2

The Classical Sequence Spaces

We now turn to the classical sequence spaces ℓ_p for $1 \le p < \infty$ and c_0. The techniques developed in the previous chapter will prove very useful in this context. These Banach spaces are, in a sense, the simplest of all Banach spaces and their structure has been well understood for many years. However, if $p \neq 2$, there can still be surprises and there remain intriguing open questions.

To avoid some complicated notation we will write a typical element of ℓ_p or c_0 as $\xi = (\xi(n))_{n=1}^{\infty}$. Let us note at once that the spaces ℓ_p and c_0 are equipped with a canonical monotone Schauder basis $(e_n)_{n=1}^{\infty}$ given by $e_n(k) = 1$ if $k = n$ and 0 otherwise. It is useful, and now fairly standard, to use c_{00} to denote the subspace of all sequences of scalars $\xi = (\xi(n))_{n=1}^{\infty}$ such that $\xi(n) = 0$ except for finitely many n.

One feature of the canonical basis of the ℓ_p-spaces and c_0 that is useful to know is that $(e_n)_{n=1}^{\infty}$is equivalent to the basis $(a_n e_n)_{n=1}^{\infty}$ whenever $0 < \inf_n |a_n| \le \sup_n |a_n| < \infty$. This property is equivalent to the *unconditionality* of the basis, but we will not formally introduce this concept until the next chapter.

2.1 The isomorphic structure of the ℓ_p-spaces and c_0

We first ask ourselves a very simple question: are the ℓ_p-spaces distinct (i.e., mutually nonisomorphic) Banach spaces? This question may seem absurd because they look different, but recall that $L_2[0,1]$ and ℓ_2 are actually the same space in two different disguises. We can observe, for instance, that c_0 and ℓ_1 are nonreflexive while the spaces ℓ_p for $1 < p < \infty$ are reflexive; further the dual of c_0 (i.e., ℓ_1) is separable but the dual of ℓ_1 (i.e., ℓ_∞) is nonseparable.

To help answer our question we need the following lemma:

Lemma 2.1.1. *Let $(u_n)_{n=1}^{\infty}$ be a normalized block basic sequence in c_0 or in ℓ_p for some $1 \le p < \infty$. Then $(u_n)_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of the space and $[u_n]$ is the range of a contractive projection.*

Proof. Let us treat the case when (u_n) is a block basic sequence in ℓ_p for $1 \leq p < \infty$ and leave the modifications for the c_0 case to the reader. Let us suppose that

$$u_k = \sum_{j=r_{k-1}+1}^{r_k} a_j e_j, \qquad k \in \mathbb{N},$$

where $0 = r_0 < r_1 < r_2 < \ldots$ are positive integers and $(a_j)_{j=1}^{\infty}$ are scalars such that

$$\|u_k\|^p = \sum_{j=r_{k-1}+1}^{r_k} |a_j|^p = 1, \qquad k \in \mathbb{N}.$$

Then, given any $m \in \mathbb{N}$ and any scalars $b_1, \ldots, b_m$ we have

$$\begin{aligned} \Big\| \sum_{k=1}^m b_k u_k \Big\| &= \Big\| \sum_{k=1}^m \sum_{j=r_{k-1}+1}^{r_k} b_k a_j e_j \Big\| \\ &= \Big(\sum_{k=1}^m |b_k|^p \sum_{j=r_{k-1}+1}^{r_k} |a_j|^p \Big)^{1/p} \\ &= \Big(\sum_{k=1}^m |b_k|^p \Big)^{1/p}. \end{aligned}$$

This establishes isometric equivalence.

We shall construct a contractive projection onto $[u_n]_{n=1}^{\infty}$. Here we suppose $1 < p < \infty$ and leave both cases c_0 and ℓ_1 to the reader. For each k we select scalars $(b_j)_{j=r_{k-1}+1}^{r_k}$ so that

$$\sum_{j=r_{k-1}+1}^{r_k} |b_j|^q = 1$$

and

$$\sum_{j=r_{k-1}+1}^{r_k} b_j a_j = 1.$$

Put

$$u_k^* = \sum_{j=r_{k-1}+1}^{r_k} b_j e_j^*.$$

Clearly, $(u_n^*)_{n=1}^{\infty}$ is biorthogonal to $(u_n)_{n=1}^{\infty}$ and $\|u_n^*\| = \|u_n\| = 1$. Our aim is to see that the operator

$$P(\xi) = \sum_{k=1}^{\infty} u_k^*(\xi) u_k, \qquad \xi \in \ell_p,$$

defines a norm-one projection from ℓ_p onto $[u_k]$. We will show that $\|P\xi\| \leq \|\xi\|$ when $\xi \in c_{00}$ and then observe that P extends by density to a contractive projection.

For each $\xi \in c_{00}$,

$$\begin{aligned} |u_k^*(\xi)| &= \Big| \sum_{j=r_{k-1}+1}^{r_k} b_j \xi(j) \Big| \\ &\le \Big(\sum_{j=r_{k-1}+1}^{r_k} |b_j|^q \Big)^{\frac{1}{q}} \Big(\sum_{j=r_{k-1}+1}^{r_k} |\xi(j)|^p \Big)^{\frac{1}{p}} \\ &= \Big(\sum_{j=r_{k-1}+1}^{r_k} |\xi(j)|^p \Big)^{\frac{1}{p}}. \end{aligned}$$

Then, using the isometric equivalence of $(u_n)_{n=1}^\infty$ and $(e_n)_{n=1}^\infty$, we have

$$\begin{aligned} \|P(\xi)\| &= \Big(\sum_{k=1}^\infty |u_k^*(\xi)|^p \Big)^{\frac{1}{p}} \\ &\le \Big(\sum_{k=1}^\infty \sum_{j=r_{k-1}+1}^{r_k} |\xi(j)|^p \Big)^{\frac{1}{p}} \\ &= \|\xi\|. \end{aligned}$$

□

Remark 2.1.2. Notice that if (u_n) is not normalized but satisfies instead an inequality

$$0 < a \le \|u_n\| \le b < \infty, \quad n \in \mathbb{N},$$

for some constants a, b (in which case (u_n) is said to be *seminormalized*), then we can apply the previous lemma to $(u_n/\|u_n\|)$ and we obtain that $(u_n)_{n=1}^\infty$ is equivalent to $(e_n)_{n=1}^\infty$ (but not isometrically) and $[u_n]$ is complemented by a contractive projection.

Although the preceding lemma was quite simple it already leads to a powerful conclusion:

Proposition 2.1.3. *Let $(x_n)_{n=1}^\infty$ be a normalized sequence in ℓ_p for $1 \le p < \infty$ [respectively, c_0] such that for each $j \in \mathbb{N}$ we have $\lim_{n\to\infty} x_n(j) = 0$ (for example suppose $(x_n)_{n=1}^\infty$ is weakly null). Then there is a subsequence $(x_{n_k})_{k=1}^\infty$ which is a basic sequence equivalent to the canonical basis of ℓ_p and such that $[x_{n_k}]_{k=1}^\infty$ is complemented in ℓ_p [respectively, c_0].*

Proof. Proposition 1.3.10 (using the "gliding hump" technique) yields a subsequence $(x_{n_k})_{k=1}^\infty$ and a block basic sequence $(u_k)_{k=1}^\infty$ of $(e_n)_{n=1}^\infty$ such that $(x_{n_k})_{k=1}^\infty$ is basic, equivalent to $(u_k)_{k=1}^\infty$ and such that $[x_{n_k}]_{k=1}^\infty$ is complemented whenever $[u_k]_{k=1}^\infty$ is. By Lemma 2.1.1 we are done.

□

Now let us prove a classical result from the 1930s (Pitt [189]).

