Chapter 2

THREE OPTIMIZATION PROBLEMS IN MASS TRANSPORTATION THEORY

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Abstract We give a model for the description of an urban transportation network and we consider the related optimization problem which consists in finding the design of the network which has the best transportation performances. This will be done by introducing, for every admissible network, a suitable metric space with a distance that inserted into the Monge-Kantorovich cost functional provides the criterion to be optimized. Together with the optimal design of an urban transportation network, other kinds of optimization problems related to mass transportation can be considered. In particular we will illustrate some models for the optimal design of a city, and for the optimal pricing policy on a given transportation network.

Keywords: Monge problem, transport map, optimal pricing policies

1. Introduction

In this paper we present some models of optimization problems in mass transportation theory; they are related to the optimal design of urban structures or to the optimal management of structures that already exist. The models we present are very simple and do not pretend to give a careful description of the urban realities; however, adding more parameters to fit more realistic situations, will certainly increase the computational difficulties but does not seem to modify the theoretical scheme in an essential way. Thus we remain in the simplest framework, since our goal is to stress the fact that mass transportation theory is the right tool to attack this kind of problems.

The problems we will present are of three kinds; all of them require the use of the Wasserstein distances W_p between two probabilities f^+ and f^-, that we will introduce in the next section. Let us illustrate here shortly the problems that we are going to study later in more details.

Optimal transportation networks. In a given urban area Ω, with two given probabilities f^+ and f^-, which respectively represent the density of residents and the density of services, a transportation network has to be designed in an optimal way. A cost functional has to be introduced through a suitable Wasserstein distance between f^+ and f^-, which takes into account the cost of residents to move outside the network (by their own means) and on the network (for instance by paying a ticket). The admissible class of networks where the minimization will be performed consists of all closed connected one-dimensional subsets of Ω with prescribed total length.

Optimal pricing policies. With the same framework as above (Ω, f^+, f^- given) we also consider the network as prescribed. The unknown is here the ticket pricing policy the manager of the network has to choose, and the goal is to maximize the total income. Of course, a too low ticket price policy will not be optimal, but also a too high ticket price policy will push customers to use their own transportation means, decreasing in this way the total income of the company.

Optimal design of an urban area. In this case the urban area Ω is still considered as prescribed, whereas f^+ and f^- are the unknowns of the problem that have to be determined in an optimal way taking into account the following facts:

- there is a transportation cost for moving from the residential areas to the services poles;
- people desire not to live in areas where the density of population is too high;
- services need to be concentrated as much as possible, in order to increase efficiency and decrease management costs.

2. The Wasserstein distances

Mass transportation theory goes back to Gaspard Monge (1781) when he presented a model in a paper on *Académie des Sciences de Paris*. The elementary work to move a particle x into $T(x)$, as in Figure 2.1, is given by $|x - T(x)|$, so that the total work is

$$\int_{\text{remblais}} |x - T(x)|\, dx\ .$$

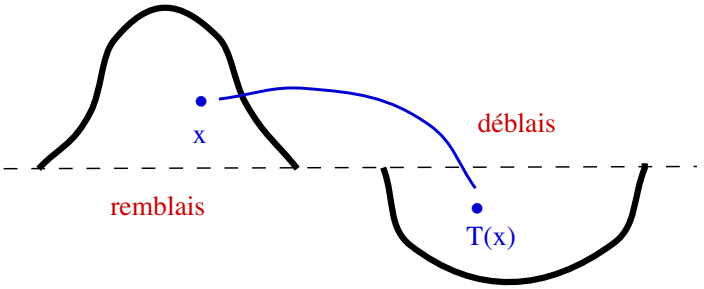

Figure 2.1. The Monge problem.

A map T is called *admissible transport map* if it maps "remblais" into "déblais". The Monge problem is then

$$\min\Big\{\int_{\text{remblais}} |x - T(x)|\,dx \;:\; T \text{ admissible}\Big\}.$$

It is convenient to consider the Monge problem in the framework of metric spaces:

- (X, d) is a metric space;
- f^+, f^- are two probabilities on X (f^+ represents the "remblais", f^- the "déblais");
- T is an admissible transport map if it maps f^+ onto f^-, that is $T^{\#} f^+ = f^-$.

