Chapter 2

EXTREMALS OF NONAUTONOMOUS PROBLEMS

In this chapter we show that the turnpike property is a general phenomenon which holds for a large class of nonautonomous variational problems with nonconvex integrands. We consider the complete metric space of integrands $\mathcal{M}$ introduced in Section 1.1 and establish the existence of a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}$ such that for each $f \in \mathcal{F}$ and each $z \in R^n$ the following properties hold:

(i) there exists an (f)-overtaking optimal function $Z^f : [0, \infty) \to R^n$ satisfying $Z^f(0) = z$;

(ii) the integrand f has the turnpike property with the trajectory $\{Z^f(t) : t \in [0, \infty)\}$ being the turnpike.

Moreover we show that the turnpike property holds for approximate solutions of variational problems with a generic integrand f and that the turnpike phenomenon is stable under small pertubations of a generic integrand f.

2.1. Main results

Let $a > 0$ be a constant and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\psi(t) \to +\infty \text{ as } t \to \infty.$$

Denote by $|\cdot|$ the Euclidean norm in R^n. We consider the space of integrands $\mathcal{M}$ introduced in Section 1.1. This space consists of all continuous functions $f : [0, \infty) \times R^n \times R^n \to R^1$ which satisfy the following assumptions:

A(i) for each $(t, x) \in [0, \infty) \times R^n$ the function $f(t, x, \cdot) : R^n \to R^1$ is convex;

A(ii) the function f is bounded on $[0, \infty) \times E$ for any bounded set $E \subset R^n \times R^n$;

A(iii)

$$f(t, x, u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a$$

for each $(t, x, u) \in [0, \infty) \times R^n \times R^n$;

A(iv) for each pair of positive numbers M, ϵ there exist $\Gamma, \delta > 0$ such that if $t \in [0, \infty)$ and if $u_1, u_2, x_1, x_2 \in R^n$ satisfy

$$|x_i| \leq M, \ |u_i| \geq \Gamma, \ i = 1, 2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta,$$

then

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon \max\{f(t, x_1, u_1), f(t, x_2, u_2)\};$$

A(v) for each pair of positive numbers M, ϵ there is a positive number δ such that if $t \in [0, \infty)$ and if $u_1, u_2, x_1, x_2 \in R^n$ satisfy

$$|x_i|, |u_i| \leq M, \ i = 1, 2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta,$$

then

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon.$$

We equip the set $\mathcal{M}$ with two topologies where one is weaker than the other. We refer to them as the weak and the strong topologies, respectively. For the set $\mathcal{M}$ we consider the uniformity determined by the following base:

$$E_s(\epsilon) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : |f(t, x, u) - g(t, x, u)| \leq \epsilon$$

$$\text{for each } t \in [0, \infty) \text{ and each } x, u \in R^n\},$$

where $\epsilon > 0$. It is not difficult to see that the uniform space $\mathcal{M}$ with this uniformity is metrizable and complete. This uniformity generates in $\mathcal{M}$ the strong topology.

We also equip the set $\mathcal{M}$ with the uniformity which is determined by the following base:

$$E(N, \epsilon, \lambda) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : |f(t, x, u) - g(t, x, u)| \leq \epsilon$$

$$\text{for each } t \in [0, \infty) \text{ and each } x, u \in R^n \text{ satisfying } |x|, |u| \leq N,$$

$$(|f(t, x, u)| + 1)(|g(t, x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for each } t \in [0, \infty) \text{ and each } x, u \in R^n \text{ satisfying } |x| \leq N\},$$

where $N > 0$, $\epsilon > 0$, $\lambda > 1$. This uniformity which was introduced in Section 1.2, generates in $\mathcal{M}$ the weak topology. By Proposition 1.3.2 the space $\mathcal{M}$ with this uniformity is complete.

We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(t, x(t), x'(t))dt \tag{1.1}$$

where $f \in \mathcal{M}$, $0 \le T_1 < T_2 < +\infty$ and $x : [T_1, T_2] \to R^n$ is an a.c. function.

For each $f \in \mathcal{M}$, each pair of vectors $y, z \in R^n$, each $T_1 \ge 0$ and each $T_2 > T_1$ we set

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) : \ x : \ [T_1, T_2] \to R^n \tag{1.2}$$

$$\text{is an a.c. function satisfying } x(T_1) = y, \ x(T_2) = z\},$$

$$\sigma^f(T_1, T_2, y) = \inf\{U^f(T_1, T_2, y, u) : \ u \in R^n\}. \tag{1.3}$$

It is not difficult to see that $U^f(T_1, T_2, y, z)$ is finite for each $f \in \mathcal{M}$, each $y, z \in R^n$ and all numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Recall the definition of an overtaking optimal function given in Section 1.1 and the definition of a good function introduced in Section 1.2.

Let $f \in \mathcal{M}$. An a.c. function $x : [0, \infty) \to R^n$ is called (f)-overtaking optimal if for any a.c. function $y : [0, \infty) \to R^n$ satisfying $y(0) = x(0)$,

$$\limsup_{T \to \infty} \int_0^T [f(t, x(t), x'(t)) - f(t, y(t), y'(t))]dt \le 0.$$

Let $f \in \mathcal{M}$. We say that an a.c. function $x : [0, \infty) \to R^n$ is an (f)-*good function* if for any a.c. function $y : [0, \infty) \to R^n$, the function

$$T \to I^f(0, T, y) - I^f(0, T, x), \ T \in (0, \infty)$$

is bounded from below.

In this chapter we establish the existence of a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{M}$ such that the following theorems are valid.

THEOREM 2.1.1 *1. For each $g \in \mathcal{F}$ and each pair of (g)-good functions $v_i : [0, \infty) \to R^n$, $i = 1, 2$,*

$$|v_2(t) - v_1(t)| \to 0 \ \textit{as} \ t \to \infty.$$

2. For each $g \in \mathcal{F}$ and each $y \in R^n$ there exists a (g)-overtaking optimal function $Y : [0, \infty) \to R^n$ satisfying $Y(0) = y$.

3. Let $g \in \mathcal{F}$, $\epsilon > 0$ and $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. Then there exists a neighborhood $\mathcal{U}$ of g in $\mathcal{M}$ with the weak topology such that the following property holds:

If $h \in \mathcal{U}$ and if $v : [0, \infty) \to R^n$ is an (h)-good function, then

$$|v(t) - Y(t)| \leq \epsilon \text{ for all large } t.$$

THEOREM 2.1.2 *Let $g \in \mathcal{F}$, $M, \epsilon > 0$ and let $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. Then there exists a neighborhood $\mathcal{U}$ of g in $\mathcal{M}$ with the weak topology and a number $\tau > 0$ such that for each $h \in \mathcal{U}$ and each (h)-overtaking optimal function $v : [0, \infty) \to R^n$ satisfying $|v(0)| \leq M$,*

$$|v(t) - Y(t)| \leq \epsilon \text{ for all } t \in [\tau, \infty).$$

Theorems 2.1.1 and 2.1.2 establish the existence of (g)-overtaking optimal functions and describe the asymptotic behavior of (g)-good functions for $g \in \mathcal{F}$.

THEOREM 2.1.3 *Let $g \in \mathcal{F}$, $S_1, S_2, \epsilon > 0$ and let $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. Then there exists a neighborhood $\mathcal{U}$ of g in $\mathcal{M}$ with the weak topology, a number $L > 0$ and an integer $Q \geq 1$ such that if $h \in \mathcal{U}$, $T_1 \in [0, \infty)$, $T_2 \in [T_1 + LQ, \infty)$ and if an a.c. function $v : [T_1, T_2] \to R^n$ satisfies one of the following relations below:*

$$\text{(a) } |v(T_i)| \leq S_1,\ i = 1, 2, \quad I^h(T_1, T_2, v) \leq U^h(T_1, T_2, v(T_1), v(T_2)) + S_2;$$

$$\text{(b) } |v(T_1)| \leq S_1, \quad I^h(T_1, T_2, v) \leq \sigma^h(T_1, T_2, v(T_1)) + S_2,$$

then the following property holds:

There exist sequences of numbers $\{d_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [T_1, T_2]$ such that

$$q \leq Q,\ b_i < d_i \leq b_i + L,\ i = 1, \dots, q,$$

$$|v(t) - Y(t)| \leq \epsilon \text{ for each } t \in [T_1, T_2] \setminus \cup_{i=1}^q [b_i, d_i].$$

THEOREM 2.1.4 *Let $g \in \mathcal{F}$, $S, \epsilon > 0$ and let $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. Then there exist a neighborhood $\mathcal{U}$ of g in $\mathcal{M}$ with the weak topology and numbers $\delta, L > 0$ such that for each $h \in \mathcal{U}$, each pair of numbers $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 2L, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies one of the following relations below:*

$$\text{(a) } |v(T_i)| \leq S,\ i = 1, 2, \quad I^h(T_1, T_2, v) \leq U^h(T_1, T_2, v(T_1), v(T_2)) + \delta;$$

$$\text{(b) } |v(T_1)| \leq S, \quad I^h(T_1, T_2, v) \leq \sigma^h(T_1, T_2, v(T_1)) + \delta$$

the inequality $|v(t) - Y(t)| \le \epsilon$ *is valid for all* $t \in [T_1 + L, T_2 - L]$.

Theorem 2.1.4 establishes the turnpike property for any $g \in \mathcal{F}$.

The results of this chapter have been established in [103].

In the sequel we use the notation

$$B(x,r) = \{y \in R^n : \ |y - x| \le r\}, \ x \in R^n, \ r > 0, \tag{1.4}$$

$$B(r) = B(0,r), \ r > 0.$$

Chapter 2 is organized as follows. In Section 2.2 for a given $f \in \mathcal{M}$ and a given neighborhood of f in $\mathcal{M}$ with the strong topology we construct an integrand f^* which belongs to this neighborhood and establishes the turnpike property for f^*. We also study the structure of approximate solutions of variational problems with integrands belonging to a small neighborhood of f^* in the weak topology. Theorems 2.1.1-2.1.4 are proved in Section 2.3. In Section 2.4 we discuss analogs of Theorems 2.1.1-2.1.4 for a class of periodic variational problems. In Section 2.5 we show that Theorems 2.1.1-2.1.4 also hold for certain subspaces of $\mathcal{M}$ which consist of smooth integrands. In Section 2.6 we consider an example of an integrand which has the turnpike property and an example of an integrand which does not have the turnpike property.

2.2. Preliminary lemmas

Fix $f \in \mathcal{M}$ and $z_* \in R^n$. Let $\epsilon > 0$, $M > |z_*|$ and let an a.c. function $Z_*^f : [0,\infty) \to R^n$ be as guaranteed by Theorem 1.2.1. We have that Z_*^f is an (f)-good function, $Z_*^f(0) = z_*$ and for each $T_1 \ge 0$, $T_2 > T_1$,

$$U^f(T_1, T_2, Z_*^f(T_1), Z_*^f(T_2)) = I^f(T_1, T_2, Z_*^f). \tag{2.1}$$

First we define functions f_r^M for $r > 0$ such that $f_r^M \to f$ as $r \to 0^+$ in the strong topology and such that each f_r^M has the turnpike property.

Fix a continuous bounded function $\phi^M : [0,\infty) \times R^n \to [0,\infty)$ which satisfies the following assumptions:

$$\text{B(i)} \ \{(t,x) \in [0,\infty) \times R^n : \ \phi^M(t,x) = 0\} = \{(t, Z_*^f(t)) :$$
$$t \in [0,\infty)\} \cup \{(t,x) \in [0,\infty) \times R^n : \ |x| \ge M + 2\};$$

B(ii) for any $\delta > 0$ there is a positive number γ such that if $t \in [0,\infty)$ and if $x_1, x_2 \in R^n$ satisfy $|x_1 - x_2| \le \gamma$, then

$$|\phi^M(t,x_1) - \phi^M(t,x_2)| \le \delta;$$

B(iii) for any positive number δ there is $\gamma > 0$ such that if $t \in [0, \infty)$ and if $x \in R^n$ satisfies

$$|x - Z_*^f(t)| \geq \delta \text{ and } |x| \leq M + 1,$$

then $\phi^M(t, x) \geq \gamma$.

REMARK 2.2.1 *Consider a continuous function $\theta : R^1 \to [0, 1]$ for which*

$$\theta(t) = 1,\ t \in (-\infty, M + 1],\ \theta(t) = 0,\ t \in [M + 2, \infty),$$

$$\theta(t) > 0,\ t \in (M + 1, M + 2).$$

Let q be a natural number.

Define a bounded continuous function $\phi^M : [0, \infty) \times R^n \to R^1$ by

$$\phi^M(t, x) = |x - Z_*^f(t)|^q \theta(|x|), \quad t \in [0, \infty),\ x \in R^n. \qquad (2.2).$$

It is easy to verify that the function ϕ^M satisfies assumption (B).

Define a function $f_\epsilon^M : [0, \infty) \times R^n \times R^n \to R^1$ by

$$f_\epsilon^M(t, x, u) = f(t, x, u) + \epsilon\phi^M(t, x),\ t \in [0, \infty),\ x, u \in R^n. \qquad (2.3)$$

It is easy to verify that $f_\epsilon^M \in \mathcal{M}$ and to prove the following result.

