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Deterministic and Stochastic Differential Games

In game theory, strategic behavior and decision making are modeled in terms of the characteristics of players, the objective or payoff function of each individual, the actions open to each player throughout the game, the order of such actions, and the information available at each stage of play. Optimal decisions are then determined under different assumptions regarding the availability and transmission of information, and the opportunities and possibilities for individuals to communicate, negotiate, collude, make threats, offer inducements, and enter into agreements which are binding or enforceable to varying degrees and at varying costs. Significant contributions to general game theory include von Neumann and Morgenstern (1944), Nash (1950, 1953), Vorob'ev (1972), Shapley (1953) and Shubik (1959a, 1959b). Dynamic optimization techniques are essential in the derivation of solutions to differential games.

2.1 Dynamic Optimization Techniques

Consider the dynamic optimization problem in which the single decision-maker:

$$\max_u \left\{ \int_{t_0}^{T} g\left[s, x\left(s\right), u\left(s\right)\right] ds + q\left(x\left(T\right)\right) \right\}, \tag{2.1}$$

subject to the vector-valued differential equation:

$$\dot{x}\left(s\right) = f\left[s, x\left(s\right), u\left(s\right)\right] ds, \quad x\left(t_0\right) = x_0, \tag{2.2}$$

where $x\left(s\right) \in X \subset R^m$ denotes the state variables of game, and $u \in U$ is the control.

The functions $f\left[s, x, u\right]$, $g\left[s, x, u\right]$ and $q\left(x\right)$ are differentiable functions.

Dynamic programming and optimal control are used to identify optimal solutions for the problem (2.1)–(2.2).

2.1.1 Dynamic Programming

A frequently adopted approach to dynamic optimization problems is the technique of dynamic programming. The technique was developed by Bellman (1957). The technique is given in Theorem 2.1.1 below.

Theorem 2.1.1. *(Bellman's Dynamic Programming) A set of controls $u^*(t) = \phi^*(t,x)$ constitutes an optimal solution to the control problem (2.1)–(2.2) if there exist continuously differentiable functions $V(t,x)$ defined on $[t_0,T] \times R^m \to R$ and satisfying the following Bellman equation:*

$$
\begin{aligned}
-V_t(t,x) &= \max_u \{g[t,x,u] + V_x(t,x) f[t,x,u]\} \\
&= \{g[t,x,\phi^*(t,x)] + V_x(t,x) f[t,x,\phi^*(t,x)]\}, \\
V(T,x) &= q(x).
\end{aligned}
$$

Proof. Define the maximized payoff at time t with current state x as a value function *in the form:*

$$
\begin{aligned}
V(t,x) &= \max_u \left[\int_t^T g(s,x(s),u(s))\,ds + q(x(T)) \right] \\
&= \int_t^T g[s,x^*(s),\phi^*(s,x^*(s))]\,ds + q(x^*(T))
\end{aligned}
$$

satisfying the boundary condition

$$
V(T,x^*(T)) = q(x^*(T)),
$$

and

$$
\dot{x}^*(s) = f[s,x^*(s),\phi^*(s,x^*(s))], \quad x^*(t_0) = x_0.
$$

If in addition to $u^*(s) \equiv \phi^*(s,x)$, we are given another set of strategies, $u(s) \in U$, with the corresponding terminating trajectory $x(s)$, then Theorem 2.1.1 implies

$$
\begin{aligned}
&g(t,x,u) + V_x(t,x) f(t,x,u) + V_t(t,x) \le 0, \text{ and} \\
&g(t,x^*,u^*) + V_{x^*}(t,x^*) f(t,x^*,u^*) + V_t(t,x^*) = 0.
\end{aligned}
$$

Integrating the above expressions from t_0 to T, we obtain

$$
\begin{aligned}
&\int_{t_0}^T g(s,x(s),u(s))\,ds + V(T,x(T)) - V(t_0,x_0) \le 0, \text{ and} \\
&\int_{t_0}^T g(s,x^*(s),u^*(s))\,ds + V(T,x^*(T)) - V(t_0,x_0) = 0.
\end{aligned}
$$

Elimination of $V(t_0,x_0)$ yields

$$\int_{t_0}^{T} g(s, x(s), u(s))\, ds + q(x(T)) \leq \int_{t_0}^{T} g(s, x^*(s), u^*(s))\, ds + q(x^*(T)),$$

from which it readily follows that u^* is the optimal strategy.

Upon substituting the optimal strategy $\phi^*(t,x)$ into (2.2) yields the dynamics of optimal state trajectory as:

$$\dot{x}(s) = f[s, x(s), \phi^*(s, x(s))]\, ds, \qquad x(t_0) = x_0. \tag{2.3}$$

Let $x^*(t)$ denote the solution to (2.3). The optimal trajectory $\{x^*(t)\}_{t=t_0}^{T}$ can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^{t} f[s, x^*(s), \psi^*(s, x^*(s))]\, ds. \tag{2.4}$$

For notational convenience, we use the terms $x^*(t)$ and x_t^* interchangeably. The value function $V(t,x)$ where $x = x_t^*$ can be expressed as

$$V(t, x_t^*) = \int_{t}^{T} g[s, x^*(s), \phi^*(s)]\, ds + q(x^*(T)).$$

Example 2.1.1. Consider the dynamic optimization problem:

$$\max_u \left\{ \int_0^T \exp[-rs] \left[-x(s) - cu(s)^2 \right] ds + \exp[-rT]\, qx(T) \right\} \tag{2.5}$$

subject to

$$\dot{x}(s) = a - u(s)(x(s))^{1/2}, \qquad x(0) = x_0, \qquad u(s) \geq 0, \tag{2.6}$$

where a, c, x_0 are positive parameters.

Invoking Theorem 2.1.1 we have

$$-V_t(t,x) = \max_u \left\{ \left[-x - cu^2 \right] \exp[-rt] + V_x(t,x) \left[a - ux^{1/2} \right] \right\},$$

and

$$V(T,x) = \exp[-rT]\, qx. \tag{2.7}$$

Performing the indicated maximization in (2.7) yields:

$$\phi(t,x) = \frac{-V_x(t,x)\, x^{1/2}}{2c} \exp[rt].$$

Substituting $\phi(t,x)$ into (2.7) and upon solving (2.7), one obtains:

$$V(t,x) = \exp[-rt]\, [A(t)\, x + B(t)],$$

where $A(t)$ and $B(t)$ satisfy:

$$\dot{A}(t) = rA(t) - \frac{A(t)^2}{4c} + 1,$$
$$\dot{B}(t) = rB(t) - aA(t),$$
$$A(T) = q \text{ and } B(T) = 0.$$

The optimal control can be solved explicitly as

$$\phi(t,x) = \frac{-A(t)\, x^{1/2}}{2c} \exp[rt].$$

Now, consider the infinite-horizon dynamic optimization problem with a constant discount rate:

$$\max_u \left\{ \int_{t_0}^{\infty} g[x(s), u(s)] \exp[-r(s-t_0)]\, ds \right\}, \tag{2.8}$$

subject to the vector-valued differential equation:

$$\dot{x}(s) = f[x(s), u(s)]\, ds, \qquad x(t_0) = x_0. \tag{2.9}$$

Since s does not appear in $g[x(s), u(s)]$ and the state dynamics explicitly, the problem (2.8)–(2.9) is an autonomous problem.

Consider the alternative problem:

$$\max_u \int_{t}^{\infty} g[x(s), u(s)] \exp[-r(s-t)]\, ds, \tag{2.10}$$

subject to

$$\dot{x}(s) = f[x(s), u(s)], \qquad x(t) = x. \tag{2.11}$$

The infinite-horizon autonomous problem (2.10)–(2.11) is independent of the choice of t and dependent only upon the state at the starting time, that is x.

Define the value function to the problem (2.8)–(2.9) by

$$V(t,x) = \max_u \left\{ \int_{t}^{\infty} g[x(s), u(s)] \exp[-r(s-t_0)]\, ds \,\middle|\, x(t) = x = x_t^* \right\},$$

where x_t^* is the state at time t along the optimal trajectory. Moreover, we can write

$$V(t,x) =$$
$$\exp[-r(t-t_0)] \max_u \left\{ \int_{t}^{\infty} g[x(s), u(s)] \exp[-r(s-t)]\, ds \,\middle|\, x(t) = x = x_t^* \right\}.$$

Since the problem

$$\max_u \left\{ \int_t^\infty g\left[x(s), u(s)\right] \exp\left[-r(s-t)\right] ds \middle| x(t) = x = x_t^* \right\}$$

depends on the current state x only, we can write:

$$W(x) = \max_u \left\{ \int_t^\infty g\left[x(s), u(s)\right] \exp\left[-r(s-t)\right] ds \middle| x(t) = x = x_t^* \right\}.$$

It follows that:

$$\begin{aligned} &V(t,x) = \exp\left[-r(t-t_0)\right] W(x), \\ &V_t(t,x) = -r \exp\left[-r(t-t_0)\right] W(x), \text{ and} \\ &V_x(t,x) = -r \exp\left[-r(t-t_0)\right] W_x(x). \end{aligned} \tag{2.12}$$

Substituting the results from (2.12) into Theorem 2.1.1 yields

$$rW(x) = \max_u \left\{ g[x,u] + W_x(x) f[x,u] \right\}. \tag{2.13}$$

Since time is not explicitly involved (2.13), the derived control u will be a function of x only. Hence one can obtain:

Theorem 2.1.2. *A set of controls $u = \phi^*(x)$ constitutes an optimal solution to the infinite-horizon control problem (2.10)–(2.11) if there exists continuously differentiable function $W(x)$ defined on $R^m \to R$ which satisfies the following equation:*

$$\begin{aligned} rW(x) &= \max_u \left\{ g[x,u] + W_x(x) f[x,u] \right\} \\ &= \left\{ g[x, \phi^*(x)] + W_x(x) f[x, \phi^*(x)] \right\}. \end{aligned}$$

Substituting the optimal control in Theorem 2.1.2 into (2.9) yields the dynamics of the optimal state path as:

$$\dot{x}(s) = f\left[x(s), \phi^*(x(s))\right] ds, \qquad x(t_0) = x_0.$$

Solving the above dynamics yields the optimal state trajectory $\{x^*(t)\}_{t \geq t_o}$ as

$$x^*(t) = x_0 + \int_{t_0}^t f\left[x^*(s), \psi^*(x^*(s))\right] ds, \quad \text{for } t \geq t_0.$$

We denote term $x^*(t)$ by x_t^*. The optimal control to the infinite-horizon problem (2.8)–(2.9) can be expressed as $\psi^*(x_t^*)$ in the time interval $[t_0, \infty)$.

Example 2.1.2. Consider the infinite-horizon dynamic optimization problem:

$$\max_u \int_0^\infty \exp[-rs] \left[-x(s) - cu(s)^2 \right] ds \tag{2.14}$$

subject to dynamics (2.6).

Invoking Theorem 2.1.2 we have

$$rW(x) = \max_u \left\{ \left[-x - cu^2 \right] + W_x(x) \left[a - ux^{1/2} \right] \right\}. \tag{2.15}$$

Performing the indicated maximization in (2.15) yields:

$$\phi^*(x) = \frac{-V_x(x)\, x^{1/2}}{2c}.$$

Substituting $\phi(x)$ into (2.15) and upon solving (2.15), one obtains:

$$V(t,x) = \exp[-rt]\,[Ax + B],$$

where A and B satisfy:

$$0 = rA - \frac{A^2}{4c} + 1 \quad \text{and} \quad B = \frac{-a}{r}A.$$

Solving A to be $2c\left[r \pm \left(r^2 + c^{-1}\right)^{1/2}\right]$. For a maximum, the negative root of A holds. The optimal control can be obtained as

$$\phi^*(x) = \frac{-Ax^{1/2}}{2c}.$$

Substituting $\phi^*(x) = -Ax^{1/2}/(2c)$ into (2.6) yields the dynamics of the optimal trajectory as:

$$\dot{x}(s) = a + \frac{A}{2c}(x(s)), \qquad x(0) = x_0.$$

Upon the above dynamical equation yields the optimal trajectory $\{x^*(t)\}_{t \ge t_0}$ as

$$x^*(t) = \left[x_0 + \frac{2ac}{A}\right] \exp\left(\frac{A}{2c}t\right) - \frac{2ac}{A} = x_t^*, \qquad \text{for } t \ge t_0.$$

The optimal control of problem (2.14)–(2.15) is then

$$\phi^*(x_t^*) = \frac{-A\left(x_t^*\right)^{1/2}}{2c}.$$

2.1.2 Optimal Control

The maximum principle of optimal control was developed by Pontryagin (details in Pontryagin et al (1962)). Consider again the dynamic optimization problem (2.1)–(2.2).