Theorem 2.1.4 (Pitt's Theorem). *Suppose* $1 \leq p < r < \infty$. *If* X *is a closed subspace of* ℓ_r *and* $T : X \to \ell_p$ *is a bounded operator then* T *is compact.*

Proof. ℓ_r is reflexive, hence X is reflexive and so B_X is weakly compact. Therefore in order to prove that T is compact it suffices to show that $T|_{B_X}$ is weak-to-norm continuous. Since the weak topology of X restricted to B_X is metrizable (Lemma 1.4.1 (ii)) it suffices to see that whenever $(x_n)_{n=1}^{\infty} \subset B_X$ is weakly convergent to some x in B_X then $(T(x_n))_{n=1}^{\infty}$ converges in norm to Tx.

We need only show that if $(x_n)_{n=1}^{\infty}$ is a weakly null sequence in X then $\lim_{n\to\infty} \|Tx_n\| = 0$. If this fails, we may suppose the existence of a weakly null sequence $(x_n)_{n=1}^{\infty}$ with $\|x_n\| = 1$ such that $\|Tx_n\| \geq \delta > 0$ for all n. By passing to a subsequence we may suppose that $(x_n)_{n=1}^{\infty}$ is a basic sequence equivalent to the canonical ℓ_r-basis (Proposition 2.1.3). But then, since $(Tx_n)_{n=1}^{\infty}$ is also weakly null, by passing to a further subsequence we may suppose that $(Tx_n/\|Tx_n\|)_{n=1}^{\infty}$, and hence $(Tx_n)_{n=1}^{\infty}$, is basic and equivalent to the canonical ℓ_p-basis. Since T is bounded we have effectively shown that the identity map $\iota : \ell_r \to \ell_p$ is bounded, which is absurd. Or, alternatively, there exist constants C_1 and C_2 such that the following inequalities hold simultaneously for all n:

$$\Big\| \sum_{k=1}^{n} x_k \Big\|_r \leq C_1 n^{\frac{1}{r}} \quad \text{and} \quad \Big\| \sum_{k=1}^{n} Tx_k \Big\|_p \geq C_2 n^{\frac{1}{p}},$$

which contradicts the boundedness of T. Thus the theorem is proved.

□

Remark 2.1.5. (a) Essentially the same proof works with c_0 replacing ℓ_r; although c_0 is nonreflexive, Lemma 1.4.1 can still be used to show that B_X is at least weakly metrizable, and the weak-to-norm continuity of $T|_{B_X}$ is enough to show that the image is relatively norm-compact.

(b) We would like to single out the following crucial ingredient in the proof of Pitt's theorem. *Suppose* $T : \ell_r \to \ell_p$ *is a bounded operator with* $1 \leq p < r < \infty$. *Then whenever* (x_n) *is a weakly null sequence in* ℓ_r *we have* $\|Tx_n\|_p \to 0$. *In particular* $\|Te_n\|_p \to 0$. *The same is true for any operator* $T : c_0 \to \ell_p$.

Corollary 2.1.6. *The spaces of the set* $\{c_0\} \cup \{\ell_p : 1 \leq p < \infty\}$ *are mutually nonisomorphic. In fact, if* X *is an infinite-dimensional subspace of one of the spaces* $\{c_0\} \cup \{\ell_p : 1 \leq p < \infty\}$, *then it is not isomorphic to a subspace of any other.*

This suggests the following definition:

Definition 2.1.7. Two infinite-dimensional Banach spaces X, Y are said to be *totally incomparable* if they have no infinite-dimensional subspaces in common (up to isomorphism).

What can be said for bounded operators $T : \ell_p \longrightarrow \ell_r$ for $p < r$? First, notice that in this case Pitt's theorem is not true. Take, for example, the natural inclusion $\iota : \ell_p \hookrightarrow \ell_r$. ι is a norm-one operator which is not compact since the image of the canonical basis of ℓ_p is a sequence contained in $\iota(B_{\ell_p})$ with no convergent subsequences.

Definition 2.1.8. A bounded operator T from a Banach space X into a Banach space Y is *strictly singular* if there is no infinite-dimensional subspace $E \subset X$ such that $T|_E$ is an isomorphism onto its range.

Theorem 2.1.9. *If $p < r$, every $T : \ell_p \longrightarrow \ell_r$ is strictly singular.*

Proof. This is immediate from Corollary 2.1.6.

□

2.2 Complemented subspaces of ℓ_p $(1 \le p < \infty)$ and c_0

The results of this section are due to Pełczyński (1960) [169]; they demonstrate the power of basic sequence techniques.

Proposition 2.2.1. *Every infinite-dimensional closed subspace Y of ℓ_p $(1 \le p < \infty)$ [respectively, c_0] contains a closed subspace Z such that Z is isomorphic to ℓ_p [respectively, c_0] and complemented in ℓ_p [respectively, c_0].*

Proof. Since Y is infinite-dimensional, for every n there is $y_n \in Y$, $\|y_n\| = 1$, such that $e_k^*(y_n) = 0$ for $1 \le k \le n$. If not, for some $N \in \mathbb{N}$ the projection $S_N(\sum_{n=1}^\infty a_n e_n) = \sum_{n=1}^N a_n e_n$ restricted to Y would be injective (since $0 \ne y \in Y$ would imply $S_N(y) \ne 0$) and so $S_N|_Y$ would be an isomorphism onto its image, which is impossible because Y is infinite-dimensional. By Proposition 2.1.3 the sequence $(y_n)_{n=1}^\infty$ has a subsequence $(y_{n_k})_{k=1}^\infty$ which is basic, equivalent to the canonical basis of the space and such that the subspace $Z = [y_{n_k}]$ is complemented.

□

Since c_0 and ℓ_1 are nonreflexive and every closed subspace of a reflexive space is reflexive, using Proposition 2.2.1 we obtain:

Proposition 2.2.2. *Let Y be an infinite-dimensional closed subspace of either c_0 or ℓ_1. Then Y is not reflexive.*

Suppose now that Y is itself complemented in ℓ_p $(1 \le p < \infty)$ [respectively, c_0]. Proposition 2.2.1 certainly tells us that Y contains a complemented copy of ℓ_p [respectively, c_0]. Can we say more? Remarkably, Pełczyński discovered a trick which enables us, by rather "soft" arguments, to do quite a bit better. This trick is nowadays known as the *Pełczyński decomposition technique* and has proved very useful in different contexts.

The situation is: we have two Banach spaces X and Y so that Y is isomorphic to a complemented subspace of X and X is isomorphic to a complemented subspace of Y. We would like to deduce that X and Y are isomorphic. This is known (by analogy with a similar result for cardinals) as the *Schroeder-Bernstein problem* for Banach spaces. The next theorem gives two criteria where the Schroeder-Bernstein problem has a positive solution. To this end we need to introduce the spaces $\ell_p(X)$ for $1 \le p < \infty$ and $c_0(X)$, where X is a given Banach space.

For $1 \le p < \infty$, the space $\ell_p(X) = (X \oplus X \oplus \dots)_p$ called the *infinite direct sum of* X *in the sense of* ℓ_p, consists of all sequences $x = (x(n))_{n=1}^\infty$ with values in X so that $(\|x(n)\|)_{n=1}^\infty \in \ell_p$, with the norm

$$\|x\| = \|(\|x(n)\|)_{n=1}^\infty\|_p.$$

Similarly, the *infinite direct sum of* X *in the sense of* c_0, $c_0(X) = (X \oplus X \oplus \dots)_{c_0}$ is the space of X-valued sequences $x = (x(n))_{n=1}^\infty$ so that $\lim_{n\to\infty} \|x(n)\| = 0$ under the norm

$$\|x\| = \max_{1 \le n < \infty} \|x(n)\|.$$

Notice that $\ell_p(\ell_p)$ can be identified with $\ell_p(\mathbb{N} \times \mathbb{N})$ and hence is isometric to ℓ_p. Analogously, $c_0(c_0)$ is isometric to c_0.