LEMMA 2.2.1 *Let $M > |z_*|$ and V be a neighborhood of f in $\mathcal{M}$ with the strong topology. Then there exists a number $r_0 > 0$ such that $f_r^M \in V$ for every number $r \in (0, r_0)$.*

Fix a natural number p. It follows from Theorem 1.2.1 and Theorem 1.2.2 that there exist a number $M^f > 0$ and a neighborhood W^f of f in $\mathcal{M}$ with the weak topology such that

$$|z_*|,\ \sup\{|Z_*^f(t)| :\ t \in [0, \infty)\} < M^f \qquad (2.4)$$

and

$$\limsup_{t \to \infty} |x(t)| < M^f \qquad (2.5)$$

for each $g \in W^f$ and each (g)-good function $x : [0, \infty) \to R^n$. There exist a positive number $M_0(f, p)$ and an open neighborhood $W_0(f, p)$ of f in $\mathcal{M}$ with the weak topology such that

$$W_0(f, p) \subset W^f,\ M_0(f, p) > 2M^f + 2p + 2 \qquad (2.6)$$

and Theorem 1.2.3 holds with

$$M_1, M_2 = 2M^f + 2p + 2, c = 4^{-1},\ S = M_0(f, p),\ \mathcal{U} = W_0(f, p). \qquad (2.7)$$

There exists a neighborhood $W(f,p)$ of f in $\mathcal{M}$ with the weak topology and a number $M(f,p)$ such that

$$W(f,p) \subset W_0(f,p),\ M(f,p) > 2M_0(f,p) + 2 \tag{2.8}$$

and Theorem 1.2.3 holds with

$$M_1, M_2 = 2M_0(f,p) + 2,\ c = 4^{-1},\ S = M(f,p),\ \mathcal{U} = W(f,p). \tag{2.9}$$

It follows from Lemma 2.2.1 that there is a positive number $r(f,p)$ such that

$$f_r^{M(f,p)+1} \in W(f,p) \text{ for each } r \in (0, r(f,p)). \tag{2.10}$$

Fix $r \in (0, r(f,p))$ and set

$$f^* = f_r^{M(f,p)+1}. \tag{2.11}$$

We study the structure of approximate solutions of variational problems with integrands belonging to a small neighborhood of f^* in the strong topology. We show that the integrand f^* has the turnpike property and the function Z_*^f is its turnpike. The next lemma establishes that each approximate solution defined on an interval $[T_1, T_2]$ is close enough to the turnpike Z_*^f at a certain point of $[T_1, T_2]$ if the integrand is close enough to f^* and $T_2 - T_1$ is large enough.

LEMMA 2.2.2 *Let $\epsilon_0 \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology and an integer $N \geq 8$ such that if $g \in \mathcal{U}$, $T \geq 0$ and if an a.c. function $v : [T, T+N] \to R^n$ satisfies*

$$\max\{|v(T)|, |v(T+N)|\} \leq 2M_0(f,p) + 2, \tag{2.12}$$

$$I^g(T, T+N, v) \leq U^g(T, T+N, v(T), v(T+N)) + 2M_0(f,p) + 2,$$

then there is an integer $i_0 \in [0, N-6]$ such that

$$|v(t) - Z_*^f(t)| \leq \epsilon_0,\ t \in [i_0 + T, i_0 + T + 6]. \tag{2.13}$$

Proof. It follows from Proposition 1.3.6 that there exist a positive number S_0 and an open neighborhood $\mathcal{U}_0$ of f^* in $\mathcal{M}$ with the weak topology such that

$$\begin{gathered}\mathcal{U}_0 \subset W(f,p),\ |U^g(T_1, T_2, y_1, y_2)| + 1 < S_0 \text{ for each } g \in \mathcal{U}_0, \\ \text{each } T_1 \in [0, \infty),\ T_2 \in [T_1 + 4^{-1}, T_1 + 8] \qquad (2.14) \\ \text{and each } y_1, y_2 \in B(M(f,p)+1),\ i = 1, 2.\end{gathered}$$

By Theorem 1.2.1 there is a positive number S_1 such that the inequality

$$I^f(T_1, T_2, Z_*^f) \le I^f(T_1, T_2, v) + S_1$$

holds for each $T_1 \ge 0$, $T_2 > T_1$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies $|v(T_1)| \le M(f, p) + 1$.

(2.4), Proposition 1.3.6 and Assertion 4 of Theorem 1.2.1 imply that there exists

$$S_2 > \sup\{|I^f(T_1, T_2, Z_*^f)| :$$
$$T_1 \in [0, \infty),\ T_2 \in [T_1 + 4^{-1}, T_1 + 8]\} + 2M(f, p). \tag{2.15}$$

By Proposition 1.3.4 there exists $\delta \in (0, 8^{-1})$ such that for each $g \in \mathcal{M}$, each $T_1, T_2 \in [0, \infty)$ satisfying $4^{-1} \le T_2 - T_1 \le 8$ and each a.c. function $v : [T_1, T_2] \to R^n$ satisfying

$$I^g(T_1, T_2, v) \le 2S_0 + 2S_1 + 2S_2 + 2, \tag{2.16}$$

the following property holds:

If $t_1, t_2 \in [T_1, T_2]$ and if $|t_1 - t_2| \le \delta$, then

$$|v(t_1) - v(t_2)| \le 16^{-1}\epsilon_0. \tag{2.17}$$

There exists a number $\epsilon_1 \in (0, 4^{-1}\epsilon_0)$ such that Assumption B(iii) holds with $M = M(f, p) + 1$, $\delta = 4^{-1}\epsilon_0$, $\gamma = \epsilon_1$. Fix a natural number $N > 48$ for which

$$4^{-1}(6^{-1}N - 8)\delta\epsilon_1 r > 2M(f, p) + 2S_0 + 6a + 4 + S_1 \tag{2.18}$$

(recall a in Assumption A(iii)). By Proposition 1.3.6 there exist a number $S_3 > 0$ and a neighborhood $\mathcal{U}_1$ of f^* in $\mathcal{M}$ with the weak topology such that

$$\mathcal{U}_1 \subset \mathcal{U}_0,$$
$$|U^g(T_1, T_2, y_1, y_2)| + 1 < S_3 \text{ for each } g \in \mathcal{U}_1 \text{ each } T_1 \in [0, \infty), \tag{2.19}$$
$$T_2 \in [T_1 + 4^{-1}, T_1 + N + 4] \text{ and each } y_1, y_2 \in B(M(f, p) + 2),\ i = 1, 2.$$

By Propositions 1.3.8 and 1.3.9 there is an open neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology such that $\mathcal{U} \subset \mathcal{U}_1$ and that for each $g \in \mathcal{U}$ and each $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 4^{-1}, T_1 + N + 4]$ the following properties hold:

a) if an a.c. function $v : [T_1, T_2] \to R^n$ satisfies

$$\min\{I^{f^*}(T_1, T_2, v),\ I^g(T_1, T_2, v)\} \le S_3 + 2M(f, p) + 2,$$

then $|I^{f^*}(T_1, T_2, v) - I^g(T_1, T_2, v)| \le 4^{-1}$;

b)

$$|U^{f^*}(T_1,T_2,y_1,y_2) - U^g(T_1,T_2,y_1,y_2)| \le 4^{-1}$$

for each $y_1, y_2 \in B(M(f,p)+2)$, $i = 1, 2$.

Assume that $g \in \mathcal{U}$, $T \in [0,\infty)$ and $v : [T, T+N] \to R^n$ is an a.c. function which satisfies (2.12). We show that (2.13) holds with an integer $i_0 \in [0, N-6]$.

Let us assume the converse. Then for any integer $i \in [0, N-6]$,

$$\sup\{|v(t) - Z_*^f(t)| : \ t \in [i+T, i+T+6]\} > \epsilon_0. \tag{2.20}$$

By the inequalities (2.12), Theorem 1.2.3 and the choice of $W(f,p)$, $M(f,p)$ (see (2.8) and (2.9)),

$$|v(t)| \le M(f,p), \ t \in [T, T+N]. \tag{2.21}$$

(2.12), (2.19) and (2.21) imply that

$$\begin{aligned} I^g(T, T+N, v) &\le U^g(T, T+N, v(T), v(T+N)) + 2M_0(f,p) + 2 \\ &\le 2M_0(f,p) + S_3 + 2. \end{aligned} \tag{2.22}$$

It follows from property (b) and the inequality (2.12) that

$$|U^{f^*}(T, T+N, v(T), v(T+N)) - U^g(T, T+N, v(T), v(T+N))| \le 4^{-1}. \tag{2.23}$$

Combined with the inequality (2.22) the property (a) implies that

$$|I^{f^*}(T, T+N, v) - I^g(T, T+N, v)| \le 4^{-1}. \tag{2.24}$$

By (2.12), (2.23) and (2.24),

$$I^{f^*}(T, T+N, v) \le U^{f^*}(T, T+N, v(T), v(T+N)) + 2M_0(f,p) + 3. \tag{2.25}$$

There exists an integer j_1 such that

$$j_1 - 2 < T \le j_1 - 1. \tag{2.26}$$

Fix an integer $i \in [0, N-6]$. By (2.20) there exists a number t_i such that

$$t_i \in [i+T, i+T+6], \ |v(t_i) - Z_*^f(t_i)| > \epsilon_0. \tag{2.27}$$

The inequality (2.15) implies that

$$|I^f(T+i, T+i+6, Z_*^f)| \le S_2. \tag{2.28}$$

It follows from (2.12), (2.21) and (2.14) that

$$I^g(T+i, T+i+6, v)$$

$$\leq U^g(T+i, T+i+6, v(T+i), v(T+i+6)) + 2M_0(f,p) + 2 \quad (2.29)$$
$$\leq 2M_0(f,p) + S_0 + 2.$$

By (2.28), (2.29), (2.15), (2.27) and the definition of δ (see (2.16), (2.17)), for each

$$t \in [i+T, i+T+6] \cap [t_i - \delta, t_i + \delta] \quad (2.30)$$

the inequalitites

$$|v(t_i) - v(t)| \leq 16^{-1}\epsilon_0, \ |Z_*^f(t_i) - Z_*^f(t)| \leq 16^{-1}\epsilon_0, \ |v(t) - Z_*^f(t)| \geq 3 \cdot 4^{-1}\epsilon_0$$

are true. By these inequalities, the definition of ϵ_1, (2.21) and assumption B(iii),

$$\phi^{M(f,p)+1}(t, v(t)) \geq \epsilon_1 \quad (2.31)$$

for each integer $i \in [0, N-6]$ and each number t sastisfying (2.30).

(2.11), (2.3) and (2.21) imply that

$$I^{f^*}(T, T+N, v) = I^f(T, T+N, v) + r\int_T^{T+N} \phi^{M(f,p)+1}(t, v(t))dt.$$

By this equality and (2.31) which is true for each integer $i \in [0, N-6]$ and each t satisfying (2.30),

$$I^{f^*}(T, T+N, v) \geq I^f(T, T+N, v) + r(N6^{-1} - 2)\delta\epsilon_1. \quad (2.32)$$

By Corollary 1.3.1 there exists an a.c. function $w : [T, T+N] \to R^n$ for which

$$w(T) = v(T), \ w(T+N) = v(T+N), \ w(t) = Z_*^f(t), \ t \in [j_1, j_1+N-3], \quad (2.33)$$
$$I^{f^*}(T, j_1, w) = U^{f_*}(T, j_1, w(T), w(j_1)), \quad I^{f^*}(j_1+N-3, T+N, w)$$
$$= U^{f_*}(j_1+N-3, T+N, w(j_1+N-3), w(T+N)).$$

Combined with the inequality (2.25) the relations (2.33) imply that

$$I^{f^*}(T, T+N, v) \leq I^{f^*}(T, T+N, w) + 2M_0(f,p) + 3. \quad (2.34)$$

By (2.33), (2.21), (2.4) and (2.14),

$$I^{f^*}(T, T+N, w) \leq I^{f^*}(j_1, j_1+N-3, Z_*^f) + 2S_0. \quad (2.35)$$

It follows from Assumption A(iii), the definition of S_1 and the inequalities (2.26) and (2.21) that

$$I^f(T, T+N, v) \geq I^f(j_1, j_1+N-3, v) - 6a$$

$$\geq -6a + I^f(j_1, j_1 + N - 3, Z_*^f) - S_1. \tag{2.36}$$

Combined with (2.32) and (2.35) the inequality (2.36) implies that

$$I^{f^*}(T, T+N, v) \geq r(N6^{-1} - 2)\delta\epsilon_1 - 6a + I^f(j_1, j_1 + N - 3, Z_*^f) - S_1$$
$$\geq I^{f^*}(T, T+N, w) + r(N6^{-1} - 2)\delta\epsilon_1 - 6a - S_1 - 2S_0.$$

Together with (2.34) this inequality implies that

$$2M_0(f,p) + 3 \geq r(N6^{-1} - 2)\delta\epsilon_1 - 6a - S_1 - 2S_0.$$

This is contradictory to (2.18). The contradiction we have reached proves the lemma.