Theorem 2.1.3. *(Pontryagin's Maximum Principle) A set of controls* $u^*(s) = \zeta^*(s, x_0)$ *provides an optimal solution to control problem (2.1)–(2.2), and* $\{x^*(s), t_0 \le s \le T\}$ *is the corresponding state trajectory, if there exist costate functions* $\Lambda(s) : [t_0, T] \to R^m$ *such that the following relations are satisfied:*

$$\zeta^*(s, x_0) \equiv u^*(s) = \arg\max_u \{g[s, x^*(s), u(s)] + \Lambda(s) f[s, x^*(s), u(s)]\},$$
$$\dot{x}^*(s) = f[s, x^*(s), u^*(s)], \quad x^*(t_0) = x_0,$$
$$\dot{\Lambda}(s) = -\frac{\partial}{\partial x}\{g[s, x^*(s), u^*(s)] + \Lambda(s) f[s, x^*(s), u^*(s)]\},$$
$$\Lambda(T) = \frac{\partial}{\partial x^*} q(x^*(T)).$$

Proof. First define the function (Hamiltonian)

$$H(t, x, u) = g(t, x, u) + V_x(t, x) f(t, x, u).$$

From Theorem 2.1.1, we obtain

$$-V_t(t, x) = \max_u H(t, x, u).$$

This yields the first condition of Theorem 2.1.1. Using u^* *to denote the payoff maximizing control, we obtain*

$$H(t, x, u^*) + V_t(t, x) \equiv 0,$$

which is an identity in x*. Differentiating this identity partially with respect to* x *yields*

$$V_{tx}(t, x) + g_x(t, x, u^*) + V_x(t, x) f_x(t, x, u^*) + V_{xx}(t, x) f(t, x, u^*)$$
$$+ [g_u(t, x, u^*) + V_x(t, x) f_u(t, x, u^*)] \frac{\partial u^*}{\partial x} = 0.$$

If u^* *is an interior point, then* $[g_u(t, x, u^*) + V_x(t, x) f_u(t, x, u^*)] = 0$ *according to the condition* $-V_t(t, x) = \max_u H(t, x, u)$*. If* u^* *is not an interior point, then it can be shown that*

$$[g_u(t, x, u^*) + V_x(t, x) f_u(t, x, u^*)] \frac{\partial u^*}{\partial x} = 0$$

(because of optimality, $[g_u(t, x, u^*) + V_x(t, x) f_u(t, x, u^*)]$ *and* $\partial u^*/\partial x$ *are orthogonal; and for specific problems we may have* $\partial u^*/\partial x = 0$*). Moreover, the expression* $V_{tx}(t, x) + V_{xx}(t, x) f(t, x, u^*) \equiv V_{tx}(t, x) + V_{xx}(t, x)\dot{x}$ *can be written as* $[dV_x(t, x)](dt)^{-1}$*. Hence, we obtain:*

$$\frac{dV_x(t, x)}{dt} + g_x(t, x, u^*) + V_x(t, x) f_x(t, x, u^*) = 0.$$

By introducing the costate vector*,* $\Lambda(t) = V_{x^*}(t, x^*)$*, where* x^* *denotes the state trajectory corresponding to* u^**, we arrive at*

$$\frac{dV_x(t,x^*)}{dt} = \dot{\Lambda}(s) = -\frac{\partial}{\partial x}\{g[s,x^*(s),u^*(s)] + \Lambda(s) f[s,x^*(s),u^*(s)]\}.$$

Finally, the boundary condition for $\Lambda(t)$ is determined from the terminal condition of optimal control in Theorem 2.1.1 as

$$\Lambda(T) = \frac{\partial V(T,x^*)}{\partial x} = \frac{\partial q(x^*)}{\partial x}.$$

Hence Theorem 2.1.3 follows.

Example 2.1.3. Consider the problem in Example 2.1.1. Invoking Theorem 2.1.3, we first solve the control $u(s)$ that satisfies

$$\arg\max_u \left\{ \left[-x^*(s) - cu(s)^2 \right] \exp[-rs] + \Lambda(s) \left[a - u(s) x^*(s)^{1/2} \right] \right\}.$$

Performing the indicated maximization:

$$u^*(s) = \frac{-\Lambda(s) x^*(s)^{1/2}}{2c} \exp[rs]. \tag{2.16}$$

We also obtain

$$\dot{\Lambda}(s) = \exp[-rs] + \frac{1}{2}\Lambda(s) u^*(s) x^*(s)^{-1/2}. \tag{2.17}$$

Substituting $u^*(s)$ from (2.16) into (2.6) and (2.17) yields a pair of differential equations:

$$\begin{aligned} \dot{x}^*(s) &= a + \frac{1}{2c}\Lambda(s)(x^*(s)) \exp[rs], \\ \dot{\Lambda}(s) &= \exp[-rs] + \frac{1}{4c}\Lambda(s)^2 \exp[rs], \end{aligned} \tag{2.18}$$

with boundary conditions:

$$x^*(0) = x_0 \text{ and } \Lambda(T) = \exp[-rT] q.$$

Solving (2.18) yields

$$\begin{aligned} \Lambda(s) = 2c &\left(\theta_1 - \theta_2 \frac{q - 2c\theta_1}{q - 2c\theta_2} \exp\left[\frac{\theta_1 - \theta_2}{2}(T-s) \right] \right) \exp(-rs) \\ &\div \left(1 - \frac{q - 2c\theta_1}{q - 2c\theta_2} \exp\left[\frac{\theta_1 - \theta_2}{2}(T-s) \right] \right), \text{ and} \\ x^*(s) = \varpi(0,s) &\left[x_0 + \int_0^s \varpi^{-1}(0,t)\, a\, dt \right], \qquad \text{for } s \in [0,T], \end{aligned}$$

where

$$\theta_1 = r - \sqrt{r^2 + \frac{1}{c}} \text{ and } \theta_2 = r + \sqrt{r^2 + \frac{1}{c}};$$

$$\varpi(0,s) = \exp\left[\int_0^s H(\tau)\,d\tau\right], \text{ and}$$

$$H(\tau) = \left(\theta_1 - \theta_2 \frac{q - 2c\theta_1}{q - 2c\theta_2} \exp\left[\frac{\theta_1 - \theta_2}{2}(T - \tau)\right]\right)$$
$$\div \left(1 - \frac{q - 2c\theta_1}{q - 2c\theta_2} \exp\left[\frac{\theta_1 - \theta_2}{2}(T - \tau)\right]\right).$$

Upon substituting $\Lambda(s)$ and $x^*(s)$ into (2.16) yields $u^*(s) = \zeta^*(s, x_0)$ which is a function of s and x_0.

Now, consider the infinite-horizon dynamic optimization problem (2.8)–(2.9). The Hamiltonian function can be expressed as

$$H(t,x,u) = g(x,u)\exp[-r(t-t_0)] + \Lambda(t) f(x,u).$$

Define $\lambda(t) = \Lambda(t)\exp[r(t-t_0)]$ and the current value Hamiltonian

$$\begin{aligned}\hat{H}(t,x,u) &= H(t,x,u)\exp[r(t-t_0)] \\ &= g(x,u) + \lambda(t) f(x,u).\end{aligned} \tag{2.19}$$

Substituting (2.19) into Theorem 2.1.3 yields the maximum principle for the game (2.10)–(2.11).

Theorem 2.1.4. *A set of controls $u^*(s) = \zeta^*(s, x_t)$ provides an optimal solution to the infinite-horizon control problem (2.10)–(2.11), and $\{x^*(s),\ s \ge t\}$ is the corresponding state trajectory, if there exist costate functions $\lambda(s) : [t,\infty) \to R^m$ such that the following relations are satisfied:*

$$\zeta^*(s,x_t) \equiv u^*(s) = \arg\max_u \{g[x^*(s), u(s)] + \lambda(s) f[x^*(s), u(s)]\},$$
$$\dot{x}^*(s) = f[x^*(s), u^*(s)], \qquad x^*(t) = x_t,$$
$$\dot{\lambda}(s) = r\lambda(s) - \frac{\partial}{\partial x}\{g[x^*(s), u^*(s)] + \lambda(s) f[x^*(s), u^*(s)]\}.$$

Example 2.1.4. Consider the infinite-horizon problem in Example 2.1.2.

Invoking Theorem 2.1.4 we have

$$\zeta^*(s,x_t) \equiv u^*(s) =$$
$$\arg\max_u \left\{\left[-x^*(s) - cu(s)^2\right] + \lambda(s)\left[a - u(s)x^*(s)^{1/2}\right]\right\},$$
$$\dot{x}^*(s) = a - u^*(s)(x^*(s))^{1/2}, \qquad x^*(t) = x_t,$$
$$\dot{\lambda}(s) = r\lambda(s) + \left[1 + \frac{1}{2}\lambda(s)u^*(s)x^*(s)^{-1/2}\right]. \tag{2.20}$$

Performing the indicated maximization yields

$$u^*(s) = \frac{-\lambda(s)\, x^*(s)^{1/2}}{2c}.$$

Substituting $u^*(s)$ into (2.20), one obtains

$$\dot{x}^*(s) = a + \frac{\lambda(s)}{2c} u^*(s)\, x^*(s), \quad x^*(t) = x_t,$$
$$\dot{\lambda}(s) = r\lambda(s) + \left[1 - \frac{1}{4c}\lambda(s)^2\right]. \tag{2.21}$$

Solving (2.21) in a manner similar to that in Example 2.1.3 yields the solutions of $x^*(s)$ and $\lambda(s)$. Upon substituting them into $u^*(s)$ gives the optimal control of the problem.

2.1.3 Stochastic Control

Consider the dynamic optimization problem in which the single decision-maker:

$$\max_u E_{t_0} \left\{ \int_{t_0}^{T} g^i[s, x(s), u(s)]\, ds + q(x(T)) \right\}, \tag{2.22}$$

subject to the vector-valued stochastic differential equation:

$$dx(s) = f[s, x(s), u(s)]\, ds + \sigma[s, x(s)]\, dz(s), \quad x(t_0) = x_0, \tag{2.23}$$

where E_{t_0} denotes the expectation operator performed at time t_0, and $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\, \sigma[s, x(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$.

The technique of stochastic control developed by Fleming (1969) can be applied to solve the problem.

Theorem 2.1.5. *A set of controls $u^*(t) = \phi^*(t, x)$ constitutes an optimal solution to the problem (2.22)–(2.23), if there exist continuously differentiable functions $V(t, x) : [t_0, T] \times R^m \to R$, satisfying the following partial differential equation:*

$$-V_t(t, x) - \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t, x)\, V_{x^h x^\zeta}(t, x) =$$
$$\max_u \left\{ g^i[t, x, u] + V_x(t, x)\, f[t, x, u] \right\}, \text{ and}$$
$$V(T, x) = q(x).$$

Proof. *Substitute the optimal control $\phi^*(t, x)$ into the (2.23) to obtain the optimal state dynamics as*

$$dx(s) = f[s, x(s), \phi^*(s, x(s))]\,ds + \sigma[s, x(s)]\,dz(s),$$
$$x(t_0) = x_0. \tag{2.24}$$

The solution to (2.24), denoted by $x^*(t)$, can be expressed as

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[s, x^*(s), \psi_1^{(t_0)*}(s, x^*(s)), \psi_2^{(t_0)*}(s, x^*(s))\right] ds + \int_{t_0}^{t} \sigma[s, x^*(s)]\,dz(s). \tag{2.25}$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (2.25). The term x_t^* is used to denote an element in the set X_t^*.

Define the maximized payoff at time t with current state x_t^* as a value function *in the form*

$$V(t, x_t^*) = \max_u E_{t_0}\left\{ \int_t^T g^i[s, x(s), u(s)]\,ds + q(x(T)) \,\middle|\, x(t) = x_t^* \right\}$$
$$= E_{t_0}\left\{ \int_t^T g[s, x^*(s), \phi^*(s, x^*(s))]\,ds + q(x^*(T)) \right\}$$

satisfying the boundary condition

$$V(T, x^*(T)) = q(x^*(T)).$$

One can express $V(t, x_t^*)$ as

$$\begin{aligned} &V(t, x_t^*) \\ &= \max_u E_{t_0}\left\{ \int_t^T g^i[s, x(s), u(s)]\,ds + q(x(T)) \,\middle|\, x(t) = x_t^* \right\} \\ &= \max_u E_{t_0}\left\{ \int_t^{t+\Delta t} g^i[s, x(s), u(s)]\,ds + V(t+\Delta t, x_t^* + \Delta x_t^*) \,\middle|\, x(t) = x_t^* \right\}. \end{aligned} \tag{2.26}$$

where

$$\Delta x_t^* = f[t, x_t^*, \phi^*(t, x_t^*)]\,\Delta t + \sigma[t, x_t^*]\,\Delta z_t + o(\Delta t),$$
$$\Delta z_t = z(t+\Delta t) - z(t), \text{ and } E_t[o(\Delta t)]/\Delta t \to 0 \text{ as } \Delta t \to 0.$$

With $\Delta t \to 0$, applying Ito's lemma equation (2.26) can be expressed as:

$$V(t, x_t^*) = \max_u E_{t_0}\Bigg\{ g^i[t, x_t^*, u]\,\Delta t + V(t, x_t^*) + V_t(t, x_t^*)\,\Delta t$$

$$+ V_{x_t}(t, x_t^*) f[t, x_t^*, \phi^*(t, x_t^*)] \Delta t + V_{x_t}(t, x_t^*) \sigma[t, x_t^*] \Delta z_t$$
$$\left. + \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t, x) V_{x^h x^\zeta}(t, x) \Delta t + o(\Delta t) \right\} . \tag{2.27}$$

Dividing (2.27) throughout by Δt, with $\Delta t \to 0$, and taking expectation yields

$$-V_t(t, x_t^*) - \frac{1}{2} \sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t, x) V_{x^h x^\zeta}(t, x) =$$
$$\max_u \left\{ g^i[t, x_t^*, u] + V_{x_t}(t, x_t^*) f[t, x_t^*, \phi^*(t, x_t^*)] \right\}, \tag{2.28}$$

with boundary condition

$$V(T, x^*(T)) = q(x^*(T)) .$$

Hence Theorem 2.1.5.

Example 2.1.5. Consider the stochastic control problem

$$E_{t_0} \left\{ \int_{t_0}^{T} \left[u(s)^{1/2} - \frac{c}{x(s)^{1/2}} u(s) \right] \exp[-r(s - t_0)] \, ds \right.$$
$$\left. + \exp[-r(T - t_0)] \, q x(T)^{1/2} \right\}, \tag{2.29}$$

subject to

$$dx(s) = \left[a x(s)^{1/2} - b x(s) - u(s) \right] ds + \sigma x(s) \, dz(s) ,$$
$$x(t_0) = x_0 \in X, \tag{2.30}$$

where c, a, b and σ are positive parameters.