Theorem 2.2.3 (The Pełczyński decomposition technique [169]). *Let X and Y be Banach spaces so that X is isomorphic to a complemented subspace of Y and Y is isomorphic to a complemented subspace of X. Suppose further that either:*
(a) $X \approx X^2 = X \oplus X$ and $Y \approx Y^2$, or
(b) $X \approx c_0(X)$ or $X \approx \ell_p(X)$ for some $1 \le p < \infty$.
Then X is isomorphic to Y.

Proof. Let us put $X \approx Y \oplus E$ and $X \approx Y \oplus F$. If (a) holds then we have

$$X \approx Y \oplus Y \oplus E \approx Y \oplus X,$$

and by a symmetrical argument $Y \approx X \oplus Y$. Hence $Y \approx X$.

If X satisfies (b) in particular we have $X \approx X^2$ so as in part (a) we obtain $Y \approx X \oplus Y$. On the other hand,

$$\ell_p(X) \approx \ell_p(Y \oplus E) \approx \ell_p(Y) \oplus \ell_p(E).$$

Hence if $X \approx \ell_p(X)$,

$$X \approx Y \oplus \ell_p(Y) \oplus \ell_p(E) \approx Y \oplus \ell_p(X) \approx Y \oplus X.$$

The proof is analogous if $X \approx c_0(X)$.

□

We are ready to prove a beautiful theorem due to Pełczyński (1960) [169] which had a profound influence on the development of Banach space theory.

Theorem 2.2.4. *Suppose Y is a complemented infinite-dimensional subspace of ℓ_p where $1 \le p < \infty$ [respectively, c_0]. Then Y is isomorphic to ℓ_p [respectively, c_0].*

Proof. Proposition 2.2.1 gives an infinite-dimensional subspace Z of Y such that Z is isomorphic to ℓ_p [respectively, c_0] and Z is complemented in ℓ_p [respectively, c_0]. Obviously Z is also complemented in Y, therefore ℓ_p [respectively, c_0] is (isomorphic to) a complemented subspace in Y. Since $\ell_p(\ell_p) = \ell_p$ [respectively, $c_0(c_0) = c_0$], (b) of Theorem 2.2.3 applies and we are done.

□

At this point let us discuss where this theorem leads. First, the alert reader may ask whether it is true that *every* subspace of ℓ_p is actually complemented. Certainly this is true when $p = 2$! This is a special case of:

The complemented subspace problem. *If X is a Banach space such that every closed subspace is complemented, is X isomorphic to a Hilbert space?*

This problem was settled positively by Lindenstrauss and Tzafriri in 1971 [135]. We will later discuss its general solution but, at the moment, let us point out that it is not so easy to demonstrate the answer even for the ℓ_p-spaces when $p \neq 2$. In this chapter we will show that ℓ_1 has an uncomplemented subspace.

Another way to approach the complemented subspace problem is to demonstrate that ℓ_p has a subspace which is not isomorphic to the whole space. Here we meet another question dating back to Banach:

The homogeneous space problem. *Let X be a Banach space which is isomorphic to every one of its infinite-dimensional closed subspaces. Is X isomorphic to a Hilbert space?*

This problem was finally solved, again positively, by Komorowski and Tomczak-Jaegermann [115] in 1996 (using an important ingredient by Gowers [70]).

Oddly enough, the ℓ_p-spaces for $p \neq 2$ are not as regular as one would expect. In fact, for every $p \neq 2$, ℓ_p contains a subspace without a basis. For $p > 2$ this was proved by Davie in 1973 [34]; for general p it was obtained by Szankowski [211] a few years later. However, the construction of such subspaces is far from easy and will not be covered in this book. Notice that this provides an example of a separable Banach space without a basis.

One natural idea that comes out of Theorem 2.2.4 is the notion that the ℓ_p-spaces and c_0 are the building blocks from which Banach spaces are constructed; by analogy they might play the role of primes in number theory. This thinking is behind the following definition:

Definition 2.2.5. A Banach space X is called *prime* if every complemented infinite-dimensional subspace of X is isomorphic to X.

Thus the ℓ_p-spaces and c_0 are prime. Are there other primes? One may immediately ask about ℓ_∞ and, indeed, this is a (nonseparable) prime space as was shown by Lindenstrauss in 1967 [129]; we will show this later. The quest for other prime spaces has proved difficult, some candidates have been found but in general it is very hard to prove that a particular space is prime. Eventually another prime space was found by Gowers and Maurey [72] but the construction is very involved and the space is far from being "natural." In fact the Gowers-Maurey prime space has the property that the only complemented subspaces of infinite dimension are of *finite* codimension. One can say that this space is prime only because it has very few complemented subspaces at all!

2.3 The space ℓ_1

The space ℓ_1 has a special role in Banach space theory. In this section we develop some of its elementary properties. We start by proving a universal property of ℓ_1 with respect to separable spaces due to Banach and Mazur [9] from 1933.

Theorem 2.3.1. *If X is a separable Banach space then there exists a continuous operator $Q : \ell_1 \to X$ from ℓ_1 onto X.*

Proof. It suffices to show that X admits of a continuous operator $Q : \ell_1 \to X$ such that $Q\{\xi \in \ell_1 : \|\xi\|_1 < 1\} = \{x \in X : \|x\| < 1\}$.

Let $(x_n)_{n=1}^\infty$ be a dense sequence in B_X and define $Q : \ell_1 \to X$ by $Q(\xi) = \sum_{n=1}^\infty \xi(n)x_n$. Notice that Q is well defined: for every $\xi = (\xi(n)) \in \ell_1$ the series $\sum_{n=1}^\infty \xi(n)x_n$ is absolutely convergent in X. Q is clearly linear and has norm one since

$$\|Q(\xi)\| = \Big\| \sum_{n=1}^\infty \xi(n)x_n \Big\| \le \sum_{n=1}^\infty |\xi(n)| = \|(\xi(n))\|_1 .$$

$Q(B_{\ell_1})$ is a dense subset of B_X, hence given $x \in B_X$ and $0 < \epsilon < 1$ there exists $\xi_1 \in B_{\ell_1}$ such that $\|x - T\xi_1\| < \epsilon$. Next we find $\xi_2' \in B_{\ell_1}$ such that $\|\frac{1}{\epsilon}(x - Q\xi_1) - Q\xi_2'\| < \epsilon$. If we let $\xi_2 = \epsilon\xi_2'$ we obtain

$$\|x - Q(\xi_1 + \xi_2)\| < \epsilon^2.$$

Iterating we find a sequence (ξ_n) in B_{ℓ_1} satisfying $\|\xi_n\|_1 < \epsilon^{n-1}$ and $\|x - Q(\xi_1 + \cdots + \xi_n)\| < \epsilon^n$. Let $\xi = \sum_{n=1}^\infty \xi_n$. Then $\|\xi\|_1 \le (1-\epsilon)^{-1}$ and $Q\xi = x$. Since $0 < \epsilon < 1$ is arbitrary, by scaling we deduce that $Q\{\xi \in \ell_1 : \|\xi\|_1 < 1\} = \{x \in X : \|x\| < 1\}$.

□

Corollary 2.3.2. *If X is a separable Banach space then X is isometrically isomorphic to a quotient of ℓ_1.*

Proof. Let $Q : \ell_1 \to X$ be the quotient map in the proof of Theorem 2.3.1. Then it follows that $\ell_1/\ker Q$ is isometrically isomorphic to X.

□

Corollary 2.3.3. *ℓ_1 has an uncomplemented closed subspace.*

Proof. Take X a separable Banach space which is not isomorphic to ℓ_1. Theorem 2.3.1 yields an operator Q from ℓ_1 onto X whose kernel is a closed subspace of ℓ_1. If $\ker Q$ were complemented in ℓ_1 then we would have $\ell_1 = \ker Q \oplus M$ for some closed subspace M of ℓ_1 and therefore

$$X = \ell_1/\ker Q \approx M.$$

But this can only occur if X is isomorphic to ℓ_1 by Theorem 2.2.4.

□

Definition 2.3.4. A Banach space X has the *Schur property* (or X is a *Schur space*) if weak and norm sequential convergence coincide in X, i.e., a sequence $(x_n)_{n=1}^\infty$ in X converges to 0 weakly if and only if $(x_n)_{n=1}^\infty$ converges to 0 in norm.

