

Chapter 2

Equations of Motion

Here, we consider the equations of motion briefly, primarily for the sake of completeness, and to establish a consistent notation. Full derivations can be found in a number of places, notably in Batchelor (1967) and Currie (1974).

We consider only flows of compressible gases where the continuum hypothesis is valid. That is, the smallest volume of interest (a fluid element) will always contain a sufficient number of molecules for statistical averages to be meaningful. The continuum hypothesis fails when the mean free path is comparable to the smallest significant dimension. This can happen under normal temperatures and pressures when the body dimensions are very small, as for the flow around a very thin wire, such as the sensitive element of a hot-wire probe where typical diameters range from 1 to 5 μm , or at very high altitudes, where the densities are very small and the mean free path can be very large. The relevant nondimensional quantity is the Knudsen number, which is the ratio of the mean free path λ to a typical flow dimension. When the Knudsen number exceeds unity we expect rarified gas effects to become important. For a hot-wire in a supersonic flow, the Knudsen number for the wire filament is about 0.2 at normal temperatures and pressures, and for the region occupied by the shock wave it approaches unity. It is difficult to justify the continuum hypothesis within the shock because for shocks of moderate strength the thickness is equal to a few mean free paths in the downstream gas. Nevertheless, the equations of motion for a continuum gas predict shock structures that agree well with experiment. For the flow over the filament itself, however, slip-flow effects associated with a breakdown in the continuum hypothesis have been observed when the Knudsen number based on the filament diameter is of order 0.1 (Davis and Davies, 1972). For laminar boundary layer flows over a vehicle, the ratio of the mean free path to the boundary layer thickness δ can be expressed in terms of the Mach and Reynolds numbers:

$$\frac{\lambda}{\delta} \simeq \frac{M}{\sqrt{Re}}$$

(Kuethe and Schetzer, 1967), indicating that the continuum approximation

will break down at high Mach numbers and low Reynolds numbers, conditions that are typically experienced only in the upper reaches of the atmosphere.

When Knudsen number effects are negligible, and the flow behavior can be described in term of its macroscopic properties such as its pressure, density, and velocity, the flow of a compressible fluid is described completely by the continuity, momentum, and energy equations, together with an equation of state and a suitable set of boundary conditions.

2.1 Continuity

For a compressible fluid the continuity equation is given by:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \quad (2.1)$$

or

$$\nabla \cdot \mathbf{V} = -\frac{1}{\rho} \frac{D\rho}{Dt}. \quad (2.2)$$

Here $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is the velocity and ρ is the density. The total derivative D/Dt represents the rate of change of density following a particle of fluid in an Eulerian system. A fluid particle has a fixed mass m , and therefore a variable volume. In terms of the volume v ($= m/\rho$),

$$-\frac{1}{\rho} \frac{D\rho}{Dt} = -\frac{v}{m} \frac{D(v/m)}{Dt} = \frac{1}{v} \frac{Dv}{Dt}, \quad (2.3)$$

and so the divergence of the velocity is equal to the fractional rate of change of volume of a fluid element of given mass. This is called the rate of dilatation, or simply the dilatation. In tensor notation we have:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \quad (2.4)$$

2.2 Momentum

The momentum equation for a compressible fluid is given by:

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j}, \quad (2.5)$$

where τ_{ij} is the (symmetric) stress tensor representing the effects of surface forces (buoyancy forces as well as other body forces have been neglected because they are rarely important in high-speed flows). The three diagonal components of τ_{ij} are the normal stresses and the six off-diagonal components are the tangential or shearing stresses. Note that the surface stresses will only

contribute to the fluid acceleration when their gradients are nonzero. When $\partial\tau_{ij}/\partial x_i = 0$, the effect of surface forces on a material element of fluid is to tend to deform it without changing its momentum.

The stress tensor in a fluid at rest (or in rigid body motion) must be isotropic (see, for example, Batchelor (1967)), and the contribution to the force per unit area exerted on the surface of a control volume with a unit normal vector \mathbf{n} is $\frac{1}{3}\tau_{ij}n_j$. That is, $\tau_{ij} = -p\delta_{ij}$, where $p (= -\frac{1}{3}\tau_{ii})$ is called the static-fluid pressure and it acts equally in all directions. The static-fluid pressure is the same as the thermodynamic pressure, which is the pressure measured in a system in equilibrium, and it is a state variable. A negative sign is included because the outward-facing normal of any control volume is defined as a positive vector and by convention a positive pressure is taken to be compressive. The stress tensor in a fluid at rest must therefore be of the form

$$\tau_{ij} = -p\delta_{ij} + d_{ij}, \quad (2.6)$$

where d_{ij} depends on the motion of the fluid only and it is called the shear stress tensor, or the deviatoric stress tensor.