Invoking Theorem 2.1.5 we have

$$-V_t(t, x) - \frac{1}{2} \sigma^2 x^2 V_{xx}(t, x) =$$
$$\max_u \left\{ \left[u^{1/2} - \frac{c}{x^{1/2}} u \right] \exp[-r(t - t_0)] + V_x(t, x) \left[a x^{1/2} - b x - u \right] \right\}, \text{ and}$$
$$V(T, x) \exp[-r(T - t_0)] \, q x^{1/2}. \tag{2.31}$$

Performing the indicated maximization in (2.31) yields

$$\phi^*(t, x) = \frac{x}{4 \left[c + V_x \exp[r(t - t_0)] \, x^{1/2} \right]^2}. \tag{2.32}$$

Substituting $\phi^*(t, x)$ from (2.32) into (2.31) and upon solving (2.31) yields the value function:

$$V(t,x)\exp[-r(t-t_0)]\left[A(t)x^{1/2}+B(t)\right],$$

where $A(t)$ and $B(t)$ satisfy:

$$\begin{aligned}\dot{A}(t) &= \left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]A(t)-\frac{1}{2[c+A(t)/2]}\\ &\qquad +\frac{c}{4[c+A(t)/2]^2}+\frac{A(t)}{8[c+A(t)/2]^2},\\ \dot{B}(t) &= rB(t)-\frac{a}{2}A(t),\\ A(T) &= q, \text{ and } B(T)=0.\end{aligned}$$

The optimal control for the problem (2.29)–(2.30) can be obtained as

$$\phi^*(t,x)=\frac{x}{4\left[c+\frac{A(t)}{2}\right]^2}.$$

Now, consider the infinite-horizon stochastic control problem with a constant discount rate:

$$\max_u E_{t_0}\left\{\int_{t_0}^{\infty} g^i[x(s),u(s)]\exp[-r(s-t_0)]\,ds\right\}, \tag{2.33}$$

subject to the vector-valued stochastic differential equation:

$$dx(s)=f[x(s),u(s)]\,ds+\sigma[x(s)]\,dz(s), \qquad x(t_0)=x_0, \tag{2.34}$$

Since s does not appear in $g[x(s),u(s)]$ and the state dynamics explicitly, the problem (2.33)–(2.34) is an autonomous problem.

Consider the alternative problem:

$$\max_u E_t\left\{\int_t^{\infty} g^i[x(s),u(s)]\exp[-r(s-t)]\,ds\right\}, \tag{2.35}$$

subject to the vector-valued stochastic differential equation:

$$dx(s)=f[x(s),u(s)]\,ds+\sigma[x(s)]\,dz(s), \qquad x(t)=x_t. \tag{2.36}$$

The infinite-horizon autonomous problem (2.35)–(2.36) is independent of the choice of t and dependent only upon the state at the starting time, that is x_t.

Define the value function to the problem (2.35)–(2.36) by

$$V(t,x_t^*)=\max_u E_{t_0}\left\{\int_t^{\infty} g[x(s),u(s)]\exp[-r(s-t_0)]\,ds\,\middle|\,x(t)=x_t^*\right\},$$

where x_t^* is an element belonging to the set of feasible values along the optimal state trajectory at time t. Moreover, we can write

$$V\left(t, x_t^*\right) =$$
$$\exp\left[-r\left(t-t_0\right)\right] \max_u E_{t_0}\left\{\int_t^{\infty} g\left[x\left(s\right), u\left(s\right)\right] \exp\left[-r\left(s-t\right)\right] ds \middle| x\left(t\right) = x_t^*\right\}.$$

Since the problem

$$\max_u E_{t_0}\left\{\int_t^{\infty} g\left[x\left(s\right), u\left(s\right)\right] \exp\left[-r\left(s-t\right)\right] ds \middle| x\left(t\right) = x_t^*\right\}$$

depends on the current state x_t^* only, we can write

$$W\left(x_t^*\right) = \max_u E_{t_0}\left\{\int_t^{\infty} g\left[x\left(s\right), u\left(s\right)\right] \exp\left[-r\left(s-t\right)\right] ds \middle| x\left(t\right) = x_t^*\right\}.$$

It follows that:

$$\begin{aligned}
&V\left(t, x_t^*\right) = \exp\left[-r\left(t-t_0\right)\right] W\left(x_t^*\right), \\
&V_t\left(t, x_t^*\right) = -r \exp\left[-r\left(t-t_0\right)\right] W\left(x_t^*\right), \\
&V_{x_t}\left(t, x_t^*\right) = -r \exp\left[-r\left(t-t_0\right)\right] W_{x_t}\left(x_t^*\right), \text{ and} \\
&V_{x_t x_t}\left(t, x_t^*\right) = -r \exp\left[-r\left(t-t_0\right)\right] W_{x_t x_t}\left(x_t^*\right).
\end{aligned} \tag{2.37}$$

Substituting the results from (2.37) into Theorem 2.1.5 yields

$$rW\left(x\right) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}\left(t, x\right) W_{x^h x^\zeta}\left(t, x\right) = \max_u \left\{g\left[x, u\right] + W_x\left(x\right) f\left[x, u\right]\right\}. \tag{2.38}$$

Since time is not explicitly involved (2.38), the derived control u will be a function of x only. Hence one can obtain:

Theorem 2.1.6. *A set of controls $u = \phi^*\left(x\right)$ constitutes an optimal solution to the infinite-horizon stochastic control problem (2.33)–(2.34) if there exists continuously differentiable function $W\left(x\right)$ defined on $R^m \to R$ which satisfies the following equation:*

$$\begin{aligned}
rW\left(x\right) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}\left(t, x\right) W_{x^h x^\zeta}\left(t, x\right) &= \max_u \left\{g\left[x, u\right] + W_x\left(x\right) f\left[x, u\right]\right\} \\
&= \left\{g\left[x, \phi^*\left(x\right)\right] + W_x\left(x\right) f\left[x, \phi^*\left(x\right)\right]\right\}.
\end{aligned}$$

Substituting the optimal control in Theorem 2.1.6 into (2.34) yields the dynamics of the optimal state path as

$$dx\left(s\right) = f\left[x\left(s\right), \phi^*\left(x\left(s\right)\right)\right] ds + \sigma\left[x\left(s\right)\right] dz\left(s\right), \qquad x\left(t_0\right) = x_0.$$

Solving the above vector-valued stochastic differential equations yields the optimal state trajectory $\left\{x^*\left(t\right)\right\}_{t \geq t_0}$ as

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[x^*(s), \psi^*(x^*(s))\right] ds + \int_{t_0}^{t} \sigma\left[x^*(s)\right] dz(s). \tag{2.39}$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (2.39). The term x_t^* is used to denote an element in the set X_t^*.

Given that x_t^* is realized at time t, the optimal control to the infinite-horizon problem (2.33)–(2.34) can be expressed as $\psi^*(x_t^*)$.

Example 2.1.6. Consider the infinite-horizon problem

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u(s)^{1/2} - \frac{c}{x(s)^{1/2}}u(s)\right]\exp\left[-r(s-t_0)\right] ds\right\}, \tag{2.40}$$

subject to

$$\begin{aligned} dx(s) &= \left[ax(s)^{1/2} - bx(s) - u(s)\right] ds + \sigma x(s)\, dz(s), \\ x(t_0) &= x_0 \in X, \end{aligned} \tag{2.41}$$

where c, a, b and σ are positive parameters.

Invoking Theorem 2.1.6 we have

$$\begin{aligned} &rW(x) - \frac{1}{2}\sigma^2 x^2 W_{xx}(x) = \\ &\quad \max_u \left\{\left[u^{1/2} - \frac{c}{x^{1/2}}u\right] + W_x(x)\left[ax^{1/2} - bx - u\right]\right\}. \end{aligned} \tag{2.42}$$

Performing the indicated maximization in (2.42) yields the control:

$$\phi^*(x) = \frac{x}{4\left[c + W_x(x)\, x^{1/2}\right]^2}.$$

Substituting $\phi^*(t, x)$ into (2.42) above and upon solving (2.42) yields the value function

$$W(x) = \left[Ax^{1/2} + B\right],$$

where A and B satisfy:

$$\begin{aligned} 0 &= \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]A - \frac{1}{2\left[c + A/2\right]} + \frac{c}{4\left[c + A/2\right]^2} + \frac{A}{8\left[c + A/2\right]^2}, \\ B &= \frac{a}{2r}A. \end{aligned}$$

The optimal control can then be expressed as

$$\phi^*(x) = \frac{x}{4\left[c + A/2\right]^2}.$$

Substituting $\phi^*(x) = x/\left\{4[c+A/2]^2\right\}$ into (2.41) yields the dynamics of the optimal trajectory as

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - \frac{x(s)}{4[c+A/2]^2}\right] ds + \sigma x(s)\, dz(s),$$
$$x(t_0) = x_0 \in X.$$

Upon the above dynamical equation yields the optimal trajectory $\{x^*(t)\}_{t \geq t_0}$ as

$$x^*(t) = \varpi(t_0, t)^2 \left[x_0^{1/2} + \int_{t_0}^{t} \varpi^{-1}(t_0, s)\, H_1 \; ds\right]^2,$$
$$\text{for } \; t \geq t_0, \tag{2.43}$$

where

$$\varpi(t_0, t) = \exp\left[\int_{t_0}^{t} \left[H_2 - \frac{\sigma^2}{8}\right] dv + \int_{t_0}^{t} \frac{\sigma}{2} dz(v)\right],$$
$$H_1 = \frac{1}{2}a, \;\text{ and }\; H_2 = -\left[\frac{1}{2}b + \frac{1}{4[c+A/2]^2} + \frac{\sigma^2}{8}\right].$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (2.43). The term x_t^* is used to denote an element in the set X_t^*. Given that x_t^* is realized at time t, the optimal control to the infinite-horizon problem (2.40)–(2.41) can be expressed as $\psi^*(x_t^*)$.

2.2 Differential Games and their Solution Concepts

One particularly complex – but fruitful – branch of game theory is dynamic or differential games, which investigates interactive decision making over time under different assumptions regarding pre-commitment (of actions), information, and uncertainty. The origin of differential games traces back to the late 1940s. Rufus Isaacs (whose work was published in 1965) formulated missile versus enemy aircraft pursuit schemes in terms of descriptive and navigation variables (state and control), and established a fundamental principle: the *tenet of transition*. The seminal contributions of Isaacs together with the classic research of Bellman on dynamic programming and Pontryagin et al. on optimal control laid the foundations of deterministic differential games. Early research in differential games centers on the extension of control theory problems. Berkovitz (1964) developed a variational approach to differential games, Leitmann and Mon (1967) studies the geometric aspects of differential games, Pontryagin (1966) solved differential games solution with his maximum principle, while Zaubermann (1975) accounted for various developments in Soviet literature prior to the early 1970s.

Contributions to differential games continue to appear in many fields and disciplines. In particular, applications in economics and management sciences have been growing rapidly, with reference to which a detailed account of these applications can be found in Dockner et al. (2000).

Differential games or continuous-time infinite dynamic games study a class of decision problems, under which the evolution of the state is described by a differential equation and the players act throughout a time interval.

In particular, in the general n-person differential game, Player i seeks to:

$$\max_{u_i} \int_{t_0}^{T} g^i \left[s, x\left(s\right), u_1\left(s\right), u_2\left(s\right), \ldots, u_n\left(s\right)\right] ds + q^i\left(x\left(T\right)\right), \tag{2.44}$$
$$\text{for } \; i \in N = \{1, 2, \ldots, n\},$$

subject to the deterministic dynamics

$$\dot{x}\left(s\right) = f\left[s, x\left(s\right), u_1\left(s\right), u_2\left(s\right), \ldots, u_n\left(s\right)\right], \qquad x\left(t_0\right) = x_0, \tag{2.45}$$

where $x\left(s\right) \in X \subset R^m$ denotes the state variables of game, and $u_i \in U^i$ is the control of Player i, for $i \in N$.

The functions $f\left[s, x, u_1, u_2, \ldots, u_n\right]$, $g^i\left[s, \cdot, u_1, u_2, \ldots, u_n\right]$ and $q^i\left(\cdot\right)$, for $i \in N$, and $s \in [t_0, T]$ are differentiable functions.

A set-valued function $\eta^i\left(\cdot\right)$ defined for each $i \in N$ as

$$\eta^i\left(s\right) = \left\{x\left(t\right), \quad t_0 \le t \le \epsilon_s^i\right\}, \qquad t_0 \le \epsilon_s^i \le s,$$

where ϵ_s^i is nondecreasing in s, and $\eta^i\left(s\right)$ determines the state information gained and recalled by Player i at time $s \in [t_0, T]$. Specification of $\eta^i\left(\cdot\right)$(in fact, ϵ_s^i in this formulation) characterizes the *information structure* of Player i and the collection (over $i \in N$) of these information structures is the *information structure* of the game.

A sigma-field N_s^i in S_0 generated for each $i \in N$ by the cylinder sets $\{x \in S_0,\, x\left(t\right) \in B\}$ where B is a Borel set in S^0 and $0 \le t \le \epsilon_s$. N_s^i, $s \ge t_0$, is called the *information field* of Player i.

A pre-specified class Γ^i of mappings $\upsilon_i : [t_0, T] \times S_0 \to S^i$, with the property that $u_i\left(s\right) = \upsilon_i\left(s, x\right)$ is n_s^i-measurable (i.e. it is adapted to the information field N_s^i). U^i is the strategy space of Player i and each of its elements υ_i is a permissible strategy for Player i.