The following auxiliary result shows that an approximate solution of a variational problem defined on an interval $[T_1, T_2]$ is close to the turnpike Z_*^f at any point of $[T_1, T_2]$ if it is close enough to the turnpike at the points T_1 and T_2.

LEMMA 2.2.3 *For each $\epsilon \in (0,1)$ there is $\delta \in (0,\epsilon)$ such that the following property holds:*

If $T_1 \in [0,\infty)$, $T_2 \in [T_1 + 1, \infty)$ and if an a.c. function $v : [T_1, T_2] \to R^n$ satisfies

$$|v(T_i) - Z_*^f(T_i)| \leq \delta,\ i = 1,2,$$
$$I^{f^*}(T_1, T_2, v) \leq U^{f^*}(T_1, T_2, v(T_1), v(T_2)) + \delta, \tag{2.37}$$

then

$$|v(t) - Z_*^f(t)| \leq \epsilon,\ t \in [T_1, T_2]. \tag{2.38}$$

Proof. Let $\epsilon \in (0,1)$. It follows from Proposition 1.3.6 that there exists a positive number S_0 such that

$$|U^{f^*}(T_1, T_2, y_1, y_2)| + 6 < S_0 \text{ for each } T_1 \in [0,\infty),\ T_2 \in [T_1 + 4^{-1}, T_1 + 10] \tag{2.39}$$
$$\text{and each } y_1, y_2 \in B(M(f,p) + 1),\ i = 1,2.$$

It follows from (2.4), Proposition 1.3.6 and Assertion 4 of Theorem 1.2.1 that there is a positive number S_1 such that

$$|I^f(T_1, T_2, Z_*^f)| + 1 < S_1 \text{ for each } T_1 \in [0,\infty),\ T_2 \in [T_1 + 4^{-1}, T_1 + 10]\}. \tag{2.40}$$

By Proposition 1.3.4 there is $\delta_0 \in (0, 8^{-1})$ such that for each $g \in \mathcal{M}$, each pair of numbers $T_1, T_2 \in [0,\infty)$ satisfying $4^{-1} \leq T_2 - T_1 \leq 10$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$I^g(T_1, T_2, v) \leq 2S_0 + 2S_1 + 2, \tag{2.41}$$

the following property holds:

$$|v(t_1) - v(t_2)| \leq 16^{-1}\epsilon \tag{2.42}$$

for each $t_1, t_2 \in [T_1, T_2]$ satisfying $|t_1 - t_2| \leq \delta_0$.

There exists a number $\epsilon_1 \in (0, 4^{-1}\epsilon)$ such that Assumption B(iii) holds with $M = M(f,p) + 1$, $\delta = 4^{-1}\epsilon$, $\gamma = \epsilon_1$. By Proposition 1.3.7 there exists a number

$$\delta \in (0, \inf\{\delta_0, 8^{-1}\epsilon_1, 8^{-1}\epsilon_1\delta_0 r\}) \tag{2.43}$$

such that if $T_1 \geq 0$, $T_2 \in [T_1 + 4^{-1}, T_1 + 10]$ and if

$$y_i, x_i \in B(M(f,p) + 2),\ i = 1, 2,\ \max\{|y_1 - y_2|,\ |x_1 - x_2|\} \leq \delta, \tag{2.44}$$

then

$$|U^f(T_1, T_2, y_1, x_1) - U^f(T_1, T_2, y_2, x_2)|, \tag{2.45}$$
$$|U^{f^*}(T_1, T_2, y_1, x_1) - U^{f^*}(T_1, T_2, y_2, x_2)| \leq 2^{-7}\epsilon_1\delta_0 r.$$

Assume that $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 1, \infty)$ and an a.c. function $v : [T_1, T_2] \to R^n$ satisfies (2.37). We show that the inequality (2.38) is valid.

Let us assume the converse. Then there exists a number t_1 for which

$$t_1 \in [T_1, T_2],\ |v(t_1) - Z_*^f(t_1)| > \epsilon. \tag{2.46}$$

(2.37), (2.7), Theorem 1.2.3 and (2.4) imply that

$$|v(t)| \leq M_0(f,p),\ t \in [T_1, T_2]. \tag{2.47}$$

It is not difficult to see that there exist $d_1, d_2 \in [T_1, T_2]$ such that

$$d_2 - d_1 = 1,\ t_1 \in [d_1, d_2]. \tag{2.48}$$

It follows from the inequalities (2.37), (2.47) and (2.39) that

$$I^{f^*}(d_1, d_2, v) \leq U^{f^*}(d_1, d_2, v(d_1), v(d_2)) + \delta \leq S_0 + 1. \tag{2.49}$$

The inequality (2.40) implies that

$$I^f(d_1, d_2, Z_*^f) < S_1. \tag{2.50}$$

By the choice of δ_0 (see (2.41), (2.42)), (2.50) and (2.49)

$$|v(t) - v(t_1)| \leq 16^{-1}\epsilon,\ |Z_*^f(t) - Z_*^f(t_1)| \leq 16^{-1}\epsilon$$

for each

$$t \in [d_1, d_2] \cap [t_1 - \delta_0, t_1 + \delta_0].$$

Combined with (2.46) this fact implies that

$$|v(t) - Z_*^f(t)| \geq 3 \cdot 4^{-1}\epsilon,\ t \in [d_1, d_2] \cap [t_1 - \delta_0, t_1 + \delta_0].$$

It follows from this inequality, assumption B(iii), the definition of ϵ_1 and the inequality (2.47) that

$$\phi^{M(f,p)+1}(t, v(t)) \geq \epsilon_1 \text{ for each } t \in [d_1, d_2] \cap [t_1 - \delta_0, t_1 + \delta_0]. \quad (2.51)$$

We prove that the inequality

$$|U^g(T_1, T_2, v(T_1), v(T_2)) - U^g(T_1, T_2, Z_*^f(T_1), Z_*^f(T_2))| \leq \delta + 32^{-1}\epsilon_1\delta_0 r \quad (2.52)$$

is true with $g = f, f^*$. Corollary 1.3.1 implies that for $g = f, f^*$ there exist a.c. functions $v_i^g : [T_1, T_2] \to R^n$, $i = 1, 2$ such that

$$v_1^g(T_i) = Z_*^f(T_i),\ i = 1, 2,\ v_1^g(t) = v(t),\ t \in [T_1 + 2^{-1}, T_2 - 2^{-1}], \quad (2.53)$$

$$I^g(S, S+2^{-1}, v_1^g) = U^g(S, S+2^{-1}, v_1^g(S), v_1^g(S+2^{-1})),\ S = T_1, T_2-2^{-1},$$

$$v_2^g(T_i) = v(T_i),\ i = 1, 2,\ v_2^g(t) = Z_*^f(t),\ t \in [T_1 + 2^{-1}, T_2 - 2^{-1}],$$

$$I^g(S, S+2^{-1}, v_2^g) = U^g(S, S+2^{-1}, v_2^g(S), v_2^g(S+2^{-1})),\ S = T_1, T_2-2^{-1}.$$

By the definition of δ (see (2.43)) and the inequalities (2.53), (2.47), (2.4) and (2.37),

$$|U^g(S, S+2^{-1}, v_2^g(S), v_2^g(S+2^{-1})) - U^g(S, S+2^{-1}, Z_*^f(S), Z_*^f(S+2^{-1}))| \quad (2.54)$$
$$\leq 2^{-7}\epsilon_1\delta_0 r, \quad g = f, f^*,\ S = T_1, T_2 - 2^{-1},$$

$$|U^g(S, S + 2^{-1}, v_1^g(S), v_1^g(S + 2^{-1})) - U^g(S, S + 2^{-1}, v(S), v(S + 2^{-1}))| \quad (2.55)$$
$$\leq 2^{-7}\epsilon_1\delta_0 r, \quad g = f, f^*,\ S = T_1, T_2 - 2^{-1}.$$

It follows from (2.3), (2.53), the inequalities (2.54) and (2.55) and Assertion 4 of Theorem 1.2.1 that for $g = f, f^*$,

$$U^g(T_1, T_2, v(T_1), v(T_2)) - U^g(T_1, T_2, Z_*^f(T_1), Z_*^f(T_2)) \quad (2.56)$$
$$\leq I^g(T_1, T_2, v_2^g) - I^g(T_1, T_2, Z_*^f)$$
$$= I^g(T_1, T_1 + 2^{-1}, v_2^g) - I^g(T_1, T_1 + 2^{-1}, Z_*^f)$$
$$+I^g(T_2 - 2^{-1}, T_2, v_2^g) - I^g(T_2 - 2^{-1}, T_2, Z_*^f)$$
$$= U^g(T_1, T_1 + 2^{-1}, v_2^g(T_1), v_2^g(T_1 + 2^{-1}))$$
$$-U^g(T_1, T_1 + 2^{-1}, Z_*^f(T_1), Z_*^f(T_1 + 2^{-1}))$$

$$+U^g(T_2 - 2^{-1}, T_2, v_2^g(T_2 - 2^{-1}), v_2^g(T_2))$$
$$-U^g(T_2 - 2^{-1}, T_2, Z_*^f(T_2 - 2^{-1}), Z_*^f(T_2))$$
$$\leq 2^{-6}\epsilon_1\delta_0 r.$$

(2.55), (2.37) and (2.53) imply that

$$U^{f^*}(T_1, T_2, v(T_1), v(T_2)) - U^{f^*}(T_1, T_2, Z_*^f(T_1), Z_*^f(T_2)) \tag{2.57}$$
$$\geq I^{f^*}(T_1, T_2, v) - \delta - I^{f^*}(T_1, T_2, v_1^{f^*}) = -\delta + I^{f^*}(T_1, T_1 + 2^{-1}, v)$$
$$-I^{f^*}(T_1, T_1 + 2^{-1}, v_1^{f^*}) + I^{f^*}(T_2 - 2^{-1}, T_2, v) - I^{f^*}(T_2 - 2^{-1}, T_2, v_1^{f^*})$$
$$\geq -\delta + U^{f^*}(T_1, T_1 + 2^{-1}, v(T_1), v(T_1 + 2^{-1}))$$
$$-U^{f^*}(T_1, T_1 + 2^{-1}, v_1^{f^*}(T_1), v_1^{f^*}(T_1 + 2^{-1}))$$
$$+U^{f^*}(T_2 - 2^{-1}, T_2, v(T_2 - 2^{-1}), v(T_2))$$
$$-U^{f^*}(T_2 - 2^{-1}, T_2, v_1^{f^*}(T_2 - 2^{-1}), v_1^{f^*}(T_2)) \geq -\delta - 2^{-6}\epsilon_1\delta_0 r.$$

It follows from Corollary 1.3.1 that there are a.c. functions $v_i : [T_1, T_2] \to R^n$, $i = 3, 4$ such that

$$v_3(T_i) = v(T_i),\ i = 1, 2, \quad I^f(T_1, T_2, v_3) = U^f(T_1, T_2, v(T_1), v(T_2)), \tag{2.58}$$
$$v_4(T_i) = Z_*^f(T_i),\ i = 1, 2,\ v_4(t) = v_3(t),\ t \in [T_1 + 2^{-1}, T_2 - 2^{-1}],$$
$$I^f(S, S+2^{-1}, v_4) = U^f(S, S+2^{-1}, v_4(S), v_4(S+2^{-1})),\ S = T_1, T_2 - 2^{-1}.$$

By Theorem 1.2.3, (2.9), (2.58) and (2.47),

$$|v_3(t)| \leq M(f, p),\ t \in [T_1, T_2]. \tag{2.59}$$

By the definition of δ (see (2.45) and (2.44)), (2.58), (2.37), (2.4) and (2.59),

$$|U^f(S, S+2^{-1}, v_3(S), v_3(S+2^{-1})) - U^f(S, S+2^{-1}, v_4(S), v_4(S+2^{-1}))| \tag{2.60}$$
$$\leq 2^{-7}\epsilon_1\delta_0 r,\ S = T_1,\ T_2 - 2^{-1}.$$

Combined with (2.58) the inequality (2.60) implies that

$$U^f(T_1, T_2, v(T_1), v(T_2)) - U^f(T_1, T_2, Z_*^f(T_1), Z_*^f(T_2))$$
$$\geq I^f(T_1, T_2, v_3) - I^f(T_1, T_2, v_4)$$
$$= U^f(T_1, T_1 + 2^{-1}, v_3(T_1), v_3(T_1 + 2^{-1}))$$
$$+U^f(T_2 - 2^{-1}, T_2, v_3(T_2 - 2^{-1}), v_3(T_2))$$

$$-U^f(T_1, T_1 + 2^{-1}, v_4(T_1), v_4(T_1 + 2^{-1}))$$
$$-U^f(T_2 - 2^{-1}, T_2, v_4(T_2 - 2^{-1}), v_4(T_2))$$
$$\geq -2^{-6}\epsilon_1\delta_0 r.$$

By these relations, (2.57) and (2.56) which holds for $g = f, f^*$, the inequality (2.52) is valid with $g = f, f^*$. Combined with Assertion 4 of Theorem 1.2.1, (2.3) and (2.37) this implies that

$$U^f(T_1, T_2, v(T_1), v(T_2)) \geq U^f(T_1, T_2, Z_*^f(T_1), Z_*^f(T_2)) - \delta - 32^{-1}\epsilon_1\delta_0 r \tag{2.61}$$
$$= I^f(T_1, T_2, Z_*^f) - \delta - 32^{-1}\epsilon_1\delta_0 r$$
$$\geq U^{f^*}(T_1, T_2, Z_*^f(T_1), Z_*^f(T_2)) - \delta - 32^{-1}\epsilon_1\delta_0 r$$
$$\geq U^{f^*}(T_1, T_2, v(T_1), v(T_2)) - 2(\delta + 32^{-1}\epsilon_1\delta_0 r)$$
$$\geq I^{f^*}(T_1, T_2, v) - \delta - 2(\delta + 32^{-1}\epsilon_1\delta_0 r).$$

It follows from (2.3), (2.11), (2.47) and (2.51) that

$$I^{f^*}(T_1, T_2, v) \geq I^f(T_1, T_2, v) + \epsilon_1\delta_0 r.$$

This inequality and (2.61) imply that

$$U^f(T_1, T_2, v(T_1), v(T_2)) \geq I^f(T_1, T_2, v) + \epsilon_1\delta_0 r - \delta - 2(\delta + 32^{-1}\epsilon_1\delta_0 r)$$
$$\geq U^f(T_1, T_2, v(T_1), v(T_2)) - 3\delta + 15 \cdot 16^{-1}\epsilon_1\delta_0 r.$$

This is contradictory to (2.43). The contradiction we have reached proves the lemma.

