3, ..., n.$$

By using the quasi-permutation signs introduced in Chapter I, we can give some regular conditions of functions.

If we write the element $\sum_A a_A e_A$ in $\mathcal{A}$ as

$$\sum_A a_A e_A = \sum_B (a_B + a_{2B} e_2) e_B = \sum_B I_B e_B,$$

where $B = \{\alpha_1, ..., \alpha_k\} \subseteq \{3, 4, ..., n\}, 3 \le \alpha_1 < \cdots < \alpha_k \le n$, $I_B \in \mathbf{C}$ (the complex plane). It is evident that we can obtain the following theorem.

Theorem 1.1 *The sufficient and necessary condition for* $\sum_A a_A e_A = 0$ *is that for all* $B = \{\alpha_1, ..., \alpha_k\} \subseteq \{3, ..., n\}$, $3 \le \alpha_1 < \cdots < \alpha_k \le n$, *the following equality holds:*

$$I_B = a_B + a_{2B} e_2 = 0. \tag{1.1}$$

Moreover, for a function whose value is in the real Clifford algebra $\mathcal{A}_n(\mathbf{R})$*:*

$$f(x) = \sum_A f_A(x) e_A :\ \Omega \to \mathcal{A}_n(\mathbf{R}),$$

we can write it as

$$f(x) = \sum_B I_B e_B :\ \Omega \to \mathcal{A}_{n-1}(\mathbf{C}),$$

where $I_B : \Omega \to \mathbf{C}$, $\mathcal{A}_{n-1}(\mathbf{C})$ *is the complex Clifford algebra.*

Theorem 1.2 *A function whose value is in the real Clifford algebra* $\mathcal{A}$*:*

$$\begin{gathered} f(x) = \sum_A f_A(x) e_A = \sum_B I_B e_B, \\ (A = \{\beta_1, ..., \beta_k\} \subseteq \{2, 3, ..., n\},\ 2 \le \beta_1 < \cdots < \beta_k \le n, \\ B = \{\alpha_1, ..., \alpha_k\} \subseteq \{3, 4, ..., n\}, 3 \le \alpha_1 < \cdots < \alpha_k \le n) \end{gathered} \tag{1.2}$$

is regular in Ω *if and only if*

$$\overline{\partial}_{12} I_B = \sum_{m=3}^{n} \delta_{\overline{mB}} \overline{I}_{\overline{mB}} x_m,$$

where $I_{\overline{mB}x_m} = \partial I_{\overline{mB}}/\partial x_m$, $\overline{I}_{\overline{mB}x_m}$ *is the conjugate of* $I_{\overline{mB}x_m}$, *and* $\overline{mB}$ *is the quasi-permutation for* mB, $\delta_{\overline{mB}}$ *is the sign of quasi-permutation* $\overline{mB}$. *In addition, a function* f *is harmonic in* Ω *if and only if every* I_B *is harmonic.*

Proof It is clear that $e_m I_B = a_B e_m - a_{2B} e_2 e_m = \overline{I}_B e_m$, so

$$\overline{\partial} f(x) = \sum_B \overline{\partial}_{12} I_B e_B + \sum_B \sum_{m=3}^{n} \overline{I}_{Bx_m} e_m e_B.$$

Denote $B = \{\alpha_1, ..., \alpha_k\}$, $\hat{B}_p = \{\alpha_1, ..., \alpha_{p-1},\ \alpha_{p+1}, ..., \alpha_k\}$, $mB = m\alpha_1$ $...\alpha_k$, $Bm = \alpha_1\ ...\alpha_k m$. When m is some α_p among B, we have

$$e_{\alpha_p} \overline{I}_{\hat{B}_p x_{\alpha_p}} e_{\hat{B}_p} = -\overline{I}_{\hat{B}_p x_{\alpha_p}} (-1)^p e_B = -\overline{I}_{\overline{\alpha_p B} x_{\alpha_p}} \delta_{\overline{\alpha_p B}} e_B.$$

When $m < \alpha_1$, we have

$$e_m \overline{I}_{mBx_m} e_{mB} = -\overline{I}_{\overline{mB}x_m} e_B = -\overline{I}_{\overline{mB}x_m} \delta_{\overline{mB}} e_B.$$

Similarly, when $m > \alpha_k$, we get

$$e_m \overline{I}_{Bmx_m} e_{Bm} = -\overline{I}_{\overline{mB}x_m} \delta_{\overline{mB}} e_B,$$

and

$$\sum_B \sum_{m=3}^{n} \overline{I}_{Bm\,x_m} e_m e_B = -\sum_B \sum_{m=3}^{n} [\overline{I}_{\overline{mB}x_m} \delta_{\overline{mB}}] e_B,$$

hence

$$\overline{\partial} f(x) = \sum_B \overline{\partial}_{12} I_B e_B - \sum_B \sum_{m=3}^{n} \overline{I}_{\overline{mB}x_m} \delta_{\overline{mB}} e_B.$$

Thus according to Theorem 1.1, the function f is regular if and only if

$$\overline{\partial}_{12} I_B = \sum_{m=3}^{n} \overline{I}_{\overline{mB}x_m} \delta_{\overline{mB}}.$$

This completes the proof.

In order to derive another sufficient and necessary requirement of the generalized Cauchy-Riemann condition, we divide the function

$$f(x) = \sum_A f_A e_A = \sum_B I_B e_B$$

into the two parts

$$f(x) = f^{(1)} + f^{(2)} = \sum_B{}' I'_B e'_B + \sum_B{}'' I''_B e''_B,$$

where B in the sum $\sum_B{}'$ obtained from $(3,4,...,n)$ is a combination with odd integers; it is called the first suffix, the rest is called the second suffix. The corresponding sum is denoted by $\sum_B{}''$, $I_B(x)$ whose B is got from the first suffix and is denoted by $I'_B(x)$, e_B whose B is derived from the second suffix is denoted by e'_B, and $I_B(x), e_B$ whose B is derived from the second suffix are denoted by I''_B, e''_B respectively. In addition, we call $f^{(i)}$ the ith part of f, and denote $f^{(i)} = J_i f (i = 1.2)$. According to this and the above theorem, we can get the following theorem.

Theorem 1.3 *A function f whose value is in real Clifford algebra $\mathcal{A}_n(R)$ is regular in domain Ω, if and only if*

$$\begin{cases} \overline{\partial}_{12} I''_B = \sum\limits_{m=3}^{n} \delta'_{\overline{mB}} \overline{I}'_{\overline{mB}} x_m, \\ \overline{\partial}_{12} I'_B = \sum\limits_{m=3}^{n} \delta''_{\overline{mB}} \overline{I}''_{\overline{mB}} x_m, \end{cases} \tag{1.3}$$

in which $f(x) = \sum\limits_A f_A e_A = \sum\limits_B I_B e_B = \sum\limits_B{}' I'_B e'_B + \sum\limits_B{}'' I''_B e''_B$, $B = \{\alpha_1, ..., \alpha_k\} \subset \{3, 4, ..., n\}$, and $\overline{\partial}_{12} I'_B$, $\overline{I}'_{\overline{mB}x_m}$, $\delta'_{\overline{mB}}$ denote the corresponding part of $\overline{\partial}_{12} I_B$, $\overline{I}_{\overline{mB}x_m}$, $\delta_{\overline{mB}}$ respectively, when B, $\overline{mB}$ are derived from the first suffix. The rest is the corresponding part which is derived from the second suffix.

Let $x = (x_1, x_2, ..., x_n) \subset R^n$, and x^T be the transpose of x, Ω : $xx^T = \sum\limits_{i=1}^{n} x_i^2 < 1$ represent a unit ball, and $\partial\Omega$: $xx^T = 1$ be a unit sphere, whose area $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

Definition 1.1 If $\sum\limits_B{}' u'_B e'_B$ is continuous in $\partial\Omega$, we find a function $f(x)$ to be regular in Ω, and continuous in $\overline{\Omega} = \Omega \cup \partial\Omega$ with the condition

$$J_1 f(\xi) = \sum_B{}' u'_B(\xi) e'_B, \ \xi \in \partial\Omega. \tag{1.4}$$

The above problem is called the Dirichlet boundary value problem in the unit ball, and we denote it by Problem D.

Theorem 1.4 *Let $\sum\limits_B{}' u'_B e'_B$ be continuous on the sphere $\partial\Omega$. Then Problem D in the ball Ω is solvable, and the solution can be represented by*

$$f(x) = \sum_B{}' I'_B(x) e'_B + \sum_B{}'' I''_B(x) e''_B, \tag{1.5}$$

where

$$I'_B(x) = \frac{1}{\omega_n} \underbrace{\int \cdots \int}_{\xi\xi^T=1} P(x,\xi) u'_B(\xi)\dot{\xi}, \tag{1.6}$$

in which $P(x,\xi) = (1-xx^T)/(1-2\xi x^T + xx^T)^{\frac{n}{2}}$ *is called the Poisson kernel,* $\dot{\xi}$ *is the area element of the sphere* $\xi\xi^T = 1$,

$$I''_B(x) = T_{12}R''_B(x) + Q''_B(x), \tag{1.7}$$

$$R''_B(x) = \frac{1}{2\omega_n} \underbrace{\int \cdots \int}_{\xi\xi^T=1} \sum_{m=3}^{n} P_{x_m} \delta'_{\overline{mB}} \overline{u'_{\overline{mB}}(\xi)}\dot{\xi}, \tag{1.8}$$

and $Q''_B(x)$ *satisfies the following relations:*

$$\partial_{\overline{z}_{12}} Q''_B(x) = 0, \tag{1.9}$$

$$\overline{\partial}_{12} I'_B(x) = \sum_{k=3}^{n} \delta''_{\overline{kB}} [\overline{T}_{12} \overline{R''_{\overline{kB}x_k}(x)} + \overline{Q''_{\overline{kB}x_k}(x)}]. \tag{1.10}$$

The operators T_{12}, $\overline{T}_{12}$ *and* $\overline{\partial}_{\overline{z}_{12}}$ *can be seen below.*

Proof Firstly, we find an expression of the solution. Suppose that $f(x)$ is a solution of Problem D in the ball Ω. Then from Theorem 2.1, Chapter I, and Theorem 1.2, we can derive that $I'_B(x)$ is harmonic in Ω. By [26]1) we obtain

$$I'_B(x) = \frac{1}{\omega_n} \underbrace{\int \cdots \int}_{\xi\xi^T=1} \frac{1-xx^T}{(1-2\xi x^T + xx^T)^{\frac{n}{2}}} u'_B(\xi)\dot{\xi},$$

which satisfies $I'_B(\xi) = u'_B(\xi)$ $(\xi \in \partial\Omega)$, where $\underbrace{\int \cdots \int}_{\xi\xi^T=1} \dot{\xi} = \omega_n$.

Denote $z_{12} = x_1 + x_2 e_2, \zeta_{12} = \xi_1 + \xi_2 e_2, \partial_{z_{12}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2}\right), \partial_{\overline{z}_{12}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2}\right)$. It follows that $\partial_{12} = 2\partial_{z_{12}}$, $\overline{\partial}_{12} = 2\partial_{\overline{z}_{12}}$.When $xx^T < 1$ we introduce two operators:

$$T_{12} f_B(x) = \frac{1}{\pi} \underbrace{\int \cdots \int}_{\xi_1^2+\xi_2^2<1-x_3^2-\cdots-x_n^2} \frac{f_B(\xi_1,\xi_2,x_3,x_4,...,x_n)}{z_{12}-\zeta_{12}} d\xi_1 d\xi_2,$$

$$\overline{T}_{12} f_B(x) = \frac{1}{\pi} \underbrace{\int \cdots \int}_{\xi_1^2+\xi_2^2<1-x_3^2-\cdots-x_n^2} \frac{f_B(\xi_1,\xi_2,x_3,x_4,...,x_n)}{\overline{z}_{12}-\overline{\zeta}_{12}} d\xi_1 d\xi_2.$$

By using the results in [77] and the first expression of Theorem 1.3, we can obtain

$$I''_B(x) = \frac{1}{2}T_{12}\sum_{m=3}^{n}\delta'_{\overline{mB}}\overline{I}'_{\overline{mB}x_m} + Q''_B(x),$$

where $\partial_{\overline{z}_{12}}Q''_B(x) = 0$ $(x \in \Omega)$. By using [26]1), we get

$$R''_B(x) = \frac{1}{2\omega_n}\underbrace{\int\cdots\int}_{\xi\xi^T=1}\sum_{m=3}^{n}\frac{-2x_m(1-2\xi x^T + xx^T) - 2^{-1}n\xi_m}{(1-2\xi x^T + xx^T)^{\frac{n}{2}}}\delta_{\overline{mB}}\overline{u'_{\overline{mB}}(\xi)}\dot{\xi}.$$

Therefore,

$$I''_B(x) = T_{12}R''_B(x) + Q''_B(x).$$

Substituting (1.7) into the second expression of Theorem 1.3, we can derive

$$\overline{\partial}_{12}I'_B(x) = \sum_{k=3}^{n}\delta''_{\overline{kB}}[\overline{T}_{12}\overline{R''_{\overline{kB}x_k}(x)} + \overline{Q''_{\overline{kB}x_k}(x)}].$$

From the overdetermined system (1.10) and $\partial_{\overline{z}_{12}}Q''_B(x) = 0$ $(x \in \Omega)$ and by [76], we can find $Q''_B(x)$. Thus, from (1.7) again, $I''_B(x)$ can also be found. That is to say, if $f(x)$ is a solution of Problem D, then the expressions (1.5) – (1.10) hold.