The instantaneous relative velocities in the neighborhood of a point in the fluid are given by the velocity gradient tensor $\partial u_i/\partial x_j$. This tensor describes the motion of all neighboring points. In other words, $\partial u_i/\partial x_j$ describes how an infinitesimal fluid particle deforms in time, and it provides a complete description of the relative state of motion at that point. The velocity gradient tensor can be split into two parts, a rate-of-strain tensor S_{ij} and a rate-of-rotation tensor R_{ij} :

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + R_{ij}, \quad (2.7)$$

where

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{and} \quad R_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \quad (2.8)$$

We see that S_{ij} describes the deformation of a fluid element, whereas R_{ij} describes the instantaneous rate-of-rotation of the fluid element if it were to rotate as a rigid body without deformation. Note that the instantaneous angular velocity is given by ω_k where $R_{ij} = \varepsilon_{ijk}\omega_k$. The vector $\vec{\omega}$ is related to the vorticity $\mathbf{\Omega}$ by $\vec{\omega} = \frac{1}{2}\nabla \times \mathbf{V} = \frac{1}{2}\mathbf{\Omega}$.

The rate-of-strain tensor is the only part of the velocity gradient tensor governed by relative motions among fluid particles. The simplest stress-strain rate relationship is a linear one, and fluids that follow this relationship are called Newtonian. For isotropic Newtonian fluids, the shear stress tensor becomes

$$d_{ij} = \lambda S_{kk}\delta_{ij} + 2\mu S_{ij} \quad (2.9)$$

(see, for example, Currie (1974)), where μ and λ are constants: μ is the dynamic viscosity, and λ is usually written in terms of the bulk viscosity

μ'' , where $\mu'' = \lambda + \frac{2}{3}\mu$. The viscosity coefficients relate the shear stress tensor to the rate-of-strain tensor. They are material properties of a fluid, and may be directly related to the molecular interactions that occur inside the fluid. They may therefore be considered as thermodynamic properties in the macroscopic sense, varying with pressure and temperature. They are proper scalars, independent of direction. Hence:

$$\tau_{ij} = -p\delta_{ij} + \mu''S_{kk}\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right). \quad (2.10)$$

This is the constitutive equation for stress in a Newtonian fluid. The linear dependence holds over a surprising variety of conditions for compressible and incompressible fluids, and it is assumed to describe all the flows considered here.

In a fluid at rest only normal stresses are exerted, and the normal stress is independent of the direction of the surface element across which it acts. We see that in a fluid in motion both the normal and the tangential stresses depend on viscosity.

The dynamic viscosity of a gas is the result of momentum exchange among molecules with the same average molecular velocities but with different bulk velocities. Because the interactions will occur within a distance comparable to the mean free path, the dynamic viscosity must depend on the average molecular speed, the number density and the mean free path. The magnitude is very small but the associated stress μS_{ij} can take large values, especially near a solid wall where the velocity gradients are large. Also, as the temperature of the gas increases, the number of collisions will increase and therefore the dynamic viscosity will increase with temperature. It is nearly independent of pressure. The variation with temperature, between 150°K to 500°K, may be approximated by the formula

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{0.76}. \quad (2.11)$$

For a wider range of temperatures, between 100°K and 1900°K, Sutherland's formula is more accurate:

$$\frac{\mu}{\mu_0} = \frac{T_0 + 110.3}{T + 110.3} \left(\frac{T}{T_0}\right)^{3/2} \quad (2.12)$$

(Ames Research Staff, 1953).

With respect to the bulk viscosity, we can introduce the *mechanical* pressure \bar{p} which is the pressure measured in a moving fluid. That is,

$$p - \bar{p} = \mu'' \frac{\partial u_k}{\partial x_k} = \mu'' \nabla \cdot \mathbf{V}.$$

Now $\bar{p} = -\frac{1}{3}\tau_{ii}$, which defines the mechanical pressure in a moving fluid (naturally, for a fluid at rest, $\bar{p} = p$). The mechanical pressure in a moving fluid

differs from the thermodynamic pressure by a term proportional to the volumetric dilatation with a coefficient of proportionality called the bulk viscosity. Note that for an incompressible fluid we need not consider the bulk viscosity—it only becomes important for compressible fluid flow.

The pressure \bar{p} is a measure of the *translational* energy of the molecules only, whereas the thermodynamic pressure is a measure of the *total* energy, which includes vibrational and rotational modes as well as the translational modes. Different modes have different relaxation times, so that energy may be transferred between modes, and the bulk viscosity is a measure of this transfer. For a monatomic gas the only mode of molecular energy is translational and the bulk viscosity is always zero. For polyatomic gases the bulk viscosity is never zero, and in fact it may be orders of magnitude larger than the dynamic viscosity (see, for example, Thompson (1972)). For instance, during the passage of a polyatomic gas through a shock wave, the vibrational modes of energy of the molecules are excited at the expense of the translational modes, so that the bulk viscosity of the gas will be nonzero. However, the stress associated with the bulk viscosity is $\mu''\nabla \cdot \mathbf{V}$, and in many cases of interest the stress is small enough to be neglected (this is usually called Stokes's hypothesis). Here, we assume that $\mu''\nabla \cdot \mathbf{V}$ is negligible.