Example 2.3.5. Neither of the spaces ℓ_p for $1 < p < \infty$ nor c_0 have the Schur property since the canonical basis is weakly null but cannot converge to 0 in norm.

The next result was discovered in an equivalent form by Schur in 1920 [205].

Theorem 2.3.6. *ℓ_1 has the Schur property.*

Proof. Suppose (x_n) is a weakly null sequence in ℓ_1 that does not converge to 0 in norm. Using Proposition 2.1.3, (x_n) contains a subsequence which is basic and equivalent to the canonical basis; this gives a contradiction because the canonical basis of ℓ_1 is clearly not weakly null.

□

Theorem 2.3.7. *Let X be a Banach space with the Schur property. Then a subset W of X is weakly compact if and only if W is norm compact.*

Proof. Suppose W is weakly compact and consider a sequence $(x_n)_{n=1}^\infty$ in W. By the Eberlein-Šmulian theorem W is weakly sequentially compact, so $(x_n)_{n=1}^\infty$ has a subsequence $(x_{n_k})_{k=1}^\infty$ that converges weakly to some $x \in W$. Since X has the Schur property, $(x_{n_k})_{k=1}^\infty$ converges to x in norm as well. Therefore W is compact for the norm topology.

□

Corollary 2.3.8. *If X is a reflexive Banach space with the Schur property then X is finite-dimensional.*

Proof. If a reflexive Banach space X has the Schur property then its unit ball is norm-compact by Theorem 2.3.7 and so X is finite-dimensional.

□

Definition 2.3.9. A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is *weakly Cauchy* if $\lim_{n\to\infty} x^*(x_n)$ exists for every x^* in X^*.

Any weakly Cauchy sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is norm-bounded by the Uniform Boundedness principle. If X is reflexive, by Corollary 1.6.4, $(x_n)_{n=1}^{\infty}$ will have a weak cluster point, x, and so $(x_n)_{n=1}^{\infty}$ will converge weakly to x. If X is nonreflexive, however, there may be sequences which are weakly Cauchy but not weakly convergent.

Definition 2.3.10. A Banach space X is said to be *weakly sequentially complete* (wsc) if every weakly Cauchy sequence in X converges weakly.

Example 2.3.11. In the space c_0 consider the sequence $x_n = e_1 + \dots + e_n$, where (e_n) is the unit vector basis. $(x_n)_{n=1}^{\infty}$ is obviously weakly Cauchy but it does not converge weakly in c_0. $(x_n)_{n=1}^{\infty}$ converges weak* in the bidual, ℓ_∞, to the element $(1, 1, \dots, 1, \dots)$. Thus c_0 is not weakly sequentially complete.

Proposition 2.3.12. *Any Banach space with the Schur property (in particular ℓ_1) is weakly sequentially complete.*

Proof. Suppose $(x_n)_{n=1}^{\infty}$is weakly Cauchy. Then for any two strictly increasing sequences of integers $(n_k)_{k=1}^{\infty}, (m_k)_{k=1}^{\infty}$ the sequence $(x_{m_k} - x_{n_k})_{k=1}^{\infty}$ is weakly null and so $\lim_{k\to\infty} \|x_{m_k} - x_{n_k}\| = 0$. Thus, being norm-Cauchy, $(x_n)_{n=1}^{\infty}$is norm-convergent and hence weak-convergent.

□

2.4 Convergence of series

Definition 2.4.1. Let $(x_n)_{n=1}^{\infty}$ be a sequence in a Banach space X. A (formal) series $\sum_{n=1}^{\infty} x_n$ in X is said to be *unconditionally convergent* if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of $\mathbb{N}$.

We will see in Chapter 8 that except in finite-dimensional spaces, unconditional convergence is weaker than *absolute convergence*, i.e., convergence of $\sum_{n=1}^{\infty} \|x_n\|$.

Lemma 2.4.2. *Given a series $\sum_{n=1}^{\infty} x_n$ in a Banach space X, the following are equivalent:*

(a) $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent;

(b) The series $\sum_{k=1}^{\infty} x_{n_k}$ *converges for every increasing sequence of integers* $(n_k)_{k=1}^{\infty}$*;*
(c) The series $\sum_{n=1}^{\infty} \epsilon_n x_n$ *converges for every choice of signs* (ϵ_n)*;*
(d) For every $\epsilon > 0$ *there exists an* n *so that if* F *is any finite subset of* $\{n+1, n+2, \ldots\}$ *then*

$$\Big\| \sum_{j\in F} x_j \Big\| < \epsilon.$$

Proof. We will establish only $(a) \Rightarrow (d)$ and leave the other easier implications to the reader. Suppose that (d) fails. Then there exists $\epsilon > 0$ so that for every n we can find a finite subset F_n of $\{n+1, \ldots\}$ with

$$\Big\| \sum_{j\in F_n} x_j \Big\| \geq \epsilon.$$

We will build a permutation π of $\mathbb{N}$ so that $\sum_{n=1}^{\infty} x_{\pi(n)}$ diverges.

Take $n_1 = 1$ and let $A_1 = F_{n_1}$. Next pick $n_2 = \max A_1$ and let $B_1 = \{n_1+1, \ldots, n_2\} \setminus A_1$. Now repeat the process taking $A_2 = F_{n_2}$, $n_3 = \max A_2$ and $B_2 = \{n_2+1, \ldots, n_3\} \setminus A_2$. Iterating we generate a sequence $(n_k)_{k=1}^{\infty}$ and a partition $\{n_k+1, \ldots, n_{k+1}\} = A_k \cup B_k$. Define π so that π permutes the elements of $\{n_k+1, \ldots, n_{k+1}\}$ in such a way that A_k precedes B_k. Then the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ is divergent because the Cauchy condition fails.

□

Definition 2.4.3. A (formal) series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is *weakly unconditionally Cauchy* (WUC) or *weakly unconditionally convergent* if for every $x^* \in X^*$ $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$.

Proposition 2.4.4. *Suppose the series* $\sum_{n=1}^{\infty} x_n$ *converges unconditionally to some* x *in a Banach space* X*. Then*

(i) $\sum_{n=1}^{\infty} x_{\pi(n)} = x$ *for every permutation* π*.*
(ii) $\sum_{n\in\mathbb{A}} x_n$ *converges unconditionally for every infinite subset* $\mathbb{A}$ *of* $\mathbb{N}$*.*
(iii) $\sum_{n=1}^{\infty} x_n$ *is WUC.*

Proof. Parts (i) and (ii) are immediate. For (iii), given $x^* \in X^*$ the scalar series $\sum_{n=1}^{\infty} x^*(x_{\pi(n)})$ converges for every permutation π. It is a classical theorem of Riemann that for scalar sequences the series $\sum_{n=1}^{\infty} a_n$ converges unconditionally if and only if it converges absolutely, i.e., $\sum_{n=1}^{\infty} |a_n| < \infty$. Thus we have $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$.

□

Let us notice that the name “weakly unconditionally convergent” series can be misleading because such series need not be weakly convergent; we will therefore use the term weakly unconditionally Cauchy or more usually its abbreviation (WUC).

Example 2.4.5. The series $\sum_{n=1}^{\infty} e_n$ in c_0, where $(e_n)_{n=1}^{\infty}$ is the canonical basis of the space, is WUC but fails to converge weakly (and so it cannot converge unconditionally). In fact, this is in a certain sense the only counterexample as we shall see.

In Proposition 2.4.7 we shall prove that WUC series are in a very natural correspondence with bounded operators on c_0. Let us first see a lemma.