Definition 2.2.1. *A set of strategies* $\{\upsilon_1^*\left(s\right), \upsilon_2^*\left(s\right), \ldots, \upsilon_n^*\left(s\right)\}$ *is said to constitute a* non-cooperative Nash equilibrium solution *for the n-person differential game (2.44)–(2.45), if the following inequalities are satisfied for all* $\upsilon_i\left(s\right) \in U^i$, $i \in N$*:*

$$\int_{t_0}^{T} g^1 \left[s, x^*(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^1\left(x^*(T)\right) \geq$$
$$\int_{t_0}^{T} g^1 \left[s, x^{[1]}(s), \upsilon_1(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^1\left(x^{[1]}(T)\right),$$
$$\int_{t_0}^{T} g^2 \left[s, x^*(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^2\left(x^*(T)\right) \geq$$
$$\int_{t_0}^{T} g^2 \left[s, x^{[2]}(s), \upsilon_1^*(s), \upsilon_2(s), \upsilon_3^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^2\left(x^{[2]}(T)\right),$$
$$\vdots \qquad\qquad \vdots$$
$$\vdots \qquad\qquad \vdots$$
$$\int_{t_0}^{T} g^n \left[s, x^*(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right] ds + q^n\left(x^*(T)\right) \geq$$
$$\int_{t_0}^{T} g^n \left[s, x^{[n]}(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_{n-1}^*(s), \upsilon_n(s)\right] ds + q^n\left(x^{[n]}(T)\right);$$

where on the time interval $[t_0, T]$*:*

$$\dot{x}^*(s) = f\left[s, x^*(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right], \quad x^*(t_0) = x_0,$$
$$\dot{x}^{[1]}(s) = f\left[s, x^{[1]}(s), \upsilon_1(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\right], \quad x^{[1]}(t_0) = x_0,$$
$$\dot{x}^{[2]}(s) = f\left[s, x^{[2]}(s), \upsilon_1^*(s), \upsilon_2(s), \upsilon_3^*(s), \ldots, \upsilon_n^*(s)\right], \quad x^{[2]}(t_0) = x_0,$$
$$\vdots \qquad\qquad \vdots$$
$$\vdots \qquad\qquad \vdots$$
$$\dot{x}^{[n]}(s) = f\left[s, x^{[n]}(s), \upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_{n-1}^*(s), \upsilon_n(s)\right], \quad x^{[n]}(t_0) = x_0.$$

The set of strategies $\{\upsilon_1^*(s), \upsilon_2^*(s), \ldots, \upsilon_n^*(s)\}$ *is known as a* Nash equilibrium *of the game.*

2.2.1 Open-loop Nash Equilibria

If the players choose to commit their strategies from the outset, the players' information structure can be seen as an *open-loop* pattern in which $\eta^i(s) = \{x_0\}$, $s \in [t_0, T]$. Their strategies become functions of the initial state x_0 and time s, and can be expressed as $\{u_i(s) = \vartheta_i(s, x_0), \text{ for } i \in N\}$. In particular, an open-loop Nash equilibria for the game (2.44) and (2.45) is characterized as:

Theorem 2.2.1. *A set of strategies* $\{u_i^*(s) = \zeta_i^*(s, x_0), \text{ for } i \in N\}$ *provides an open-loop Nash equilibrium solution to the game (2.44)–(2.45), and*

$\{x^*(s), t_0 \le s \le T\}$ is the corresponding state trajectory, if there exist m costate functions $\Lambda^i(s) : [t_0, T] \to R^m$, for $i \in N$, such that the following relations are satisfied:

$$\zeta_i^*(s, x_0) \equiv u_i^*(s) =$$
$$\arg\max_{u_i \in U^i} \left\{ g^i \left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right.$$
$$\left. + \Lambda^i(s) f\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right\},$$
$$\dot{x}^*(s) = f[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)], \quad x^*(t_0) = x_0,$$
$$\dot{\Lambda}^i(s) = -\frac{\partial}{\partial x^*}\left\{ g^i[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)] \right.$$
$$\left. + \Lambda^i(s) f[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)] \right\},$$
$$\Lambda^i(T) = \frac{\partial}{\partial x^*} q^i(x^*(T)); \qquad \text{for } i \in N.$$

Proof. Consider the i^{th} equality in Theorem 2.1.1, which states that $\upsilon_i^*(s) = u_i^*(s) = \zeta_i^*(s, x_0)$ maximizes

$$\int_{t_0}^{T} g^i\left[s, x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] ds$$
$$+ q^i(x(T)),$$

over the choice of $\upsilon_i(s) \in U^i$ subject to the state dynamics:

$$\dot{x}(s) = f\left[s, x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right],$$
$$x(t_0) = x_0, \qquad \text{for } i \in N.$$

This is standard optimal control problem for Player i, since $u_j^*(s)$, for $j \in N$ and $j \ne i$, are open-loop controls and hence do not depend on $u_i^*(s)$. These results then follow directly from the maximum principle of Pontryagin as stated in Theorem 2.1.3.

Derivation of open-loop equilibria in nonzero-sum deterministic differential games first appeared in Berkovitz (1964) and Ho et al. (1965), with open-loop and feedback Nash equilibria in nonzero-sum deterministic differential games being presented in Case (1967, 1969) and Starr and Ho (1969a and 1969b). A detailed account of applications of open-loop equilibria in economic and management science can be found in Dockner et al. (2000).

2.2.2 Closed-loop Nash Equilibria

Under the memoryless perfect state information, the players' information structures follow the pattern $\eta^i(s) = \{x_0, x(s)\}$, $s \in [t_0, T]$. The players' strategies become functions of the initial state x_0, current state $x(s)$ and current time s, and can be expressed as $\{u_i(s) = \vartheta_i(s, x, x_0), \text{ for } i \in N\}$. The following theorem provides a set of necessary conditions for any closed-loop no-memory Nash equilibrium solution to satisfy.

Theorem 2.2.2. *A set of strategies $\{u_i(s) = \vartheta_i(s, x, x_0), \text{ for } i \in N\}$ provides a closed-loop no memory Nash equilibrium solution to the game (2.44)–(2.45), and $\{x^*(s), t_0 \le s \le T\}$ is the corresponding state trajectory, if there exist N costate functions $\Lambda^i(s) : [t_0, T] \to R^m$, for $i \in N$, such that the following relations are satisfied:*

$$\vartheta_i^*(s, x^*, x_0) \equiv u_i^*(s) =$$

$$\arg\max_{u_i \in U^i} \left\{ g^i \left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right.$$
$$\left. + \Lambda^i(s) f\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right\},$$

$$\dot{x}^*(s) = f\left[s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right], \quad x^*(t_0) = x_0,$$

$$\begin{aligned}\dot{\Lambda}^i(s) = -\frac{\partial}{\partial x^*} \{ & g^i [s, x^*(s), \vartheta_1^*(s, x^*, x_0), \vartheta_2^*(s, x^*, x_0), \ldots \\ & \ldots, \vartheta_{i-1}^*(s, x^*, x_0), u_i^*(s), \vartheta_{i+1}^*(s, x^*, x_0), \ldots, \vartheta_n^*(s, x^*, x_0)] \\ & + \Lambda^i(s) f[s, x^*(s), \vartheta_1^*(s, x^*, x_0), \vartheta_2^*(s, x^*, x_0), \ldots \\ & \ldots, \vartheta_{i-1}^*(s, x^*, x_0), u_i^*(s), \vartheta_{i+1}^*(s, x^*, x_0), \ldots, \vartheta_n^*(s, x^*, x_0)]\},\end{aligned}$$

$$\Lambda^i(T) = \frac{\partial}{\partial x^*} q^i(x^*(T)); \qquad \text{for } i \in N.$$

Proof. *Consider the i^{th} equality in Theorem 2.2.2, which fixed all players' strategies (except those of the i^{th} player) at $u_j^*(s) = \vartheta_j^*(s, x^*, x_0)$, for $j \ne i$ and $j \in N$, and constitutes an optimal control problem for Player i. Therefore, the above conditions follow from the maximum principle of Pontryagin, and Player i maximizes*

$$\int_{t_0}^{T} g^i \left[s, x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] ds$$
$$+ q^i(x(T)),$$

over the choice of $\upsilon_i(s) \in U^i$ subject to the state dynamics:

$$\dot{x}(s) = f\left[s, x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right],$$
$$x(t_0) = x_0, \qquad \text{for } i \in N.$$

Note that the partial derivative with respect to x in the costate equations of Theorem 2.2.2 receives contributions from dependence of the other $n-1$ players' strategies on the current value of x. This is a feature absent from the costate equations of Theorem 2.2.1. The set of equations in Theorem 2.2.2 in general admits of an uncountable number of solutions, which correspond to "*informationally nonunique*" Nash equilibrium solutions of differential games under memoryless perfect state information pattern. Derivation of nonunique closed-loop Nash equilibria can be found in Basar (1977c) and Mehlmann and Willing (1984).

2.2.3 Feedback Nash Equilibria

To eliminate information nonuniqueness in the derivation of Nash equilibria, one can constrain the Nash solution further by requiring it to satisfy the feedback Nash equilibrium property. In particular, the players' information structures follow either a *closed-loop perfect state* (CLPS) pattern in which $\eta^i(s) = \{x(s), t_0 \le t \le s\}$ or a *memoryless perfect state* (MPS) pattern in which $\eta^i(s) = \{x_0, x(s)\}$. Moreover, we require the following feedback Nash equilibrium condition to be satisfied.

Definition 2.2.2. *For the n-person differential game (2.1)–(2.2) with MPS or CLPS information, an n-tuple of strategies* $\left\{u_i^*(s) = \phi_i^*(s,x) \in U^i,\right.$ *for* $\left. i \in N\right\}$ *constitutes a* feedback Nash equilibrium solution *if there exist functionals* $V^i(t,x)$ *defined on* $[t_0,T] \times R^m$ *and satisfying the following relations for each* $i \in N$*:*

$$V^i(T,x) = q^i(x),$$

$$V^i(t,x) =$$

$$\int_t^T g^i\left[s, x^*(s), \phi_1^*(s,\eta_s), \phi_2^*(s,\eta_s), \ldots, \phi_n^*(s,\eta_s)\right] ds + q^i(x^*(T)) \ge$$

$$\int_t^T g^i\Big[s, x^{[i]}(s), \phi_1^*(s,\eta_s), \phi_2^*(s,\eta_s), \ldots$$

$$\ldots, \phi_{i-1}^*(s,\eta_s), \phi_i(s,\eta_s), \phi_{i+1}^*(s,\eta_s), \ldots, \phi_n^*(s,\eta_s)\Big] ds + q^i\left(x^{[i]}(T)\right),$$

$$\forall \phi_i(\cdot,\cdot) \in \Gamma^i, x \in R^n$$

where on the interval $[t_0,T]$*,*

$$\dot{x}^{[i]}(s) = f\Big[s, x^{[i]}(s), \phi_1^*(s,\eta_s), \phi_2^*(s,\eta_s), \ldots$$

$$\ldots, \phi_{i-1}^*(s,\eta_s), \phi_i(s,\eta_s), \phi_{i+1}^*(s,\eta_s), \ldots, \phi_n^*(s,\eta_s)\Big], \quad x^{[1]}(t) = x;$$

$$\dot{x}^*(s) = f\left[s, x^*(s), \phi_1^*(s,\eta_s), \phi_2^*(s,\eta_s), \ldots, \phi_n^*(s,\eta_s)\right], \quad x(s) = x;$$

and η_s *stands for either the data set* $\{x(s), x_0\}$ *or* $\{x(\tau), \tau \le s\}$*, depending on whether the information pattern is MPS or CLPS.*

One salient feature of the concept introduced above is that if an n-tuple $\{\phi_i^*;\, i \in N\}$ provides a feedback Nash equilibrium solution (FNES) to an N-person differential game with duration $[t_0, T]$, its restriction to the time interval $[t, T]$ provides an FNES to the same differential game defined on the shorter time interval $[t, T]$, with the initial state taken as $x(t)$, and this being so for all $t_0 \le t \le T$. An immediate consequence of this observation is that, under either MPS or CLPS information pattern, feedback Nash equilibrium strategies will depend only on the time variable and the current value of the state, but not on memory (including the initial state x_0). Therefore the players' strategies can be expressed as $\{u_i(s) = \phi_i(s, x)\,,\ \text{for}\ \ i \in N\}$. The following theorem provides a set of necessary conditions characterizing a feedback Nash equilibrium solution for the game (2.44) and (2.45) is characterized as follows:

Theorem 2.2.3. *An n-tuple of strategies $\left\{u_i^*(s) = \phi_i^*(t,x) \in U^i,\ \text{for}\ \ i \in N\right\}$ provides a feedback Nash equilibrium solution to the game (2.44)–(2.45) if there exist continuously differentiable functions $V^i(t,x) : [t_0, T] \times R^m \to R$, $i \in N$, satisfying the following set of partial differential equations:*

$$\begin{aligned}
-V_t^i(t,x) = \max_{u_i} \Big\{ & g^i\big[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \ldots \\
& \qquad \ldots, \phi_{i-1}^*(t,x), u_i(t,x), \phi_{i+1}^*(t,x), \ldots, \phi_n^*(t,x)\big] \\
& + V_x^i(t,x)\, f\big[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \ldots \\
& \qquad \ldots, \phi_{i-1}^*(t,x), u_i(t,x), \phi_{i+1}^*(t,x), \ldots, \phi_n^*(t,x)\big]\Big\} \\
= \Big\{ & g^i\left[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \ldots, \phi_n^*(t,x)\right] \\
& + V_x^i(t,x)\, f\left[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \ldots, \phi_n^*(t,x)\right]\Big\}, \\
V^i(T,x) = q^i(x)\,, & \quad i \in N.
\end{aligned}$$

Proof. By Theorem 2.1.1, $V^i(t,x)$ is the value function associated with the optimal control problem of Player i, $i \in N$. Together with the i^{th} expression in Definition 2.2.1, the conditions in Theorem 2.2.3 imply a Nash equilibrium.

Consider the two-person zero-sum version of the game (2.44)–(2.45) in which the payoff of Player 1 is the negative of that of Player 2. Under either MPS or CLPS information pattern, a feedback saddle-point is characterized as follows.