The final form of the momentum equation, known as the Navier-Stokes equation, is given by:

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \right). \quad (2.13)$$

We now have two equations, the continuity and momentum equations—Equations 2.5 and 2.13, respectively—with three unknowns: p , u_i , and ρ (it is assumed that μ is a known function of the state variables).

For a flow in thermodynamic equilibrium, an extra equation is provided by the equation of state. For the flows considered here it is assumed that the pressure, temperature and density are related to each other by the ideal gas law; that is,

$$p = \rho RT, \quad (2.14)$$

where R is the gas constant ($= 287.03 \text{ m}^2\text{s}^2\text{K}$ for air). We are interested in systems where velocity, temperature, and pressure gradients are present, systems that may not be in perfect equilibrium. When the rates of change are large, the flow does not have time to achieve local equilibrium, and then the equations of state will contain time as a variable. Lagging internal processes such as dissociation, ionization, evaporation, chemical reactions, and the transfer of energy between molecular modes are called relaxation processes. We have already indicated how the transfer of energy among molecular modes can give rise to a nonzero bulk viscosity coefficient. Nevertheless, variables of state can still be used if the rates of change are not too large. If we restrict our attention to the flow of nonhypersonic continuum gases, and we do not consider the flow

inside shock waves, the experimental evidence suggests that the assumption of thermodynamic equilibrium appears to be reasonable.

To close this system of equations we need to consider the energy of the system.

2.3 Energy

The energy equation for a compressible fluid may be written as:

$$\rho \frac{\partial}{\partial t} \left(e + \frac{1}{2} V^2 \right) + \rho u_j \frac{\partial}{\partial x_j} \left(e + \frac{1}{2} V^2 \right) = \frac{\partial \tau_{ij} u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}, \quad (2.15)$$

where $V^2 = u^2 + v^2 + w^2$. The left-hand side represents the rate of change of the total energy per unit volume following a fluid particle, and the right-hand side is the sum of the rate of work done on the fluid by the surface forces and the rate of heat added to the fluid by conduction. The work done by the body forces is neglected because buoyancy effects are rarely important in high-speed flows, and temperature differences are assumed to be small enough to neglect radiative heat transfer.

The total energy is the sum of the internal energy e and the kinetic energy of the bulk motion $\frac{1}{2} V^2$. The internal energy is a measure of the energy contained in the translational, rotational and vibrational motions of the gas molecules (as well as electron energies), and it is a state variable. It is Galilean invariant (that is, it is independent of translational motion of the observer) whereas the kinetic energy of the bulk motion is not. (Note that any term involving the absolute value of the velocity will depend on the observer, whereas terms that depend on velocity differences will not.)

The rate of work done by the surface forces may be written as:

$$\frac{\partial \tau_{ij} u_i}{\partial x_j} = u_i \frac{\partial \tau_{ij}}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j}. \quad (2.16)$$

The first part on the right-hand side arises from small differences in stresses on opposite sides of a fluid element, and it contributes to the gain in kinetic energy of the bulk motion of the fluid. The second part is associated with small differences in velocity on opposite sides of the fluid element, and it represents the rate of work done in deforming the element without a change in its bulk velocity. The work done in deforming the element is the only work term that contributes to an increase in the internal energy.

By convention, \mathbf{q} is the conductive heat flux leaving the fluid element (the rate of heat outflow per unit area). Heat conduction contributes to the change in internal energy of the fluid, and makes no contribution to the kinetic energy of the bulk motion. It is assumed that the heat flux is given by the Fourier

heat conduction relationship:

$$q_j = -k \frac{\partial T}{\partial x_j}. \quad (2.17)$$

This is a constitutive equation in that it relates a heat flux to a temperature gradient. A single coefficient of proportionality, the heat conductivity k , is used, which is a property of the fluid and depends on the temperature and pressure.