Moreover, we verify that the function satisfying expressions (1.5) – (1.10) is a solution of Problem D. In fact, since $f^{(1)}|_{\partial\Omega} = \sum_B{}' I'_B e'_B$, $\xi \in \partial\Omega$, by using [27], we have $I'_B(\xi) = u'_B(\xi)$, and then

$$f^{(1)}|_{\partial\Omega} = \sum_B{}' u'_B(\xi)e'_B.$$

From (1.7), (1.9) and $\overline{\partial}_{12} = 2\partial_{\overline{z}_{12}}$, we immediately derive

$$\overline{\partial}_{12}I''_B = \overline{\partial}_{12}T_{12}R''_B + 2\partial\overline{z}_{12}Q''_B = \overline{\partial}_{12}T_{12}R''_B = 2\partial_{\overline{z}_{12}}T_{12}R''_B.$$

In addition, from (1.6) and (1.8), we have

$$\begin{aligned} 2\partial_{\overline{z}_{12}}T_{12}R''_B &= \sum_{m=3}^{n}\frac{\delta'_{\overline{mB}}}{\omega_n}\underbrace{\int\cdots\int}_{\xi\xi^T=1}P_{x_m}(x,\xi)\overline{u'_{\overline{mB}}(\xi)}\dot{\xi} \\ &= \sum_{m=3}^{n}\delta'_{\overline{mB}}\overline{I}'_{\overline{mB}x_m}. \end{aligned}$$

Thus, from (1.10) and (1.7), we get

$$\overline{\partial_{12}} I'_B = \sum_{m=3}^{n} \delta''_{\overline{mB}} \overline{I}''_{\overline{mB}x_m}.$$

From Theorem 1.3 again, it is easy to see that the function $f(x)$ satisfying (1.5) – (1.10) is regular in Ω. To sum up, $f(x)$ is a solution of Problem D. This proof is completed.

Next, we discuss the pseudo-modified Dirichlet problem. In order to discuss the uniqueness of its solution, we first consider the sectional domains of Ω. Cutting Ω by "the planes":

$$\begin{cases} x_3 = a_3, \\ x_4 = a_4, \\ \quad \cdots \\ x_n = a_n, \end{cases}$$

we obtain a sectional domain G_a in the x_1x_2 plane:

$$x_a x_a^T = x_1^2 + x_2^2 + \sum_{m=3}^{n} a_m^2 < 1.$$

Let $\Gamma_a : \xi_a \xi_a^T = \xi_1^2 + \xi_2^2 + \sum\limits_{m=3}^{n} a_m^2 = 1$ be the boundary of G_a, and its center be denoted by $O_a = (0, 0, a_3, a_4, ..., a_n)$.

For given continuous functions $\sum\limits_B{}' u'_B(\xi) e'_B$ $(\xi \in \partial\Omega), \phi''_B(\xi_a)$ $(\xi_a \in \Gamma_a)$ and the complex constants d''_{Ba}, we find a regular function $f(x) = \sum\limits_B{}' I'_B e'_B + \sum\limits_B{}'' I''_B e''_B$ in Ω, which is continuous in $\overline{\Omega}$ with the following pseudo-modified conditions:

$$\begin{cases} J_1 f(\xi) & = & \sum\limits_B{}' u'_B(\xi) e'_B,\ \xi \in \partial\Omega, \\ \mathrm{Re} I''_B|_{\Gamma_a} & = & \phi''_B(\xi_a) + h''_B(\xi_a),\ \xi_a \in \Gamma_a, \\ I''_B(O_a) & = & d''_{Ba}, \end{cases}$$

where $h''_B(\xi_a) = h''_{Ba}$ $(\xi_a \in \Gamma_a)$ are all unknown real constants to be determined appropriately, and $\mathrm{Re} I''_B = \mathrm{Re}(F''_B + F''_{2B} e_2) = F''_B$. The above problem will be denoted by Problem D^*.

Theorem 1.5 *Suppose that* $\sum\limits_B{}' u'_B e'_B$ *is continuous on* $\partial\Omega$, *and for any fixed* $a_3, a_4, ..., a_n$, *the function* $\phi''_B(\xi_a)$ *is continuous on* Γ_a, *here*

$\xi = (\xi_1, \xi_2, a_3, ..., a_n)$. *Then there exists a unique solution of Problem* D^*. *Moreover the solution possesses the expressions* (1.5) – (1.10) *and satisfies*

$$\mathrm{Re}Q''_B(\xi_a) = -\mathrm{Re}[T_{12}R''_B(\xi_a)] + \phi''_B(\xi_a) + h''_B(\xi_a),\ \xi_a \in \Gamma_a, \tag{1.11}$$

$$Q''_B(O_a) = -T_{12}R''_B(O_a) + d''_{Ba}. \tag{1.12}$$

Proof Evidently, on the basis of the proof of Theorem 1.4, it is sufficient to add the following proof.

Firstly, we find the integral representation of the solution. Suppose that $f(x)$ is a solution of Problem D^*. From (1.7) and the boundary condition, we can derive

$$\mathrm{Re}[T_{12}R''_B(\xi_a) + Q''_B(\xi_a)] = \phi''_B(\xi_a) + h''_B(\xi_a),$$
$$T_{12}R''_B(O_a) + Q''_B(O_a) = I''_B(O_a) = d''_{Ba}.$$

Noting that

$$\partial_{\overline{z}_{12}} Q''_B(x) = 0,\ x \in \Omega,$$

it is clear that $Q''_B(x_a)$ satisfies the conditions

$$\begin{cases} \partial_{\overline{z}_{12}} Q''_B(x_a) = 0,\ x_a \in G_a, \\ \mathrm{Re}Q''_B(\xi_a) = -\mathrm{Re}[T_{12}R''_B(\xi_a)] + \phi''_B(\xi_a) + h''_B(\xi_a),\ \xi_a \in \Gamma_a, \\ Q''_B(O_a) = -T_{12}R''_B(O_a) + d''_{Ba}. \end{cases}$$

Since the modified Dirichlet problem for analytic functions has a unique solution [80]7), from (1.9), (1.11) and (1.12), we can find $Q''_B(x_a)$, $x \in G_a$, and then $Q''_B(x)$, $x \in \Omega$, because a is an arbitrary point.

That is to say, if $f(x)$ is a solution of Problem D^*, then the expressions (1.5) – (1.12) hold.

Next, we verify that the function $f(x)$ determined by the above expressions is a solution of Problem D^*. From (1.7) – (1.11), it follows that

$$\mathrm{Re}(I''_B)|_{\Gamma_a} = \mathrm{Re}[T_{12}R''_B(\xi_a)] + \mathrm{Re}[Q''_B(\xi_a)] = \phi''_B(\xi_a) + h''_B(\xi_a),\ \xi_a \in \Gamma_a,$$

and then $I''_B(O_a) = T_{12}R''_B(O_a) + Q''_B(O_a) = d''_{Ba}$. Therefore, the above function $f(x)$ is just a solution of Problem D^*.

Finally, we prove that the solution of Problem D^* is unique. Suppose that $f_1(x)$ and $f_2(x)$ are two solutions of Problem D^*, and denote $f_1(x)-$

$f_2(x)$ by $F(x)$. It is clear that $F(x)$ is regular in Ω and is a solution of the corresponding homogeneous equation of Problem D^* and $F^{(1)}|_{\partial\Omega} = \sum\limits_B{}' u'_B(\xi)e'_B - \sum\limits_B{}' u'_B(\xi)e'_B = 0$. For convenience, we shall adopt the same symbols for $f(x)$ as before, namely denote $F(x) = \sum\limits_B{}' I'_B e'_B + \sum\limits_B{}'' I''_B e''_B$. Since $F(x)$ is regular in Ω, thus it is harmonic in Ω, therefore for all B, $I'_B(x)$ are all harmonic in Ω. Since $\sum\limits_B{}' I'_B e'_B|_{\partial\Omega} = F^{(1)}|_{\partial\Omega} = 0, I'_B|_{\partial\Omega} = 0$, again by using the uniqueness of the solution of the Dirichlet problem for harmonic functions in a ball (see [26]1)), we get $I'_B \equiv 0$ in Ω, thus $J_1 F \equiv 0$, $I'_{\overline{mB}} \equiv 0$ in Ω. From the definition of $R''_B(x)$,

$$R''_B(x) = \frac{1}{2}\sum_{m=3}^{n} \delta'_{\overline{mB}} \overline{I}'_{\overline{mB}} x_m,$$

and then $R''_B \equiv 0$. Hence

$$I''_B = T_{12}R''_B(x) + Q''_B(x) = Q''_B(x). \tag{1.13}$$

Since $F(x)$ is a solution of the corresponding homogeneous equation of Problem D^*, from (1.9), (1.11) and (1.12), we derive

$$\begin{cases} \partial_{\bar{z}_{12}} Q''_B(x_a) = 0, \ x_a \in G_a, \\ \mathrm{Re} Q''_B(\xi_a) = h''_B(\xi_a), \ \xi_a \in \Gamma_a, \\ Q''_B(O_a) = 0, \ O_a \in G_a. \end{cases}$$

In addition, using the results about the existence and uniqueness of solutions of the modified Dirichlet problem for analytic functions (see [80]7)), we can obtain $Q''_B(x_a) \equiv 0, \ x_a \in G_a$. Hence $Q''_B(x) \equiv 0, \ x \in \Omega$. From (1.13), $I''_B \equiv 0, \ x \in \Omega$, and then $J_2F(x) \equiv 0$ in Ω. So $F(x) \equiv 0, \ x \in \Omega$, i.e. $f_1(x) = f_2(x), \ x \in \Omega$. This shows the uniqueness of the solution of Problem D^*.

2 The Mixed Boundary Value Problem for Generalized Regular Functions in Real Clifford Analysis

In this section, we discuss the existence and uniqueness of solutions of the so-called mixed boundary value problem (Problem *P-R-H*) for generalized regular functions in real Clifford analysis; the material is derived from Huang Sha's paper [29]6).

Definition 2.1 We assume the linear elliptic system of second order equations

$$\Delta u = \sum_{m=1}^{n} d_m u_{x_m} + fu + g, \quad x \in \Omega, \tag{2.1}$$

where $\Omega(\in \mathbf{R}^n)$ is a bounded domain and $\Omega \in C^{2,\alpha}\,(0 < \alpha < 1)$, $d_m = d_m(x) = d_m(x_1, ..., x_n) \in C^{0,\alpha}(\overline{\Omega})$, $d_m(x) \geq 0$, $x \in \overline{\Omega}$. The oblique derivative problem of equation (2.1) is to find a solution $u(x) = u(x_1, ..., x_n) \in C^2(\overline{\Omega})$ satisfying (2.1) and the boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial \nu} + \sigma(x)u(x) = \tau(x) + h, \ x \in \partial\Omega, \\ u(d) = u_0, \ d = (d_1, ..., d_n) \in \overline{\Omega}, \end{cases} \tag{2.2}$$

in which $\sigma(x), \tau(x) \in C^{1,\alpha}(\partial\Omega)$, $\sigma(x) \geq 0$, $x \in \partial\Omega$, h is an unknown real constant, u_0 is a real constant, ν is a vector on $x \in \partial\Omega$, $\cos(\nu, n_0) \geq 0$, n_0 is the outward normal vector on $x \in \partial\Omega$, and $\cos(\nu, n_0) \in C^{1,\alpha}(\partial\Omega)$. The above boundary problem is called Problem O.

Problem O is a non-regular oblique derivative problem. If the coefficients $\nu(x), \sigma(x)$ satisfy $\cos(\nu, n_0) \equiv 0$, $\sigma(x) \equiv 0$ on $\partial\Omega$, then Problem O is the Dirichlet problem. If $\nu(x), \sigma(x)$ satisfy $\cos(\nu, n_0) \equiv 1$, $\sigma(x) \equiv 0$ on $\partial\Omega$, then Problem O is the Neumann problem. If $\cos(\nu, n_0) \geq \delta > 0$, $\sigma(x) \geq 0$ on $\partial\Omega$, then Problem O is the regular oblique derivative problem. In [59], B. P. Ponejah proved the following lemma using the method of integral equations.

Lemma 2.1 *Problem O for equation (2.1) has a unique solution.*

Proof The existence and uniqueness of solutions for Problem O for equation (2.1) in the plane can be found in [80]4). Using a similar method in [80]4), we can also prove the uniqueness of the solution in Lemma 2.1. Using a priori estimates of solutions and the Leray-Schauder theorem [18], the existence of solutions in Lemma 2.1 can be proved.