Now we will prove an auxiliary result which generalizes Lemma 2.2.3. This result shows that the convergence property established in Lemma 2.2.3 for the integrand f^* is also valid for all integrands from a small neighborhood of f^*.

LEMMA 2.2.4 *For each $\epsilon \in (0, 1)$ there exist an open neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology and $\delta \in (0, \epsilon)$ such that the following property holds:*

If $g \in \mathcal{U}$, $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 1, \infty)$ and if an a.c. function $v : [T_1, T_2] \to R^n$ satisfies

$$|v(T_i) - Z_*^f(T_i)| \leq \delta, \; i = 1, 2,$$
$$I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta, \tag{2.62}$$

then

$$|v(t) - Z_*^f(t)| \leq \epsilon, \; t \in [T_1, T_2]. \tag{2.63}$$

Proof. Let $\epsilon \in (0,1)$. It follows from Lemma 2.2.3 that there exists $\delta \in (0,\epsilon)$ such that if $T_1 \in [0,\infty)$, $T_2 \in [T_1+1,\infty)$ and if an a.c. function $v : [T_1,T_2] \to R^n$ satisfies

$$|v(T_i) - Z_*^f(T_i)| \le 8\delta,\ i = 1,2,$$

$$I^{f^*}(T_1,T_2,v) \le U^{f^*}(T_1,T_2,v(T_1),v(T_2)) + 8\delta, \tag{2.64}$$

then

$$|v(t) - Z_*^f(t)| \le \epsilon,\ t \in [T_1,T_2]. \tag{2.65}$$

Lemma 2.2.2 implies that there are an integer $N \ge 8$ and an open neighborhood $\mathcal{U}_0$ of f^* in $\mathcal{M}$ with the weak topology such that

$$\mathcal{U}_0 \subset W(f,p)$$

and for each $g \in \mathcal{U}_0$, each $T \ge 0$ and each a.c. function $v : [T,T+N] \to R^n$ the following property holds:

If

$$\sup\{|v(T)|, |v(T+N)|\} \le 2M_0(f,p) + 2, \tag{2.66}$$

$$I^g(T,T+N,v) \le U^g(T,T+N,v(T),v(T+N)) + 2M_0(f,p) + 2,$$

then there exists an integer $i_0 \in [0,N-6]$ such that

$$|v(t) - Z_*^f(t)| \le \delta,\ t \in [i_0+T, i_0+T+6]. \tag{2.67}$$

By Proposition 1.3.6 there exist an open neighborhood $\mathcal{U}_1$ of f^* in $\mathcal{M}$ with the weak topology and a positive number S such that

$$\mathcal{U}_1 \subset \mathcal{U}_0,\ |U^g(T_1,T_2,y_1,y_2)| + 1 < S \text{ for each } g \in \mathcal{U}_1,\ \text{each } T_1 \in [0,\infty), \tag{2.68}$$
$$T_2 \in [T_1 + 2^{-1}, T_1 + 8N + 8] \text{ and each } y_1,y_2 \in B(M(f,p)+2),\ i = 1,2.$$

It follows from Propositions 1.3.8 and 1.3.9 that there exist an open neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology such that $\mathcal{U} \subset \mathcal{U}_1$ and that for each $g \in \mathcal{U}$, each $T_1 \in [0,\infty)$ and each $T_2 \in [T_1+4^{-1}, T_1+8N+8]$ the following properties hold:

a) If an a.c. function $v : [T_1,T_2] \to R^n$ satisfies

$$\min\{I^{f^*}(T_1,T_2,v), I^g(T_1,T_2,v)\} \le 2S + 2M(f,p) + 4, \tag{2.69}$$

then

$$|I^{f^*}(T_1,T_2,v) - I^g(T_1,T_2,v)| \le 4^{-1}\delta; \tag{2.70}$$

b) If $y_1, y_2 \in B(M(f,p)+2)$, $i = 1,2$, then

$$|U^{f^*}(T_1,T_2,y_1,y_2) - U^g(T_1,T_2,y_1,y_2)| \le 4^{-1}\delta. \tag{2.71}$$

Let $g \in \mathcal{U}$, $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 1, \infty)$ and let an a.c. function $v : [T_1, T_2] \to R^n$ satisfy (2.62). We show that the inequality (2.63) is true.

We have two cases: (i) $T_2 - T_1 \le 6N$; (ii) $T_2 - T_1 > 6N$. Consider the case (i). (2.4), (2.62) and (2.68) imply the inequality (2.69). The inequality (2.70) follows from (2.69) and property a). By property b) and (2.62),

$$|U^{f^*}(T_1, T_2, v(T_1), v(T_2)) - U^g(T_1, T_2, v(T_1), v(T_2))| \le 4^{-1}\delta.$$

Combined with (2.62) and (2.70) this inequality implies that

$$I^{f^*}(T_1, T_2, v) \le I^g(T_1, T_2, v)$$

$$+4^{-1}\delta \le U^g(T_1, T_2, v(T_1), v(T_2)) + \delta + 4^{-1}\delta$$
$$\le U^{f^*}(T_1, T_2, v(T_1), v(T_2)) + 3 \cdot 2^{-1}\delta.$$

It follows from this inequality, the inequality (2.62) and the definition of δ (see (2.64), (2.65)) that the inequality (2.63) is true.

Consider the case (ii). By Theorem 1.2.3, the definitions of $M_0(f, p)$ and $W_0(f, p)$ (see (2.7)) and the inequality (2.62),

$$|v(t)| \le M_0(f, p) + 1, \ t \in [T_1, T_2]. \tag{2.72}$$

It follows from the choice of $\mathcal{U}_0$ and N (see (2.66) and (2.67)) and the inequalities (2.72) and (2.62) that the following property holds:

For each $\tau \in [T_1, T_2 - N]$ there exists an integer $i_\tau \in [0, N - 6]$ such that

$$|v(t) - Z_*^f(t)| \le \delta, \ t \in [i_\tau + \tau, i_\tau + \tau + 6]. \tag{2.73}$$

It is easy to see that there exists a finite sequence $\{t_i\}_{i=0}^q \subset [T_1, T_2]$ such that

$$t_0 = T_1, \ t_q = T_2, \ t_{i+1} - t_i \in [6, N], \ i = 0, \dots, q-2, \ t_q - t_{q-1} \in [N, 2N], \tag{2.74}$$

$$|v(t_i) - Z_*^f(t_i)| \le \delta, \ i = 0, \dots, q. \tag{2.75}$$

Fix an integer $i \in \{0, \dots, q-1\}$. (2.62), (2.74), (2.75) and (2.68) imply that

$$I^g(t_i, t_{i+1}, v) \le U^g(t_i, t_{i+1}, v(t_i), v(t_{i+1})) + \delta \le \delta + S. \tag{2.76}$$

It follows from (2.76), (2.74), (2.72) and the properties a), b) that

$$|I^{f^*}(t_i, i_{i+1}, v) - I^g(t_i, t_{i+1}, v)| \le 4^{-1}\delta,$$

$$|U^{f^*}(t_i, t_{i+1}, v(t_i), v(t_{i+1})) - U^g(t_i, t_{i+1}, v(t_i), v(t_{i+1}))| \le 4^{-1}\delta,$$

$$I^{f^*}(t_i, i_{i+1}, v) \le I^g(t_i, t_{i+1}, v) + 4^{-1}\delta$$

$$\le U^g(t_i, t_{i+1}, v(t_i), v(t_{i+1})) + \delta + \delta/4 \le U^{f^*}(t_i, t_{i+1}, v(t_i), v(t_{i+1})) + 3\delta/2. \tag{2.77}$$

Combined with the choice of δ (see (2.64), (2.65)) and (2.75) the inequality (2.77) implies that $|v(t) - Z_*^f(t)| \le \epsilon$ for all $t \in [t_i, t_{i+1}]$. This completes the proof of the lemma.

We need the next lemma in order to establish the convergence property of Theorem 2.1.3. This lemma follows from Lemmas 2.2.4 and 2.2.2.

LEMMA 2.2.5 *For each $\epsilon \in (0,1)$ there exist an open neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology, a number $l_* \ge 8$ and an integer $q_* \ge 4$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$, $T_2 \in [T_1 + l_* q_*, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ the following property holds:*

If

$$|v(t)| \le 2M_0(f, p) + 2, \ t \in [T_1, T_2]$$

and

$$I^g(T_1, T_2, v) \le U^g(T_1, T_2, v(T_1), v(T_2)) + p, \tag{2.78}$$

then there exist sequences of numbers $\{d_i\}_{i=1}^q$, $\{\bar{d}_i\}_{i=1}^q$ such that

$$q \le q_*, \ d_i < \bar{d}_i \le d_i + l_*, \ i = 1, \dots, q, \tag{2.79}$$

$$|v(t) - Z_*^f(t)| \le \epsilon, \ t \in [T_1, T_2] \setminus \cup_{i=1}^q [d_i, \bar{d}_i]. \tag{2.80}$$

Proof. Let $\epsilon \in (0,1)$. It follows from Lemma 2.2.4 that there exist $\delta \in (0, \epsilon)$ and a neighborhood $\mathcal{U}_0$ of f^* in $\mathcal{M}$ with the weak topology such that the inequality (2.63) is true for each $g \in \mathcal{U}_0$, each $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 1, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies (2.62).

By Lemma 2.2.2 there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology and an integer $N \ge 8$ such that $\mathcal{U} \subset \mathcal{U}_0$ and if $g \in \mathcal{U}$, $T \ge 0$ and if an a.c. function $v : [T, T+N] \to R^n$ satisfies (2.66) then there exists an integer $i_0 \in [0, N-6]$ such that (2.67) is true. Fix an integer

$$q_* > 4 + \delta^{-1} p \text{ and a number } l_* \ge 2N. \tag{2.81}$$

Assume that $g \in \mathcal{U}$, $T_1 \in [0, \infty)$, $T_2 \in [T_1 + l_* q_*, \infty)$ and $v : [T_1, T_2] \to R^n$ is an a.c. function satisfying (2.78). It follows from the definition of $\mathcal{U}$ and (2.78) that for each $\tau \in [T_1, T_2 - N]$ there is an integer $i_\tau \in [0, N-6]$ such that

$$|v(t) - Z_*^f(t)| \le \delta \text{ for all } t \in [i_\tau + \tau, i_\tau + \tau + 6].$$

This implies that there exists a sequence of numbers $t_0, \dots, t_G$ such that

$$t_0 = T_1,\ t_G = T_2,\ t_{i+1} - t_i \in [3, N],\ i = 0, \dots, G-1, \tag{2.82}$$

$$|v(t_i) - Z_*^f(t_i)| \le \delta,\ i = 1, \dots, G-1. \tag{2.83}$$

Set

$$C = \{i \in \{1, \dots, G-2\} :\ I^g(t_i, t_{i+1}, v) > U^g(t_i, t_{i+1}, v(t_i), v(t_{i+1})) + \delta\} \tag{2.84}$$

and denote by $\mathrm{Card}(C)$ the cardinality of C. By (2.78),

$$p \ge I^g(T_1, T_2, v) - U^g(T_1, T_2, v(T_1), v(T_2)) \ge \sum_{i \in C} [I^g(t_i, t_{i+1}, v) \tag{2.85}$$

$$-U^g(t_i, t_{i+1}, v(t_i), v(t_{i+1}))] \ge \delta\ \mathrm{Card}(C), \quad \mathrm{Card}(C) \le \delta^{-1} p.$$

Let $i \in \{1, \dots, G-2\} \setminus C$. It follows from the definition of δ, $\mathcal{U}_0$ and (2.84), (2.83), (2.82) that $|v(t) - Z_*^f(t)| \le \epsilon$ for all $t \in [t_i, t_{i+1}]$. Therefore

$$|v(t) - Z_*^f(t)| \le \epsilon,$$

$$t \in [t_i, t_{i+1}],\ i \in \{1, \dots, G-2\} \setminus C.$$

It is easy to see that

$$[T_1, T_2] \setminus \cup\{[t_i, t_{i+1}] :\ i \in \{1, \dots, G-2\} \setminus C\}$$

$$\subset \cup\{[t_i, t_{i+1}] :\ i \in C \cup \{0, G-1\}\}$$

and by (2.82), (2.81), (2.85),

$$t_{i+1} - t_i \le N \le l_*,\ i = 0, \dots, G-1, \ \ \mathrm{Card}(C \cup \{0, G-1\}) \le q_*.$$

This completes the proof of the lemma.