Theorem 2.2.4. *A pair of strategies $\{\phi_i^*(t,x);\, i = 1, 2\}$ provides a feedback saddle-point solution to the zero-sum version of the game (2.44)–(2.45) if there exists a function $V : [t_0, T] \times R^m \to R$ satisfying the partial differential equation:*

$$-V_t(t,x) = \min_{u_1 \in S^1} \max_{u_2 \in S^2} \left\{ g\left[t, x, u_1(t), u_2(t)\right] + V_x f\left[t, x, u_1(t), u_2(t)\right] \right\}$$

$$= \max_{u_2 \in S^2} \min_{u_1 \in S^1} \{g[t, x, u_1(t), u_2(t)] + V_x f[t, x, u_1(t), u_2(t)]\}$$
$$= \{g[t, x, \phi_1^*(t, x), \phi_2^*(t, x)] + V_x f[t, x, \phi_1^*(t, x), \phi_2^*(t, x)]\},$$
$$V(T, x) = q(x).$$

Proof. This result follows as a special case of Theorem 2.2.3 by taking $n = 2$, $g^1(\cdot) = -g^2(\cdot) \equiv g(\cdot)$, and $q^1(\cdot) = -q^2(\cdot) \equiv q(\cdot)$, in which case $V^1 = -V^2 \equiv V$ and existence of a saddle point is equivalent to interchangeability of the min max operations.

The partial differential equation in Theorem 2.2.4 was first obtained by Isaacs (see, Isaacs, 1965), and is therefore called the Isaacs equation.

While open-loop solutions are relatively tractable and more widely applied, feedback solutions avoid the time-inconsistency problem at the expense of intractability. Examples of differential games solved in feedback Nash solution include Clemhout and Wan (1974), Fershtman (1987), Fershtman and Kamien (1987), Jørgensen (1985), Jørgensen and Sorger (1990), Leitmann and Schmitendorf (1978), Lukes (1971a and 1971b), Sorger (1989), and Yeung (1987, 1989, 1992 and 1994).

In general, feedback Nash equilibrium solution and open-loop Nash equilibrium solution do not coincide. A feedback Nash equilibrium solution of a differential game is a degenerate feedback Nash equilibrium solution if it coincides with the game's open-loop Nash equilibrium solution. In particular, the degenerate Nash equilibrium strategies depend on time only and therefore $\phi_i^*(t, x) = \varphi_i^*(t) = \vartheta_i^*(t, x_0)$, for $i \in N$.

Clemhout and Wan (1974), Leitmann and Schmitendorf (1978), Reinganum (1982a and 1982b), Dockner, Feichtinger and Jørgensen (1985), Jørgensen (1985) and Plourde and Yeung (1989), Yeung (1987 and 1992) identified and examined several classes of games in which the feedback Nash equilibrium and open-loop Nash equilibrium coincide. Fershtman (1987) proposed a technique to identify classes of games for which the open-loop Nash equilibrium is a degenerate feedback Nash equilibrium, while Yeung (1994) established a lemma demonstrating that for every differential game which yields a degenerate/non-degenerate FNE, there exists a class of non-degenerate/degenerate correspondences.

2.3 Application of Differential Games in Economics

In this section we consider application of differential games in competitive advertising.

2.3.1 Open-loop Solution in Competitive Advertising

Consider the competitive dynamic advertising game in Sorger (1989). There are two firms in a market and the profit of firm 1 and that of 2 are respectively:

$$\int_0^T \left[q_1 x(s) - \frac{c_1}{2} u_1(s)^2 \right] \exp(-rs)\, ds + \exp(-rT)\, S_1 x(T)$$

and

$$\int_0^T \left[q_2 (1 - x(s)) - \frac{c_2}{2} u_2(s)^2 \right] \exp(-rs)\, ds + \exp(-rT)\, S_2 [1 - x(T)], \tag{2.46}$$

where r, q_i, c_i, S_i, for $i \in \{1, 2\}$, are positive constants,
$x(s)$ is the market share of firm 1 at time s, $[1 - x(s)]$ is that of firm 2's,
$u_i(s)$ is advertising rate for firm $i \in \{1, 2\}$.

It is assumed that market potential is constant over time. The only marketing instrument used by the firms is advertising. Advertising has diminishing returns, since there are increasing marginal costs of advertising as reflected through the quadratic cost function. The dynamics of firm 1's market share is governed by

$$\dot{x}(s) = u_1(s) [1 - x(s)]^{1/2} - u_2(s)\, x(s)^{1/2}, \qquad x(0) = x_0, \tag{2.47}$$

There are saturation effects, since u_i operates only on the buyer market of the competing firm j.

Consider that the firms would like to seek an open-loop solution. Using open-loop strategies requires the firms to determine their action paths at the outset. This is realistic only if there are restrictive commitments concerning advertising. Invoking Theorem 2.2.1, an open-loop solution to the game (2.46)–(2.47) has to satisfy the following conditions:

$$\begin{aligned}
u_1^*(s) = \arg\max_{u_1} \Big\{ & \left[q_1 x^*(s) - \frac{c_1}{2} u_1(s)^2 \right] \exp(-rs) \\
& + \Lambda^1(s) \left(u_1(s) [1 - x^*(s)]^{1/2} - u_2(s)\, x^*(s)^{1/2} \right) \Big\},
\end{aligned}$$

$$\begin{aligned}
u_2^*(s) = \arg\max_{u_2} \Big\{ & \left[q_2 (1 - x^*(s)) - \frac{c_2}{2} u_2(s)^2 \right] \exp(-rs) \\
& + \Lambda^2(s) \left(u_1(s) [1 - x^*(s)]^{1/2} - u_2(s)\, x^*(s)^{1/2} \right) \Big\},
\end{aligned}$$

$$\dot{x}^*(s) = u_1^*(s) [1 - x^*(s)]^{1/2} - u_2^*(s)\, x^*(s)^{1/2}, \qquad x^*(0) = x_0,$$

$$\dot{\Lambda}^1(s) =$$

$$\left\{ -q_1 \exp(-rs) + \Lambda^1(s) \left(\frac{1}{2} u_1^*(s) [1 - x^*(s)]^{-1/2} + \frac{1}{2} u_2^*(s)\, x^*(s)^{-1/2} \right) \right\},$$

$$\dot{\Lambda}^2(s) =$$

$$\left\{ q_2 \exp(-rs) + \Lambda^2(s) \left(\frac{1}{2} u_1^*(s) [1 - x^*(s)]^{-1/2} + \frac{1}{2} u_2^*(s)\, x^*(s)^{-1/2} \right) \right\},$$

$$\Lambda^1(T) = \exp(-rT) S_1,$$
$$\Lambda^2(T) = -\exp(-rT) S_2. \tag{2.48}$$

Using (2.48), we obtain

$$u_1^*(s) = \frac{\Lambda^1(s)}{c_1} [1 - x^*(s)]^{1/2} \exp(rs), \quad \text{and}$$
$$u_2^*(s) = \frac{\Lambda^2(s)}{c_2} [x^*(s)]^{1/2} \exp(rs).$$

Upon substituting $u_1^*(s)$ and $u_2^*(s)$ into (2.48) yields:

$$\dot{\Lambda}^1(s) = \left\{ -q_1 \exp(-rs) + \left(\frac{\left[\Lambda^1(s)\right]^2}{2c_1} + \frac{\Lambda^1(s)\Lambda^2(s)}{2c_2} \right) \right\},$$
$$\dot{\Lambda}^2(s) = \left\{ q_2 \exp(-rs) + \left(\frac{\left[\Lambda^2(s)\right]^2}{2c_2} + \frac{\Lambda^1(s)\Lambda^2(s)}{2c_1} \right) \right\}, \tag{2.49}$$

with boundary conditions

$$\Lambda^1(T) = \exp(-rT) S_1 \ \text{ and } \ \Lambda^2(T) = -\exp(-rT) S_2.$$

The game equilibrium state dynamics becomes:

$$\dot{x}^*(s) = \frac{\Lambda^1(s)\exp(rs)}{c_1} [1 - x^*(s)] - \frac{\Lambda^2(s)\exp(rs)}{c_2} x^*(s),$$
$$x^*(0) = x_0. \tag{2.50}$$

Solving the block recursive system of differential equations (2.49)–(2.50) gives the solutions to $x^*(s)$, $\Lambda^1(s)$ and $\Lambda^2(s)$. Upon substituting them into $u_1^*(s)$ and $u_2^*(s)$ yields the open-loop game equilibrium strategies.

2.3.2 Feedback Solution in Competitive Advertising

A feedback solution which allows the firm to choose their advertising rates contingent upon the state of the game is a realistic approach to the problem (2.46)–(2.47). Invoking Theorem 2.2.3, a feedback Nash equilibrium solution to the game (2.46)–(2.47) has to satisfy the following conditions:

$$-V_t^1(t,x) = \max_{u_1} \left\{ \left[q_1 x - \frac{c_1}{2} u_1^2 \right] \exp(-rt) \right.$$
$$\left. + V_x^1(t,x) \left(u_1 [1 - x]^{1/2} - \phi_2^*(t,x) x^{1/2} \right) \right\},$$
$$-V_t^2(t,x) = \max_{u_2} \left\{ \left[q_2 (1 - x) - \frac{c_2}{2} u_2^2 \right] \exp(-rt) \right.$$
$$\left. + V_x^2(t,x) \left(\phi_1^*(t,x) [1 - x]^{1/2} - u_2 x^{1/2} \right) \right\},$$
$$V^1(T,x) = \exp(-rT) S_1 x,$$
$$V^2(T,x) = \exp(-rT) S_2 (1 - x). \tag{2.51}$$

Performing the indicated maximization in (2.51) yields:

$$\phi_1^*(t,x) = \frac{V_x^1(t,x)}{c_1}[1-x]^{1/2}\exp(rt) \quad \text{and}$$
$$\phi_2^*(t,x) = \frac{V_x^2(t,x)}{c_2}[x]^{1/2}\exp(rt).$$

Upon substituting $\phi_1^*(t,x)$ and $\phi_2^*(t,x)$ into (2.51) and solving (2.51) we obtain the value functions:

$$V^1(t,x) = \exp[-r(t)][A_1(t)x + B_1(t)] \quad \text{and}$$
$$V^2(t,x) = \exp[-r(t)][A_2(t)(1-x) + B_2(t)] \tag{2.52}$$

where $A_1(t)$, $B_1(t)$, $A_2(t)$ and $B_2(t)$ satisfy:

$$\dot{A}_1(t) = rA_1(t) - q_1 + \frac{A_1(t)^2}{2c_1} + \frac{A_1(t)A_2(t)}{2c_2},$$
$$\dot{A}_2(t) = rA_2(t) - q_2 + \frac{A_2(t)^2}{2c_2} + \frac{A_1(t)A_2(t)}{2c_1},$$
$$A_1(T) = S_1,\ B_1(T) = 0,\ A_2(T) = S_2 \text{ and } B_2(T) = 0.$$

Upon substituting the relevant partial derivatives of $V^1(t,x)$ and $V^2(t,x)$ from (2.52) into (2.51) yields the feedback Nash equilibrium strategies

$$\phi_1^*(t,x) = \frac{A_1(t)}{c_1}[1-x]^{1/2} \quad \text{and} \quad \phi_2^*(t,x) = \frac{A_2(t)}{c_2}[x]^{1/2}. \tag{2.53}$$

2.4 Infinite-Horizon Differential Games

Consider the infinite-horizon autonomous game problem with constant discounting, in which T approaches infinity and where the objective functions and state dynamics are both autonomous. In particular, the game becomes:

$$\max_{u_i} \int_{t_0}^{\infty} g^i[x(s), u_1(s), u_2(s), \ldots, u_n(s)] \exp[-r(s-t_0)]\, ds,$$
$$\text{for } i \in N, \tag{2.54}$$

subject to the deterministic dynamics

$$\dot{x}(s) = f[x(s), u_1(s), u_2(s), \ldots, u_n(s)], \quad x(t_0) = x_0, \tag{2.55}$$

where r is a constant discount rate.

2.4.1 Game Equilibrium Solutions

Now consider the alternative game to (2.54)–(2.55)

$$\max_{u_i} \int_t^\infty g^i \left[x(s), u_1(s), u_2(s), \ldots, u_n(s)\right] \exp\left[-r(s-t)\right] ds,$$
$$\text{for } i \in N, \tag{2.56}$$

subject to the deterministic dynamics

$$\dot{x}(s) = f\left[x(s), u_1(s), u_2(s), \ldots, u_n(s)\right], \quad x(t) = x. \tag{2.57}$$

The infinite-horizon autonomous game (2.56)–(2.57) is independent of the choice of t and dependent only upon the state at the starting time, that is x.

In the infinite-horizon optimization problem in Section 2.1.1, the control is shown to be a function of the state variable x only. With the validity of the game equilibrium $\left\{u_i^*(s) = \phi_i^*(x) \in U^i, \text{ for } i \in N\right\}$ to be verified later, we define

Definition 2.4.1. *For the n-person differential game (2.54)–(2.55) with MPS or CLPS information, an n-tuple of strategies*

$$\left\{u_i^*(s) = \phi_i^*(x) \in U^i, \textit{ for } i \in N\right\}$$

constitutes a feedback Nash equilibrium solution *if there exist functionals $V^i(t,x)$ defined on $[t_0, \infty) \times R^m$ and satisfying the following relations for each $i \in N$:*

$$V^i(t,x) = \int_t^\infty g^i\left[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)\right] \exp\left[-r(s-t_0)\right] ds \geq$$
$$\int_t^\infty g^i\left[x^{[i]}(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_{i-1}^*(\eta_s)\, \phi_i(\eta_s)\, \phi_{i+1}^*(\eta_s), \ldots, \phi_n^*(\eta_s)\right]$$
$$\times \exp\left[-r(s-t_0)\right] ds$$

$$\forall \phi_i(\cdot,\cdot) \in \Gamma^i, \quad x \in R^n,$$

where on the interval $[t_0, \infty)$,

$$\dot{x}^{[i]}(s) =$$
$$f\left[s^{[i]}(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_{i-1}^*(\eta_s)\, \phi_i(\eta_s)\, \phi_{i+1}^*(\eta_s), \ldots, \phi_n^*(\eta_s)\right],$$
$$x^{[1]}(t) = x;$$
$$\dot{x}^*(s) = f\left[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)\right], \qquad x^*(s) = x;$$

and η_s stands for either the data set $\{x(s), x_0\}$ or $\{x(\tau), \tau \leq s\}$, depending on whether the information pattern in MPS or CLPS.