For convenience, we order all $\omega_A(x)$ with numbers of the form 2^{n-1} in $\omega(x) = \sum\limits_A \omega_A(x)e_A$ according to the following method, and denote them by $\omega_1, \omega_2, \omega_2, ..., \omega_{2^{n-1}}$.

1) If none of the suffixes in $\omega_A = \omega_{h_1,...,h_r}$ is $h_i = n$, but there exists some suffix $k_j = n$ in $\omega_B = \omega_{k_1,...,k_s}$, then we arrange ω_A before ω_B.

2) If none of the suffixes in $\omega_A = \omega_{h_1,...,h_r}$, $\omega_B = \omega_{k_1,...,k_s}$ is n, then when $r < s$, we order ω_A before ω_B. When $\omega_A = \omega_{h_1,...,h_r}, \omega_C = \omega_{\alpha_1,...,\alpha_r}$,

and $h_1 + \cdots + h_r < \alpha_1 + \cdots + \alpha_r$, we order ω_A before ω_C. When $h_1 + \cdots + h_r = \alpha_1 + \cdots + \alpha_r$ and $(h_1, ..., h_r) \neq (\alpha_1, ..., \alpha_r)$, if the first unequal suffix is $h_i < \alpha_i$, we also order ω_A before ω_C.

3) If there exists some suffix in $\omega_A = \omega_{h_1,...,h_r}$ and $\omega_B = \omega_{k_1,...,k_s}$ is n, then $\omega_A = \omega_{h_1,...,h_{r-1},n}$, $\omega_B = \omega_{k_1,...,k_{s-1},n}$, we order $\omega_A = \omega_{h_1,...,h_{r-1}}$ and $\omega_B = \omega_{k_1,...,k_{s-1}}$ by using the method as in 2), and regard them as the order of $\omega_A = \omega_{h_1,...,h_r}$ and $\omega_B = \omega_{k_1,...,k_s}$.

We have ordered all suffixes with numbers in the form 2^{n-1} through 1), 2), 3), then we can denote them by $\omega_1, ..., \omega_{2^{n-1}}$.

Definition 2.2 The oblique derivative problem for generalized regular functions in Ω is to find a solution $\omega(x) \in C^{1,\alpha}(\overline{\Omega}) \bigcap C^2(\Omega)$for the elliptic system of first order equations

$$\bar{\partial}\omega = a\omega + b\bar{\omega} + l \tag{2.3}$$

satisfying the boundary conditions

$$\begin{cases} \dfrac{\partial \omega_k}{\partial \nu_k} + \sigma_k(x)\omega_k(x) = \tau_k(x) + h_k, \ x \in \partial\Omega, \\ \omega_k(d) = u_k, \ 1 \leq k \leq 2^{n-1} \end{cases} \tag{2.4}$$

in which $\sigma_k(x), \tau_k(x) \in C^{1,\alpha}(\partial\Omega), \sigma_k(x) \geq 0$ *on* $\partial\Omega$, h_k is an unknown real constant, u_k is a real constant, ν_k is the vector on $\partial\Omega$, n_0 is the outward normal vector on $\partial\Omega$; moreover, $\cos(\nu_k, n_0) \in C^{1,\alpha}(\partial\Omega)$. The above boundary value problem will be called Problem P.

Let $\omega(x) = \sum\limits_A \omega_A(x)e_A$ be a solution of Problem P. Then according to Property 3.3 and Property 3.4, Chapter I, we know that the following equalities for arbitrary index A are true:

$$\begin{aligned} \sum_{m=1,\overline{mB}=A}^{n} \delta_{\overline{mB}}\omega_{Bx_m} = &\sum_{C,\overline{C\underline{M}}=A} (a_C+b_C)\omega_{\underline{M}}\delta_{\overline{C\underline{M}}} \\ &+ \sum_{C,\overline{C\underline{\underline{M}}}} (a_C-b_C)\omega_{\underline{\underline{M}}}\delta_{\overline{C\underline{\underline{M}}}}+l_A, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \Delta\omega_D = &\sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{C\underline{M}}}(a_C + b_C)_{x_j}\omega_{\underline{M}} + \sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{C\underline{M}}}(a_C + b_C)\omega_{\underline{M}x_j} \\ &+ \sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{C\underline{\underline{M}}}}(a_C - b_C)_{x_j}\omega_{\underline{\underline{M}}} + \sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{C\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}x_j} \\ &+ \sum_{j=1}^{n} \delta_{\overline{jD}}l_{Ax_j}, \end{aligned} \tag{2.6}$$

where $j = 1, 2, ..., n$, $\overline{jD} = A$, $\overline{C\underline{M}} = A$, $\overline{C\underline{\underline{M}}} = A$; $\underline{M}$, $\underline{\underline{M}}$ denote two kinds of indices of M respectively (see [29]6)).

Suppose that the equalities

$$(a_C + b_C)_{x_j} = 0,\ (a_C - b_C)_{x_j} = 0,\ \overline{jD} = A,\ \overline{CM} = A,\ C \neq j \tag{2.7}$$

in (2.6) are true. Especially, when $n = 3$, set $C = 23$, then according to the condition (2.7), we get $(a_{23} - b_{23})_{x_j} = 0$, $1 \leq j \leq 3$, namely $(a_{23} - b_{23})$ is a constant. If $C = 3$, by the condition (2.7), we get $(a_3 - b_3)_{x_j} = 0$, $j = 1, 2$, i.e. $(a_3 - b_3)$ only depends on x_3.

Noting the condition (2.7), the equality (2.6) can be written as

$$\begin{aligned}\Delta\omega_D = &\sum_j \delta_{\overline{jD}}\delta_{\overline{j\underline{M}}}(a_j+b_j)_{x_j}\omega_{\underline{M}} + \sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{C\underline{M}}}(a_C+b_C)\omega_{\underline{M}x_j}\\ &+\sum_j \delta_{\overline{jD}}\delta_{\overline{j\underline{\underline{M}}}}(a_j-b_j)_{x_j}\omega_{\underline{\underline{M}}} + \sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{C\underline{\underline{M}}}}(a_C-b_C)\omega_{\underline{\underline{M}}x_j}\\ &+\sum_{j=1}^{n} \delta_{\overline{jD}} l_{Ax_j},\end{aligned} \tag{2.8}$$

in which $\overline{C\underline{M}} = A$, $\overline{C\underline{\underline{M}}} = A$, $\overline{jD} = A$.

Suppose that D is the $\underline{A}$-type index. In the first term of the right side in (2.8), we see $\overline{j\underline{M}} = \overline{jD}$, $\underline{M} = D$, and then $\delta_{\overline{jD}}\delta_{\overline{j\underline{M}}} = 1$; and in the third term of the right side in (2.8), if the equality $\overline{j\underline{\underline{M}}} = \overline{j\underline{M}}$ holds, then $D = \underline{\underline{M}}$ is the $\underline{\underline{A}}$-type index. This is a contradiction. Hence when D is the $\underline{A}$-type index, the third term of the equality (2.8) disappears. Thus the equality (2.8) can be written as

$$\begin{aligned}\Delta\omega_D = &\sum_{j=1}^{n}(a_j + b_j)_{x_j}\omega_D + \sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{C\underline{M}}}(a_C + b_C)\omega_{\underline{M}x_j}\\ &+\sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{C\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}x_j} + \sum_{j=1}^{n} \delta_{\overline{jD}} l_{Ax_j},\end{aligned} \tag{2.9}$$

in which $\overline{C\underline{M}} = A, \overline{C\underline{\underline{M}}} = A, \overline{jD} = A$. Especially, when $n = 3, D = 1$, the equality (2.9) possesses the form

$$\begin{aligned}\Delta\omega_1 = &\sum_{j=1}^{3}(a_j + b_j)\omega_{1x_j} + \sum_{j=1}^{3}(a_j + b_j)_{x_j}\omega_1 + (a_1 - b_1)\omega_{2x_2}\\ &+(a_1 - b_1)\omega_{3x_3} - (a_2 - b_2)\omega_{2x_2} - (a_2 - b_2)\omega_{23x_3}\\ &-(a_3 - b_3)\omega_{3x_3} + (a_3 - b_3)\omega_{23x_2} - (a_{23} - b_{23})\omega_{23x_1}\\ &-(a_{23} - b_{23})\omega_{3x_2} + (a_{23} - b_{23})\omega_{2x_3} + \sum_{j=1}^{3} l_j x_j.\end{aligned} \tag{2.10}$$

Noting $\underline{M} = D = 1$, $\underline{\underline{M}} \neq D = 1$, it is clear that the third term of the equality (2.8): $\sum_{j=1}^{n}(a_j - b_j)_{x_j}\omega_{\underline{\underline{M}}}$ in (2.10) disappears. When $\underline{M} = D$, we have $C = \overline{\underline{M}A} = \overline{\underline{M}jD} = \overline{DjD} = j$, $\delta_{\overline{jD}}\delta_{\overline{C\underline{M}}} = \delta_{\overline{jD}}\delta_{\overline{CD}} = \delta_{\overline{jD}}\delta_{\overline{jD}} = 1$, and then the equality (2.9) can be rewritten as

$$\begin{aligned}\Delta\omega_D = &\sum_j (a_j+b_j)_{x_j}\omega_D + \sum_j (a_j+b_j)\omega_{Dx_j} + \sum_{j,C}\delta_{\overline{jD}}\delta_{\overline{C\underline{M}}}(a_C+b_C)\omega_{\underline{M}x_j}\\ &+ \sum_{j,C}\delta_{\overline{jD}}\delta_{\overline{C\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}x_j} + \sum_j \delta_{\overline{jD}} l_{Ax_j},\end{aligned} \tag{2.11}$$

where $\underline{M} \neq D$, and A is the index satisfying $A = \overline{jD}$.

In addition, we first write equation (2.5) in the form

$$\begin{aligned}&\sum_{j=1}^{n}\delta_{\overline{j\underline{M}}}\omega_{\underline{M}x_j} + \sum_{j=1}^{n}\delta_{\overline{j\underline{\underline{M}}}}\omega_{\underline{\underline{M}}x_j}\\ &= \sum_E (a_E+b_E)\omega_{\underline{M}}\delta_{\overline{E\underline{M}}} + \sum_E (a_E-b_E)\delta_{\overline{E\underline{\underline{M}}}}\omega_{\underline{\underline{M}}} + l_B,\end{aligned} \tag{2.12}$$

where $\overline{E\underline{M}} = B$, $\overline{B\underline{\underline{M}}} = B$, $\overline{j\underline{M}} = B$, and B runs all indexes. Moreover we use $\delta_{\overline{CD}}(a_C - b_C)$ to multiply every equation in (2.12), herein $\overline{CD} = B$, D is the fixed index, and sum according to the index B, we obtain

$$\begin{aligned}&\sum_{B,j}\delta_{\overline{CD}}\delta_{\overline{j\underline{M}}}(a_C - b_C)\omega_{\underline{M}}x_j + \sum_{B,j}\delta_{\overline{CD}}\delta_{\overline{j\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}}x_j\\ &= \sum_{B,E}(a_C - b_C)(a_E + b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{M}}}\omega_{\underline{M}}\\ &\quad + \sum_{B,E}(a_C - b_C)(a_E - b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{\underline{M}}}}\omega_{\underline{\underline{M}}} + \sum_B (a_C - b_C)\delta_{\overline{CD}} l_B\end{aligned} \tag{2.13}$$

where $\overline{CD} = B, \overline{E\underline{M}} = B, \overline{E\underline{\underline{M}}} = B, \overline{j\underline{M}} = B$.

According to Property 3.6 about the quasi-permutation in Section 3, Chapter I, $C = \overline{DB}$ and the arbitrariness of B, we know that C can run all indexes, so the second term in the left-hand side in (2.13) can also be written as

$$\begin{aligned}\sum_{B,j}\delta_{\overline{CD}}\delta_{\overline{j\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}}x_j &= \sum_{C,j}\delta_{\overline{CD}}\delta_{\overline{j\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}}x_j\\ &= \sum_{j,C}\mu_{jDC}\delta_{\overline{jD}}\delta_{\overline{C\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}}x_j,\end{aligned} \tag{2.14}$$

where $\overline{j\underline{\underline{M}}} = B$, $\overline{CD} = B$.