For many applications it is more useful to express the energy equation in terms of the enthalpy h , where $h = e + p/\rho$, or the stagnation enthalpy, or the total enthalpy h_0 , where

$$h_0 = h + \frac{1}{2}V^2. \quad (2.18)$$

For constant pressure flows, the enthalpy and total enthalpy have conservation properties that make them attractive variables to use. By using the continuity equation, we can write an equation for the total enthalpy:

$$\rho \frac{\partial h_0}{\partial t} + \rho u_j \frac{\partial h_0}{\partial x_j} = \frac{\partial p}{\partial t} - \frac{\partial q_i}{\partial x_i} + \frac{\partial d_{ij} u_i}{\partial x_j}. \quad (2.19)$$

We see that the total enthalpy can change only through the actions of heat transfer, unsteady pressure variations, or diffusion of kinetic energy by viscosity. Hence the total enthalpy is constant for the flow of a frictionless non-conducting fluid with a steady pressure distribution. Even for a turbulent boundary layer in the steady flow of air, it is a matter of observation that the total enthalpy is nearly constant if there is no heat transfer.

For one-dimensional steady flow, Equations 2.18 and 2.19 are equivalent, and 2.18 is usually called the one-dimensional energy equation. Under these conditions, Equation 2.19 can be integrated between two positions along a streamline, and the total enthalpy will have the same value at these two points if the gradients of velocity and temperature vanish at these locations (see, for example, Batchelor (1967)). This result is particularly useful when considering shock waves or other discontinuities in the flow. The total enthalpy within the shock can vary along a streamline but its value will be the same at all points that lie outside the shock. In other words, the increase and decrease of h_0 along the streamline due to viscous forces and heat conduction exactly cancel over the particular range covered by the shock. Of course, the entropy always increases.

An equation for the enthalpy may be obtained by using the momentum equation to simplify Equation 2.19. Hence,

$$\rho \frac{\partial h}{\partial t} + \rho u_j \frac{\partial h}{\partial x_j} = \frac{\partial p}{\partial t} + u_j \frac{\partial p}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + d_{ij} \frac{\partial u_i}{\partial x_j}. \quad (2.20)$$

The last term on the right-hand side involves the viscous stresses and it is usually called the dissipation function Φ (= dissipation rate per unit volume).

The dissipation function represents that part of the viscous work going into deformation, rather than acceleration of the fluid particle. If we assume Stokes's hypothesis,

$$\Phi = d_{ij} \frac{\partial u_j}{\partial x_i} = \frac{1}{2} \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2, \quad (2.21)$$

and we see that Φ is always positive.

To write the energy equation in terms of temperature, it is usually assumed that the gas is a perfect gas with constant specific heats. That is, $h = C_p T$ and $h_0 = C_p T_0$, where T_0 is called the *stagnation* or *total temperature*. Then,

$$\rho C_p \frac{DT}{Dt} = \frac{Dp}{Dt} - \frac{\partial q_i}{\partial x_i} + \Phi, \quad (2.22)$$

which shows that the temperature of a fluid particle can change by variations in pressure in the direction of the flow, by heat transfer, and by dissipation of kinetic energy by viscosity and thermal conduction.

The energy equation can also be written in terms of entropy. Entropy is another useful parameter primarily because it may also be taken as constant in many flows. By writing the enthalpy in terms of the entropy,

$$T ds = dh - \frac{1}{\rho} dp,$$

we obtain

$$\rho T \frac{Ds}{Dt} = - \frac{\partial q_i}{\partial x_i} + \Phi. \quad (2.23)$$

That is,

$$\frac{Ds}{Dt} = \frac{\Phi}{\rho T} + \frac{1}{\rho T} \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{\partial x_j} \right). \quad (2.24)$$

The heat conduction term can be written as the sum of a flux term (that is, a divergence of a temperature gradient), and a term that is quadratic in temperature gradients (positive definite). That is,

$$\frac{Ds}{Dt} - \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(k \frac{\partial \ln T}{\partial x_j} \right) = \frac{\Phi}{\rho T} + \frac{k}{\rho T^2} \left(\frac{\partial T}{\partial x_j} \right)^2. \quad (2.25)$$

The left-hand side is the sum of a rate of change of entropy per unit mass, plus the net outflow of entropy per unit mass. The right-hand side is zero for a reversible process, and positive for an irreversible process, so that it represents the rate of entropy production per unit mass. It depends on the square of velocity and temperature gradients, and therefore, if the gradients are small, the entropy is changed only by heat conduction.

When there is no heat conduction, the flow is called *adiabatic*. Sometimes, in considering the flow in and out of a control volume, the flow is called *adiabatic* if there is no heat conduction across the boundaries of the control volume.

When there is no heat conduction and the entropy production (called dissipation) is also negligible, then the flow is *isentropic*. A flow is called *homentropic* if this is true for all particles. For an isentropic flow of a simple system (such as a perfect gas), any state variable can be expressed as a unique function of any one other state variable. Isentropic flows are therefore *barotropic*, that is, density is only a function of pressure. Furthermore, $dh = dp/\rho$, and p varies as ρ^γ (for a perfect gas with constant specific heats).