We can write

$$V^i(t,x) = \exp[-r(t-t_0)] \int_t^\infty g^i[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)] \times \exp[-r(s-t)]\, ds,$$
$$\text{for } x(t) = x = x_t^* = x^*(t).$$

Since

$$\int_t^\infty g^i[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)] \exp[-r(s-t)]\, ds$$

depends on the current state x only, we can write:

$$W^i(x) = \int_t^\infty g^i[x^*(s), \phi_1^*(\eta_s), \phi_2^*(\eta_s), \ldots, \phi_n^*(\eta_s)] \exp[-r(s-t)]\, ds.$$

It follows that:

$$\begin{aligned} &V^i(t,x) = \exp[-r(t-t_0)]\, W^i(x), \\ &V_t^i(t,x) = -r\exp[-r(t-t_0)]\, W^i(x), \text{ and} \\ &V_x^i(t,x) = \exp[-r(t-t_0)]\, W_x^i(x), \text{ for } i \notin N. \end{aligned} \tag{2.58}$$

A feedback Nash equilibrium solution for the infinite-horizon autonomous game (2.56) and (2.57) can be characterized as follows:

Theorem 2.4.1. *An n-tuple of strategies $\{u_i^*(s) = \phi_i^*(\cdot) \in U^i;\ \text{for } i \in N\}$ provides a feedback Nash equilibrium solution to the infinite-horizon game (2.56) and (2.57) if there exist continuously differentiable functions $W^i(x) : R^m \to R,\ i \in N$, satisfying the following set of partial differential equations:*

$$\begin{aligned} &rW^i(x) = \\ &\max_{u_i} \left\{ g^i\left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_{i-1}^*(x), u_i, \phi_{i+1}^*(x), \ldots, \phi_n^*(x)\right] \right. \\ &\quad \left. + W_x^i(x) f\left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \ldots, \phi_n^*(x)\right] \right\} \\ &= \left\{ g^i[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)] \right. \\ &\quad \left. + W_x^i(x) f[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)] \right\}, \qquad \text{for } i \in N. \end{aligned}$$

Proof. *By Theorem 2.1.2, $W^i(x)$ is the value function associated with the optimal control problem of Player i, $i \in N$. Together with the i^{th} expression in Definition 2.4.1, the conditions in Theorem 2.4.1 imply a Nash equilibrium.*

Since time s is not explicitly involved the partial differential equations in Theorem 2.4.1, the validity of the feedback Nash equilibrium $\{u_i^* = \phi_i^*(x),$ for $i \in N\}$ are functions independent of time is obtained.

Substituting the game equilibrium strategies in Theorem 2.4.1 into (2.55) yields the game equilibrium dynamics of the state path as:

$$\dot{x}(s) = f\left[x(s), \phi_1^*(x(s)), \phi_2^*(x(s)), \ldots, \phi_n^*(x(s))\right], \qquad x(t_0) = x_0.$$

Solving the above dynamics yields the optimal state trajectory $\{x^*(t)\}_{t \geq t_0}$ as

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[x^*(s), \phi_1^*(x^*(s)), \phi_2^*(x^*(s)), \ldots, \phi_n^*(x^*(s))\right] ds,$$
$$\text{for } \; t \geq t_0.$$

We denote term $x^*(t)$ by x_t^*. The feedback Nash equilibrium strategies for the infinite-horizon game (2.54)–(2.55) can be obtained as

$$\left[\phi_1^*(x_t^*), \phi_2^*(x_t^*), \ldots, \phi_n^*(x_t^*)\right], \qquad \text{for } \; t \geq t_0.$$

Following the above analysis and using Theorems 2.1.4 and 2.2.1, we can characterize an open loop equilibrium solution to the infinite-horizon game (2.56) and (2.57) as:

Theorem 2.4.2. *A set of strategies* $\{u_i^*(s) = \zeta_i^*(s, x_t), \text{ for } \; i \in N\}$ *provides an open-loop Nash equilibrium solution to the infinite-horizon game (2.56) and (2.57), and* $\{x^*(s), \; t \leq s \leq T\}$ *is the corresponding state trajectory, if there exist* m *costate functions* $\Lambda^i(s) : [t, T] \to R^m$, *for* $i \in N$, *such that the following relations are satisfied:*

$$\begin{aligned}
&\zeta_i^*(s, x) \equiv u_i^*(s) = \\
&\quad \arg\max_{u_i \in U^i} \left\{ g^i \left[x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right. \\
&\quad \left. +\lambda^i(s) f\left[x^*(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] \right\}, \\
&\dot{x}^*(s) = f\left[x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right], \qquad x^*(t) = x_t, \\
&\dot{\lambda}^i(s) = r\lambda(s) - \frac{\partial}{\partial x^*} \left\{ g^i \left[x^*(s), u_1^*(s), u_2^*(s), \ldots \right.\right. \\
&\qquad\qquad \left.\left. \ldots, u_n^*(s)\right] + \lambda^i(s) f\left[x^*(s), u_1^*(s), u_2^*(s), \ldots, u_n^*(s)\right] \right\}, \\
&\qquad \textit{for } \; i \in N.
\end{aligned}$$

Proof. Consider the i^{th} *equality in Theorem 2.4.2, which states that* $\upsilon_i^*(s) = u_i^*(s) = \zeta_i^*(s, x_t)$ *maximizes*

$$\int_{t_0}^{\infty} g^i \left[x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right] ds,$$

over the choice of $\upsilon_i(s) \in U^i$ *subject to the state dynamics:*

$$\dot{x}(s) = f\left[x(s), u_1^*(s), u_2^*(s), \ldots, u_{i-1}^*(s), u_i(s), u_{i+1}^*(s), \ldots, u_n^*(s)\right],$$
$$x(t) = x_t, \qquad \textit{for } i \in N.$$

This is the infinite-horizon optimal control problem for Player i, since $u_j^*(s)$, for $j \in N$ and $j \neq i$, are open-loop controls and hence do not depend on $u_i^*(s)$. These results are stated in Theorem 2.1.4.

2.4.2 Infinite-Horizon Duopolistic Competition

Consider a dynamic duopoly in which there are two publicly listed firms selling a homogeneous good. Since the value of a publicly listed firm is the present value of its discounted expected future earnings, the terminal time of the game, T, may be very far in the future and nobody knows when the firms will be out of business. Therefore setting $T = \infty$ may very well be the best approximation for the true game horizon. Even if the firm's management restricts itself to considering profit maximization over the next year, it should value its asset positions at the end of the year by the earning potential of these assets in the years to come. There is a lag in price adjustment so the evolution of market price over time is assumed to be a function of the current market price and the price specified by the current demand condition. In particular, we follow Tsutsui and Mino (1990) and assume that

$$\dot{P}(s) = k\left[a - u_1(s) - u_2(s) - P(s)\right], \quad P(t_0) = P_0, \tag{2.59}$$

where $P(s)$ is the market price at time s, $u_i(s)$ is output supplied firm $i \in \{1, 2\}$, current demand condition is specified by the instantaneous inverse demand function $P(s) = [a - u_1(s) - u_2(s)]$, and $k > 0$ represents the price adjustment velocity.

The payoff of firm i is given as the present value of the stream of discounted profits:

$$\int_{t_0}^{\infty} \left\{P(s)\, u_i(s) - c u_i(s) - (1/2)\left[u_i(s)\right]^2\right\} \exp\left[-r(s - t_0)\right] ds,$$
$$\text{for } i \in \{1, 2\}, \tag{2.60}$$

where $c u_i(s) + (1/2)\left[u_i(s)\right]^2$ is the cost of producing output $u_i(s)$ and r is the interest rate.

Once again, we consider the alternative game

$$\max_{u_i} \int_{t_0}^{\infty} \left\{P(s)\, u_i(s) - c u_i(s) - (1/2)\left[u_i(s)\right]^2\right\} \exp\left[-r(s - t)\right] ds,$$
$$\text{for } i \in \{1, 2\}, \tag{2.61}$$

subject to

$$\dot{P}(s) = k\left[a - u_1(s) - u_2(s) - P(s)\right], \quad P(t) = P. \tag{2.62}$$

The infinite-horizon game (2.61)–(2.62) has autonomous structures and a constant rate. Therefore, we can apply Theorem 2.4.1 to characterize a feedback Nash equilibrium solution as:

$$
\begin{aligned}
rW^i(P) = \max_{u_i} \Big\{ & \left[Pu_i - cu_i - (1/2)(u_i)^2 \right] \\
& + W_P^i \left[k \left(a - u_i - \phi_j^*(P) - P \right) \right] \Big\}, \quad \text{for } i \in \{1, 2\}. \qquad (2.63)
\end{aligned}
$$

Performing the indicated maximization in (2.63), we obtain:

$$
\phi_i^*(P) = P - c - kW_P^i(P), \quad \text{for } i \in \{1, 2\}. \qquad (2.64)
$$

Substituting the results from (2.64) into (2.63), and upon solving (2.63) yields:

$$
W^i(P) = \frac{1}{2} A P^2 - BP + C, \qquad (2.65)
$$

where

$$
\begin{aligned}
A &= \frac{r + 6k - \sqrt{(r+6k)^2 - 12k^2}}{6k^2}, \\
B &= \frac{-akA + c - 2kcA}{r - 3k^2 A + 3k}, \quad \text{and} \\
C &= \frac{c^2 + 3k^2 B^2 - 2kB(2c + a)}{2r}.
\end{aligned}
$$

Again, one can readily verify that $W^i(P)$ in (2.65) indeed solves (2.63) by substituting $W^i(P)$ and its derivative into (2.63) and (2.64).

The game equilibrium strategy can then be expressed as:

$$
\phi_i^*(P) = P - c - k(AP - B), \quad \text{for } i \in \{1, 2\}.
$$

Substituting the game equilibrium strategies above into (2.59) yields the game equilibrium state dynamics of the game (2.59)–(2.60) as:

$$
\dot{P}(s) = k \left[a - 2(c + kB) - (3 - kA) P(s) \right], \quad P(t_0) = P_0.
$$

Solving the above dynamics yields the optimal state trajectory as

$$
P^*(t) = \left[P_0 - \frac{k[a - 2(c + kB)]}{k(3 - kA)} \right] \exp\left[-k(3 - kA)t \right] + \frac{k[a - 2(c + kB)]}{k(3 - kA)}.
$$

We denote term $P^*(t)$ by P_t^*. The feedback Nash equilibrium strategies for the infinite-horizon game (2.59)–(2.60) can be obtained as

$$
\phi_i^*(P_t^*) = P_t^* - c - k(AP_t^* - B), \quad \text{for } \{1, 2\}.
$$

An open loop solution for the game is left for the readers to do in Problem 2.4 in Section 2.8.

2.5 Stochastic Differential Games and their Solutions

One way to incorporate stochastic elements in differential games is to introduce stochastic dynamics. A stochastic formulation for quantitative differential games of prescribed duration involves a vector-valued stochastic differential equation

$$\begin{aligned} &dx(s) = f[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)]\,ds + \sigma[s, x(s)]\,dz(s), \\ &x(t_0) = x_0. \end{aligned} \tag{2.66}$$

which describes the evolution of the state and N objective functionals

$$E_{t_0}\left\{\int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)]\,ds + q^i(x(T))\right\}, \quad \text{for } i \in N, \tag{2.67}$$

with $E_{t_0}\{\cdot\}$ denoting the expectation operation taken at time t_0, $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\,\sigma[s, x(s)]'$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$. Moreover, $E[dz_\varpi] = 0$ and $E[dz_\varpi dt] = 0$ and $E\left[(dz_\varpi)^2\right] = dt$, for $\varpi \in [1, 2, \ldots, \Theta]$; and $E[dz_\varpi dz_\omega] = 0$, for $\varpi \in [1, 2, \ldots, \Theta]$, $\omega \in [1, 2, \ldots, \Theta]$ and $\varpi \neq \omega$. Given the stochastic nature, the information structures must follow the MPS pattern or CLPS pattern or the *feedback perfect state* (FB) pattern in which $\eta^i(s) = \{x(s)\}$, $s \in [t_0, T]$.

A Nash equilibrium of the stochastic game (2.66)–(2.67) can be characterized as:

Theorem 2.5.1. *An n-tuple of feedback strategies $\left\{\phi_i^*(t, x) \in U^i;\ i \in N\right\}$ provides a Nash equilibrium solution to the game (2.66)–(2.67) if there exist suitably smooth functions $V^i : [t_0, T] \times R^m \to R$, $i \in N$, satisfying the semilinear parabolic partial differential equations*

$$\begin{aligned} -V_t^i - \frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(t, x)\, V^i_{x_h x_\zeta} = \\ \max_{u_i} \big\{ g^i [t, x, \phi_1^*(t, x), \phi_2^*(t, x), \ldots \\ \ldots, \phi_{i-1}^*(t, x), u_i(t), \phi_{i+1}^*(t, x), \ldots, \phi_n^*(t, x)] \\ + V_x^i(t, x) f[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \ldots \\ \ldots, \phi_{i-1}^*(t, x), u_i(t), \phi_{i+1}^*(t, x), \ldots, \phi_n^*(t, x)] \big\} \\ = \big\{ g^i [t, x, \phi_1^*(t, x), \phi_2^*(t, x), \ldots, \phi_n^*(t, x)] \\ + V_x^i(t, x) f[t, x, \phi_1^*(t, x), \phi_2^*(t, x), \ldots, \phi_n^*(t, x)] \big\}, \\ V^i(T, x) = q^i(x), \quad i \in N. \end{aligned}$$

Proof. This result follows readily from the definition of Nash equilibrium and from Theorem 2.1.5, since by fixing all players' strategies, except the i^{th} one's, at their equilibrium choices (which are known to be feedback by hypothesis), we arrive at a stochastic optimal control problem of the type covered by Theorem 2.1.5 and whose optimal solution (if it exists) is a feedback strategy.