Since D is given, we get $A = \overline{jD}\,(j = 1, 2, ..., n)$ and $C = \overline{\underline{M}A} = \overline{\underline{M}\overline{jD}}$. Suppose that the coefficients corresponding to C which do not conform to the condition $\overline{jD} = \overline{C\underline{\underline{M}}} = A$ satisfy

$$a_C - b_C = 0, \tag{2.15}$$

and the coefficients corresponding to $C = \overline{BD}$ with the condition $\mu_{jDC} = 1$ satisfy

$$a_C - b_C = 0; \tag{2.16}$$

then the equality (2.14) can also be written in the form

$$\sum_{\substack{B,j \\ (\overline{CD},\overline{j\underline{\underline{M}}}=B)}} \delta_{\overline{CD}}\delta_{\overline{j\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}x_j} = -\sum_{\substack{j,C \\ (\overline{jD}=A,\overline{C\underline{\underline{M}}}=A)}} \delta_{\overline{jD}}\delta_{\overline{j\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}x_j}. \tag{2.17}$$

In addition, from the equality (2.13), we have

$$\begin{aligned}
&\sum_{j,C} \delta_{\overline{jD}}\delta_{\overline{j\underline{\underline{M}}}}(a_C - b_C)\omega_{\underline{\underline{M}}x_j} \\
&= \sum_{\substack{j,B \\ (\overline{CD}=B,\overline{j\underline{M}}=B)}} \delta_{\overline{CD}}\delta_{\overline{j\underline{M}}}(a_C - b_C)\omega_{\underline{M}x_j} - \sum_{\substack{E,B \\ (\overline{CD}=B,\overline{E\underline{M}}=B)}} (a_C - b_C)(a_E + b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{M}}}\omega_{\underline{M}} \\
&\quad - \sum_{\substack{E,B \\ (\overline{CD}=B,\overline{E\underline{M}}=B)}} (a_C - b_C)(a_E - b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{\underline{M}}}}\omega_{\underline{\underline{M}}} - \sum_{\substack{B \\ (\overline{CD}=B)}} (a_C - b_C)\delta_{\overline{CD}}l_B.
\end{aligned} \tag{2.18}$$

Substituting the equality (2.18) into the equality (2.11), we get

$$\begin{aligned}
\Delta\omega_D = &\sum_{\substack{j \\ (\overline{jD}=A)}} (a_j + b_j)_{x_j}\omega_D + \sum_{\substack{j \\ (\overline{jD}=A)}} (a_j + b_j)\omega_{Dx_j} \\
&+ \sum_{\substack{j,C \\ (\overline{jD}=A) \\ \overline{C\underline{M}}=A,\underline{M}\neq D}} \delta_{\overline{jD}}\delta_{\overline{C\underline{M}}}(a_C + b_C)\omega_{\underline{M}}x_j + \sum_{\substack{B,j \\ (\overline{j\underline{M}}=B,\overline{CD}=B)}} \delta_{\overline{CD}}\delta_{\overline{j\underline{M}}}(a_C - b_C)\omega_{\underline{M}}x_j \\
&- \sum_{\substack{B,E \\ (\overline{CD}=B,\overline{E\underline{M}}=B)}} (a_C - b_C)(a_E - b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{M}}}\omega_{\underline{M}} \\
&- \sum_{\substack{B,E \\ (\overline{CD}=B,\overline{E\underline{\underline{M}}}=B)}} (a_C - b_C)(a_E - b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{\underline{M}}}}\omega_{\underline{\underline{M}}} \\
&- \sum_{\substack{B \\ (\overline{CD}=B)}} (a_C - b_C)\delta_{\overline{CD}}l_B + \sum_{\substack{j \\ (\overline{jD}=A)}} \delta_{\overline{jD}}l_{Ax_j}.
\end{aligned} \tag{2.19}$$

In the fourth term of the equality (2.19), when $\underline{M} = D$, we have $\overline{CD} = B = \overline{j\underline{M}} = \overline{jD}$, and $C = j$, thus $\delta_{\overline{CD}} = \delta_{\overline{j\underline{M}}} = \delta_{\overline{jD}} = 1$. In the fifth term of the equality (2.19), when $\underline{M} = D$, in accordance with $\overline{CD} = \overline{E\underline{M}} = B$, we have $C = E$, and then $\delta_{\overline{CD}}\delta_{\overline{E\underline{M}}} = \delta_{\overline{CD}}\delta_{\overline{CD}} = 1$. So, the equality (2.19) can be written as

$$\begin{aligned}
&\Delta\omega_D = \\
&\left[\sum_{j,(\overline{jD}=A)}(a_j+b_j)_{x_j} - \sum_{C,(\overline{jD}=B)}(a_C^2-b_C^2)\right]\omega_D + \sum_{\substack{j\\ \overline{jD}=A,\overline{CD}=B}} 2a_C\omega_{Dx_j}\\
&+\sum_{\substack{j,C\\ \overline{jD}=A,\overline{C\underline{M}}=A,\underline{M}\neq D}}\delta_{\overline{jD}}\delta_{\overline{C\underline{M}}}(a_C+b_C)\omega_{\underline{M}x_j}+\sum_{\substack{B,j\\ \overline{j\underline{M}}=B,\overline{CD}=B,\underline{M}\neq D}}\delta_{\overline{CD}}\delta_{\overline{j\underline{M}}}(a_C-b_C)\omega_{\underline{M}x_j}\\
&-\sum_{\substack{B,E\\ \overline{CD}=B,\overline{E\underline{M}}=B,\underline{M}\neq D}}(a_C-b_C)(a_E+b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{M}}}\omega_{\underline{M}}\\
&-\sum_{\substack{B,E\\ \overline{CD}=B,\overline{E\underline{\underline{M}}}=B}}(a_C-b_C)(a_E-b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{\underline{M}}}}\omega_{\underline{\underline{M}}}\\
&-\sum_{B,\ \overline{CD}=B}(a_C-b_C)\delta_{\overline{CD}}l_B+\sum_{j,\ \overline{jD}=A}\delta_{\overline{jD}}l_{Ax_j}.
\end{aligned} \tag{2.20}$$

In the sixth term of the right-hand side of equality (2.20), when $E\neq C$ and $\mu_{CDE}=-1$, by Property 3.6 in Section 3, Chapter I, we get $(a_C-b_C)(a_E-b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{\underline{M}}}} = -(a_C-b_C)(a_E-b_E)\delta_{\overline{ED}}\delta_{\overline{C\underline{\underline{M}}}} = -(a_E-b_E)(a_C-b_C)\delta_{\overline{ED}}\delta_{\overline{C\underline{\underline{M}}}}$, hence $\sum_{\substack{B,E\\ E\neq C,\mu_{CDE}=-1}}(a_C-b_C)(a_E-b_E)\times\delta_{\overline{CD}}\delta_{\overline{E\underline{\underline{M}}}}\omega_{\underline{\underline{M}}} = 0$;.
While $E\neq C$, $\mu_{CDE}=-1$, suppose that the term corresponding to the sixth term of the right-hand side in (2.20) satisfies

$$(a_C-b_C)(a_E-b_E)=0. \tag{2.21}$$

When $E=C$, we have $\overline{CD}=\overline{E\underline{\underline{M}}}=\overline{C\underline{\underline{M}}}$, so $D=\underline{\underline{M}}$. This contradicts that D is an $\underline{A}$-type index. Hence this condition does not hold, hence the equality (2.20) can also be written as

$$\begin{aligned}
\Delta\omega_D = &\left[\sum_{j,(\overline{jD}=A)}(a_j+b_j)_{x_j} - \sum_{C,(\overline{jD}=B)}(a_C^2-b_C^2)\right]\omega_D + \sum_j 2a_C\omega_{Dx_j}\\
&+\sum_{\substack{j,C\\ \overline{jD}=A\\ \overline{C\underline{M}}=A,\underline{M}\neq D}}\delta_{\overline{jD}}\delta_{\overline{C\underline{M}}}(a_C+b_C)\omega_{\underline{M}x_j}\\
&+\sum_{\substack{B,j\\ \overline{j\underline{M}}=B\\ \overline{CD}=B,\underline{M}\neq D}}\delta_{\overline{CD}}\delta_{\overline{j\underline{M}}}(a_C-b_C)\omega_{\underline{M}x_j}\\
&-\sum_{\substack{B,E\\ \overline{CD}=B,\overline{E\underline{M}}=B\\ \underline{M}\neq D,E\neq C}}(a_C-b_C)(a_E+b_E)\delta_{\overline{CD}}\delta_{\overline{E\underline{M}}}\omega_{\underline{M}}
\end{aligned}$$

$$-\sum_{B,(\overline{CD}=B)}(a_C-b_C)\delta_{\overline{CD}}l_B+\sum_{j,(\overline{jD}=A)}\delta_{\overline{jD}}l_{Ax_j}. \tag{2.22}$$

Let the third, fourth, and fifth terms in the right-hand side of (2.22) satisfy

$$a_C+b_C=0,\ a_C-b_C=0,\ (a_C-b_C)(a_E+b_E)=0, \tag{2.23}$$

respectively. Then equality (2.22) can be written in the form

$$\begin{aligned}\Delta\omega_D=&\Big[\sum_{j,(\overline{jD}=A)}(a_j+b_j)_{x_j}-\sum_C(a_C^2-b_C^2)\Big]\omega_D+\sum_{\substack{j\\ \overline{CD}=B,\overline{jD}=A}}2a_C\omega_{Dx_j}\\ &-\sum_{B,(\overline{CD}=B)}(a_C-b_C)\delta_{\overline{CD}}l_B+\sum_{j,(\overline{jD}=A)}\delta_{\overline{jD}}l_{Ax_j}.\end{aligned} \tag{2.24}$$

When $D=1$, we simply write the equality (2.24) as

$$\Delta\omega_1=\sum_{m=1}^n d_{m_1}\omega_{1x_m}+f_1\omega_1+g_1, \tag{2.25}$$

here $d_{m_1}=d_{m_1}(x)=d_{m_1}(x_1,...,x_n)$, $f_1=f_1(x)=f_1(x_1,...,x_n)$, $g_1=g_1(x)=g_1(x_1,...,x_n)$. Let

$$f_1(x)\geq 0. \tag{2.26}$$

Then according to Lemma 2.1, there exists a unique function $\omega_1(x)$ satisfying the boundary condition (2.4). After we get $\omega_1(x)$, we can consider that $\omega_1(x)$ in equation (2.5) is known as well as the known coefficients $a_A(x), b_A(x), l_A(x)$. Applying the same method concluding with the equality (2.25), we get the equality (2.24) when $D=2$, and write it simply as

$$\Delta\omega_2=\sum_{m=1}^n d_{m_2}(x)\omega_{2x_m}+f_2\omega_2+g_2, \tag{2.27}$$

where $d_{m_2}=d_{m_2}(x), f_2=f_2(x), g_2=g_2(x,\omega_1(x))$. Set $f_2(x)\geq 0$; on the basis of Lemma 2.1, there exists a unique $\omega_2(x)$ satisfying the boundary condition (2.4). After getting $\omega_2(x)$, we can regard $\omega_1(x)$, $\omega_2(x)$ as the known functions. Using the above method, we can get

$$\Delta\omega_3=\sum_{m=1}^n d_{m_3}\omega_{3x_m}+f_3\omega_3+g_3, \tag{2.28}$$

in which $d_{m_3} = d_{m_3}(x)$, $f_3 = f_3(x)$, $g_3 = g_3(x, \omega_1(x), \omega_2(x))$. In accordance with the above steps and the order of $\omega_1, ..., \omega_{2^{n-1}}$, we proceed until $\omega_{2^{n-2}}$. At last, we get the unique $\omega_1, ..., \omega_{2^{n-2}}$ satisfying

$$\Delta \omega_k = \sum_{m=1}^{n} d_{m_k} \omega_{k x_m} + f_k \omega_k + g_k, \tag{2.29}$$

where

$$d_{m_k} = d_{m_k}(x),\ f_k = f_k(x),\ g_k = g_k(x, \omega_1(x),\ \omega_2(x), ..., \omega_{k-1}(x)),$$
$$f_k(x) \geq 0,\ 1 \leq k \leq 2^{n-2}.$$

For simplicity, denote by $\overline{U}$ the conditions (2.7), (2.15), (2.16), (2.17), (2.21), (2.23),... and $f_k(x) \geq 0\,(1 \leq k \leq 2^{n-2})$ of system (2.3).

From the above discussion, by means of Lemma 2.1, if system (2.3) satisfies the condition $\overline{U}$, then there exists a unique solution ω_k, , $1 \leq k \leq 2^{n-2}$.

In order to further discuss $\omega_k\,(2^{n-2}+1 \leq k \leq 2^{n-1})$, we need to study the system (2.5). Suppose that the suffix A of l_A in the equality (2.5) has been arranged according to the above method, and l_A has been written $l_1, ..., l_{2^{n-2}}, l_{2^{n-2}+1}, ..., l_{2^{n-1}}$. We may only discuss the system of equations corresponding to $l_{2^{n-2}+1}, l_{2^{n-2}+2}, ..., l_{2^{n-1}}$ in the equality (2.5), namely

$$\begin{aligned} \sum_{\substack{m=1 \\ \overline{mB}=A}}^{n} \delta_{\overline{mB}} \omega_{B x_m} = & \sum_{C,(\overline{C\underline{M}}=A)} (a_C + b_C) \omega_{\underline{M}} \delta_{\overline{C\underline{M}}} \\ & + \sum_{C,(\overline{C\underline{\underline{M}}}=A)} (a_C - b_C) \omega_{\underline{\underline{M}}} \delta_{\overline{C\underline{\underline{M}}}} + l_A, \end{aligned} \tag{2.30}$$

where $l_A = l_k$, $2^{n-2} + 1 \leq k \leq 2^{n-1}$, $\omega(x) = \sum\limits_{A} \omega_A(x) e_A$, $x \in \Omega$. The following assumption is called the condition $\overline{V}$. Set $n = 2m$, and denote $x_{2k-1} + x_{2k} i = z_k$, $k = 1, ..., m$, i is the imaginary unit, and $\omega_{2k-1} + i\omega_{2k} = \overline{\omega_k}$, $k = 1, ..., 2^{n-2}$. Let $\Omega = G_1 \times \cdots \times G_m$ be a multiply circular cylinder about complex variables $z_1, ..., z_m$. Then we regard $\omega_1, ..., \omega_{2^{n-2}}$ as the known functions, by using the result in [80]4), the elliptic system of first order equations: (2.30) about $\omega_{2^{n-2}+1}, ..., \omega_{2^{n-1}}$ can be written as

$$\frac{\partial \overline{\omega}_k}{\partial \overline{z}_k} = f_{kl}(z_1, ..., z_m, \overline{\omega}_{2^{n-3}+1}, \overline{\omega}_{2^{n-3}+2}, ..., \overline{\omega}_{2^{n-2}}), \tag{2.31}$$

in which $k = 2^{n-3} + 1, ..., 2^{n-2}$, $l = 1, ..., m$.