2.4 Summary

The equations of motion for a compressible, viscous, heat-conducting fluid are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0, \quad (2.26)$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial d_{ij}}{\partial x_j}, \quad (2.27)$$

$$\rho \frac{\partial h}{\partial t} + \rho u_j \frac{\partial h}{\partial x_j} = \frac{Dp}{Dt} - \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) + d_{ij} \frac{\partial u_i}{\partial x_j}, \quad (2.28)$$

which are, respectively, the continuity, momentum, and energy equations. The deviatoric stress tensor d_{ij} is given by

$$d_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right). \quad (2.29)$$

With the addition of an equation of state and the known variation of viscosity and thermal conductivity with temperature and pressure, this is a complete set of equations describing compressible fluid flow, provided that: the continuum hypothesis holds, the fluid particles are in local thermodynamic equilibrium, body forces can be neglected, the fluid is Newtonian, heat conduction follows Fourier's law, and radiative heat transfer can be neglected.

The general formulation of boundary conditions in compressible flow is a specialized topic that is not addressed here, and we only consider the specific conditions introduced by the presence of a wall. The no-slip condition means that the fluid in contact with a solid surface has no relative velocity with respect to the surface. That is, at the wall

$$\mathbf{V} = \mathbf{V}_w. \quad (2.30)$$

When there is heat transfer from the surface, there is an analogue to the no-slip condition in that the temperature of the fluid in contact with the solid surface has the same temperature as the surface itself. That is,

$$T = T_{surface}. \quad (2.31)$$

Alternatively, the heat flux must be specified:

$$k \frac{\partial T}{\partial n} = q_w, \quad (2.32)$$

where q_w is the heat flux from the surface to the fluid and n is the direction normal to the surface.

The no-slip condition breaks down at scales comparable to the mean free path of the molecules, and therefore the assumption of continuum flow generally implies that the no-slip condition holds. Under certain conditions, the no-slip condition may break down, even if the fluid still behaves as a continuum. These conditions are usually only found at high Mach numbers, where $U_{slip}/U_\infty \approx MC_f$. A further discussion of the no-slip condition is given by White (1991), and a fascinating historical note may be found in Goldstein (1938), p. 676.

2.5 Compressible Couette Flow

To illustrate the use of the equations of motion, and to introduce some of the concepts encountered in compressible viscous flows, we consider compressible, laminar Couette flow in some detail. The flow configuration is shown in Figure 2.1. Air flows between two parallel plates, where the upper plate moves at a constant velocity u_e relative to the lower plate. By the no-slip condition, the velocity of the air in contact with the upper plate is also u_e relative to the air in contact with the lower plate. The other boundary conditions are as shown. The temperature of the upper and lower plates are T_e and T_w , respectively, and there is a heat flux q_w from the lower plate into the flow and a shear stress τ_w also acts on the lower plate. The flow is two-dimensional so that $\partial/\partial z = 0$; it is fully-developed so that $\partial/\partial x = 0$, and it is steady so that $\partial/\partial t = 0$. We assume that the gas has constant specific heats.

The continuity equation gives:

$$\frac{\partial \rho v}{\partial y} = 0, \quad (2.33)$$

where $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Hence, $v = 0$ by symmetry, and the flow is parallel.

The momentum equation gives:

$$\frac{\partial \tau_{12}}{\partial y} = 0; \quad \text{that is,} \quad \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) = 0. \quad (2.34)$$

Therefore the shear stresses and pressure are constant throughout the flow-field. If the flow were incompressible, the viscosity would be constant and Equation 2.34 could be integrated to show that the velocity profile is linear.

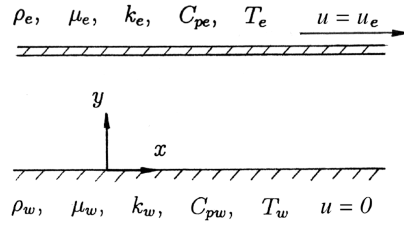


Figure 2.1. Notation for compressible Couette flow.

When the flow is compressible, variations in temperature will produce variations in viscosity, and we need to solve for the temperature distribution before Equation 2.34 can be integrated.

The energy equation gives:

$$\frac{\partial}{\partial y} \left(\mu u \frac{\partial u}{\partial y} + k \frac{\partial T}{\partial y} \right) = 0. \quad (2.35)$$

That is, the diffusion of kinetic energy and heat are in equilibrium. If we write the energy equation in terms of the enthalpy, we find:

$$\Phi - \frac{\partial q_i}{\partial x_i} = 0. \quad (2.36)$$

That is, the dissipation and the heat flux are in equilibrium. In fact, the two relations give the same result because:

$$\frac{\partial u_i \tau_{ij}}{\partial x_j} = \Phi + u_i \frac{\partial \tau_{ij}}{\partial x_j}$$

and the viscous stress is constant. It follows that in this fully developed flow the temperature distribution depends only on how the kinetic energy is dissipated and how the heat is diffused.