Lemma 2.4.6. *Let $\sum_{n=1}^{\infty} x_n$ be a formal series in a Banach space X. Then the following are equivalent:*

(i) $\sum_{n=1}^{\infty} x_n$ *is WUC.*
(ii) There exists $C > 0$ such that for all $(\xi(n)) \in c_{00}$ we have

$$\Big\| \sum_{n=1}^{\infty} \xi(n) x_n \Big\| \leq C \max_n |\xi(n)|.$$

(iii) There exists $C' > 0$ such that

$$\Big\| \sum_{n \in F} \epsilon_n x_n \Big\| \leq C'$$

for any finite subset F of $\mathbb{N}$ and all $\epsilon_n = \pm 1$.

Proof. $(i) \Rightarrow (ii)$. Put

$$S = \Big\{ \sum_{n=1}^{\infty} \xi(n) x_n \in X : \ \xi = (\xi(n)) \in c_{00}, \ \|\xi\|_\infty \leq 1 \Big\}.$$

The WUC property implies that S is weakly bounded, therefore it is norm-bounded by the Uniform Boundedness principle.

Obviously, (ii) implies (iii). For $(iii) \Rightarrow (i)$, given $x^* \in X^*$ let $\epsilon_n = \operatorname{sgn} x^*(x_n)$. Then for each integer N we have

$$\sum_{n=1}^{N} |x^*(x_n)| = \Big| x^* \Big(\sum_{n=1}^{N} \epsilon_n x_n \Big) \Big| \leq C \|x^*\|$$

and therefore the series $\sum_{n=1}^{\infty} |x^*(x_n)|$ converges.

□

Proposition 2.4.7. *Let $\sum_{n=1}^{\infty} x_n$ be a series in a Banach space X. Then $\sum_{n=1}^{\infty} x_n$ is WUC if and only if there is a bounded operator $T : c_0 \to X$ with $Te_n = x_n$.*

Proof. If $\sum_{n=1}^{\infty} x_n$ is WUC then the operator $T : c_{00} \to X$ defined by $T\xi = \sum_{n=1}^{\infty} \xi(n) x_n$ is bounded for the c_0-norm by Lemma 2.4.6. By density T extends to a bounded operator $T : c_0 \to X$.

For the converse, let $T : c_0 \to X$ be a bounded operator with $Te_n = x_n$ for all n. For each $x^* \in X^*$ we have

$$\sum_{n=1}^{\infty} |x^*(x_n)| = \sum_{n=1}^{\infty} |x^*(Te_n)| = \sum_{n=1}^{\infty} |T^*(x^*)(e_n)|,$$

which is finite since $\sum_{n=1}^{\infty} e_n$ is WUC.

$\square$

Proposition 2.4.8. *Let $\sum_{n=1}^{\infty} x_n$ be a WUC series in a Banach space X. Then $\sum_{n=1}^{\infty} x_n$ converges unconditionally in X if and only if the operator $T : c_0 \to X$ such that $Te_n = x_n$ is compact.*

Proof. Suppose $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent. We will show that $\lim_{n\to\infty} \|T - TS_n\| = 0$, where $(S_n)_{n=1}^{\infty}$ are the partial sum projections associated to the canonical basis (e_n) of c_0. Thus, being a uniform limit of finite-rank operators, T will be compact.

Given $\epsilon > 0$ we use Lemma 2.4.2 to find $n = n(\epsilon)$ so that if F is a finite subset of $\{n+1, n+2, \dots\}$ then $\|\sum_{j\in F} x_j\| \le \epsilon/2$. For every $x^* \in X^*$ with $\|x^*\| \le 1$ we have

$$\sum_{\{j\in F\ :\ x^*(x_j)\ge 0\}} x^*(x_j) \le \frac{\epsilon}{2},$$

therefore

$$\sum_{j\in F} |x^*(x_j)| \le \epsilon.$$

Hence if $\xi \in c_{00}$ with $\|\xi\|_\infty \le 1$ it follows that $|x^*(T - TS_m)\xi| \le \epsilon$ for $m \ge n$ and all $x^* \in X^*$. By density we conclude that $\|T - TS_m\| \le \epsilon$.

Assume, conversely, that T is compact. Let us consider

$$T^{**} : c_0^{**} = \ell_\infty \longrightarrow X \subset X^{**}.$$

The restriction of T^{**} to B_{ℓ_∞} is weak*-to-norm continuous because on a norm compact set the weak* topology agrees with the norm topology. Since $\sum_{n=1}^{\infty} e_{\pi(n)}$ converges weak* in ℓ_∞ for every permutation π, $\sum_{n=1}^{\infty} x_n$ also converges unconditionally in X.

$\square$

Note that the above argument also implies the following stability property of unconditionally convergent series with respect to the multiplication by bounded sequences. The proof is left as an exercise.

Proposition 2.4.9. *A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is unconditionally convergent if and only if $\sum_{n=1}^{\infty} t_n x_n$ converges (unconditionally) for all $(t_n) \in \ell_\infty$.*

The next theorem and its consequences are essentially due to Bessaga and Pełczyński in their 1958 paper [12] and represent some of the earliest applications of the basic sequence methods.

Theorem 2.4.10. *Suppose $T : c_0 \to X$ is a bounded operator. Then the following conditions on T are equivalent:*

(i) T is compact,
(ii) T is weakly compact,
(iii) T is strictly singular.

Proof. $(i) \Rightarrow (ii)$ is obvious. For $(ii) \Rightarrow (iii)$, let us suppose that T fails to be strictly singular. Then there exists an infinite-dimensional subspace Y of c_0 such that $T|_Y$ is an isomorphism onto its range. If T is weakly compact this forces Y to be reflexive, contradicting Proposition 2.2.2.

We now consider $(iii) \Rightarrow (i)$. Assume that T fails to be compact. Then, by Proposition 2.4.8, $\sum_{n=1}^{\infty} Te_n$ does not converge unconditionally so, by Lemma 2.4.2, there exists $\epsilon > 0$ and a sequence of disjoint finite subsets of integers $(F_n)_{n=1}^{\infty}$ so that $\|\sum_{k\in F_n} Te_k\| \geq \epsilon$ for every n. Let $x_n = \sum_{k\in F_n} Te_k$. $(x_n)_{n=1}^{\infty}$ is weakly null in X since $\sum_{k\in F_n} e_k$ is weakly null in c_0. Using Proposition 1.3.10 we can, by passing to a subsequence of $(x_n)_{n=1}^{\infty}$, assume it is basic in X with basis constant K, say. Then for $\xi = (\xi(n))_{n=1}^{\infty} \in c_{00}$,

$$\Big\|\sum_{n=1}^{\infty} \xi(n)x_n\Big\| = \Big\|T\Big(\sum_{n=1}^{\infty} \xi(n) \sum_{k\in F_n} e_k\Big)\Big\| \leq \|T\| \max_{n\in\mathbb{N}} |\xi(n)|.$$

On the other hand,

$$\max_{n\in\mathbb{N}} |\xi(n)| \leq 2K\Big\|\sum_{n=1}^{\infty} \xi(n)x_n\Big\|.$$

Thus $(x_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of c_0 and therefore to $(\sum_{k\in F_n} e_k)_{n=1}^{\infty}$. We conclude that T cannot be strictly singular.

□

From now on, whenever we say that a Banach space X *contains a copy of a Banach space* Y we mean that X contains a closed subspace E which is isomorphic to Y. Using Theorem 2.4.10 we obtain a very nice characterization of spaces that contain a copy of c_0.

Theorem 2.4.11. *In order that every WUC series in a Banach space X be unconditionally convergent it is necessary and sufficient that X contains no copy of c_0.*

Proof. Suppose that X contains no copy of c_0 and that $\sum_{n=1}^{\infty} x_n$ is a WUC series in X. By Proposition 2.4.7 there exists a bounded operator $T : c_0 \to X$ such that $Te_n = x_n$ for all n. T must be strictly singular since every infinite-dimensional subspace of c_0 contains a copy of c_0 (Proposition 2.2.1) so T is compact by Theorem 2.4.10. Hence the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally by Proposition 2.4.8. The converse follows trivially from Example 2.4.5.