Denote (see [43])

$$N_1^{\perp}=\left\{ f \left| \begin{array}{l} f \text{ is a Hölder continuous function defined on characteristic} \\ \text{manifold } \partial G_1 \times \cdots \times \partial G_m, \text{ whose real index is } \beta, \text{ and} \\ \displaystyle\int_{\partial G_1} \cdots \int_{\partial G_m} \frac{f(\zeta_1,...,\zeta_m)}{\zeta_1^{J_1}...\zeta_m^{J_m}} d\zeta_1...d\zeta_m = 0, \\ J_l > -k_l,\ l=1,...,m \end{array} \right. \right\}.$$

If (i) f_{kl} is continuous with respect to $z = (z_1,...,z_m) \in \overline{\Omega}$, $\overline{\omega} = (\overline{\omega}_{2^{n-3}+1}, \overline{\omega}_{2^{n-3}+2}, ..., \overline{\omega}_{2^{n-2}}) \in B_\theta$, here $B_\theta = \{\omega | |\overline{\omega}_j| < \theta,\ j = 2^{n-3}+1, 2^{n-3}+2, ..., 2^{n-2}\}$, $\theta > 0$, moreover, f_{kl} is holomorphic about $\overline{\omega} \in B_\theta$, and has continuous mixed partial derivatives until $m-1$ order for different $\overline{z}_j$ $(j \neq l)$, namely $\dfrac{\partial^\lambda f_{kl}}{\partial \overline{z}_{i_1}...\partial \overline{z}_{i_\lambda}}$ is continuous about $\overline{\omega} \in B_\theta, z \in \overline{\Omega}$, here $\lambda \leq m-1$, $1 \leq i_1 < \cdots < i_\lambda \leq m$, $i_k \neq l$, $k = 1,...,\lambda$.

(ii) the system (2.31) is completely integrable, that is

$$\frac{\partial f_{kl}}{\partial \overline{z}_j} + \sum_{p=2^{n-3}+1}^{2^{n-2}} \frac{\partial f_{kl}}{\partial \overline{\omega}_p} f_{p_j} = \frac{\partial f_{kj}}{\partial \overline{z}_l} + \sum_{p=2^{n-3}+1}^{2^{n-2}} \frac{\partial f_{kj}}{\partial \overline{\omega}_p} f_{pl},$$

$$k = 2^{n-3}+1, ..., 2^{n-2},\ 1 \leq j,\ l \leq m.$$

(iii) the set

$$M_\theta=\left\{ \overline{\omega} \left| \begin{array}{l} \overline{\omega} \text{ are several complex variable functions defined on } \overline{\Omega}, \\ \text{with continuous mixed partial derivatives up to order } m \\ \text{for different } \overline{z}_j, \text{ and satisfy} \\ |\overline{\omega}_j| < \theta,\ |\dfrac{\partial^\lambda \overline{\omega}_j}{\partial \overline{z}_{i_1}...\partial \overline{z}_{i_\lambda}}| \leq \theta,\ j = 2^{n-3}+1, ..., 2^{n-2}, \\ 1 \leq i_1 < \cdots < i_\lambda \leq m,\ 1 \leq \lambda \leq m,\ z \in \overline{\Omega} \end{array} \right. \right\}$$

is defined. When $\overline{\omega} \in M_\theta$, the composite function f_{kl} and its continuous mixed partial derivatives up to order m for different $\overline{z}_j$ $(j \neq l)$ are uniformly bounded, we denote its bound by K_θ. Moreover, for arbitrary $\overline{\omega}, \tilde{\overline{\omega}} \in M_\theta$, the composite function f_{kl} and its mixed partial derivatives satisfy the Lipschitz condition, that is

$$\left| \frac{\partial^\lambda f_{kl}(z_1,...,z_m, \overline{\omega}_{2^{n-3}+1}, ..., \overline{\omega}_{2^{n-2}})}{\partial \overline{z}_{i_1}...\partial \overline{z}_{i_\lambda}} \right.$$
$$\left. - \frac{\partial^\lambda f_{kl}(z_1,...,z_m, \tilde{\overline{\omega}}_{2^{n-3}+1}, ..., \tilde{\overline{\omega}}_{2^{n-2}})}{\partial \overline{z}_{i_1}...\partial \overline{z}_{i_\lambda}} \right|$$

$$\leq L_{\theta} \max_{\substack{2^{n-3}+1\leq\varepsilon\leq 2^{n-2}\\ 1\leq j_1<\cdots<j_\alpha\leq m}} \sup_{\Omega} \left|\frac{\partial^\alpha(\overline{\omega}_E-\tilde{\overline{\omega}}_\varepsilon)}{\partial\overline{z}_{j_1}...\partial\overline{z}_{j_\alpha}}\right|,$$

$$1\leq i_1<\cdots<i_\lambda\leq m, i_p\neq l.$$

(iv) the real functions $\psi_j(z_1, ..., z_m)$ on $\partial G_1\times\cdots\times\partial G_m$ satisfy the Hölder condition, namely $\psi_j(z_1, ..., z_m)\in C^\beta(\partial G_1\times\cdots\times\partial G_m)$. In addition, we assume an unknown real function h_j on $N_1^\perp$. The above conditions are called the condition $\overline{V}$.

Definition 2.3 For the solution $\omega = \sum_{k=1}^{2^{n-1}}\omega_k(x)e_k$ of Problem P, if $\overline{\omega}_{2^{n-3}+1}, ..., \overline{\omega}_{2^{n-2}}(\overline{\omega}_k=\omega_{2k-1}+i\omega_{2k})$ satisfy generalized Riemann-Hilbert boundary condition on $\partial G_1\times\cdots\times\partial G_m$ (see [43]):

$$\begin{aligned}&\mathrm{Re}[z_1^{-k_1}, ..., z_m^{-k_m}, \overline{\omega}_j(z_1, ..., z_m)]=\psi_j(z_1, ..., z_m)+h_j,\\ &j=2^{n-3}+1, ..., 2^{n-2}, z=(z_1, ..., z_m)\in\partial G_1\times\cdots\times\partial G_m,\end{aligned}\tag{2.32}$$

then the problem for generalized regular functions is called the mixed boundary problem, which will be denoted by Problem P-R-H.

On the basis of the result in [43], under the condition $\overline{V}$, when $K_j<0\,(j=1, ..., m)$, and $K_\theta, L_\theta, \sum_{j=2^{n-3}+1}^{2^{n-2}} C_\beta(\psi_j)$ are small enough, there exists a unique solution $(\overline{\omega}_{2^{n-3}+1}, ..., \overline{\omega}_{2^{n-2}})$ for the modified problem (2.31), (2.32), so $\omega_{2^{n-2}+1}, ..., \omega_{2^{n-1}}$ are uniquely determined.

From the above discussion, we get the existence and uniqueness of the solution of Problem P-R-H for generalized regular functions in real Clifford analysis.

Theorem 2.2 *Under the condition $\overline{U}$, $\overline{V}$, when $K_j<0$ $(j=1,2, ..., m)$, and $K_\theta, L_\theta, \sum_{j=2^{n-3}+1}^{2^{n-2}} C_\beta(\psi_j)$ are small enough, there exists a unique solution $\omega(x)=\sum_A\omega_A(x)e_A=\sum_{k=1}^{2^{n-1}}\omega_k(x)e_k\,(x\in\Omega)$ of Problem $P-R-H$ for generalized regular functions, where $\omega_1(x), ..., \omega_{2^{n-2}}(x)$ satisfy equation (2.3) and the boundary condition (2.4) of Problem P. Denote $\overline{\omega}_k=\omega_{2k-1}+i\omega_{2k}\,(k=2^{n-3}+1, ..., 2^{n-2})$, then $\overline{\omega}_{2^{n-3}+1}, ..., \overline{\omega}_{2^{n-2}}$ satisfy equation (2.31), and the corresponding functions $\omega_{2^{n-2}+1}$, $\omega_{2^{n-2}+2}$, ..., $\omega_{2^{n-1}}$ satisfy equation (2.3) and the generalized $R-H$ boundary condition (2.32).*

3 A Nonlinear Boundary Value Problem With Haseman Shift for Regular Functions in Real Clifford Analysis

This section deals with the nonlinear boundary value problem with Haseman shift $d(t)$ in real Clifford analysis, whose boundary condition is as follows:

$$\begin{aligned} & a(t)\Phi^+(t) + b(t)\Phi^+(d(t)) + c(t)\Phi^-(t) \\ = \ & g(t) \cdot f(x, \Phi^+(t), \Phi^-(t), \Phi^+(d(t)), \Phi^-(d(t))). \end{aligned} \tag{3.1}$$

We shall prove the existence of solutions for the problem (3.1) by using the Schauder fixed point theorem (see [29]1)). It is easy to see that when $a(t) = g(t) \equiv 0$, $b(t) \equiv 1$, the problem (3.1) becomes the Haseman problem

$$\Phi^+(d(t)) = G(t)\Phi^-(t). \tag{3.2}$$

The problem (3.2) was first solved by C. Haseman [22]. In general, all boundary value conditions for holomorphic functions can be expressed as the pasting condition of the unknown functions, hence the boundary value problem can be regarded as the conformal pasting problem [87] in function theory. But the method of conformal pasting cannot be used to handle all problems of multiple elements. In 1974, A. M. Hekolaeshuk [23] gave an example, i.e. for the boundary value problem

$$a(t)\Phi^+(t) + b(t)\Phi^-(d(t)) + c(t)\Phi^-(t) = g(t), \tag{3.3}$$

the method of conformal pasting cannot be eliminated the shift $d(t)$. For the problem (3.1) discussed in this section, we choose the linear case of (3.3) as its example.

Firstly, we reduce the problem to the integral equation problem, and then use the fixed point theorem to prove the existence of solutions for the problem.

Assume a connected open set $\Omega \in \mathbf{R^n}$, whose boundary $\partial\Omega$ is a smooth, oriented, compact Liapunov surface (see Section 2, Chapter 1). Suppose that $a(t), b(t), c(t), d(t), g(t)$ are given on $\partial\Omega$, and $d(t)$ is a homeomorphic mapping, which maps $\partial\Omega$ onto $\partial\Omega$. Denote $\Omega^+ = \Omega$, $\Omega^- = R^n\backslash\overline{\Omega}$, $\overline{\Omega} = \Omega\bigcup\partial\Omega$; we shall find a regular function $\Phi(x)$ in Ω^+, which is continuous on $\Omega^\pm\bigcup\partial\Omega$, and satisfies $\Phi^-(\infty) = 0$ and the nonlinear boundary condition (3.1) with Haseman shift. The above problem is called Problem SR. Set

$$\Phi(x) = \frac{1}{\omega_n}\int_{\partial\Omega} E(x,t)m(t)\varphi(t)ds_t, \tag{3.4}$$

where $\omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the surface area of a unit ball in $\mathbf{R}^n$, $E(x,t) = \frac{\overline{t-x}}{|t-x|^n}$, $m(t) = \sum\limits_{j=1}^{n} e_j \cos(m, e_j)$ is the outward normal unit vector on $\partial\Omega$, and ds_t is the area element, and $\varphi(t)$ is the unknown Hölder continuous function on $\partial\Omega$. According to the Plemelj formula (2.24) in Section 2, Chapter I,

$$\Phi^+(x) = \frac{\varphi(x)}{2} + P\varphi(x),\ x\in\partial\Omega, \tag{3.5}$$

$$\Phi^-(x) = -\frac{\varphi(x)}{2} + P\varphi(x),\ x\in\partial\Omega, \tag{3.6}$$

in which the operator

$$P\varphi(x) = \frac{1}{\omega_n}\int_{\partial\Omega} E(x,t)m(t)\varphi(t)dS_t,\ x\in\partial\Omega.$$

In addition

$$\Phi^+(d(t)) = \frac{\varphi_1(x)}{2} + P_1\varphi(x),\ \ x\in\partial\Omega, \tag{3.7}$$

$$\Phi^-(d(t)) = -\frac{\varphi_1(x)}{2} + P_1\varphi(x),\ \ x\in\partial\Omega, \tag{3.8}$$

where $\varphi_1(x) = \varphi(d(x))$, and

$$\begin{aligned} P_1\varphi(x) &= P\varphi(d(x)) \\ &= \frac{1}{\omega_n}\int_{\partial\Omega} E(d(x),t)m(t)\varphi(t)dS_t. \end{aligned} \tag{3.9}$$

Substituting (3.5) – (3.8) into (3.1), we get

$$a(\frac{\varphi}{2}+P\varphi) + b(\frac{\varphi_1}{2}+P_1\varphi) + c(-\frac{\varphi}{2}+P\varphi) = g\cdot f. \tag{3.10}$$

Introducing the operator

$$F\varphi = (a+c)(-\frac{\varphi}{2}+P\varphi) + b(\frac{\varphi_1}{2}+P_1\varphi) + (1+a)\varphi - gf,$$

the equation (3.10) becomes

$$\varphi = F\varphi. \tag{3.11}$$

Thus the problem SR is reduced to solving the integral equation (3.11).