Integrating Equation 2.35 from the lower wall outwards gives:

$$\mu u \frac{\partial u}{\partial y} + \frac{\mu C_p}{P} \frac{\partial T}{\partial y} = -q_w.$$

Here we have introduced the Prandtl number $P = \mu C_p / k$. By using the fact that the stress is constant and by assuming that the Prandtl number is constant (this is a good approximation for many gases) we find that:

$$\frac{\partial}{\partial y} \left(C_p T + \frac{1}{2} P u^2 \right) = -P \frac{q_w}{\tau_w} \frac{\partial u}{\partial y}.$$

By further integration:

$$C_p(T - T_w) + \frac{1}{2}Pu^2 = -P\frac{q_w}{\tau_w}u, \quad (2.37)$$

and using the boundary conditions at the upper wall, we obtain:

$$T_w = T_e + \frac{P}{C_p} \left(\frac{1}{2}u_e^2 + \frac{q_w}{\tau_w}u_e \right). \quad (2.38)$$

Consider the case where the lower wall is adiabatic ($q_w = 0$). Here,

$$T_w = T_r = T_e + \frac{1}{2}\frac{Pu_e^2}{C_p} = T_e \left(1 + P\frac{\gamma - 1}{2}M_e^2 \right),$$

where M_e is the Mach number based on the velocity and temperature of the upper wall, and T_r is called the *recovery temperature* or *adiabatic wall temperature*. Because the velocity of the air in contact with the lower wall is zero, T_w is equal to the total temperature at that point. The total temperature at the upper wall is $T_{0e} = T_e + \frac{1}{2}u_e^2/C_p$. We see that if the Prandtl number is not equal to one, the temperature of the stationary wall is not equal to the total temperature at the upper boundary: the kinetic energy is not exactly “recovered” in the form of heat at the lower boundary. The ratio of the recovered energy to the external energy is called the recovery factor r , where

$$r = \frac{T_r - T_e}{T_{0e} - T_e}. \quad (2.39)$$

In the case of laminar Couette flow, the recovery factor is simply equal to the Prandtl number. Note that when the Prandtl number is less than one (for air $P = 0.72$ at NTP), the temperature of the stationary wall is lower than the total temperature evaluated at the moving boundary.

Equation 2.38 gives the solution for the heat transfer from the lower wall:

$$q_w = \frac{\tau_w C_p}{Pu_e} (T_w - T_r), \quad (2.40)$$

which shows that the heat transfer is proportional to the temperature difference $T_w - T_r$, and not $T_w - T_e$ as in low-speed flows. Also, we see that the heat transfer is proportional to the wall shear. That is, for the lower wall,

$$C_h = \frac{1}{2P}C_f, \quad (2.41)$$

where

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} \quad \text{and} \quad C_h = \frac{q_w}{\rho_e u_e C_p (T_w - T_r)}.$$

C_f is the skin-friction coefficient and C_h is the heat-transfer coefficient, or *Stanton number*. Relationships such as Equation 2.41 which connect the heat-transfer and skin-friction coefficients are called Reynolds analogues, and the ratio $s = 2C_h/C_f$ is called the Reynolds Analogy Factor.

We can find the skin-friction coefficient by integrating the momentum equation (Equation 2.34). If we assume that the viscosity is proportional to the temperature (the actual variation is given by Equations 2.11 and 2.12), we obtain:

$$C_f = \frac{1}{R_e} \left(1 + \frac{T_w}{T_e} + \frac{2}{3} P (\gamma - 1) M^2 \right). \quad (2.42)$$

For compressible Couette flow, the skin-friction coefficient increases with heating and Mach number (this is not necessarily true for other wall-bounded flows such as boundary layer flows).

In addition, we obtain from Equations 2.37 and 2.38:

$$\frac{T}{T_e} = \frac{T_w}{T_e} + \frac{T_r - T_w}{T_e} \left(\frac{u}{u_e} \right) - \frac{\gamma - 1}{2} P M_e^2 \left(\frac{u}{u_e} \right)^2. \quad (2.43)$$

This temperature-velocity relationship, derived for compressible Couette flow, is often used in the analysis of laminar and turbulent wall-bounded flows by replacing the Prandtl number in Equation 2.43 with the recovery factor appropriate to those flows. In that form, it is often called the Crocco relation or Crocco's law (see Chapter 5 for further details).