□

Remark 2.4.12. This theorem of Bessaga and Pełczyński is a prototype for exclusion theorems which say that if we can exclude a certain subspace from a Banach space then it will have a particular property. It had considerable influence in suggesting that such theorems might be true. In Chapter 10 we will see a similar and much more difficult result for Banach spaces not containing ℓ_1 (due to Rosenthal [197]) which when combined with the Bessaga-Pełczyński theorem gives a very elegant pair of bookends in Banach space theory. It is also worth noting that the hypothesis that a Banach space fails to contain c_0 becomes ubiquitous in the theory precisely because of Theorem 2.4.11.

We have seen that a series $\sum_{n=1}^{\infty} x_n$ in a Banach space X converges unconditionally in norm if and only if each subseries $\sum_{k=1}^{\infty} x_{n_k}$ does. In particular every subseries of an unconditionally convergent series is weakly convergent. The Orlicz-Pettis theorem establishes that the converse is true as well. First we see an auxiliary result.

Lemma 2.4.13. *Let m_0 be the set of all sequences of scalars assuming only finitely many different values. Then m_0 is dense in ℓ_∞.*

Proof. Let $a = (a_n)_{n=1}^{\infty}$ be a sequence of scalars with $\|a\|_\infty \le 1$. For any $\epsilon > 0$ pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then the sequence $b = (b_n)_{n=1}^{\infty} \in m_0$ given by

$$b_n = (\operatorname{sgn} a_n)\frac{j}{N} \quad \text{if} \quad \frac{j}{N} \le |a_n| \le \frac{j+1}{N}, \quad j = 1, \dots, N$$

satisfies $\|a - b\|_\infty \le \frac{1}{N} < \epsilon$.

□

Theorem 2.4.14 (The Orlicz-Pettis Theorem). *Suppose $\sum_{n=1}^{\infty} x_n$ is a series in a Banach space X for which every subseries $\sum_{k=1}^{\infty} x_{n_k}$ converges weakly. Then $\sum_{n=1}^{\infty} x_n$ converges unconditionally in norm.*

Proof. The hypothesis easily yields that $\sum_{n=1}^{\infty} x_n$ is a WUC series so, by Proposition 2.4.7, there exists a bounded operator $T : c_0 \to X$ with $Te_n = x_n$ for all n. We will show that T is actually compact.

Let us look at $T^{**} : \ell_\infty \to X^{**}$. For every $A \subset \mathbb{N}$ let us denote by $\chi_A = (\chi_A(k))_{k=1}^{\infty}$ the element of ℓ_∞ such that $\chi_A(k) = 1$ if $k \in A$ and 0 otherwise. By hypothesis $\sum_{n\in A} x_n$ converges weakly in X and it follows that $T^{**}(\chi_A) \in X$. The linear span of all such χ_A consists of the space of scalar sequences taking only finitely many different values, m_0, which by Lemma 2.4.13 is dense in ℓ_∞. Hence T^{**} maps ℓ_∞ into X. This means that T is a weakly compact operator. Now Theorem 2.4.10 implies that T is a compact operator and Proposition 2.4.8 completes the proof.

□

Now, as a corollary, we can give a reciprocal of Proposition 2.4.4 (*iii*).

Corollary 2.4.15. *If a Banach space X is weakly sequentially complete then every WUC series in X is unconditionally convergent.*

Proof. If $\sum_{n=1}^{\infty} x_n$ is WUC then $\sum_{n=1}^{\infty} x^*(x_n)$ is absolutely convergent for every $x^* \in X^*$, which is equivalent to saying that $\sum_{k=1}^{\infty} x^*(x_{n_k})$ converges for each subseries $\sum_{k=1}^{\infty} x_{n_k}$ and each $x^* \in X^*$. Hence $\sum_{k=1}^{\infty} x_{n_k}$ is weakly Cauchy and therefore weakly convergent by hypothesis. We deduce that $\sum_{n=1}^{\infty} x_n$ converges unconditionally in norm by the Orlicz-Pettis theorem.

□

The Orlicz-Pettis theorem predates basic sequence techniques. It was first proved by Orlicz in 1929 [162] and referenced in Banach's book [8]. He attributes the result to Orlicz in the special case when X is weakly sequentially complete so that every WUC series has the property of the theorem. However, it seems that Orlicz did know the more general statement. Independently, Pettis published a proof in 1938 [178]. Pettis was interested in such a result as a by-product of the study of vector measures. If Σ is a σ-algebra of sets and $\mu : \Sigma \to X$ is a map such that for every $x^* \in X^*$ the set function $x^* \circ \mu$ is a (countably additive) measure then the Orlicz-Pettis theorem implies that μ is countably additive in the norm topology. Thus weakly countably additive set functions are norm countably additive.

This is an attractive theorem and as a result it has been proved, reproved, and generalized many times since then. It is not clear that there is much left to say on this subject! We will suggest some generalizations in the Problems.

2.5 Complementability of c_0

Let us discuss the following extension problem. Suppose that X and Y are Banach spaces and that E is a subspace of X. Let $T : E \to Y$ be a bounded operator. Can we extend T to a bounded operator $\tilde{T} : X \to Y$? If we consider the special case when $Y = E$ and T is the identity map on E, we are asking simply if E is the range of a projection on X, i.e., if E is complemented in X.

The Hahn-Banach theorem asserts that if Y has dimension one then such an extension is possible with preservation of norm. However, in general such an extension is not possible and we have discussed the fact that there are noncomplemented subspaces in almost all Banach spaces. For instance we have seen that ℓ_1 must have an uncomplemented subspace, but the construction of this subspace as the kernel of a certain quotient map means that it is rather difficult to see exactly what it is. In this section we will study a very natural example. Let us formalize the notion of an injective Banach space.

Definition 2.5.1. A Banach space Y is called *injective* if whenever X is a Banach space, E is a closed subspace of X, and $T : E \to Y$ is a bounded operator then there is a bounded linear operator $\tilde{T} : X \to Y$ which is an extension of T. Y is called *isometrically injective* if $\tilde{T}$ can be additionally chosen to have $\|\tilde{T}\| = \|T\|$.

We will defer our discussion of injective spaces to later and restrict ourselves to one almost trivial observation:

Proposition 2.5.2. *The space ℓ_∞ is an isometrically injective space. Hence, if a Banach space X has a subspace E isomorphic to ℓ_∞, then E is necessarily complemented in X.*

Proof. Suppose E is a subspace of X and $T : E \to \ell_\infty$ is bounded. Then $Te = (e_n^*(e))_{n=1}^\infty$ for some sequence $(e_n^*)_{n=1}^\infty$ in E^*; clearly $\|T\| = \sup_n \|e_n^*\|$. By the Hahn-Banach theorem we choose extensions $x_n^* \in X^*$ with $\|x_n^*\| = \|e_n^*\|$ for each n. By letting $\tilde{T}x = (x_n^*(x))_{n=1}^\infty$ we are done.

□

c_0 is a subspace of ℓ_∞ (its bidual) and it is easy to see that c_0 will be injective if and only if it is complemented in ℓ_∞. Must a Banach space be complemented in its bidual? Certainly this is true for any space which is the dual of another space since for any Banach space X the space X^* is always complemented in its bidual, X^{***}. To see this consider the natural embedding $j : X \to X^{**}$. Then $j^* : X^{***} \to X^*$ is a norm-one operator. Denote by J the canonical injection of X^* into X^{***}. We claim that j^*J is the identity I_{X^*} on X^*. Indeed, suppose $x^* \in X^*$ and that $x \in X$. Then $\langle x, j^*J(x^*)\rangle = \langle jx, Jx^*\rangle = \langle x, x^*\rangle$. Thus j^* is a norm-one projection of X^{***} onto X^*. If X is isomorphic (but not necessarily isometric) to a dual space we leave for the reader the details to check that X will still be complemented in its bidual. So we may also ask if c_0 is isomorphic to a dual space.