LEMMA 2.2.6 *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology, a number $l_* \ge 8$, an integer $q_* \ge 4$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$, $T_2 \in [T_1 + l_* q_*, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies one of the following relations below:*

$$(i) |v(T_i)| \le 2M^f + 2p + 2,\ i = 1, 2,$$

$$I^g(T_1, T_2, v) \le U^g(T_1, T_2, v(T_1), v(T_2)) + p;$$

$$(ii) |v(T_1)| \le 2M^f + 2p + 2,\ I^g(T_1, T_2, v) \le \sigma^g(T_1, T_2, v(T_1)) + p,$$

there are sequences of numbers $\{d_i\}_{i=1}^q$, $\{\bar{d}_i\}_{i=1}^q$ for which (2.79) and (2.80) hold.

Lemma 2.2.6, which is an extension of Lemma 2.2.5, now follows from Lemma 2.2.5, Theorem 1.2.3 and the definition of $W_0(f,p)$, $M_0(f,p)$ (see (2.6), (2.7)).

The next auxiliary result which follows from Lemmas 2.2.4 and 2.2.2 will be used in order to establish the convergence property of Theorems 2.1.1 and 2.1.2.

LEMMA 2.2.7 *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology such that, for each $g \in \mathcal{U}$ and each (g)-good function $v : [0,\infty) \to R^n$, the inequality $|v(t) - Z_*^f(t)| \le \epsilon$ is valid for all sufficiently large t.*

Proof. It follows from Lemma 2.2.4 that there exist a number $\delta \in (0,\epsilon)$ and a neighborhood $\mathcal{U}_0$ of f^* in $\mathcal{M}$ with the weak topology such that the inequality (2.63) is true for each $g \in \mathcal{U}_0$, each $T_1 \in [0,\infty)$, $T_2 \in [T_1+1,\infty)$ and each a.c. function $v : [T_1,T_2] \to R^n$ which satisfies (2.62).

It follows from Lemma 2.2.2 that there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology and an integer $N \ge 8$ such that

$$\mathcal{U} \subset \mathcal{U}_0 \cap W(f,p)$$

and that the following property holds:

If $g \in \mathcal{U}$, $T \ge 0$ and if an a.c. function $v : [T, T+N] \to R^n$ satisfies (2.12), then there is an integer $i_0 \in [0, N-6]$ such that

$$|v(t) - Z_*^f(t)| \le \delta,\ t \in [i_0+T, i_0+T+6]. \tag{2.86}$$

Assume that $g \in \mathcal{U}$ and $v : [0,\infty) \to R^n$ is a (g)-good function. By the definition of W^f (see (2.5)),

$$|v(t)| \le M^f \text{ for all large } t. \tag{2.87}$$

Since v is a (g)-good function there exists $T_0 > 0$ such that

$$I^g(t_1,t_2,v) \le U^g(t_1,t_2,v(t_1),v(t_2)) + \delta \tag{2.88}$$

for each $t_1 \ge T_0$, $t_2 > t_1$. We may assume that $|v(t)| \le M^f$ for all $t \in [T_0,\infty)$.

Let $T \ge T_0$. It follows from the definition of $\mathcal{U}$, N and (2.88) which holds with $t_1 = T$, $t_2 = T+N$ that there exists an integer $i_T \in [0, N-6]$ such that

$$|v(t) - Z_*^f(t)| \le \delta,\ t \in [i_T+T, i_T+T+6].$$

Therefore there exists a sequence $\{T_i\}_{i=1}^{\infty} \subset (T_0,\infty)$ such that

$$T_{i+1} - T_i \in [6,N],\ i = 0,1,\ldots,\ |v(T_i) - Z_*^f(T_i)| \le \delta,\ i = 1,2,\ldots.$$

It follows from these relations, (2.88) which holds with $t_1 = T_i$, $t_2 = T_{i+1}$ and the definition of $\mathcal{U}_0$, δ that

$$|v(t) - Z_*^f(t)| \le \epsilon,\ t \in [T_i, T_{i+1}],\ i = 1, 2, \ldots.$$

The lemma is proved.

The next lemma plays a crucial role in the proof of Theorem 2.1.4. Its proof is based on Lemmas 2.2.4 and 2.2.2.

LEMMA 2.2.8 *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology, $\delta \in (0,\epsilon)$ and $\Delta > 1$ such that the following property holds:*

For each $g \in \mathcal{U}$, each $T_1 \in [0,\infty)$, $T_2 \in [T_1 + 2\Delta, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(t)| \le M_0(f,p),\ t \in [T_1, T_2],$$

$$I^g(T_1, T_2, v) \le U^g(T_1, T_2, v(T_1), v(T_2)) + \delta, \tag{2.89}$$

there exist $\tau_1 \in [T_1, T_1 + \Delta]$ and $\tau_2 \in [T_2 - \Delta, T_2]$ such that

$$|v(t) - Z_*^f(t)| \le \epsilon,\ t \in [\tau_1, \tau_2]. \tag{2.90}$$

Moreover if $|v(T_1) - Z_*^f(T_1)| \le \delta$, then $\tau_1 = T_1$.

Proof. By Lemma 2.2.4 there exist $\delta \in (0,\epsilon)$ and a neighborhood $\mathcal{U}_0$ of f^* in $\mathcal{M}$ with the weak topology such that for each $g \in \mathcal{U}_0$, each $T_1 \in [0,\infty)$, $T_2 \in [T_1 + 1, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies (2.62), relation (2.63) holds.

By Lemma 2.2.2 there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology and an integer $N \ge 8$ such that $\mathcal{U} \subset \mathcal{U}_0$ and for each $g \in \mathcal{U}$, each $T \ge 0$ and each a.c. function $v : [T, T+N] \to R^n$ which satisfies (2.12), there is an integer $i_0 \in [0, N-6]$ for which (2.86) holds. Set

$$\Delta = 2N. \tag{2.91}$$

Assume that $g \in \mathcal{U}$, $T_1 \in [0,\infty)$, $T_2 \in [T_1 + 2\Delta, \infty)$ and $v : [T_1, T_2] \to R^n$ is an a.c. function satisfying (2.89). It follows from the definition of $\mathcal{U}, N$ and (2.89) that for each $T \in [T_1, T_2 - N]$ there is an integer $i_0 \in [0, N-6]$ for which (2.86) holds. Therefore there exists a sequence of numbers $\{t_i\}_{i=0}^G \subset [T_1, T_2]$ such that

$$t_0 = T_1,\ t_{i+1} - t_i \in [6, N],\ i = 0, \ldots, G-1,\ T_2 - t_G \le N, \tag{2.92}$$

$$|v(t_i) - Z_*^f(t_i)| \le \delta,\ i = 1, \ldots, G. \tag{2.93}$$

It follows from the definition of δ, $\mathcal{U}_0$, (2.89), (2.93), (2.92) that

$$|v(t) - Z_*^f(t)| \leq \epsilon, \ t \in [t_i, t_{i+1}] \tag{2.94}$$

for all $i = 1, \ldots, G-1$ and if $|v(T_1) - Z_*^f(T_1)| \leq \delta$, then (2.94) holds for $i = 0, \ldots, G-1$. This completes the proof of the lemma.

The following lemma is an extension of Lemma 2.2.8.

LEMMA 2.2.9 *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology, $\delta \in (0, \epsilon)$, $\Delta > 1$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 2\Delta, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ the following properties hold:*

(i) If $|v(T_i)| \leq 2M^f + 2 + 2p, \ i = 1, 2$

and

$$I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta,$$

then (2.90) holds with $\tau_1 \in [T_1, T_1 + \Delta]$ and $\tau_2 \in [T_2 - \Delta, T_2]$. Moreover if $|v(T_1) - Z_*^f(T_1)| \leq \delta$, then $\tau_1 = T_1$.

(ii) If $|v(T_1)| \leq 2M^f + 2 + 2p$, $I^g(T_1, T_2, v) \leq \sigma^g(T_1, T_2, v(T_1)) + \delta$, then (2.90) holds with $\tau_1 \in [T_1, T_1 + \Delta]$ and $\tau_2 \in [T_2 - \Delta, T_2]$. Moreover if $|v(T_1) - Z_*^f(T_1)| \leq \delta$, then $\tau_1 = T_1$.

Lemma 2.2.9 now follows from Lemma 2.2.8, Theorem 1.2.3 and the definition of $W_0(f,p)$, $M_0(f,p)$ (see (2.6), (2.7)).

LEMMA 2.2.10 *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of f^* in $\mathcal{M}$ with the weak topology, $\Delta > 1$ such that for each $g \in \mathcal{U}$ and each (g)-overtaking optimal function $v : [0, \infty) \to R^n$ satisfying $|v(0)| \leq 2M^f + 2 + 2p$ the relation $|v(t) - Z_*^f(t)| \leq \epsilon$ holds for all $t \in [\Delta, \infty)$.*

Lemma 2.2.10 follows from the definition of W^f, M^f (see (2.5)) and Lemma 2.2.9.

2.3. Proofs of Theorems 2.1.1-2.1.4

Construction of the set $\mathcal{F}$. Fix $z_* \in R^n$ and an integer $p \geq 1$. For each $f \in \mathcal{M}$ we define a function $Z_*^f : [0, \infty) \to R^n$, numbers M^f, $M_0(f,p)$, $M(f,p)$, a function $\phi^{M(f,p)+1}$, a number $r(f,p) > 0$ and neighborhoods W^f, $W_0(f,p)$, $W(f,p)$ of f in $\mathcal{M}$ with the weak topology as in Section 2.2 (see (2.3)-(2.10)). Set

$$E_p = \{f_r^{M(f,p)+1} : \ f \in \mathcal{M}, \ r \in (0, r(f,p))\}. \tag{3.1}$$

Clearly for each $f \in \mathcal{M}$ and each $r \in (0, r(f,p))$ Lemmas 2.2.2-2.2.10 hold with $f^* = f_r^{M(f,p)+1}$. By Lemma 2.2.1 E_p is everywhere dense in $\mathcal{M}$ with the strong topology.

For each $f \in \mathcal{M}$, each $r \in (0, r(f,p))$ and each integer $k \geq 1$ there exist an open neighborhood $V(f,p,r,k)$ of $f_r^{M(f,p)+1}$ in $\mathcal{M}$ with the weak topology, an integer $q(f,p,r,k) \geq 4$, numbers

$$\delta(f,p,r,k) \in (0,(4k)^{-1}),\ l(f,p,r,k) \geq 8,\ \Delta(f,p,r,k) > 1$$

such that:

(i) Lemma 2.2.6 holds with

$$f^* = f_r^{M(f,p)+1},\ \epsilon = (4k)^{-1},\ \mathcal{U} = V(f,p,r,k),$$

$$q_* = q(f,p,r,k),\ l_* = l(f,p,r,k);$$

(ii) Lemma 2.2.7 holds with

$$f^* = f_r^{M(f,p)+1},\ \epsilon = (4k)^{-1},\ \mathcal{U} = V(f,p,r,k);$$

(iii) Lemmas 2.2.9 and 2.2.10 hold with

$$f^* = f_r^{M(f,p)+1},\ \mathcal{U} = V(f,p,r,k),\ \epsilon = (4k)^{-1},$$

$$\delta = \delta(f,p,r,k),\ \Delta = \Delta(f,p,r,k).$$

We define

$$\mathcal{F}_p = \cap_{k=1}^{\infty} \cup \{V(f,p,r,k) :\ f \in \mathcal{M},\ r \in (0, r(f,p))\}, \tag{3.2}$$

$$\mathcal{F} = \cap_{p=1}^{\infty} \mathcal{F}_p. \tag{3.3}$$

Clearly $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{M}$.

Proof of Theorem 2.1.1. We prove Assertion 1. Assume that $g \in \mathcal{F}$ and $v_i : [0,\infty) \to R^n$, $i = 1,2$ are (g)-good functions. Let $\epsilon > 0$. Fix an integer $k \geq 4\epsilon^{-1}$. There exist $f \in \mathcal{M}$ and $r \in (0, r(f,1))$ such that $g \in V(f,1,r,k)$. By condition (ii) and Lemma 2.2.7,

$$|v_2(t) - v_1(t)| \leq (2k)^{-1} < \epsilon \text{ for all large } t.$$

Since ϵ is an arbitrary positive number we conclude that

$$|v_2(t) - v_1(t)| \to 0 \text{ as } t \to \infty.$$

Assertion 1 is proved.