One conclusion we can draw from this example is that in the absence of pressure gradients the kinetic energy of a supersonic flow is of such magnitude that when it is transformed into heat and redistributed in space by molecular diffusion significant temperature variations can occur. This can also be understood from the definition of Mach number: $M = u/\sqrt{\gamma RT}$. Here the numerator (squared) is proportional to the kinetic energy and the denominator (squared) is proportional to the internal energy. When the Mach number is of order unity the two forms of energy are of the same order of magnitude, and changes in the kinetic energy will result in changes in the temperature and its distribution in space.

2.6 Vorticity

Equations 2.26 to 2.28 represent a complete set of equations describing compressible fluid flow. However, the concept of vorticity can be very useful for gaining further insight, especially in turbulent flows where rotational fluid is often distributed in space in rather compact forms such as vortex tubes and sheets.

For a compressible fluid, the vorticity transport equation is written in terms of the specific vorticity $\mathbf{\Omega}/\rho$. By taking the curl of the momentum equation,

we obtain:

$$\begin{aligned} \frac{\partial (\Omega_k/\rho)}{\partial t} + u_j \frac{\partial (\Omega_k/\rho)}{\partial x_j} &= \frac{\Omega_j}{\rho} \frac{\partial u_k}{\partial x_j} + \frac{1}{\rho^3} \nabla \rho \times \nabla p \\ &+ \frac{1}{\rho} \varepsilon_{ijk} \frac{\partial^2}{\partial x_i \partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right). \end{aligned} \quad (2.44)$$

The first term on the right-hand side of Equation 2.44 is sometimes called the *stretching and tilting* term: part of this term describes the exchange of vorticity between components because of the rotation of vortex lines due to presence of velocity gradients; another part gives the rate of change of vorticity due to stretching of vortex lines. The third term on the right-hand side represents the rate of change of vorticity due to molecular diffusion. The second term is the baroclinic term and it is of particular interest here because it can be an important source term for vorticity in compressible flows. We see from the baroclinic term that in a compressible flow the pressure still appears in the vorticity transport equation. If the density is only a function of pressure, pressure and density gradients are always parallel, and this term does not contribute to the vorticity transport. Under these conditions, the pressure force acting on a fluid element will pass through the center of gravity of the fluid element and no moment results (see Figure 2.2). The flow is then called *barotropic* (for example, isentropic flow of a perfect gas). If the pressure gradients are not parallel to the density gradients, there is a contribution to the rate of change of vorticity: the pressure force does not pass through the center of gravity, a moment about the center of gravity exists, and the rotation vector will change with time. The flow is called *nonbarotropic*.

When the flow is inviscid, barotropic, and two-dimensional, the right-hand side is zero, and the specific vorticity remains constant following a fluid particle: for example, if the flow was initially irrotational, it remains irrotational. So vorticity can change by stretching and tilting, barotropic torques, and viscous diffusion.

A useful result for inviscid compressible flows may be obtained directly from the momentum equation:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla p. \quad (2.45)$$

That is,

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla \frac{1}{2} V^2 - \mathbf{V} \times \boldsymbol{\Omega} = -\frac{1}{\rho} \nabla p. \quad (2.46)$$

When the pressure is eliminated in terms of the entropy and enthalpy, we obtain Crocco's equation for inviscid flows with no body forces:

$$\mathbf{V} \times \boldsymbol{\Omega} + T \nabla s = \nabla \left(h + \frac{1}{2} V^2 \right) + \frac{\partial \mathbf{V}}{\partial t}. \quad (2.47)$$

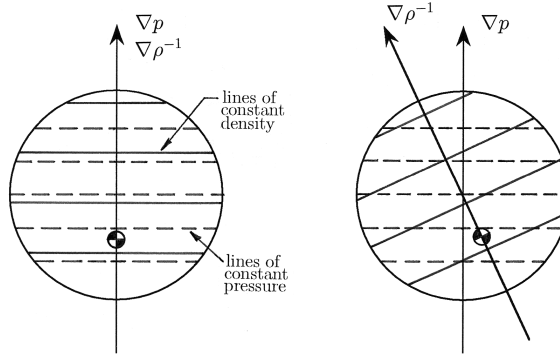


Figure 2.2. Generation of vorticity through baroclinic torques.

If the flow is steady and adiabatic,

$$\mathbf{V} \times \boldsymbol{\Omega} + T \nabla s = \nabla h_0 \quad (2.48)$$

and the total enthalpy will be constant along each streamline. Hence ∇h_0 is a vector that is perpendicular to the streamlines. Because $\mathbf{V} \times \boldsymbol{\Omega}$ is also perpendicular to the streamlines, the remaining vector $T \nabla s$ is also. Equation 2.48 can then be written in scalar form:

$$V \Omega + T \frac{ds}{dn} = \frac{dh_0}{dn}. \quad (2.49)$$

where $V = |\mathbf{V}|$, $\Omega = |\boldsymbol{\Omega}|$, and n is the coordinate perpendicular to the streamlines. Crocco's equation shows that when the total enthalpy is constant everywhere, the vorticity in a homentropic flow must be zero. More interestingly, as a flow passes through a curved shock, the total enthalpy remains constant but the entropy will increase by an amount that depends on the local strength of the shock. Therefore an initially irrotational flow will acquire vorticity proportional to the entropy gradient generated by the curved shock. This is another example of the production of vorticity by baroclinic torques: the term $T \nabla s$ in Equation 2.48 corresponds to the baroclinic term in the vorticity transport equation.