Denote by $H(\partial\Omega, \beta)$ the set of the above Hölder continuous functions with the Hölder index $\beta\,(0 < \beta < 1)$. For arbitrary $\varphi \in H(\partial\Omega, \beta)$, the norm of φ is defined as

$$\| \varphi \|_\beta = C(\varphi, \partial\Omega) + H(\varphi, \partial\Omega, \beta),$$

where $C(\varphi, \partial\Omega) = \max_{t\in\partial\Omega} |\varphi(t)|, H(\varphi, \partial\Omega, \beta) = \sup_{t_1\neq t_2,\, t_1, t_2\in\partial\Omega} \dfrac{|\varphi(t_1)-\varphi(t_2)|}{|t_1 - t_2|^\beta}$. It is evident that $H(\partial\Omega,\, \beta)$ is a Banach space; moreover we easily verify

$$\| f + g \|_\beta \leq \| f \|_\beta + \| g \|_\beta, \quad \| f \cdot g \|_\beta \leq 2^{n-1} \| f \|_\beta \| g \|_\beta, \tag{3.12}$$

where $f, g \in H(\partial\Omega, \beta)$.

Theorem 3.1 *Suppose that the operator* $\theta : \theta\varphi = \frac{\varphi}{2} - P\varphi$ *and the function* $\varphi(t) \in H(\partial\Omega, \beta)$ *are given; then there exists a constant* J_1 *independent of* φ*, such that*

$$\| \theta\varphi \|_\beta \leq J_1 \| \varphi \|_\beta. \tag{3.13}$$

Proof On the basis of Theorem 2.7, Chapter I, we know

$$\frac{1}{\omega_n} \int_{\partial\Omega} E(x, t)m(t)ds_t = \frac{1}{2},\ x \in \partial\Omega,$$

and then

$$|(\theta\varphi)(x)| \leq \frac{1}{\omega_n} H(\varphi, \partial\Omega, \beta) \int_{\partial\Omega} \frac{ds_t}{|t-x|^{n-1-\beta}} = M_1 H(\varphi, \partial\Omega, \beta), \tag{3.14}$$

in which M_1 is a constant independent of φ.

In order to consider $H(\theta\varphi, \partial\Omega, \beta)$, we choose arbitrary $x, \hat{x} \in \partial\Omega$, and denote $\delta = |x - \hat{x}|$. Firstly, suppose $6\delta < d$ (d is the constant about a Liapunov surface in Section 2, Chapter I); we can make a sphere with the center at x and radius 3δ. The inner part of this sphere is denoted $\partial\Omega_1$ and the remaining part is denoted $\partial\Omega_2$, thus we have

$$\begin{aligned} & |(\theta\varphi)(x) - (\theta\varphi)(\hat{x})| \leq \frac{1}{\omega_n} \left| \int_{\partial\Omega_1} E(x, t)m(t)(\varphi(x) - \varphi(t))ds_t \right| \\ + & \frac{1}{\omega_n} \left| \int_{\partial\Omega_1} E(\hat{x}, t)m(t)(\varphi(\hat{x}) - \varphi(t))ds_t \right| \\ + & \frac{1}{\omega_n} \left| \int_{\partial\Omega_2} E(x, t)m(t)(\varphi(x) - \varphi(t))ds_t - \int_{\partial\Omega_2} E(\hat{x}, t)m(t)(\varphi(\hat{x}) - \varphi(t))ds_t \right| \\ = & L_1 + L_2 + L_3. \end{aligned}$$

For x, we use the result about $N_0 \in \partial\Omega$ in Section 2, Chapter I, and denote by π_1 the projection field of $\partial\Omega_1$ on the tangent plane of x; then

$$L_1 \leq M_2 H(\varphi, \partial\Omega, \beta) \int_0^{3\delta} \frac{\rho_0^{n-2}}{\rho_0^{n-1-\beta}} d\rho_0 = M_3 H(\varphi, \partial\Omega, \beta)|x - \hat{x}|^\beta,$$

where M_2, M_3 are constants independent of x, $\hat{x}$. In the following we shall denote by M_i the constant having this property. Similarly, $L_2 \leq M_4 H(\varphi, \partial\Omega, \beta)|x - \hat{x}|^\beta$. Next, we estimate L_3:

$$\begin{aligned} L_3 \quad &\leq \quad \frac{1}{\omega_n} \Big| \int_{\partial\Omega_2} (E(x,t) - E(\hat{x},t)) m(t)(\varphi(x) - \varphi(t)) ds_t \Big| \\ &+ \quad \frac{1}{\omega_n} \Big| \int_{\partial\Omega_2} E(\hat{x},t) m(t)(\varphi(x) - \varphi(\hat{x})) ds_t \Big| = O_1 + O_2. \end{aligned}$$

By using Hile's lemma (see Section 2, Chapter 1), we get

$$\begin{aligned} |E(x,t) - E(\widehat{x},t)| &= \left| \frac{\overline{t-x}}{|t-x|^n} - \frac{\overline{t-\hat{x}}}{|t-\hat{x}|^n} \right| \\ &\leq \sum_{k=0}^{n-2} \left| \frac{t-x}{t-\hat{x}} \right|^{-(k+1)} |t-\hat{x}|^{-n} |x-\hat{x}|. \end{aligned}$$

For arbitrary $t \in \partial\Omega_2$, we have $|t - \hat{x}| \geq 2\delta$, and then

$$\frac{1}{2} \leq \Big|\frac{t-x}{t-\hat{x}}\Big| \leq 2.$$

Thus $O_1 \leq M_5 H(\varphi, \partial\Omega, \beta)|x-\hat{x}|^\beta$. Noting that $\varphi \in H(\varphi, \partial\Omega)$, it is easy to see that $O_2 \leq M_6 H(\varphi, \partial\Omega, \beta)|x - \hat{x}|^\beta$. Hence

$$L_3 \leq M_7 H(\varphi, \partial\Omega, \beta)|x - \hat{x}|^\beta.$$

From the above discussion, when $6|x - \hat{x}| < d$, we have

$$|(\theta\varphi)(x) - (\theta\varphi)(\hat{x})| \leq M_8 H(\varphi, \partial\Omega, \beta)|x - \hat{x}|^\beta. \tag{3.15}$$

On the basis of the results in [53], we obtain the above estimation for $6|x-\hat{x}| \geq d$. Moreover, according to (3.14),(3.15), there exists a positive constant J_1, such that $\| \theta\varphi \|_\beta \leq J_1 \| \varphi \|_\beta$. This completes the proof.

Taking into account

$$P\varphi = \frac{\varphi}{2} - \theta\varphi,$$

we get

$$\| P\varphi \|_\beta \leq \frac{1}{2} \| \varphi \|_\beta + \| \theta\varphi \|_\beta \leq (\frac{1}{2} + J_1) \| \varphi \|_\beta. \tag{3.16}$$

Similarly, it is easy to prove the following corollary.

Corollary 3.2 *For arbitrary $\varphi \in H(\partial\Omega, \beta)$, there exists a constant J_2 independent of φ, such that*

$$\| \frac{\varphi}{2} + P\varphi \|_{\beta} \leq J_2 \| \varphi \|_{\beta}. \tag{3.17}$$

Theorem 3.3 *Let the shift $d = d(x)\,(x \in \partial\Omega)$ satisfy the Lipschitz condition on $\partial\Omega$. Then for arbitrary $x, \hat{x} \in \partial\Omega$, we have*

$$|d(x) - d(\hat{x})| \leq J_3 |x - \hat{x}|. \tag{3.18}$$

We introduce the operator

$$G\varphi = \frac{\varphi_1}{2} + P_1\varphi = \frac{\varphi(d(x))}{2} + P\varphi(d(x)),$$

then for arbitrary $\varphi \in H(\partial\Omega, \beta)$, there exists a constant J_6 independent of φ, such that

$$\| G\varphi \|_{\beta} \leq J_6 \| \varphi \|_{\beta}. \tag{3.19}$$

Proof According to (3.16), we get

$$C(G\varphi, \partial\Omega) \leq \frac{\| \varphi \|_{\beta}}{2} + (\frac{1}{2} + J_1) \| \varphi \|_{\beta} = (1 + J_1) \| \varphi \|_{\beta}. \tag{3.20}$$

Similar to the proof of (3.15), we have

$$|P\varphi(d(x)) - P\varphi(d(\hat{x}))| \leq J_4 \| \varphi \|_{\beta} |d(x) - d(\hat{x})|^{\beta} \leq J_5 \| \varphi \|_{\beta} |x - \hat{x}|^{\beta}. \tag{3.21}$$

From (3.20), (3.21), it follows that the inequality $\| G\varphi \|_{\beta} \leq J_6 \| \varphi \|_{\beta}$ holds.

Corollary 3.4 *Under the same condition as in Theorem 3.3, the following inequality holds:*

$$\| \frac{-\varphi_1}{2} + P_1\varphi \|_{\beta} \leq J_7 \| \varphi \|_{\beta}. \tag{3.22}$$

Theorem 3.5 *Suppose that the shift $d = d(x)$ in Problem SR satisfies the condition (3.18) and $a(t)$, $b(t)$, $c(t)$, $g(t) \in H(\partial\Omega,\, \beta)$. Then, if the function $f(t, \Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \Phi^{(4)})$ is Hölder continuous for the arbitrary fixed Clifford numbers $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \Phi^{(4)}$ about fixed $t \in \partial\Omega$*

and satisfies the Lipschitz condition for the arbitrary fixed $t \in \partial\Omega$ *about* $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \Phi^{(4)}$, *namely*

$$\begin{aligned}&|f(t_1, \Phi_1^{(1)}, \Phi_1^{(2)}, \Phi_1^{(3)}, \Phi_1^{(4)}) - f(t_2, \Phi_2^{(1)}, \Phi_2^{(2)}, \Phi_2^{(3)}, \Phi_2^{(4)})| \\ &\le J_8|t_1 - t_2|^\beta + J_9|\Phi_1^{(1)} - \Phi_2^{(1)}| + \cdots + J_{12}|\Phi_1^{(4)} - \Phi_2^{(4)}|,\end{aligned} \tag{3.23}$$

where J_i, $i = 8, ..., 12$ *are positive constants independent of* t_i, $\Phi_j^{(i)}$ $(i = 1, 2, 3, 4,\ j = 1, 2)$, $f(0, 0, 0, 0, 0) = 0$; *and if* $\| a + c \|_\beta < \varepsilon < 1$, $\| b \|_\beta < \varepsilon < 1$, $\| 1 + a \|_\beta < \varepsilon < 1$, $0 < \mu = \varepsilon \cdot 2^{n-1}(J_1 + J_6 + 1) < 1$, *and* $\| g \|_\beta < \delta$; *then when* $0 < \delta \le \dfrac{M(1-\mu)}{2^{n-1}(J_{17} + J_{18}M)}$, *Problem SR is solvable, where* M *is the given positive number* ($\| \varphi \|_\beta \le M$), J_{17}, J_{18} *are the positive numbers dependent on* $J_i, i = 1, 2, 6, 7, ..., 12$.