We prove Assertion 2. Let $g \in \mathcal{F}$ and let $y \in R^n$. By Theorem 1.2.1 there exists a (g)-good function $Y : [0, \infty) \to R^n$ such that $Y(0) = y$ and

$$I^g(0, T, Y) = U^g(0, T, Y(0), Y(T)) \tag{3.4}$$

for each $T \geq 0$. We show that Y is a (g)-overtaking optimal function.

Let us assume the converse. Then there exists a number $\epsilon > 0$ and an a.c. function $Z : [0, \infty) \to R^n$ such that $Z(0) = y$ and

$$\limsup_{T \to \infty} [I^g(0, T, Y) - I^g(0, T, Z)] > \epsilon. \tag{3.5}$$

There exists a sequence of positive numbers $\{T_i\}_{i=1}^{\infty}$ such that

$$T_i \to +\infty \text{ as } i \to \infty,$$

$$I^g(0, T_i, Y) - I^g(0, T_i, Z) > \epsilon, \ i = 1, 2, \dots. \tag{3.6}$$

By Theorem 1.2.1, Z is a bounded (g)-good function. Therefore

$$Y(t) - Z(t) \to 0 \text{ as } t \to \infty. \tag{3.7}$$

The functions Y and Z are (f)-good and bounded. Therefore we can choose a number

$$S > \sup\{|Z(t)|, Y(t)| : \ t \in [0, \infty)\}. \tag{3.8}$$

It follows from Proposition 1.3.7 that there exists $\delta > 0$ such that the following property holds:

If $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 4^{-1}, T_1 + 4]$ and if $y_1, y_2, z_1, z_2 \in R^n$ satisfy

$$|y_i|, |z_i| \leq S, \ i = 1, 2, \ |y_1 - y_2|, |z_1 - z_2| \leq \delta,$$

then

$$|U^g(T_1, T_2, y_1, z_1) - U^g(T_1, T_2, y_2, z_2| \leq 8^{-1}\epsilon. \tag{3.9}$$

Since $|Y(t) - Z(t)| \to 0$ as $t \to \infty$ there exists $\tau > 0$ such that

$$|Z(t) - Y(t)| \leq 2^{-1}\delta, \ t \in [\tau, \infty). \tag{3.10}$$

Fix a natural number j such that $T_j > \tau$. There exists an a.c. function $X : [0, \infty) \to R^n$ such that

$$X(t) = Z(t), t \in [0, T_j], \ X(t) = Y(t), \ t \in [T_j + 1, \infty), \tag{3.11}$$

$$I^g(T_j, T_j + 1, X) = U^g(T_j, T_j + 1, X(T_j), X(T_j + 1)).$$

It follows from (3.11), (3.4), (3.8), (3.10) and the definition of δ that

$$X(0) = Y(0), \ X(T_j + 1) = Y(T_j + 1),$$

$$|I^g(T_j, T_j+1, X) - I^g(T_j, T_j+1, Y)| = |U^g(T_j, T_j+1, X(T_j), X(T_j+1))$$
$$-U^g(T_j, T_j+1, Y(T_j), Y(T_j+1))| \le 8^{-1}\epsilon.$$

Together with (3.11) and (3.6) these relations imply that

$$I^g(0, T_j+1, Y) - U^g(0, T_j+1, Y(0), Y(T_j+1))$$
$$\ge I^g(0, T_j+1, Y) - I^g(0, T_j+1, X) =$$
$$I^g(0, T_j, Y) - I^g(0, T_j, Z) + I^g(T_j, T_j+1, Y) - I^g(T_j, T_j+1, X) \ge \epsilon - 8^{-1}\epsilon.$$

This is contradictory to (3.4). The obtained contradiction proves Assertion 2.

We prove Assertion 3. Let $g \in \mathcal{F}$, $\epsilon > 0$ and $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. Fix an integer $k \ge 4\epsilon^{-1}$. There exist $f \in \mathcal{M}$, $r \in (0, r(f, 1))$ such that $g \in V(f, 1, r, k)$.

Assume that $h \in V(f, 1, r, k)$ and $v : [0, \infty) \to R^n$ is an (h)-good function. It follows from condition (ii) and Lemma 2.2.7 that $|v(t) - Y(t)| < (2k)^{-1}$ for all large t. Assertion 3 is proved.

Proof of Theorem 2.1.2. Let $g \in \mathcal{F}$, $M, \epsilon > 0$ and let $Y : [0, \infty) \to R^n$ be a (g)-overtaking function. Fix integers

$$p > 2M + 2|Y(0)| + 2, \ k > 4\epsilon^{-1}. \tag{3.12}$$

There exists $f \in \mathcal{M}$ and $r \in (0, r(f, p))$ such that $g \in V(f, p, r, k)$. By condition (ii) and Lemma 2.2.7 there exists a number $\tau_0 > 0$ such that

$$|Y(t) - Z_*^f(t)| \le (4k)^{-1}, \ t \in [\tau_0, \infty). \tag{3.13}$$

Set

$$\tau = \tau_0 + \Delta(f, p, r, k) + 1. \tag{3.14}$$

Assume that $h \in V(f, p, r, k)$ and $v : [0, \infty) \to R^n$ is an (h)-overtaking optimal function such that $|v(0)| \le M$. By condition (iii), Lemma 2.2.10 and (3.12),

$$|v(t) - Z_*^f(t)| \le (4k)^{-1}, \ t \in [\Delta(f, p, r, k), \infty).$$

Together with (3.13), (3.14) and (3.12) this implies that $|v(t) - Y(t)| \le \epsilon$ for all $t \in [\tau, \infty)$. The theorem is proved.

Proof of Theorem 2.1.3. Let $g \in \mathcal{F}$, $S_1, S_2, \epsilon > 0$ and $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. By Theorem 1.2.1, Y is a (g)-good function and $\sup\{|Y(t)| : \ t \in [0, \infty)\} < \infty$. Fix integers

$$p > 2S_1 + 2S_2 + 1 + \sup\{|Y(t)| : \ t \in [0, \infty)\}, \ k > 8\epsilon^{-1}. \tag{3.15}$$

There exist $f \in \mathcal{M}$ and $r \in (0, r(f,p))$ such that $g \in V(f,p,r,k)$. By condition (ii) and Lemma 2.2.7 there exists a number $\tau_0 > 0$ such that

$$|Y(t) - Z_*^f(t)| \leq (4k)^{-1}, \ t \in [\tau_0, \infty). \tag{3.16}$$

Set

$$\mathcal{U} = V(f,p,r,k), \ L = \tau_0 + l(f,p,r,k), \ Q = q(f,p,r,k) + 1. \tag{3.17}$$

Assume that $h \in \mathcal{U}$, $T_1 \in [0, \infty)$, $T_2 \in [T_1 + LQ, \infty)$ and an a.c. function $v : [T_1, T_2] \to R^n$ satisfies one of the conditions (a), (b) of the theorem. By condition (i) and Lemma 2.2.6 there are numbers $\{d_i\}_{i=1}^q$, $\{\bar{d}_i\}_{i=1}^q$ such that

$$q \leq q(f,p,r,k), \ d_i < \bar{d}_i \leq d_i + l(f,p,r,k), \ i = 1, \dots, q,$$

$$|v(t) - Z_*^f(t)| \leq (4k)^{-1}, \ t \in [T_1, T_2] \setminus \cup_{i=1}^q [d_i, \bar{d}_i].$$

By these relations and (3.15), (3.16),

$$|v(t) - Y(t)| \leq \epsilon, \ t \in [T_1, T_2] \setminus ([0, \tau_0] \cup_{i=1}^q [d_i, \bar{d}_i]).$$

This completes the proof of the theorem.

Proof of Theorem 2.1.4. Let $g \in \mathcal{F}$, $S, \epsilon > 0$ and let $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. By Theorem 1.2.1, Y is a (g)-good function and $\sup\{|Y(t)| : t \in [0, \infty)\} < \infty$. Fix integers

$$p > 2S + 1 + \sup\{|Y(t)| : t \in [0, \infty)\}, \ k > 8\epsilon^{-1}. \tag{3.18}$$

There exist $f \in \mathcal{M}$ and $r \in (0, r(f,p))$ such that $g \in V(f,p,r,k)$. By condition (ii) and Lemma 2.2.7 there exists a number τ_0 such that (3.16) holds. Set

$$\mathcal{U} = V(f,p,r,k), \ L = \tau_0 + \Delta(f,p,r,k), \ \delta = \delta(f,p,r,k). \tag{3.19}$$

Assume that $h \in \mathcal{U}$, $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 2L, \infty)$ and an a.c. function $v : [T_1, T_2] \to R^n$ satisfies one of the conditions (a), (b) of the theorem. It follows from condition (iii) and Lemma 2.2.9 that

$$|v(t) - Z_*^f(t)| \leq (4k)^{-1}, \ t \in [T_1 + \Delta(f,p,r,k), T_2 - \Delta(f,p,r,k)].$$

By this relation, (3.16), (3.18) and (3.19), $|v(t) - Y(t)| \leq \epsilon$ for all $t \in [T_1 + L, T_2 - L]$. The theorem is proved.

2.4. Periodic variational problems

Let $a > 0$, $\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$ and let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$. Denote by $\mathcal{M}^p$ the set of continuous functions $f: [0,\infty) \times R^n \times R^n \to R^1$ which satisfy the following assumptions:

A (i) $f(t, x+q, u) = f(t,x,u)$ for each $t \in [0,\infty)$, $x, u \in R^n$, $q \in \mathbf{Z}^n$;

A (ii) for each $(t,x) \in [0,\infty) \times R^n$ the function $f(t,x,\cdot): R^n \to R^1$ is convex;

A (iii) the function f is bounded on $[0,\infty) \times R^n \times E$ for any bounded set $E \subset R^n$;

A (iv) $f(t,x,u) \geq \psi(|u|)|u| - a$ for each $(t,x,u) \in [0,\infty) \times R^n \times R^n$;

A (v) for each $\epsilon > 0$ there exist positive numbers Γ, δ such that if $t \in [0,\infty)$ and if $u_1, u_2, x_1, x_2 \in R^n$ are such that

$$|u_i| \geq \Gamma,\ i = 1,2, \quad \max\{|x_1 - x_1|, |u_1 - u_2|\} \leq \delta,$$

then

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \leq \epsilon \max\{f(t,x_1,u_1), f(t,x_2,u_2)\};$$

A (vi) for each $M, \epsilon > 0$ there exists a positive number δ such that if $t \in [0,\infty)$ and if $u_1, u_2, x_1, x_2 \in R^n$ satisfy

$$|u_i| \leq M,\ i = 1,2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta,$$

then

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \leq \epsilon.$$

We equip the set $\mathcal{M}^p$ with the uniformity which is determined by the following base:

$$E(N,\epsilon,\lambda) = \{(f,g) \in \mathcal{M}^p \times \mathcal{M}^p : |f(t,x,u) - g(t,x,u)| \leq \epsilon$$

$$\text{for each } t \in [0,\infty) \text{ and each } x, u \in R^n \text{ satisfying } |u| \leq N,$$

$$(|f(t,x,u)| + 1)(|g(t,x,u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for each } t \in [0,\infty) \text{ and each } x, u \in R^n\},$$

where $N > 0$, $\epsilon > 0$, $\lambda > 1$.

We can show that the uniform space $\mathcal{M}^p$ is metrizable and complete.

Consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(t, x(t), x'(t))dt$$

where $f \in \mathcal{M}^p$, $0 \le T_1 < T_2 < +\infty$ and $x : [T_1, T_2] \to R^n$ is an a.c. function.

For $f \in \mathcal{M}^p$, $y, z \in R^n$ and numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ we define $U^f(T_1, T_2, y, z)$ and $\sigma^f(T_1, T_2, y)$ by (1.2) and (1.3) and set

$$\tilde{U}^f(T_1, T_2, y, z) = \inf\{U^f(T_1, T_2, y, z + m) : \ m \in \mathbf{Z}^n\}.$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < +\infty$ for each $f \in \mathcal{M}^p$, each $y, z \in R^n$ and each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

For $f \in \mathcal{M}^p$ we use the notions of an (f)-good function and an (f)-overtaking optimal function.

The methods used in the proofs of Theorems 1.2.1-1.2.3 and Theorems 2.1.1-2.1.4 are applicable to the space $\mathcal{M}^p$. The following results are valid.

THEOREM 2.4.1 *For each $h \in \mathcal{M}^p$ and each $z \in R^n$ there exists an (h)-good function $Z^h : [0, \infty) \to R^n$ satisfying $Z^h(0) = z$ such that:*

1.

$$\tilde{U}^f(T_1, T_2, Z^f(T_1), Z^f(T_2)) = I^f(T_1, T_2, Z^f)$$

for each $f \in \mathcal{M}$, each $z \in R^n$ and each $T_1 \ge 0$, $T_2 > T_1$.