There are a number of other useful properties of vorticity and circulation in inviscid compressible flow. For example, the circulation Γ is defined as:

$$\Gamma = \oint \mathbf{V} \cdot d\boldsymbol{\ell}, \quad (2.50)$$

where counterclockwise contour integration is taken to be positive. That is, the circulation contained within a closed contour in a body of fluid is defined as the integral around the contour of the velocity vector which is locally tangential

to the contour itself. In the absence of shocks and other discontinuities, we can use Stokes's theorem, so that:

$$\Gamma = \iint \boldsymbol{\Omega} \cdot \mathbf{n} \, dS. \quad (2.51)$$

So the circulation can also be expressed as an area integral of the vorticity component normal to the surface enclosed by the contour, as it is in incompressible flow.

From the definition of vorticity, we have as a vector identity

$$\nabla \cdot \boldsymbol{\Omega} = \nabla \cdot \nabla \times \mathbf{V} = 0. \quad (2.52)$$

That is, there are no sources or sinks of vorticity inside the fluid, and vorticity can only be generated at the boundaries of the fluid. Therefore vortex tubes must close on themselves or terminate on the boundaries, which may be either a solid surface, or in a compressible flow, a shock wave (because it can be a source of vorticity). Furthermore, in the absence of sources or sinks of vorticity:

$$\iiint \nabla \cdot \boldsymbol{\Omega} \, dv = \iint \mathbf{n} \cdot \boldsymbol{\Omega} \, dS. \quad (2.53)$$

A vortex tube is a tube defined by vortex lines (lines which are tangent to the local vorticity vector). There is no flux of vorticity normal to the surface of the vortex tube, and Equation 2.53 shows that the flux of vorticity along the vortex tube is constant, that is, the circulation around a vortex tube in an inviscid flow is constant. This result, known as Helmholtz's theorem, will hold for compressible and incompressible flows because it is based on the vector identity given in Equation 2.52. Because the circulation remains constant along the length of the vortex tube, the vorticity will increase if the area decreases. What about the rate of change of the circulation as the vortex tube moves? Consider the circulation about a fluid circuit, that is, a circuit defined by fluid particles of fixed identity. We need to find $D\Gamma/Dt$, where the symbol D/Dt is understood to be the rate of change following a fluid circuit. For an inviscid fluid, the momentum equation can be used to show that:

$$\frac{D\Gamma}{Dt} = - \oint \frac{\delta p}{\rho}, \quad (2.54)$$

where δp is the pressure difference between adjacent fluid particles on the closed contour (see Batchelor (1967)). If the fluid is also barotropic (so that the pressure is only a function of density), the right-hand side of Equation 2.54 is zero, and the circulation around any closed material curve is invariant. This is Kelvin's theorem, and it will apply to compressible and incompressible flows as long as the flow is barotropic. Using the definition of circulation, we can also write

$$\frac{D}{Dt} \left(\iint \mathbf{n} \cdot \boldsymbol{\Omega} \, dS \right) = 0.$$

Thus the flux of vorticity across an open material surface is invariant.

For an inviscid barotropic flow we can therefore make the following observations: If at some instant the flow is without vorticity (such that the circulation around any contour is zero), Kelvin's theorem states that the circulation must remain without vorticity at any future time. In particular, a fluid initially at rest has no vorticity, and must remain without vorticity; vortex tubes are in some sense permanent: a vortex tube specified at an initial instant by the vortex lines that intersect a given closed curve in the fluid has a continuing identity. We may say that in a barotropic inviscid fluid a vortex tube moves with the fluid and its strength remains constant. Because the fluid making up the vortex tube retains its identity as the vortex tube moves and deforms, we must conserve mass. In an incompressible flow, therefore, stretching a vortex tube will inevitably increase its vorticity because the cross-sectional area will always decrease. In a compressible flow, we need to know how the density varies before we can say what is happening to the vorticity: it was clear from the vorticity transport equation that we need to consider the specific vorticity.

Finally, it is important to understand how the velocity field induced by a vortex is affected by compressibility, that is, under what conditions does the usual Biot-Savart law for incompressible flow break down. This requires a consideration of turbulence Mach numbers and fluctuating divergence, and therefore the discussion is deferred until Chapter 4.

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