Consider the two-person zero-sum version of the game (2.66)–(2.67) in which the payoff of Player 1 is the negative of that of Player 2. Under either MPS or CLPS information pattern, a Nash equilibrium solution can be characterized as follows.

Theorem 2.5.2. *A pair of strategies $\left\{\phi_i^*(t,x) \in U^i;\ i = 1,2\right\}$ provides a feedback saddle-point solution to the two-person zero-sum version of the game (2.66)–(2.67) if there exists a function $V : [t_0, T] \times R^m \to R$ satisfying the partial differential equation:*

$$
\begin{aligned}
&-V_t - \frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(t,x)\, V_{x_h x_\zeta} \\
&\qquad = \min_{u_1\in S^1}\max_{u_2\in S^2}\left\{g\left[t,x,u_1,u_2\right] + V_x f\left[t,x,u_1,u_2\right]\right\} \\
&\qquad = \max_{u_2\in S^2}\min_{u_1\in S^1}\left\{g\left[t,x,u_1,u_2\right] + V_x f\left[t,x,u_1,u_2\right]\right\} \\
&\qquad = \left\{g\left[t,x,\phi_1^*(t,x),\phi_2^*(t,x)\right] + V_x f\left[t,x(t)\,\phi_1^*(t,x),\phi_2^*(t,x)\right]\right\}, \\
&V(T,x) = q(x).
\end{aligned}
$$

Proof. This result follows as a special case of Theorem 2.5.1 by taking $n = 2$, $g^1(\cdot) = -g^2(\cdot) \equiv g(\cdot)$, and $q^1(\cdot) = -q^2(\cdot) \equiv q(\cdot)$, in which case $V^1 = -V^2 \equiv V$ and existence of a saddle point is equivalent to interchangeability of the min max operations.

Basar (1977a, 1977c and 1980) was first to derive explicit results for stochastic quadratic differential games. Subsequent examples of solvable stochastic differential games include Clemhout and Wan (1985), Kaitala (1993), Jørgensen and Yeung (1996 and 1999), and Yeung (1998, 1999 and 2001).

2.6 An Application of Stochastic Differential Games in Resource Extraction

Consider an economy endowed with a renewable resource and with $n \geq 2$ resource extractors (firms). The lease for resource extraction begins at time t_0 and ends at time T. Let $u_i(s)$ denote the rate of resource extraction of firm i at time s, $i \in N = \{1, 2, \ldots, n\}$, where each extractor controls its rate of extraction. Let U^i be the set of admissible extraction rates, and $x(s)$ the size of the resource stock at time s. In particular, we have $U^i \in R^+$ for $x > 0$, and $U^i = \{0\}$ for $x = 0$. The extraction cost for firm $i \in N$ depends on the quantity of resource extracted $u_i(s)$, the resource stock size $x(s)$, and a parameter c.

In particular, extraction cost can be specified as $C^i = cu_i(s)/x(s)^{1/2}$. The market price of the resource depends on the total amount extracted and supplied to the market. The price-output relationship at time s is given by the following downward sloping inverse demand curve $P(s) = Q(s)^{-1/2}$, where $Q(s) = \sum_{i \in N} u_i(s)$ is the total amount of resource extracted and marketed at time s. A terminal bonus $wx(T)$ is offered to each extractor and r is a discount rate which is common to all extractors. Extractor i seeks to maximize the expected payoff:

$$E_{t_0}\left\{\int_{t_0}^{T}\left[\left(\sum_{j=1}^{n} u_j(s)\right)^{-1/2} u_i(s) - \frac{c}{x(s)^{1/2}} u_i(s)\right]\exp\left[-r(t-t_0)\right] ds \right.$$
$$\left. + \exp\left[-r(T-t_0)\right] wx(T)^{1/2}\right\}, \qquad \text{for } i \in N, \tag{2.68}$$

subject to the resource dynamics:

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - \sum_{j=1}^{n} u_j(s)\right] ds + \sigma x(s)\, dz(s),$$
$$x(t_0) = x_0 \in X. \tag{2.69}$$

Invoking Theorem 2.5.1, a set of feedback strategies $\{u_i^*(t) = \phi_i^*(t,x);\ i \in N\}$ constitutes a Nash equilibrium solution for the game (2.68)–(2.69), if there exist functionals $V^i(t,x) : [t_0,T] \times R \to R$, for $i \in N$, which satisfy the following set of partial differential equations:

$$-V_t^i(t,x) - \frac{1}{2}\sigma^2 x^2 V_{xx}^i(t,x)$$
$$= \max_{u_i \in U^i}\left\{\left[u_i\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\phi_j^*(t,x) + u_i\right)^{-1/2} - \frac{c}{x^{1/2}}u_i(t)\right]\exp\left[-r(t-t_0)\right]\right.$$
$$\left. + V_x^i\left[ax^{1/2} - bx - \sum_{\substack{j=1\\ j\neq i}}^{n}\phi_j^*(t,x) - u_i\right]\right\}, \text{ and}$$
$$V^i(T,x) = \exp\left[-r(T-t_0)\right] wx^{1/2}. \tag{2.70}$$

Applying the maximization operator on the right-hand-side of the first equation in (2.70) for Player i, yields the condition for a maximum as:

$$\left[\left(\sum_{\substack{j=1\\j\neq i}}^{n}\phi_j^*(t,x)+\frac{1}{2}\phi_i^*(t,x)\right)\left(\sum_{j=1}^{n}\phi_j^*(t,x)\right)^{-3/2}-\frac{c}{x^{1/2}}\right]$$

$$\times\exp\left[-r\left(t-t_0\right)\right]-V_x^i=0,\qquad \text{for } i\in N. \tag{2.71}$$

Summing over $i=1,2,\ldots,n$ in (2.71) yields:

$$\left(\sum_{j=1}^{n}\phi_j^*(t,x)\right)^{1/2}=\left(n-\frac{1}{2}\right)\left(\sum_{j=1}^{n}\left[\frac{c}{x^{1/2}}+\exp\left[r\left(t-t_0\right)\right]V_x^j\right]\right)^{-1}. \tag{2.72}$$

Substituting (2.72) into (2.71) produces

$$\left(\sum_{\substack{j=1\\j\neq i}}^{n}\phi_j^*(t,x)+\frac{1}{2}\phi_i^*(t,x)\right)\left(n-\frac{1}{2}\right)^{-3}\left(\sum_{j=1}^{n}\left[\frac{c}{x^{1/2}}+\exp\left[r\left(t-t_0\right)\right]V_x^j\right]\right)^{3}$$

$$-\frac{c}{x^{1/2}}-\exp\left[r\left(t-t_0\right)\right]V_x^i=0,\quad \text{for } i\in N. \tag{2.73}$$

Re-arranging terms in (2.73) yields:

$$\left(\sum_{\substack{j=1\\j\neq i}}^{n}\phi_j^*(t,x)+\frac{1}{2}\phi_i^*(t,x)\right)=$$

$$\left(n-\frac{1}{2}\right)^{3}\frac{\left[c+\exp\left[r\left(t-t_0\right)\right]V_x^i x^{1/2}\right]x}{\left(\sum_{j=1}^{n}\left[c+\exp\left[r\left(t-t_0\right)\right]V_x^j x^{1/2}\right]\right)^{3}},$$

$$\text{for } i\in N. \tag{2.74}$$

Condition (2.74) represents a system of equations which is linear in $\{\phi_1^*(t,x),\phi_2^*(t,x),\ldots,\phi_n^*(t,x)\}$. Solving (2.74) yields:

$$\phi_i^*(t,x)=\frac{x\left(2n-1\right)^2}{2\left[\sum_{j=1}^{n}\left[c+\exp\left[r\left(t-t_0\right)\right]V_x^j x^{1/2}\right]\right]^3}$$

$$\times\left\{\sum_{\substack{j=1\\j\neq i}}^{n}\left[c+\frac{V_x^j x^{1/2}}{\exp\left[-r\left(t-t_0\right)\right]}\right]-\left(n-\frac{3}{2}\right)\left[c+\frac{V_x^i x^{1/2}}{\exp\left[-r\left(t-t_0\right)\right]}\right]\right\},$$

$$\text{for } i\in N. \tag{2.75}$$

Substituting $\phi_i^*(t,x)$ in (2.75) into (2.70) and upon solving yields:

Corollary 2.6.1. *The system (2.70) admits a solution*

$$V^i(t,x) = \exp\left[-r(t-t_0)\right]\left[A(t)\,x^{1/2} + B(t)\right], \tag{2.76}$$

where $A(t)$ and $B(t)$ satisfy:

$$\begin{aligned}
\dot{A}(t) &= \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right] A(t) - \frac{(2n-1)}{2n^2}\left(c + \frac{A(t)}{2}\right)^{-1} \\
&\quad + \frac{c\,(2n-1)^2}{4n^3}\left(c + \frac{A(t)}{2}\right)^{-2} + \frac{(2n-1)^2 A(t)}{8n^2\left(c + \frac{A(t)}{2}\right)^2}, \\
\dot{B}(t) &= rB(t) - \frac{a}{2}A(t), \\
A(T) &= w, \text{ and} \\
B(T) &= 0.
\end{aligned} \tag{2.77}$$

The first equation in (2.77) can be further reduced to:

$$\begin{aligned}
\dot{A}(t) = &\left\{\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)\frac{[A(t)]^3}{4} + \left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c\,[A(t)]^2\right. \\
&\left. + \left[\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c^2 + \frac{\left(4n^2 - 8n + 3\right)}{8n^2}\right]A(t) - \frac{(2n-1)\,c}{4n^3}\right\} \\
&\div \left(c + \frac{A(t)}{2}\right)^2.
\end{aligned} \tag{2.78}$$

The denominator of the right-hand-side of (2.78) is always positive. Denote the numerator of the right-hand-side of (2.78) by:

$$F[A(t)] - \frac{(2n-1)\,c}{4n^3}. \tag{2.79}$$

In particular, $F[A(t)]$ is a polynomial function in $A(t)$ of degree 3. Moreover, $F[A(t)] = 0$ for $A(t) = 0$, and for any $A(t) \in (0,\infty)$,

$$\begin{aligned}
\frac{dF[A(t)]}{dA(t)} = &\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)\frac{3\,[A(t)]^2}{4} + 2\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c\,[A(t)] \\
&+ \left[\left(r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right)c^2 + \frac{\left(4n^2 - 8n + 3\right)}{8n^2}\right] > 0.
\end{aligned} \tag{2.80}$$

Therefore, there exists a unique level of $A(t)$, denoted by A^* at which

$$F\left[A^*\right] - \frac{(2n-1)\,c}{4n^3} = 0. \tag{2.81}$$

If $A(t)$ equals A^*, $\dot{A}(t) = 0$. For values of $A(t)$ less than A^*, $\dot{A}(t)$ is negative. For values of $A(t)$ greater than A^*, $\dot{A}(t)$ is positive. A phase diagram depicting the relationship between $\dot{A}(t)$ and $A(t)$ is provided in Figure 2.1a, while the time paths of $A(t)$ in relation to A^* are illustrated in Figure 2.1b.

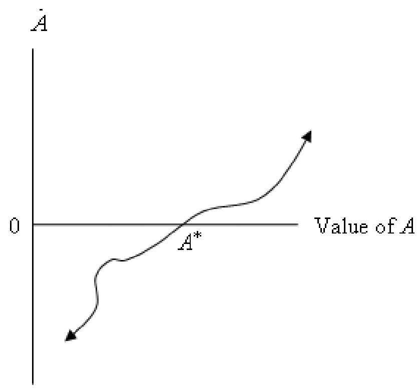

Fig. 2.1a. Phase diagram depicting the relationship between $\dot{A}$ and A.

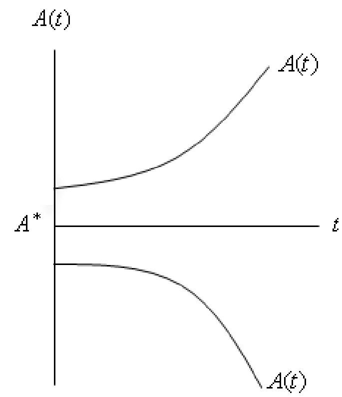

Fig. 2.1b. The time paths of $A(t)$ in relation to A^*.

For a given value of w which is less than A^*, the time path $\{A(t)\}_{t=t_0}^{T}$ will start at a value $A(t_0)$, which is greater than w and less than A^*. The value of $A(t)$ will decrease over time and reach w at time T. On the other hand, for a given value of w which is greater than A^*, the time path $\{A(t)\}_{t=t_0}^{T}$ will start at a value $A(t_0)$, which is less than w and greater than A^*. The value of $A(t)$ will increase over time and reach w at time T. Therefore $A(t)$ is a monotonic function and $A(t) > 0$ for $t \in [t_\tau, T]$.