Proof Denote by

$$T = \{\varphi | \varphi \in H(\partial\Omega,\ \beta), \| \varphi \|_\beta \le M\}$$

the subset of the continuous function space $C(\partial\Omega)$. According to (3.11), we have

$$\begin{aligned}&||F\varphi||_\beta \le 2^{n-1}\|a+c\|_\beta\|\theta\varphi\|_\beta + 2^{n-1}||b||_\beta||G\varphi||_\beta + 2^{n-1}||1+a||_\beta||\varphi||_\beta \\ &+2^{n-1}||g||_\beta \cdot ||f(t, \frac{\varphi}{2} + P\varphi, -\frac{\varphi}{2} + P\varphi, \frac{\varphi_1}{2} + P_1\varphi, -\frac{\varphi_1}{2} + P_1\varphi)||_\beta .\end{aligned}$$

From Theorems 3.1, 3.3, Corollaries 3.2, 3.4 and the condition (3.23), it follows that

$$C(f, \partial\Omega) \le J_{13} + J_{14}||\varphi||_\beta. \tag{3.24}$$

Moreover using (3.23) we have

$$\begin{aligned}&\left|f\left(t_1, \frac{\varphi(t_1)}{2} + P\ \varphi(t_1), \frac{-\varphi(t_1)}{2} + P\varphi(t_1), \frac{\varphi_1(t_1)}{2} + P_1\varphi(t_1)\right.\right., \\ &\left.-\frac{\varphi_1(t_1)}{2} + P_1\varphi(t_1)\right) - f\left(t_2, \frac{\varphi(t_2)}{2} + P\varphi(t_2)\ , \frac{-\varphi(t_2)}{2} + P\varphi(t_2),\right. \\ &\left.\left.\frac{\varphi_1(t_2)}{2} + P_1\varphi(t_2), \frac{-\varphi(t_2)}{2} + P_1\varphi(t_2)\right)\right| \\ &\le (J_{15} + J_{16}||\varphi||_\beta)|t_1 - t_2|^\beta\ (J_{15} = J_8).\end{aligned} \tag{3.25}$$

In accordance with (3.24),(3.25), we obtain

$$||f||_\beta \le J_{17} + J_{18}||\varphi||_\beta, \tag{3.26}$$

hence when $\varphi \in T$, applying the condition in this theorem, the inequality

$$\begin{aligned} ||F\varphi||_\beta &\le \mu||\varphi||_\beta + 2^{n-1}\delta(J_{17} + J_{18}||\varphi||_\beta) \\ &< \mu M + \delta 2^{n-1}(J_{17} + J_{18}M) \le M \end{aligned}$$

is concluded. This shows that F maps the set T into itself.

In the following, we shall prove that F is a continuous mapping. Choose arbitrary $\varphi^{(n)}(x) \in T$, such that $\{\varphi^{(n)}(x)\}$ uniformly converges to $\varphi(x), x \in \partial\Omega$. It is clear that for arbitrary given number $\varepsilon > 0$, when n is large enough, $||\varphi^{(n)} - \varphi||_\beta$ may be small enough. Now we consider $P\varphi^{(n)}(x) - P\varphi(x)$. Let $6\delta < d$, $\delta > 0$. Then we can make a sphere with the center at x and radius 3δ. The inner part of the sphere is denoted by $\partial\Omega_1$, and the rest part is denoted $\partial\Omega_2$. Thus we have

$$\begin{aligned} &|P\varphi^{(n)}(x) - P\varphi(x)| \le \frac{1}{\omega_n}\left|\int_{\partial\Omega} E(x,t)m(t)[\varphi^{(n)}(t) - \varphi(t)]dS_t\right| \\ = \; & \frac{1}{\omega_n}|\int_{\partial\Omega} E(x,t)m(t)[\varphi^{(n)}(t) - \varphi^{(n)}(x) + \varphi(x) - \varphi(t) + \varphi^{(n)}(x) \\ & -\varphi(x)]dS_t| \le \frac{1}{\omega_n}|\int_{\partial\Omega} E(x,t)m(t)[(\varphi^{(n)}(t) - \varphi^{(n)}(x)) \\ & +(\varphi(x) - \varphi(t))]dS_t| + \frac{1}{\omega_n}\left|\int_{\partial\Omega} E(x,t)m(t)\left(\varphi^{(n)}(x) - \varphi(x)\right)dS_t\right| \\ \le \; & \frac{1}{\omega_n}|\int_{\partial\Omega_1} E(x,t)m(t)[(\varphi^{(n)}(t) - \varphi^{(n)}(x)) + (\varphi(x) - \varphi(t))]dS_t| \\ & +\frac{1}{\omega_n}|\int_{\partial\Omega_2} E(x,t)m(t)[(\varphi^{(n)}(t) - \varphi^{(n)}(x)) + (\varphi(x) - \varphi(t))]dS_t| \\ & +\frac{1}{2}||\varphi^{(n)} - \varphi||_\beta = L_4 + L_5 + \frac{||\varphi^{(n)} - \varphi||_\beta}{2} \end{aligned}$$

where

$$\begin{aligned} L_4 &= \frac{1}{\omega_n}|\int_{\partial\Omega_1} E(x,t)m(t)[(\varphi^{(n)}(t) - \varphi^{(n)}(x)) + (\varphi(x) - \varphi(t))dS_t| \\ &\le J_{19}\int_0^{3\delta} \frac{1}{\rho_0^{n-1-\beta}}\rho_0^{n-2}d\rho_0 = J_{19}\int_0^{3\delta} \rho_0^{\beta-1}d\rho_0 = J_{20}\delta^\beta, \end{aligned}$$

and

$$\begin{aligned} L_5 &= \frac{1}{\omega_n}|\int_{\partial\Omega_2} E(x,t)m(t)[(\varphi^{(n)}(t) - \varphi(t)) - (\varphi^{(n)}(x) - \varphi(x))dS_t| \\ &\le J_{21}||\varphi^{(n)} - \varphi||_\beta, \end{aligned}$$

hence

$$|P\varphi^{(n)}(x) - P\varphi(x)| \le J_{20}\delta^\beta + J_{22}||\varphi^{(n)} - \varphi||_\beta.$$

We choose a sufficiently small positive number δ, such that $J_{20}\delta^\beta < \varepsilon/2$, and then choose a sufficiently large positive integer n, such that $J_{22}||\varphi^{(n)} - \varphi||_\beta < \frac{\varepsilon}{2}$. Thus for the arbitrary $x \in \partial\Omega$, we have

$$|P\varphi^{(n)}(x) - P\varphi(x)| < \varepsilon. \tag{3.27}$$

Similarly, when n is large enough, for arbitrary $x \in \partial\Omega$, we can derive

$$|\varphi_1^{(n)}(x) - \varphi_1(x)| < \varepsilon, \tag{3.28}$$

$$|P_1\varphi^{(n)}(x) - P_1\varphi(x)| < \varepsilon. \tag{3.29}$$

Taking into account (3.11), (3.23) and (3.27) – (3.29), we can choose a sufficiently large positive integer n, such that

$$|F\varphi^{(n)}(x) - F\varphi(x)| < \varepsilon, \text{ for arbitrary } x \in \partial\Omega.$$

This shows that F is a continuous mapping, which maps T into itself. By means of the Ascoli-Arzela theorem, we know that T is a compact set in the continuous function space $C(\partial\Omega)$. Hence the continuous mapping F maps the closed convex set T in $C(\partial\Omega)$ onto itself, and $F(T)$ is also a compact set in $C(\partial\Omega)$. By the Schauder fixed point theorem, there exists a function $\varphi_0 \in H(\partial\Omega, \beta)$ satisfying the integral equation (3.11). This shows that Problem SR is solvable.

Theorem 3.6 *If $f \equiv 1$ in Theorem 3.5, then Problem SR has a unique solution.*

In fact, for arbitrary $\varphi_1, \varphi_2 \in H(\partial\Omega, \beta)$, by using the similar method as before, we can obtain

$$||F\varphi_1 - F\varphi_2||_\beta < \mu||\varphi_1 - \varphi_2||_\beta.$$

Taking account of the condition $0 < \mu < 1$, we know that $F\varphi$ (when $f \equiv 1$) is a contracting mapping from the Banach space $H(\partial\Omega, \beta)$ into itself, hence there exists a unique fixed point $\varphi_0(x)$ of the functional equation $\varphi_0 = F\varphi_0$, i.e. Problem SR has a unique solution

$$\Phi(x) = \frac{1}{\omega_n}\int_{\partial\Omega} E(x,t)m(t)\varphi_0(t)dS_t, \ \Phi^-(\infty) = 0.$$

In 1991, Sha Huang discussed the boundary value problem with conjugate value

$$a(t)\Phi^+(t) + b(t)\overline{\Phi^+}(t) + c(t)\Phi^-(t) + d(t)\overline{\Phi^-}(t)$$
$$= g(t), \ t \in \Omega$$

for regular functions in real Clifford analysis (see [29]3)). Similarly, we can discuss the nonlinear boundary value problem with shift and conjugate value for regular functions in real Clifford analysis.

4 The Dirichlet Problem of Hyperbolic Harmonic Functions in Real Clifford Analysis

One of the generalized forms of a Cauchy-Riemann system in high dimensional space is the following system of equations:

$$\begin{cases} x_n\left(\dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u_2}{\partial x_2} - \cdots - \dfrac{\partial u_n}{\partial x_n}\right) + (n-1)u_n = 0, \\ \dfrac{\partial u_i}{\partial x_k} = \dfrac{\partial u_k}{\partial x_i}, \ i, k = 2, ..., n, \\ \dfrac{\partial u_1}{\partial x_k} = -\dfrac{\partial u_k}{\partial x_1}, \ k = 2, ..., n. \end{cases} \tag{H_n}$$

The system (H_n) appeared in a remark of H. Hasse paper [21] in 1949, but to our knowledge has not been treated so far. In 1992, H. Leutwiler established the relation between solutions for system (H_n) and classical holomorphic functions [41]. In this section, on the basis of [41], we study the Schwarz integral representation for hyperbolic harmonic functions and the existence of solutions for a kind of boundary value problems for hyperbolic harmonic functions for a high dimension ball in real Clifford analysis. We also discuss hyperbolic harmonic functions in real Clifford analysis and the relation with solutions of system (H_n). The material comes from Sha Huang's paper [29]7).

4.1 The Relation Between Solutions for System (H_n) and Holomorphic Functions

Setting $x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n$, we denote $l(x) = [\sum\limits_{k=2}^{n} x_k^2]^{\frac{1}{2}}$, $I(x) = \sum\limits_{k=2}^{n} x_k e_k / l(x)$. In the following, we shall introduce a kind of mapping from $\mathbf{R}^n$ to $\mathcal{A}_n(\mathbf{R})$. For any complex variable function $f(z) = u(x, y) + iv(x, y)$, we consider its corresponding function $\tilde{f} = \tilde{f}(x_1, x_2, ..., x_n) = u(x_1, l(x)) + I(x)v(x_1, l(x))$. In [41], H. Leutwiler gave the following result.

Theorem 4.1 *Let* $\Omega \subset (\mathbf{R}^2)^+ = \{z \,|\, z = (x, y) \in \mathbf{R}^2 , \ y > 0\}$ *be an* *open set, and* $f(z) = u + iv$ *be holomorphic in* Ω*. Then*

$$\tilde{f}(x) = \tilde{f}(x_1, x_2, ..., x_n) = u(x_1, l(x)) + I(x)v(x_1, l(x)) \tag{4.1}$$

is a solution of system (H_n) *in* $\widetilde{\Omega} = \{x = \sum\limits_{k=1}^{n} x_k e_k \in \mathbf{R}^n | x_1 + il(x) \in \Omega\}$*. Moreover, if we denote* $\tilde{f} = \sum\limits_{k=1}^{n} u_k e_k$ *in (4.1), then* $u_i x_k = u_k x_i$*,* $i, k = 2, ..., n$*.*

Proof Since $f(z)$ is holomorphic, we have

$$\begin{cases} u'_x = v'_y, \\ u'_y = -v'_x, \end{cases} \tag{4.2}$$

$$\tilde{f} = u(x_1, l(x)) + (\frac{x_2 e_2}{l(x)} + \frac{x_3 e_3}{l(x)} + \cdots + \frac{x_n e_n}{l(x)})v(x_1, l(x)),$$

and then

$$\begin{aligned} u_1(x) &= u(x_1, l(x)), \\ u_k(x) &= \frac{x_k}{l(x)} v(x_1, l(x)), \ k = 2, 3, ..., n. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial u_1}{\partial x_1} &= u'_x(x_1, l(x)), \\ \frac{\partial u_k}{\partial x_k} &= (\frac{x_k}{l(x)})'_{x_k} v(x_1, l(x)) + \frac{x_k}{l(x)}[v(x_1, l(x))]_{x_k} \\ &= \frac{l(x) - x_k l'_{x_k}(x)}{[l(x)]^2} v(x_1, l(x)) + \frac{x_k}{l(x)} v'_y(x_1, l(x)) l'_{x_k}(x) \\ &= \frac{l(x)v(x_1, l(x)) - x_k^2[(l(x))^{-1}v(x_1, l(x)) + v'_y(x_1, l(x))]}{[l(x)]^2}. \end{aligned}$$

Substitute the above equality into the first equality of system (H_n); it is obvious that the first equality holds. After a similar computation, the other equalities are all true.

Sha Huang gave the corresponding results about the above functions in [29]7).

Theorem 4.2 *Suppose we have complex constants* $a = a_1 + ib$*,* $c = c_1 + id$ *and complex variable number* $z = x_1 + iy$*. Then*

1) $\tilde{i} = I(x)$, $\tilde{z} = x = x_1 + \sum_{k=2}^{n} x_k e_k$.

2) $\tilde{a} = a_1 + I(x)b$, *specially*, $\tilde{a} = a$, $\widetilde{a+z} = \tilde{a} + \tilde{z}$, *when* a *is a real number.*

3) $\widetilde{(\frac{1}{z})} = \frac{1}{\tilde{z}}$ $\widetilde{(\frac{a}{z})} = \frac{\tilde{a}}{\tilde{z}}$.