As we will see next, c_0 is *not* complemented in ℓ_∞. This was proved essentially by Phillips [180] in 1940 although first formally observed by Sobczyk [208] the following year. Phillips in fact proved the result for the subspace c of convergent sequences. The proof we give is due to Whitley [220] and requires a simple lemma:

Lemma 2.5.3. *Every countably infinite set $\mathbb{S}$ has an uncountable family of infinite subsets $\{\mathbb{A}_i\}_{i\in\mathcal{I}}$ such that any two members of the family have finite intersection.*

Proof. The proof is very simple but rather difficult to spot! Without loss of generality we can identify $\mathbb{S}$ with the set of the rational numbers $\mathbb{Q}$. For each irrational number θ, take a sequence of rational numbers $(q_n)_{n=1}^\infty$ converging to θ. Then the sets of the form $\mathbb{A}_\theta = \{(q_n)_{n=1}^\infty : q_n \to \theta\}$ verify the lemma.

□

If $\mathbb{A}$ is any subset of $\mathbb{N}$ we denote by $\ell_\infty(\mathbb{A})$ the subspace of ℓ_∞ given by

$$\ell_\infty(\mathbb{A}) = \big\{\xi = (\xi(k))_{k=1}^\infty \in \ell_\infty : \xi(k) = 0 \text{ if } k \notin \mathbb{A}\big\}.$$

Theorem 2.5.4. *Let $T : \ell_\infty \to \ell_\infty$ be a bounded operator such that $T\xi = 0$ for all $\xi \in c_0$. Then there is an infinite subset $\mathbb{A}$ of $\mathbb{N}$ so that $T\xi = 0$ for every $\xi \in \ell_\infty(\mathbb{A})$.*

Proof. We use the family $(\mathbb{A}_i)_{i\in\mathcal{I}}$ of infinite subsets of $\mathbb{N}$ given by Lemma 2.5.3. Suppose that for every such set we can find $\xi_i \in \ell_\infty(\mathbb{A}_i)$ with $T\xi_i \neq 0$. We can assume by normalization that $\|\xi_i\|_\infty = 1$ for every $i \in \mathcal{I}$. There must exist

$n \in \mathbb{N}$ so that the set $\mathcal{I}_n = \{i \in \mathcal{I} : \ \xi_i(n) \neq 0\}$ is uncountable. Similarly, there exists $k \in \mathbb{N}$ so that the set $\mathcal{I}_{n,k} = \{i : \ |\xi_i(n)| \geq k^{-1}\}$ is also uncountable. For each $i \in \mathcal{I}_{n,k}$ choose α_i with $|\alpha_i| = 1$ and $\alpha_i \xi_i(n) = |\xi_i(n)|$.

Let $\mathbb{F}$ be a finite subset of $\mathcal{I}_{n,k}$. Consider $y = \sum_{i \in \mathbb{F}} \alpha_i \xi_i$. Since the intersection of the supports of any two distinct ξ_i is finite we can write $y = u + v$ where $\|u\|_\infty \leq 1$ and v has finite support. Thus

$$\|Ty\|_\infty = \|Tu\|_\infty \leq \|T\|,$$

and so

$$e_n^*(Ty) = \sum_{i \in \mathbb{F}} |\xi_i(n)| \leq \|T\|.$$

It follows that if $|\mathbb{F}| = m$ we have $mk^{-1} \leq \|T\|$, i.e., $m \leq k\|T\|$. Since this holds for every finite subset of $\mathcal{I}_{n,k}$ we have shown that $\mathcal{I}_{n,k}$ is in fact finite, which is a contradiction.

□

Theorem 2.5.5 (Phillips-Sobczyk, 1940-1). *There is no bounded projection from ℓ_∞ onto c_0.*

Proof. If P is such a projection we can apply Theorem 2.5.4 to $T = I - P$, with I the identity operator on ℓ_∞, and then it is clear that $P\xi = \xi$ for all $\xi \in \ell_\infty(\mathbb{A})$ for some infinite set $\mathbb{A}$, which gives a contradiction.

□

Corollary 2.5.6. *c_0 is not isomorphic to a dual space.*

Proof. If c_0 were isomorphic to a dual space then, by the comments that follow the proof of Proposition 2.5.2, c_0 should be complemented in c_0^{**}, which would lead to contradiction with Theorem 2.5.5.

□

Several comments are in order here. Theorem 2.5.4 proves more than is needed for Phillips-Sobczyk's theorem. It shows that there is no bounded, one-to-one operator from the quotient space ℓ_∞/c_0 into ℓ_∞; in other words the points of ℓ_∞/c_0 cannot be separated by countably many bounded linear functionals. (Of course, if E is a complemented subspace of a Banach space X, then X/E must be isomorphic to a subspace of X which is complementary to E.)

Now we are also in position to note that c_0 is not an injective space. Actually there are no separable injective spaces, but we will see this later, when we discuss the structure of ℓ_∞ in more detail. For the moment let us notice the dual statement of Theorem 2.3.1.

Theorem 2.5.7. *If X is a separable Banach space then X embeds isometrically into ℓ_∞.*

Proof. Let $(x_n)_{n=1}^\infty$ be a dense sequence in X. For each integer n pick $x_n^* \in X^*$ so that $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\|$. The sequence $(x_n^*)_{n=1}^\infty \subset X^*$ is norming in X. Therefore the operator $T : X \to \ell_\infty$ defined for each x in X by $T(x) = (x_n^*(x))_{n=1}^\infty$ provides the desired embedding.

□

Thus X separable can only be injective if it is isomorphic to a complemented subspace of ℓ_∞. Therefore classifying the complemented subspaces of ℓ_∞ becomes important; we will see in Chapter 5 the (already mentioned) theorem of Lindenstrauss [129] that ℓ_∞ is a prime space and this will answer our question.

In the meantime we turn to Sobczyk's main result in his 1941 paper, which gives some partial answers to these questions. The proof we present here is due to Veech [219].

Theorem 2.5.8 (Sobczyk, 1941). *Let X be a separable Banach space. If E is a closed subspace of X and $T : E \longrightarrow c_0$ is a bounded operator then there exists an operator $\tilde{T} : X \longrightarrow c_0$ such that $\tilde{T}|_E = T$ and $\|\tilde{T}\| \le 2\|T\|$.*

Proof. Without loss of generality we can assume that $\|T\| = 1$. It is immediate to realize that the operator T must be of the form

$$Tx = (f_n^*(x))_{n=1}^\infty, \qquad x \in E$$

for some $(f_n^*) \subset E^*$. Moreover $\|f_n^*\| \le 1$ for all n and (f_n^*) converges to 0 in the weak* topology of E^*. By the Hahn-Banach theorem, for each $n \in \mathbb{N}$ there exists $\varphi_n^* \in X^*$, $\|\varphi_n^*\| \le 1$, such that $\varphi_n^*|_E = f_n^*$.

X separable implies that (B_{X^*}, w^*) is metrizable (Lemma 1.4.1). Let ρ be the metric on B_{X^*} that induces the weak* topology on B_{X^*}. We claim that $\lim_{n\to\infty} \rho(\varphi_n^*, B_{X^*} \cap E^\perp) = 0$. If this is not the case, there would be some $\epsilon > 0$ and a subsequence $(\varphi_{n_k}^*)$ of (φ_n^*) such that $\rho(\varphi_{n_k}^*, B_{X^*} \cap E^\perp) \ge \epsilon$ for every k. Let $(\varphi_{n_{k_j}}^*)$ be a subsequence of $(\varphi_{n_k}^*)$ such that $\varphi_{n_{k_j}}^* \xrightarrow{w^*} \varphi^*$. Then $\varphi^* \in E^\perp \cap B_{X^*}$ since for each $e \in E$ we have

$$\varphi^*(e) = \lim_j \varphi_{n_{k_j}}^*(e) = \lim_j f_{n_{k_j}}^*(e) = 0.$$

Hence

$$\rho(\varphi_{n_{k_j}}^*, \varphi^*) \ge \epsilon \ \text{ for all } j. \tag{2.1}$$

On the other hand

$$\lim_{j\to\infty} \rho(\varphi_{n_{k_j}}^*, B_{X^*} \cap E^\perp) = \rho(\varphi^*, B_{X^*} \cap E^\perp) = 0 \tag{2.2}$$

since the function $\rho(\,\cdot\,, B_{X^*} \cap E^\perp)$ is weak* continuous on B_{X^*}. Clearly we cannot have (2.1) and (2.2) at the same time, so our claim holds.