Using $A(t)$, the solution to $B(t)$ can be readily obtained as:

$$B(t) = \exp(rt)\left(K - \int_{t_0}^{t} \frac{a}{2} A(s) \exp(-rs)\, ds\right), \tag{2.82}$$

where

$$K = \int_{t_0}^{T} \frac{a}{2} A(s) \exp(-rs)\, ds.$$

2.7 Infinite-Horizon Stochastic Differential Games

Consider the infinite-horizon autonomous game problem with constant discounting, in which T approaches infinity and where the objective functions and state dynamics are both autonomous. In particular, the game can be formulated as:

$$\max_{u_i} E_{t_0}\left\{\int_{t_0}^{\infty} g^i\left[x(s), u_1(s), u_2(s), \ldots, u_n(s)\right]\exp\left[-r(s-t_0)\right] ds\right\},$$
$$\text{for } i \in N, \tag{2.83}$$

subject to the stochastic dynamics

$$dx(s) = f\left[x(s), u_1(s), u_2(s), \ldots, u_n(s)\right] ds + \sigma\left[x(s)\right] dz(s),$$
$$x(t_0) = x_0. \tag{2.84}$$

Consider the alternative game

$$\max_{u_i} E_t\left\{\int_t^{\infty} g^i\left[x(s), u_1(s), u_2(s), \ldots, u_n(s)\right] \times \exp\left[-r(s-t)\right] ds \,\middle|\, x(t) = x_t\right\},$$
$$\text{for } i \in N, \tag{2.85}$$

subject to the stochastic dynamics

$$dx(s) = f\left[x(s), u_1(s), u_2(s), \ldots, u_n(s)\right] ds + \sigma\left[x(s)\right] dz(s),$$
$$x(t) = x_t. \tag{2.86}$$

The infinite-horizon autonomous game (2.85)–(2.86) is independent of the choice of t and dependent only upon the state at the starting time, that is, x.

Using Theorem 2.1.6 and following the analysis leading to Theorem 2.4.1, a Nash equilibrium solution for the infinite-horizon stochastic differential game (2.85)–(2.86) can be characterized as:

Theorem 2.7.1. *An n-tuple of strategies $\left\{u_i^*(s) = \phi_i^*(\cdot) \in U^i,\ \text{for } i \in N\right\}$ provides a Nash equilibrium solution to the game (2.85)–(2.86) if there exist continuously twice differentiable functions $W^i(x) : R^m \to R$, $i \in N$, satisfying the following set of partial differential equations:*

$$rW^i(x) - \frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(x)\, W^i_{x_h x_\zeta}(x) =$$
$$\max_{u_i}\left\{g^i\left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \ldots, \phi_n^*(x)\right] + W^i_x(x)\, f\left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \ldots, \phi_n^*(x)\right]\right\}$$
$$= \left\{g^i\left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)\right] + W^i_x(x)\, f\left[x, \phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)\right]\right\}, \qquad \textit{for } i \in N.$$

Proof. This result follows readily from the definition of Nash equilibrium and from Theorem 2.1.6, since by fixing all players' strategies, except the i^{th} one's, at their equilibrium choices (which are known to be feedback by hypothesis), we arrive at a stochastic optimal control problem of the type covered by Theorem 2.1.6 and whose optimal solution (if it exists) is a feedback strategy.

Example 2.7.1. Consider the infinite-horizon game in which extractor i seeks to maximize the expected payoff:

$$E_{t_0}\left\{\int_{t_0}^{T}\left[\left(\sum_{j=1}^{n}u_j(s)\right)^{-1/2}u_i(s)-\frac{c}{x(s)^{1/2}}u_i(s)\right]\exp\left[-r(s-t_0)\right]ds\right\},$$
$$\text{for } i\in N, \tag{2.87}$$

subject to the resource dynamics:

$$dx(s)=\left[ax(s)^{1/2}-bx(s)-\sum_{j=1}^{n}u_j(s)\right]ds+\sigma x(s)\,dz(s),$$
$$x(t_0)=x_0\in X. \tag{2.88}$$

Consider the alternative problem

$$E_{t}\left\{\int_{t}^{T}\left[\left(\sum_{j=1}^{n}u_j(s)\right)^{-1/2}u_i(s)-\frac{c}{x(s)^{1/2}}u_i(s)\right]\exp\left[-r(s-t)\right]ds\right\},$$
$$\text{for } i\in N, \tag{2.89}$$

subject to the resource dynamics:

$$dx(s)=\left[ax(s)^{1/2}-bx(s)-\sum_{j=1}^{n}u_j(s)\right]ds+\sigma x(s)\,dz(s),$$
$$x(t)=x\in X. \tag{2.90}$$

Invoking Theorem 2.7.1 we obtain, a set of feedback strategies $\{\phi_i^*(x),\ i\in N\}$ constitutes a Nash equilibrium solution for the game (2.89)–(2.90), if there exist functionals $V^i(x):R\to R$, for $i\in N$, which satisfy the following set of partial differential equations:

$$rW^i(x) - \frac{1}{2}\sigma^2 x^2 W^i_{xx}(x) = \max_{u_i \in U^i} \left\{ \left[u_i \left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) + u_i \right)^{-1/2} - \frac{c}{x^{1/2}} u_i \right] \right.$$
$$\left. + W^i_x \left[a x^{1/2} - bx - \sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) - u_i \right] \right\},$$
$$\text{for } \ i \in N, \tag{2.91}$$

Applying the maximization operator in (2.91) for Player i, yields the condition for a maximum as:

$$\left[\left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) + \frac{1}{2}\phi_i^*(x) \right) \left(\sum_{j=1}^n \phi_j^*(x) \right)^{-3/2} - \frac{c}{x^{1/2}} \right] - W^i_x = 0,$$
$$\text{for } \ i \in N. \tag{2.92}$$

Summing over $i = 1, 2, \ldots, n$ in (2.92) yields:

$$\left(\sum_{j=1}^n \phi_j^*(x) \right)^{1/2} = \left(n - \frac{1}{2} \right) \left(\sum_{j=1}^n \left[\frac{c}{x^{1/2}} + W^j_x \right] \right)^{-1}. \tag{2.93}$$

Substituting (2.93) into (2.92) produces

$$\left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) + \frac{1}{2}\phi_i^*(x) \right) \left(n - \frac{1}{2} \right)^{-3} \left(\sum_{j=1}^n \left[\frac{c}{x^{1/2}} + W^j_x \right] \right)^3$$
$$- \frac{c}{x^{1/2}} - W^i_x = 0, \qquad \text{for } \ i \in N. \tag{2.94}$$

Re-arranging terms in (2.94) yields:

$$\left(\sum_{\substack{j=1 \\ j \neq i}}^n \phi_j^*(x) + \frac{1}{2}\phi_i^*(x) \right) = \left(n - \frac{1}{2} \right)^3 \frac{\left[c + W^i_x x^{1/2} \right] x}{\left(\sum_{j=1}^n \left[c + W^j_x x^{1/2} \right] \right)^3},$$
$$\text{for } \ i \in N. \tag{2.95}$$

Condition (2.95) represents a system of equations which is linear in $\{\phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)\}$. Solving (2.95) yields:

$$\phi_i^*(x) = \frac{x(2n-1)^2}{2\left[\sum_{j=1}^n \left[c + W_x^j x^{1/2}\right]\right]^3} \left\{ \sum_{\substack{j=1 \\ j \neq i}}^n \left[c + W_x^j x^{1/2}\right] - \left(n - \frac{3}{2}\right)\left[c + W_x^i x^{1/2}\right] \right\},$$

$$\text{for } i \in N. \tag{2.96}$$

Substituting $\phi_i^*(t,x)$ in (2.96) into (2.91) and upon solving yields:

Corollary 2.7.1. *The system (2.91) admits a solution*

$$W^i(x) - \left[Ax^{1/2} + B\right], \tag{2.97}$$

where A and B satisfy:

$$0 = \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right] - \frac{(2n-1)}{2n^2}\left(c + \frac{A}{2}\right)^{-1} + \frac{c(2n-1)^2}{4n^3}\left(c + \frac{A}{2}\right)^{-2} + \frac{(2n-1)^2 A}{8n^2\left(c + \frac{A}{2}\right)^2},$$

$$B = \frac{a}{2r}A. \tag{2.98}$$

2.8 Problems

Problem 2.1. Consider the dynamic optimization problem

$$\int_{t_0}^T \left[u(s)^{1/2} - \frac{c}{x(s)^{1/2}}u(s)\right]\exp\left[-r(s-t_0)\right]ds + \exp\left[-r(T-t_0)\right]qx(T)^{1/2},$$

subject to

$$\dot{x}(s) = \left[ax(s)^{1/2} - bx(s) - u(s)\right], \qquad x(t_0) = x_0 \in X.$$

Use Bellman's techniques of dynamic programming to solve the problem.

Problem 2.2. Consider again the dynamic optimization problem

$$\int_{t_0}^T \left[u(s)^{1/2} - \frac{c}{x(s)^{1/2}}u(s)\right]\exp\left[-r(s-t_0)\right]ds + \exp\left[-r(T-t_0)\right]qx(T)^{1/2},$$

subject to

$$\dot{x}(s) = \left[ax(s)^{1/2} - bx(s) - u(s)\right], \qquad x(t_0) = x_0 \in X.$$

(a) If $c = 1$, $q = 2$, $r = 0.01$, $t_0 = 0$, $T = 5$, $a = 0.5$, $b = 1$ and $x_0 = 20$, use optimal control theory to solve the optimal controls, the optimal state trajectory and the costate trajectory $\{\Lambda(s)\}_{t=t_0}^{T}$.
(b) If $c = 1$, $q = 2$, $r = 0.01$, $t_0 = 0$, $T = 5$, $a = 0.5$, $b = 1$ and $x_0 = 30$, use optimal control theory to solve the optimal controls, the optimal state trajectory and the costate trajectory $\{\Lambda(s)\}_{t=t_0}^{T}$.

Problem 2.3. Consider the infinite-horizon problem

$$E_{t_0}\left\{\int_{t_0}^{\infty}\left[u(s)^{1/2} - \frac{0.5}{x(s)^{1/2}}u(s)\right]\exp[-0.02(s-t_0)]\,ds\right\},$$

subject to

$$dx(s) = \left[x(s)^{1/2} - 1.5x(s) - u(s)\right]ds + 0.05x(s)\,dz(s), \quad x(t_0) = x_0 = 50.$$

Use the technique of stochastic to solve the problem.

Problem 2.4. Consider the game in Example 2.7.1 in which Player i maximizes

$$\int_{t_0}^{\infty}\left\{P(s)u_i(s) - cu_i(s) - (1/2)[u_i(s)]^2\right\}\exp[-r(s-t_0)]\,ds,$$
$$\text{for } i \in \{1.2\},$$

subject to

$$\dot{P}(s) = k[a - u_1(s) - u_2(s) - P(s)], \qquad P(t_0) = P_0.$$

Derive an open-loop solution to the game.

Problem 2.5. Consider the game

$$\max_{u_i}\left\{\int_0^{10}\left[10u_i(s) - \frac{u_i(s)^2}{x(s)}\right]\exp[-0.05s]\,ds + \exp(-0.5)\,2x(T)\right\},$$
$$\text{for } i \in \{1,2,\ldots,6\}$$

subject to

$$\dot{x}(s) = 15 - \frac{1}{2}x(s) - \sum_{j=1}^{6}u_j(s), \qquad x(0) = 25.$$

(a) Obtain an open-loop solution for the game.
(b) Obtain a feedback Nash equilibrium for the game.

Problem 2.6. Consider the following stochastic dynamic advertising game. There are two firms in a market and the expected profit of firm 1 and that of 2 are respectively:

$$E_0\left\{\int_0^T \left[q_1 x(s) - \frac{c_1}{2} u_1(s)^2\right] \exp(-rs)\, ds + \exp(-rT)\, S_1 x(T)\right\}$$

and

$$E_0\left\{\int_0^T \left[q_2(1-x(s)) - \frac{c_2}{2} u_2(s)^2\right] \exp(-rs)\, ds \right.$$
$$\left. + \exp(-rT)\, S_2 [1-x(T)]\right\},$$

where r, q_i, c_i, S_i, for $i \in \{1,2\}$, are positive constants,
$x(s)$ is the market share of firm 1 at time s, $[1-x(s)]$ is that of firm 2's,
$u_i(s)$ is advertising rate for firm $i \in \{1,2\}$.

The dynamics of firm 1's market share is governed by the stochastic differential equation:

$$dx(s) = \left\{u_1(s)[1-x(s)]^{1/2} - u_2(s)\, x(s)^{1/2}\right\} ds + \sigma x(s)\, dz(s),$$
$$x(0) = x_0,$$

where σ is a positive constant and $z(s)$ is a standard Wiener process.

Derive a Nash equilibrium for the above stochastic differential game.

Problem 2.7. Consider an infinite-horizon version of Problem 2.6 in which the expected profit of firm 1 and that of 2 are respectively:

$$E_0\left\{\int_0^\infty \left[q_1 x(s) - \frac{c_1}{2} u_1(s)^2\right] \exp(-rs)\, ds\right\}$$

and

$$E_0\left\{\int_0^\infty \left[q_2(1-x(s)) - \frac{c_2}{2} u_2(s)^2\right] \exp(-rs)\, ds\right\},$$

where $x(s)$ is the market share of firm 1 at time s, and $u_i(s)$ is advertising rate for firm $i \in \{1,2\}$.

The dynamics of firm 1's market share is governed by the stochastic differential equation:

$$dx(s) = \left\{u_1(s)[1-x(s)]^{1/2} - u_2(s)\, x(s)^{1/2}\right\} ds + \sigma x(s)\, dz(s),$$
$$x(0) = x_0,$$

where σ is a positive constant and $z(s)$ is a standard Wiener process.

Derive a Nash equilibrium for the above infinite-horizon stochastic differential game.

http://www.springer.com/978-0-387-27620-5

Cooperative Stochastic Differential Games
Yeung, D.W.K.; Petrosjan, L.A.
2006, XIII, 242 p., Hardcover
ISBN: 978-0-387-27620-5