4) $\widetilde{(\frac{1}{a})} = \tilde{\bar{a}}|a|^{-2}$, *here*, $|a| = \sqrt{a_1^2 + b^2}$, $\widetilde{(\frac{z}{a})} = \tilde{z}\widetilde{(\frac{1}{a})}$.

5) $\widetilde{(\frac{z+a}{c-z})} = \frac{\widetilde{z+a}}{\widetilde{c-z}}$.

We can verify by direct computation that all above terms are true. Here, the proof is omitted.

Theorem 4.3 *Suppose that* $\tilde{f} = \sum\limits_{k=1}^{n} u_k e_k$ *is defined in the ball* $\widetilde{B} \subset (\mathbf{R}^n)^+$, *which does not intersect with the real axis in* $\mathbf{R}^n$, *moreover* $\tilde{f}$ *is a solution for system* (H_n) *satisfying* $u_i x_k = u_k x_i,\ i, k = 2, ..., n$. *Then there exists a holomorphic function* $f(z) = u + iv$ *defined in a circular disk* B, *such that* $\tilde{f}(x) = u(x_1, \sqrt{\sum_{k=2}^n x_k^2}) + I(x)v(x_1, \sqrt{\sum_{k=2}^n x_k^2})$, *where* $x \in \widetilde{B} = \left\{x = \sum_{k=1}^n x_k e_k \in \mathbf{R^n} | (x_1 + i\sqrt{\sum_{k=2}^n x_k^2}) \in B\right\}$.

4.2 The Integral Representation of Hyperbolic Harmonic Functions in Real Clifford Analysis

The components $u_1, ..., u_{n-1}$ of (twice continuously differentiable) solution $(u_1, ..., u_n)$ of (H_n) satisfy the hyperbolic version of the Laplace equation i.e. the hyperbolic Laplace equation in mathematics and physics is

$$x_n \Delta u - (n-1)\frac{\partial u}{\partial x_n} = 0, \tag{4.3}$$

where $u : \mathbf{R}^n \to \mathbf{R}$ is a real-valued function with n variables.

Definition 4.1 The twice continuously differentiable solution $u(x)$ of equation (4.3) is called the real hyperbolic harmonic function of n variables.

In [41], H. Leutwiler introduced the definition of hyperbolic harmonic function in real Clifford analysis.

Definition 4.2 Let $\tilde{f} = \sum\limits_{k=1}^{n} u_k(x) e_k : \mathbf{R}^n \to \mathbf{R}^n$ possess twice continuously differentiable derivatives, the components $u_1, u_2, ..., u_{n-1}$ be hyperbolic harmonic, and u_n satisfy the equation

$$x_n^2 \Delta u - (n-1)x_n \frac{\partial u}{\partial x_n} + (n-1)u = 0. \tag{4.4}$$

Then $\widetilde{f}$ is called a hyperbolic harmonic function.

Theorem 4.4 *Let $\widetilde{f} = u_1 + u_2e_2 + \cdots + u_ne_n$ be a twice continuously differentiable function. Then $\widetilde{f}$ is a solution of system (H_n) if and only if $\widetilde{f}$ and the functions*

$$x\widetilde{f}e_k + e_k\widetilde{f}x,\ k = 1, ..., n-1$$

are hyperbolic harmonic in the above sense.

Proof The Clifford numbers $\omega_k = \dfrac{1}{2}(x\widetilde{f}e_k + e_k\widetilde{f}x)$ $(k = 2, ..., n-1)$ are vectors, whose components ω_{ki} $(i = 1, ..., n)$ are given by

$$\begin{cases} \omega_{k1} = -x_1u_k - x_ku_1, \quad k = 2, ..., n-1, \\ \omega_{ki} = -x_iu_k + x_ku_i, \quad i = 2, ..., n,\ k = 2, ..., n-1,\ i \neq k, \\ \omega_{kk} = x_1u_1 - x_2u_2 - \cdots - x_nu_n, \quad k = 2, ..., n-1. \end{cases}$$

In case $k = 1$, i.e. $\omega_1 = \dfrac{1}{2}(x\widetilde{f} + \widetilde{f}x)$, we have

$$\begin{cases} \omega_{11} = x_1u_1 - x_2u_2 - \cdots - x_nu_n, \\ \omega_{1i} = x_iu_1 + x_1u_i, \quad i = 2, ..., n. \end{cases}$$

It is easy to verify that $\widetilde{f}$ and ω_k $(k = 1, ..., n-1)$ are hyperbolic harmonic if and only if $\widetilde{f}$ satisfies system (H_n).

Theorem 4.5 *Suppose that the ball $\widetilde{B}$ with the radius $R > 0$, $\widetilde{B} \subset (\mathbf{R}^n)^+$, (or $\widetilde{B} \subset (\mathbf{R}^n)^-$), $\tilde{f} = \sum\limits_{k=1}^{n} u_k(x)e_x$ is a hyperbolic harmonic function in $\widetilde{B}$ and continuous on the boundary of $\widetilde{B}$, and denote by ω_{ki} $(1 \leq k \leq n-1, 1 \leq i \leq n)$ the components of $\omega_k(x) = (x\tilde{f}e_k + e_k\tilde{f}x)/2$. Let the following four conditions hold : i) $\omega_{ki} = 0\,(2 \leq k \leq n-1,\ 2 \leq i \leq n)$; ii) The other ω_{k1} $(2 \leq k \leq n-1), \omega_{1i}\,(1 \leq i \leq n-1)$ are real-valued hyperbolic harmonic functions; iii) $x_iu_n = x_nu_i\,(2 \leq i \leq n-1)$; iv) ω_{n1} satisfies (4.4). Then, $\tilde{f}(x)$ possesses the integral representation:*

$$\tilde{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(t)\frac{\tilde{t} + x - 2\tilde{a}}{\tilde{t} - x}d\varphi + I(x)v(a), \tag{4.5}$$

where $x \in \widetilde{B}$, t belongs to the boundary of a circular disk B with the center at $a \in (\mathbf{R}^2)^+$ (see Theorem 4.1) and radius R, $(t-a) = Re^{i\varphi}$, $f(z) = u + iv$ is analytic in B, and $\mathrm{Re}f(x) = \Phi(t)$.

Proof According to the conditions i), ii), we see that $\omega_{ki}\,(1 \le k \le n-1, 1 \le i \le n-1)$ are real-valued hyperbolic harmonic. From the conditions i), iv), it follows that $\omega_{kn}(x)\,(1 \le k \le n-1)$ satisfies (4.4). Hence $\omega_k(x)\,(1 \le k \le n-1)$ are hyperbolic harmonic functions. Moreover by Theorem 4.4, we know that $\tilde{f}$ is a solution of system (H_n). It is clear that from the formula (1.7) in [41], when $2 \le k \le n-1,\ 2 \le i \le n$, we have

$$\begin{aligned}\omega_k &= \frac{1}{2}(x\tilde{f}e_k + e_k\tilde{f}x) = x\mathrm{Re}[\tilde{f}e_k] + e_k\mathrm{Re}[\tilde{f}x] - (x, e_k)\overline{\tilde{f}}\\ &= (x_1 + \sum_{i=2}^{n} x_i e_i)(-u_k) + e_k[u_1 x_1 - \sum_{i=2}^{n} u_i x_i] - x_k(u_1 - \sum_{i=2}^{n} u_i e_i)\\ &= -x_1 u_k + \sum_{i=2}^{n}(-1)x_i u_k e_i + [u_1 x_1 - \sum_{i=2}^{n} u_i x_i]e_k - x_k u_1 + \sum_{i=2}^{n} x_k u_i e_i\\ &= -(x_1 + u_k + x_k u_1) + \sum_{\substack{i=2\\(i\ne k)}}^{n}(x_k u_i - x_i u_k)e_i + [u_1 x_1 - \sum_{i=2}^{n} u_i x_i]e_k,\end{aligned}$$

and its components are

$$\omega_{ki} = -x_i u_k + x_k u_i,\ 2 \le k \le n-1,\ 2 \le i \le n, i \ne k.$$

In addition, by the conditions i), iii), we have $x_i u_k = x_k u_i\ (i, k = 2, ..., n)$ when $i = k$, hence the above equality is true. By using Theorem 4.3, there exists $f(z) = u + iv$, which is analytic in the circular disc B : $|z - a| < R, B \subset (\mathbf{R}^2)^+$, such that

$$\tilde{f}(x) = u\left(x_1, \sqrt{\sum_{k=2}^{n} x_k^2}\right) + I(x)v\left(x_1, \sqrt{\sum_{k=2}^{n} x_k^2}\right),$$

where $x \in \widetilde{B} = \{x = \sum_{k=1}^{n} x_k e_k \in \mathbf{R}^n | (x_1 + i\sqrt{\sum_{k=2}^{n} x_k^2}) \in B\}$, and in the following we denote $\widetilde{f} = u(x) + I(x)v(x)$, $[\widetilde{f}(x)]_1 = u(x)$. Finally, according to the Schwarz formula of the holomorphic function:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(t)\frac{t + z - 2a}{t - z}d\varphi + iv(a)(z \in B) \tag{4.6}$$

in which $(t - a) = Re^{i\varphi}, \Phi(t) = \mathrm{Re}f(t)$, and using Theorem 4.2 and (4.6), we obtain

$$\tilde{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(t)\frac{\tilde{t} + x - 2\tilde{a}}{\tilde{t} - x}d\varphi + I(x)v(a),$$

where $x \in \widetilde{B}, t \in B$.

4.3 The Existence and Integral Representation of Solutions for a Kind of Boundary Value Problem

Let $\Phi(t)$ be a continuous function defined on the boundary $\dot{B}$ $(|t-a|=R)$ of the disk $B:|z-a|<R$ in $(\mathbf{R}^2)^+$, and $\widetilde{B}=\{x|x=\sum_{k=1}^n x_k e_k\in\mathbf{R}^n,\ x_1+i\sqrt{\sum_{k=2}^n x_k^2}\in B\}$ and $t=t_1+ih\in\dot{B}$. Denote $h=\sqrt{\sum_{k=2}^n t_k^2}$, $\underline{t}=t_1+\sum_{k=2}^n t_k e_k$, and $\Phi(t)=\Phi(t_1,h)=\Phi(t_1,\sqrt{\sum_{k=2}^n t_k^2})=\Phi(\underline{t})$. In the following, we shall discuss the hyperbolic harmonic function $\tilde{f}:\mathbf{R}^n\to\mathbf{R}^n$ in $\widetilde{B}$, and find a solution of the problem of $Re\widetilde{f}(t_1)=\Phi(\underline{t})$; here $Re\widetilde{f}(x)=u(x_1,\,l(x))$, u is a real number. This problem is called Problem D.

Theorem 4.6 *Problem D for hyperbolic harmonic functions in a high dimension ball $\widetilde{B}$ is solvable.*

Proof On the basis of the existence of solutions of the Dirichlet problem for holomorphic functions, we see that there exists a holomorphic function $f(z)=u+iv$, such that when $z\to t$, $\mathrm{Re}f(z)\to\Phi(t)$, where $t=t_1+ih\in\dot{B}$, $z=x_1+iy\in B$. By Theorem 4.1, we know that

$$\tilde{f}(x)=u\left(x_1,\sqrt{\sum_{k=2}^n x_k^2}\right)+I(x)v\left(x_1,\sqrt{\sum_{k=2}^n x_k^2}\right)$$

is a solution of system (H_n) in $\widetilde{B}$. If we denote the above function as $\tilde{f}(x)=\sum\limits_{k=1}^n u_k v_k$, then $u_i x_k=u_k x_i$, $i,k=2,...,n$. From Theorem 4.4, it is clear that $\tilde{f}(x)$ is hyperbolic harmonic in $\widetilde{B}$. Denote $x=x_1+\sum\limits_{k=2}^n x_k e_k\in\widetilde{B}$, and when $x\to\underline{t}=t_1+\sum\limits_{k=2}^n t_k e_k$, we have $z=x_1+iy=x_1+i\sqrt{\sum_{k=2}^n x_k^2}\to t=t_1+ih=t_1+i\sqrt{\sum_{k=2}^n t_k^2}$, thus $\mathrm{Re}f(z)\to\Phi(t)$, i.e. $u(x_1,y)=u(z)\to\Phi(t)=\Phi(t_1,h)$. Again because $u(x_1,y)=u(x_1,\sqrt{\sum_{k=2}^n x_k^2})=u(x)$, $\Phi(t_1,h)=\Phi(\underline{t}\,)$, we have $Re\widetilde{f}(x)=u(x)\to\Phi(\underline{t})(x\to\underline{t})$, namely $Re\widetilde{f}(\underline{t})=\Phi(\underline{t})$. This shows that $\tilde{f}(x)$ is a solution of Problem D.

Theorem 4.7 *The solution $\tilde{f}(x)$ as in Theorem 4.6 possesses the integral representation (4.5).*

Proof In fact, the hyperbolic harmonic function $\tilde{f}(x)$ in Theorem 4.6 is a solution of Problem D, hence it is also a solution of system (H_n) satisfying (4.7). According to the proof of Theorem 4.5, we know that $\tilde{f}(x)$ possesses the integral representation (4.5).

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