Recall that $E^\perp$ is weak* closed, hence $B_{X^*} \cap E^\perp$ is weak* compact. Therefore for each n we can pick $v_n^* \in B_{X^*} \cap E^\perp$ such that

$$\rho(\varphi_n^*, v_n^*) = \rho(\varphi_n^*, B_{X^*} \cap E^\perp).$$

Let $x_n^* = \varphi_n^* - v_n^*$ and define the operator $\tilde{T}$ on X by $\tilde{T}(x) = (x_n^*(x))$. Notice that $\tilde{T}(x) \in c_0$ because $x_n^* \xrightarrow{w^*} 0$. Moreover, for each $x \in X$ we have

$$\|\tilde{T}(x)\| = \sup_n |x_n^*(x)| = \sup_n(|\varphi_n^*(x) - v_n^*(x)|) \le \sup_n(\|\varphi_n^*\| + \|v_n^*\|)\,\|x\| \le 2\,\|x\|\,,$$

so $\|\tilde{T}\| \le 2$.

□

Corollary 2.5.9. *If E is a closed subspace of a separable Banach space X and E is isomorphic to c_0, then there is a projection P from X onto E.*

Proof. Suppose that $T : E \to c_0$ is an isomorphism and let $\tilde{T} : X \to c_0$ be the extension of T given by the preceding theorem. Then $P = T^{-1}\tilde{T}$ is a projection from X onto E. (Note that since $\|\tilde{T}\| \le 2\|T\|$, if E is *isometric* to c_0 then $\|P\| \le 2$.)

□

Remark 2.5.10. It follows that if a separable Banach space X contains a copy of c_0 then X is not injective.

We finish this chapter by observing that in light of Theorem 2.5.8 it is natural to define a Banach space Y to be *separably injective* if whenever X is a separable Banach space, E is a closed subspace of X and $T : E \to Y$ is a bounded operator then T can be extended to an operator $\tilde{T} : X \to Y$. It was for a long time conjectured that c_0 is the only separable and separably injective space. This was solved by Zippin in 1977 [225], who showed that, indeed, c_0 is, up to isomorphism, the only separable space which is separably injective.

We also note that the constant 2 in Theorem 2.5.8 is the best possible (see Problem 2.7).

Problems

2.1. Let $T : X \to Y$ be an operator between the Banach spaces X, Y.

(a) Show that if T is strictly singular then in every infinite-dimensional subspace E of X there is a normalized basic sequence (x_n) with $\|Tx_n\| < 2^{-n}\|x_n\|$ for all n.

(b) Deduce that T is strictly singular if and only if every infinite-dimensional closed subspace E contains a further infinite-dimensional closed subspace F so that the restriction of T to F is compact.

2.2. Show that the sum of two strictly singular operators is strictly singular. Show also that if $T_n : X \to Y$ are strictly singular and $\|T_n - T\| \to 0$ then T is strictly singular.

2.3. Show that the set of all strictly singular operators on a Banach space X forms a closed two-sided ideal in the algebra $\mathcal{L}(X)$ of all bounded linear operators from X to X.

2.4. Show that if $1 < p < \infty$ and $T : \ell_p \to \ell_p$ is not compact then there is a complemented subspace E of ℓ_p so that T is an isomorphism of E onto a complemented subspace $T(E)$. Deduce that the Banach algebra $\mathcal{L}(\ell_p)$ contains exactly one proper closed two-sided ideal (the ideal of compact operators). Note that every strictly singular operator is compact in these spaces.

2.5. Show that $\mathcal{L}(\ell_p \oplus \ell_r)$ for $p \neq r$ contains at least two nontrivial closed two-sided ideals.

2.6. Suppose X is a Banach space whose dual is separable. Suppose that $\sum x_n^*$ is a series in X^* which has the property that every subseries $\sum x_{n_k}^*$ converges weak*. Show that $\sum x_n$ converges in norm. [*Hint*: Every $x^{**} \in X^{**}$ is the limit of a weak* converging sequence from X.]

2.7. Let c be the subspace of ℓ_∞ of converging sequences. Show that for any bounded projection P of c onto c_0 we have $\|P\| \geq 2$. This proves that 2 is the best possible constant in Sobczyk's theorem (Theorem 2.5.8).

2.8. In this exercise we will focus on the special properties of ℓ_1 as a target space for operators and show its *projectivity*.

(a) Suppose $T : X \to \ell_1$ is an operator from a Banach space X *onto* ℓ_1. Show that then X contains a complemented subspace isomorphic to ℓ_1.

(b) Prove that if Y is a separable infinite-dimensional Banach space with the property that whenever $T : X \to Y$ is a bounded surjective operator then Y is isomorphic to a complemented subspace of X, then Y is isomorphic to ℓ_1.

2.9. Let X be a Banach space.

(a) Show that for any $x^{**} \in X^{**}$ and any finite-dimensional subspace E of X^* there exists $x \in X$ such that

$$\|x\| < (1 + \epsilon)\|x^{**}\|,$$

and

$$x^*(x) = x^{**}(x^*), \qquad x^* \in E.$$

(b) Use part (a) to deduce the following result of Bessaga and Pełczyński ([12]): If X^* contains a subspace isomorphic to c_0 then X contains a complemented subspace isomorphic to ℓ_1, and hence X^* contains a subspace isomorphic to ℓ_∞. In particular, no separable dual space can contain an isomorphic copy of c_0. [This may also be used in Problem 2.6.]

2.10. For an arbitrary set Γ we define $c_0(\Gamma)$ as the space of functions $\xi : \Gamma \to \mathbb{R}$ such that for each $\epsilon > 0$ the set $\{\gamma : \ |\xi(\gamma)| > \epsilon\}$ is finite. When normed by $\|\xi\| = \max_{\gamma \in \Gamma} |\xi(\gamma)|$, the space $c_0(\Gamma)$ becomes a Banach space.

(a) Show that $c_0(\Gamma)^*$ can be identified with $\ell_1(\Gamma)$ the space of functions $\eta : \Gamma \to \mathbb{R}$ such that $\eta \in c_0(\Gamma)$ and $\|\eta\| = \sum_{\gamma \in \Gamma} |\eta(\gamma)| < \infty$.

(b) Show that $\ell_1(\Gamma)^* = \ell_\infty(\Gamma)$.

(c) Show, using the methods of Lemma 2.5.3 and Theorem 2.5.4, that $c_0(\mathbb{R})$ is isomorphic to a subspace of ℓ_∞ / c_0.

2.11. Let Γ be an infinite set and let $\mathcal{P}\Gamma$ denote its power set $\mathcal{P}\Gamma = \{A : \ A \subset \Gamma\}$.

(a) Show that $\ell_1(\mathcal{P}\Gamma)$ is isometric to a subspace of $\ell_\infty(\Gamma)$. [*Hint*: For each $\gamma \in \Gamma$ define $\varphi_\gamma \in \ell_\infty(\mathcal{P}\Gamma)$ by $\varphi_\gamma = 1$ when $\gamma \in A$ and -1 when $\gamma \notin A$.]

(b) Show that if $\ell_1(\Gamma)$ is a quotient of a subspace of X then $\ell_1(\Gamma)$ embeds into X (compare with Problem 2.8).

(c) Deduce that if $\ell_1(\Gamma)$ embeds into X then $\ell_1(\mathcal{P}\Gamma)$ embeds into X^*.

(d) Deduce that ℓ_1^{**} contains an isometric copy of $\ell_1(\mathcal{P}\mathbb{R})$.

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