2. For each $f \in \mathcal{M}^p$, each $z \in R^n$ and each a.c. function $y : [0, \infty) \to R^n$ either

$$I^f(0, T, y) - I^f(0, T, Z^f) \to +\infty \ \text{as} \ T \to \infty$$

or

$$\sup\{|I^f(0, T, y) - I^f(0, T, Z^f)| : \ T \in (0, \infty)\} < \infty \tag{4.1}$$

and if (4.1) is valid, then

$$\sup\{|y(t_1) - y(t_2)| : \ t_1 \in [0, \infty), \ t_2 \in [t_1, t_1 + 1]\} < \infty.$$

3. For each $f \in \mathcal{M}^p$ there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{M}^p$ and a number $Q > 0$ such that

$$\sup\{|Z^g(t_1) - Z^g(t_2)| : \ t_1 \in [0, \infty), \ t_2 \in [t_1, t_1 + 1]\} \le Q$$

for each $g \in \mathcal{U}$ and each $z \in R^n$.

4. For each $f \in \mathcal{M}^p$ there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{M}^p$ and a number $Q > 0$ such that

$$I^g(T_1, T_2, Z^g) \le I^g(T_1, T_2, y) + Q$$

for $g \in \mathcal{U}$, each $z \in R^n$, each $T_1 \ge 0$, $T_2 > T_1$ and each a.c. function $y : [T_1, T_2] \to R^n$.

THEOREM 2.4.2 *For each $f \in \mathcal{M}^p$ there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{M}^p$ and $M > 0$ such that if $g \in \mathcal{U}$ and if $x : [0, \infty) \to R^n$ is a (g)-good function, then*

$$\limsup_{T \to \infty} (\sup\{|x(t_1) - x(t_2)| :$$

$$t_1 \in [T, \infty),\ t_2 \in [t_1, t_1 + 1]\}) < M.$$

We can show that there exists a set $\mathcal{F} \subset \mathcal{M}^p$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}^p$ such that the following theorems are valid.

THEOREM 2.4.3 *1. For each $g \in \mathcal{F}$ and each pair of (g)-good functions $v_i : [0, \infty) \to R^n$, $i = 1, 2$,*

$$|v_2(t) - v_1(t) - m| \to 0 \text{ as } t \to \infty$$

with $m \in \mathbf{Z}^n$.

2. For each $g \in \mathcal{F}$ and each $y \in R^n$ there exists a (g)-overtaking optimal function $Y : [0, \infty) \to R^n$ such that $Y(0) = y$.

3. Let $g \in \mathcal{F}$, $\epsilon > 0$ and $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. Then there exists a neighborhood $\mathcal{U}$ of g in $\mathcal{M}^p$ such that the following property holds:

If $h \in \mathcal{U}$ and if $v : [0, \infty) \to R^n$ is an (h)-good function, then there is $m \in \mathbf{Z}^n$ such that

$$|v(t) - Y(t) - m| \leq \epsilon \text{ for all large } t.$$

THEOREM 2.4.4 *Let $g \in \mathcal{F}$, $\epsilon > 0$ and let $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. Then there exists a neighborhood $\mathcal{U}$ of g in $\mathcal{M}^p$ and a number $\tau > 0$ such that the following property holds:*

If $h \in \mathcal{U}$ and if $v : [0, \infty) \to R^n$ is an (h)-overtaking optimal function, then there exists $m \in \mathbf{Z}^n$ such that

$$|v(t) - Y(t) - m| \leq \epsilon,\ t \in [\tau, \infty).$$

THEOREM 2.4.5 *Let $g \in \mathcal{F}$, $S, \epsilon > 0$ and let $Y : [0, \infty) \to R^n$ be a (g)-overtaking optimal function. Then there exists a neighborhood $\mathcal{U}$ of g in $\mathcal{M}^p$, a number $L > 0$ and an integer $Q \geq 1$ such that for each $h \in \mathcal{U}$ and each pair of numbers $T_1 \in [0, \infty)$, $T_2 \in [T_1 + LQ, \infty)$ the following property holds:*

If an a.c. function $v : [T_1, T_2] \to R^n$ satisfies

$$I^h(T_1, T_2, v) \leq \tilde{U}^h(T_1, T_2, v(T_1), v(T_2)) + S,$$

then there exist sequences of numbers

$$\{d_i\}_{i=1}^q,\ \{b_i\}_{i=1}^q \subset [T_1, T_2]$$

such that

$$q \le Q,\ b_i < d_i \le b_i + L,\ i = 1, \dots, q$$

and that for each interval

$$\mathcal{J} \subset [T_1, T_2] \setminus \cup_{i=1}^q [b_i, d_i]$$

there is $m \in \mathbf{Z}^n$ *for which*

$$|v(t) - Y(t) - m| \le \epsilon,\ t \in \mathcal{J}.$$

THEOREM 2.4.6 *Let* $g \in \mathcal{F}$, $\epsilon > 0$ *and let* $Y : [0, \infty) \to R^n$ *be a* (g)-*overtaking optimal function. Then there exists a neighborhood* $\mathcal{U}$ *of* g *in* $\mathcal{M}^p$ *and numbers* $\delta, L > 0$ *such that for each* $h \in \mathcal{U}$, *each pair of numbers* $T_1 \in [0, \infty)$, $T_2 \in [T_1 + 2L, \infty)$ *and each a.c. function* $v : [T_1, T_2] \to R^n$ *which satisfies* $I^h(T_1, T_2, v) \le \tilde{U}^h(T_1, T_2, v(T_1), v(T_2)) + \delta$ *the following property holds:*

There exists $m \in \mathbf{Z}^n$ *such that*

$$|v(t) - Y(t) - m| \le \epsilon,\ t \in [T_1 + L, T_2 - L].$$

2.5. Spaces of smooth integrands

Consider the complete metric space $\mathcal{M}$ defined in Section 2.1. For any function $g : R^1 \times R^n \times R^n \to R^1$ denote by $\mathcal{L}(g)$ the restriction of g to $[0, \infty) \times R^{2n}$ and for an integer $k \ge 1$ denote by $C(k, \mathcal{M})$ the space of all integrands $f = f(t, x, u) \in C^k(R^{2n+1})$ such that $\mathcal{L}(f) \in \mathcal{M}$.

Let $k \ge 1$ be an integer. For $p = (p_1, \dots, p_{2n+1}) \in \{0, \dots, k\}^{2n+1}$ and $f \in C^k(R^{2n+1})$ we set

$$|p| = \sum_{i=1}^{2n+1} p_i,\ D^p f = \partial^{|p|} f / \partial y_1^{p_1} \dots \partial y_{2n+1}^{p_{2n+1}}.$$

For the set $C(k, \mathcal{M})$ we consider the uniformity which is determined by the following base:

$$E(N, \epsilon, \lambda) = \{(f, g) \in C(k, \mathcal{M}) \times C(k, \mathcal{M}) :$$

$$|D^p f(t, x, u) - D^p g(t, x, u)| \le \epsilon$$

$$\text{for each } (t,x,u) \in R^{2n+1} \text{ satisfying } |t|,|x|,|u| \le N$$

$$\text{and each } p \in \{0,\dots k\}^{2n+1} \text{ such that } |p| \le k,$$

$$|f(t,x,u) - g(t,x,u)| \le \epsilon \text{ for each } t \in [0,\infty)$$

$$\text{and each } x,u \in R^n \text{ for which } |x|,|u| \le N,$$

$$(|f(t,x,u)|+1)(|g(t,x,u)|+1)^{-1} \in [\lambda^{-1},\lambda] \text{ for each } t \in [0,\infty)$$

$$\text{and each } x,u \in R^n \text{ such that } |x| \le N\},$$

where $N > 0$, $\epsilon > 0$, $\lambda > 1$.

Clearly the uniform space $C(k,\mathcal{M})$ is Hausdorff and has a countable base. Therefore $C(k,\mathcal{M})$ is metrizable. It is easy to verify that the uniform space $C(k,\mathcal{M})$ is complete and the operator $\mathcal{L} : C(k,\mathcal{M}) \to \mathcal{M}$ is continuous. Denote by $\mathcal{M}_k$ the space of all functions $f \in C(k,\mathcal{M})$ which satisfy the following conditions:

$$f \in C^k(R^{2n+1}),\ \partial f/\partial u_i \in C^k(R^{2n+1}) \text{ for } i = 1,\dots,n; \tag{5.1}$$

the matrix $(\partial^2 f/\partial u_i \partial u_j)(t,x,u)$, $i,j = 1,\dots,n$ is positive definite (5.2)

for all $(t,x,u) \in R^{2n+1}$;

there exist a number $c_0 > 1$ and monotone increasing functions $\phi_i : [0,\infty) \to [0,\infty)$, $i = 0,1,2$ such that

$$\phi_0(t)t^{-1} \to +\infty \text{ as } t \to +\infty,$$

$$f(t,x,u) \ge \phi_0(c_0|u|) - \phi_1(|x|),\ t \in R^1,\ x,u \in R^n; \tag{5.3}$$

$$\sup\{|\partial f/\partial x_i(t,x,u)|,\ |\partial f/\partial u_i(t,x,u)|\} \le \phi_2(|x|)(1+\phi_0(|u|)), \tag{5.4}$$

$$t \in R^1,\ x,u \in R^n,\ i = 1,\dots,n.$$

Denote by $\bar{\mathcal{M}}_k$ the closure of $\mathcal{M}_k$ in $C(k,\mathcal{M})$. We consider the topological subspace $\bar{\mathcal{M}}_k \subset C(k,\mathcal{M})$ with the relative topology. In order to show that the conclusions of Theorems 2.1.1-2.1.4 also hold for a G_δ-subset of the space $\bar{\mathcal{M}}_k$ we need the following result.

PROPOSITION 2.5.1 *Let $k \ge 1$ be an integer, a number $c_0 > 1$, $\phi_i : [0,\infty) \to [0,\infty)$, $i = 0,1,2$ be monotone increasing functions and let an integrand $f : R^{2n+1} \to R^1$ satisfy (5.1)-(5.4). Assume that $T_1 \in [0,\infty)$, $T_2 > T_1$ and an a.c. function $w : [T_1,T_2] \to R^n$ satisfies*

$$I^f(T_1,T_2,w) = U^f(T_1,T_2,w(T_1),w(T_2)) < \infty. \tag{5.5}$$

Then $w \in C^{k+1}([T_1, T_2]; R^n)$ *and*

$$(d/dt)(\partial f/\partial u_i(t, w(t), w'(t)) = (\partial f/\partial x_i(t, w(t), w'(t)) \tag{5.6}$$

for each $i \in \{1, \dots, n\}$ *and each* $t \in [T_1, T_2]$.

Denote by $< x, y >$ the scalar product of $x, y \in R^n$. In the proof of Proposition 2.5.1 we need the following simple result.

LEMMA 2.5.1 *Let* $n \geq 1$ *be an integer,* $-\infty < T_1 < T_2 < +\infty$ *and let* $f \in L^1([T_1, T_2]; R^n)$ *have the following property:*

$$\int_{T_1}^{T_2} < f(t), g(t) > dt = 0$$

for every function $g \in L^\infty([T_1, T_2]; R^n)$ *such that* $\int_{T_1}^{T_2} g(t)dt = 0$.

Then there exists $d \in R^n$ *such that* $f(t) = d$ *for almost all* $t \in [T_1, T_2]$.

Proof of Proposition 2.5.1. Put

$$x = w(T_1), \ y = w(T_2).$$

In proving Proposition 2.5.1 we follow [64, Theorem 1.10.1]. For $t \in [T_1, T_2]$ we set $B(t) = (t, w(t), w'(t))$. Analogously to the proof of Theorem 1.10.1 of [64] we can show that if an a.c. function $h : [T_1, T_2] \to R^n$ satisfies

$$h(T_1) = h(T_2) = 0, \ h' \in L^\infty([T_1, T_2]; R^n), \tag{5.7}$$

then

$$\sum_{i=1}^n \partial f/\partial x_i(B(t))h_i(t) + \sum_{i=1}^n \partial f/\partial u_i(B(t))h_i'(t) \in L^1(T_1, T_2), \tag{5.8}$$

$$\int_{T_1}^{T_2} [\sum_{i=1}^n \partial f/\partial x_i(B(t))h_i(t) + \sum_{i=1}^n \partial f/\partial u_i(B(t))h_i'(t)]dt = 0. \tag{5.9}$$

It follows from (5.3)-(5.5) that the function

$$t \to |\partial f/\partial x_i(B(t))| + |\partial f/\partial u_i(B(t))|, \ t \in [T_1, T_2]$$

belongs to the space $L^1(T_1, T_2)$ for $i = 1, \dots, n$.

Consider a function $g \in L^\infty(T_1, T_2); R^n)$ such that

$$\int_{T_1}^{T_2} g(t)dt = 0 \tag{5.10}$$

and put

$$h(t) = \int_{T_1}^{t} g(\tau)d\tau,\ t \in [T_1, T_2],$$

$$E_i(t) = \int_{T_1}^{t} (\partial f/\partial x_i)(B(\tau))d\tau,\ t \in [T_1, T_2],\ i = 1, \dots, n.$$

Clearly h satisfies (5.7). Thus (5.9) and (5.8) are true. The Fubini theorem implies that for $i = 1, \dots, n$,

$$\int_{T_1}^{T_2} (\partial f/\partial x_i)(B(t))h_i(t)dt = \int_{T_1}^{T_2} g_i(\tau)(E_i(T_2) - E_i(\tau))d\tau.$$

Combined with (5.9) this equality implies that

$$\int_{T_1}^{T_2} [\sum_{i=1}^{n} \partial f/\partial u_i(B(t)) + E_i(T_2) - E_i(t)]g_i(t)dt = 0.$$

We have shown that this equality holds for every $g \in L^\infty([T_1, T_2]; R^n)$ satisfying (5.10). Therefore Lemma 2.5.1 implies that there is $d = (d_1, \dots, d_n) \in R^n$ such that

$$\partial f/\partial u_i(B(t)) + E_i(T_2) - E_i(t) = d_i \tag{5.11}$$

for each $i \in \{1, \dots, n\}$ and almost all $t \in [T_1, T_2]$.

Define a mapping $G : R^1 \times R^n \times R^n \to R^1 \times R^n \times R^n$ by

$$G(t, x, u) = (t, x, (\partial f/\partial u_i(t, x, u))_{i=1}^n).$$

Assume that $(t^i, x^i, u^i) \in R^{2n+1}$, $i = 1, 2$ and

$$G(t^1, x^1, u^1) = G(t^2, x^2, u^2).$$

Clearly $t^1 = t^2$, $x^1 = x^2$. We show that $u_1 = u_2$. For $\lambda \in [0, 1]$ we denote by $A(\lambda)$ the matrix

$$(\partial^2 f/\partial u_i \partial u_j)(t^1, x^1, u^1 + \lambda(u^2 - u^1)),\ i, j = 1, \dots, n.$$

For $i = 1, \dots, n$ we have

$$0 = \partial f/\partial u_i(t^1, x^1, u^2) - \partial f/\partial u_i(t^1, x^1, u^1)$$

$$= \int_0^1 (d/d\lambda)(\partial f/\partial u_i(t^1, x^1, u^1 + \lambda(u^2 - u^1))d\lambda$$

$$= \int_0^1 < (\partial^2 f/\partial u_i \partial u_j)(t^1, x^1, u^1 + \lambda(u^2 - u^1))_{j=1}^n, u_2 - u_1 > d\lambda.$$

This implies that

$$\int_0^1 A(\lambda)(u_2 - u_1)d\lambda = 0,$$

$$\int_0^1 < A(\lambda)(u_2 - u_1), u_2 - u_1 > d\lambda = 0.$$

By the definition of $A(\lambda)$ and (5.2) $u_2 = u_1$. Therefore the mapping G is injective.

By the inverse function theorem and the conditions of the proposition, $G(R^{2n+1})$ is an open subset of R^{2n+1}, there exists $G^{-1}: G(R^{2n+1}) \to R^{2n+1} \in C^1$ and

$$(G^{-1})'(y) = [G' \cdot G^{-1}(y)]^{-1}, \ y \in G(R^{2n+1}). \tag{5.12}$$

We show that

$$G(R^{2n+1}) = R^{2n+1}. \tag{5.13}$$

Let (t, x, U), $(t^i, x^i, u^i) \in R^{2n+1}$, $i = 1, 2, \dots$,

$$G(t^i, x^i, u^i) \to (t, x, U) \text{ as } i \to \infty. \tag{5.14}$$

It is sufficient to prove that $(t, x, U) \in G(R^{2n+1})$. Clearly

$$(t^i, x^i) \to (t, x) \text{ as } i \to \infty. \tag{5.15}$$

We show that the sequence $\{u_i\}_{i=1}^{\infty}$ is bounded. Let us assume the converse. By (5.14) and (5.15) there exists a number $M > 0$ such that

$$|G(t^i, x^i, u^i)|, \ t^i, \ |x^i| \leq M \text{ as } i = 1, 2, \dots. \tag{5.16}$$

There exist $M_0 > 0$ such that

$$|f(\tau, y, 0)|, \ |(\partial f/\partial u_j)(\tau, y, 0)| \leq M_0, \ j = 1, \dots, n, \ |\tau| \leq M, |y| \leq M. \tag{5.17}$$

We may assume that

$$|u_i| \to \infty \text{ as } i \to \infty. \tag{5.18}$$

It follows from (5.2) and (5.16)-(5.18) that for any integer $i \geq 1$

$$f(t^i, x^i, 0) \geq f(t^i, x^i, u^i) - < (\partial f/\partial u_j(t^i, x^i, u^i))_{j=1}^n, u^i >, \tag{5.19}$$

$$f(t^i, x^i, u^i) \leq M_0 + M|u^i|, \ \limsup_{i\to\infty} f(t^i, x^i, u^i)/|u^i| \leq M.$$

On the other hand by (5.16), (5.18) and (5.3) for any integer $i \geq 1$

$$f(t^i, x^i, u^i) \geq \phi_0(c_0|u^i|) - \phi_1(M),$$

$$\limsup_{i\to\infty} f(t^i, x^i, u^i)/|u^i| \geq \limsup_{i\to\infty} \phi_0(c_0|u^i|)/|u^i| = +\infty.$$

This is contradictory to (5.19). The obtained contradiction proves the boundedness of $\{u_i\}_{i=1}^{\infty}$. We may assume that $u_i \to u \in R^n$ as $i \to \infty$. Together with (5.14), (5.15) this implies that

$$(t, x, U) = \lim_{i\to\infty} G(t^i, x^i, u^i) = G(t, x, u) \in G(R^{2n+1}).$$

Therefore (5.13) holds. By (5.11) for almost all $t \in [T_1, T_2]$,

$$(t, w(t), w'(t)) = G^{-1}(t, w(t), (d_i - E_i(T_2) - E_i(t))_{i=1}^n). \tag{5.20}$$

It is now easy to see that the last relation holds for all $t \in [T_1, T_2]$ and $w \in C^2([T_1T_2]; R^n)$. (5.9) implies that for each $h \in C^1([T_1, T_2]; R^n)$ satisfying (5.7),

$$\sum_{i=1}^{n} \int_{T_1}^{T_2} h_i(t)[\partial f/\partial x_i(B(t)) - (d/dt)(\partial f/\partial u_i(B(t)))]dt = 0.$$

This implies (5.6) for each $t \in [T_1, T_2]$ and each $i \in \{1, \dots, n\}$. By (5.12) $G^{-1} \in C^k$. Together with (5.20) this implies that $w \in C^{k+1}([T_1, T_2]; R^n)$. This completes the proof of the proposition.

THEOREM 2.5.1 *Let $k \geq 1$ be an integer. Then there exists a G_δ-set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets in $\mathcal{M}$ and for which the conclusions of Theorems 2.1.1.-2.1.4 hold and a set $\mathcal{F}_k \subset \bar{\mathcal{M}}_k$ which is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}_k$ such that $\mathcal{L}(\mathcal{F}_k) \subset \mathcal{F}$.*

Proof. Fix $z_* \in R^n$. For each $f \in \mathcal{M}$ let $Z_*^f : [0, \infty) \to R^n$ be as guaranteed by Theorem 1.2.1. For each $M > 0$ there exists a function $\psi^M \in C^\infty(R^1)$ such that

$$\psi^M(t) = 1,\ t \in [-M-1, M+1],\ \psi^M(t) = 0,\ |t| \geq M+2,$$

$$\psi^M(t) \in (0, 1),$$

$$t \in (-M-2, -M-1) \cup (M+1, M+2).$$

For $f \in \mathcal{M}$ and $M > |z_*|$ we define $\phi^M : [0, \infty) \times R^n \to R^1$ as follows:

$$\phi^M(t, x) = |x - Z_*^f(t)|^2 \psi^M(|x|),\ t \in [0, \infty),\ x \in R^n.$$

By Remark 2.2.1 the function ϕ^M satisfies Assumption B.

For each $f \in \mathcal{M}$ and each integer $p \geq 1$ we define numbers M^f, $M_0(f,p)$, $M(f,p)$, $r(f,p) > 0$ and neighborhoods W^f, $W_0(f,p)$, $W(f,p)$ of f in $\mathcal{M}$ as in Section 2.2 (see (2.3)-(2.10)). Consider the set $E_p \subset \mathcal{M}$ defined by (3.1), (2.3). For each $f \in \mathcal{M}$, each pair of integers $p, q \geq 1$ and each $r \in (0, r(f,p))$ we define an open neighborhood $V(f,p,r,q)$ of $f_r^{M(f,p)+1}$ in $\mathcal{M}$ as in Section 2.3 and define a set $\mathcal{F}$ by (5.2), (5.3). In Section 2.5, Theorems 2.2.1-2.2.4 were established for the set $\mathcal{F}$.

Let $g \in \mathcal{M}_k$, $p \geq 1$ be an integer and $r \in (0, r(\mathcal{L}(g), p))$. By Theorem 1.2.1 and Proposition 2.5.1, $Z_*^{\mathcal{L}(g)} \in C^{k+1}$. There exists $Y^g : R^1 \to R^n \in C^{k+1}$ such that

$$Y^g(t) = Z_*^{\mathcal{L}(g)}(t) \text{ for } t \in [0, \infty),$$

$$(d^{k+1}Y^g/dt^{k+1})(t) = (d^{k+1}Y^g/dt^{k+1})(0) \text{ for } t \in (-\infty, 0).$$

We define a function $g_r^p : R^{2n+1} \to R^1$ as follows:

$$g_r^p(t,x,u) = g(t,x,u) + r|x - Y^g(t)|^2 \psi^S(|x|),$$

$$t \in R^1, \ x, u \in R^n$$

where $S = M(\mathcal{L}(g), p) + 1$.

It is easy to verify that $g_r^p \in \mathcal{M}_k$,

$$g_r^p \to g \text{ as } r \to 0 \text{ in } C(k, \mathcal{M}),$$

$\mathcal{L}(g_r^p) \in E_p$ for $g \in \mathcal{M}_k$, $p \geq 1$ and $r \in (0, r(\mathcal{L}(g), p))$. For each integer $p \geq 1$ we set

$$G_p = \{g_r^p : \ g \in \mathcal{M}_k, \ r \in (0, r(\mathcal{L}(g), p))\}.$$

For each $g \in \mathcal{M}_k$, each pair of integers $p, q \geq 1$ and each

$$r \in (0, r(\mathcal{L}(g), p))$$

we set

$$U(g,p,r,q) = \mathcal{L}^{-1}(V(\mathcal{L}(g), p, r, q)) \cap \bar{\mathcal{M}}_k.$$

Evidently $U(g,p,r,q)$ is an open neighborhood of g_r^p in $\bar{\mathcal{M}}_k$. We define

$$\mathcal{F}_{kp} = \cap_{q=1}^{\infty} \cup \{U(g,p,r,q) : \ g \in \mathcal{M}_k, \ r \in (0, r(\mathcal{L}(g), p))\},$$

$$\mathcal{F}_k = \cap_{p=1}^{\infty} \mathcal{F}_{kp}.$$

Clearly $\mathcal{F}_k$ is a countable intersection of open everywhere dense sets in $\bar{\mathcal{M}}_k$ and $\mathcal{L}(\mathcal{F}_k) \subset \mathcal{F}$. This completes the proof of the theorem.

2.6. Examples

Let $n \geq 1$. Fix a positive constant a and set $\psi(t) = t$, $t \in [0, \infty)$. Consider a complete metric space $\mathcal{M}$ of integrands $f : [0, \infty) \times R^1 \times R^1 \to R^1$ defined in Section 2.1 and a G_δ-subset $\mathcal{F} \subset \mathcal{M}$ constructed in Section 2.3. Define by $\mathcal{F}_0$ the set of all integrands $g \in \mathcal{M}$ for which the conclusions of Theorems 2.1.1-2.1.4 are valid. Clearly the set $\mathcal{F}$ is everywhere dense in $\mathcal{M}$ and $\mathcal{F} \subset \mathcal{F}_0$.

Example 6.1. Consider an integrand $f(t, x, u) = x^2 + u^2$, $t, x, u \in R^1$. It is easy to see that $f \in \mathcal{M}$. Applying the methods used in the proofs of Theorems 2.1.1-2.1.4 we can show that $f \in \mathcal{F}_0$.

Example 6.2. Fix a number $q > 0$ and consider an integrand

$$g(t, x, u) = qx^2(x-1)^2 + u^2, \ t, x, u \in R^1.$$

It is easy to see that $g \in \mathcal{M}$ if a is large enough. Clearly the function $v_1(t) = 0$, $v_2(t) = 1$, $t \in [0, \infty)$ are (g)-overtaking optimal. Assertion 1 of Theorem 2.1.1 implies that $g \notin \mathcal{F}_0$. It is easy to verify that $f, g \in \mathcal{M}_k$ for any integer $k \geq 1$.

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