The Monge problem is then

$$\min\Big\{\int_X d\big(x, T(x)\big)\,df^+(x) \;:\; T \text{ admissible}\Big\}.$$

The question about the existence of an optimal transport map T_{opt} for the Monge problem above is very delicate and does not belong to the purposes of the present paper (we refer the interested reader to the several papers available in the literature). Since we want to consider f^+ and f^- as general probabilities, it is convenient to reformulate the problem in a relaxed form (due to Kantorovich (Kantorovich, 1942, Kantorovich, 1948)): instead of transport maps we consider measures γ on $X \times X$ (called *transport plans*); γ is said an admissible transport plan if

$$\pi_1^{\#}\gamma = f^+, \qquad \pi_2^{\#}\gamma = f^-$$

where π_1 and π_2 respectively denote the projections of $X \times X$ on the first and second factors. In this way, the Monge-Kantorovich problem becomes:

$$\min\Big\{ \int_{X\times X} d(x,y)\, d\gamma(x,y) \ : \ \gamma \text{ admissible}\Big\}.$$

THEOREM 2.1 *There exists an optimal transport plan γ_{opt}; in the Euclidean case γ_{opt} is actually a transport map T_{opt} whenever f^+ and f^- are in L^1.*

We denote by $MK(f^+, f^-, d)$ the minimum value in the Monge-Kantorovich problem above. This defines the Wasserstein distance (of exponent 1) by

$$W_1(f^+, f^-, d) = MK(f^+, f^-, d)$$

where the metric space (X, d) is considered as fixed. The Wasserstein distances of exponent $p > 1$ are defined in a similar way:

$$W_p(f^+, f^-, d) = \min\Big\{ \Big(\int_{X\times X} d^p(x,y)\, d\gamma(x,y)\Big)^{1/p} \ : \ \gamma \text{ admissible}\Big\}.$$

When X is a compact metric space all the distances W_p are topologically equivalent, and the topology generated by them coincides with the weak* topology on the probabilities on X.

3. Optimal transportation networks

We consider here a model for the optimal planning of an urban transportation network, see (Brancolini and Buttazzo, 2003). Suppose that the following objects are given:

- a compact regular domain Ω of $\mathbb{R}^N$ $(N \geq 2)$; it represents the geographical region or urban area we are dealing with;
- a nonnegative measure f^+ on Ω; it represents the density of residents in the urban area Ω;
- a nonnegative measure f^- on Ω; it represents the density of services in the urban area Ω.

We assume that f^+ and f^- have the same mass, that we normalize to 1; so f^+ and f^- are supposed to be probability measures. The main unknown of the problem is the transportation network Σ that has to

be designed in an optimal way to transport the residents f^+ into the services f^-. The goal is to introduce a cost functional $F(\Sigma)$ and to minimize it on a class of admissible choices. We assume that Σ varies among all closed connected 1-dimensional subsets of Ω with total length bounded by a given constant L. Thus the admissible class where Σ varies is

$$\mathcal{A}_L = \big\{\Sigma \subset \Omega,\ \text{closed, connected},\ \mathcal{H}^1(\Sigma) \le L\big\}. \tag{2.1}$$

In order to introduce the optimization problem we associate to every "admissible urban network" Σ a suitable "point-to-point cost function" d_Σ which takes into account the costs for residents to move by their own means as well as by using the network. The cost functional will be then

$$F(\Sigma) = W_p(f^+, f^-, d_\Sigma) \tag{2.2}$$

for some fixed $p \ge 1$, so that the optimization problem we deal with is

$$\min\{F(\Sigma)\ :\ \Sigma \in \mathcal{A}_L\}. \tag{2.3}$$

It remains to introduce the function d_Σ (that in the realistic situations will be a semi-distance on Ω). To do that, we consider:

- a continuous and nondecreasing function $A : [0, +\infty[\to [0, +\infty[$ with $A(0) = 0$, which measures the cost for residents of traveling by their own means;
- a lower semicontinuous and nondecreasing function $B : [0, +\infty[\to [0, +\infty[$ with $B(0) = 0$, which measures the cost for residents of traveling by using the network.

More precisely, $A(t)$ represents the cost for a resident to cover a length t by his own means (walking, time consumption, car fuel, ...), whereas $B(t)$ represents the cost to cover a length t by using the transportation network (ticket, time consumption, ...). The assumptions made on the pricing policy function B allow us to consider the usual cases below: the flat rate policy of Figure 2.2 (a) as well as the multiple-zones policy of Figure 2.2 (b).

Therefore, the function d_Σ is defined by:

$$d_\Sigma(x, y) = \inf\big\{A\big(\mathcal{H}^1(\phi \setminus \Sigma)\big) + B\big(\mathcal{H}^1(\phi \cap \Sigma)\big)\ :\ \phi \in \mathcal{C}_{x,y}\big\}, \tag{2.4}$$

where $\mathcal{C}_{x,y}$ denotes the class of all curves in Ω connecting x to y.

THEOREM 2.2 *The optimization problem (2.3) admits at least a solution* Σ_{opt}.

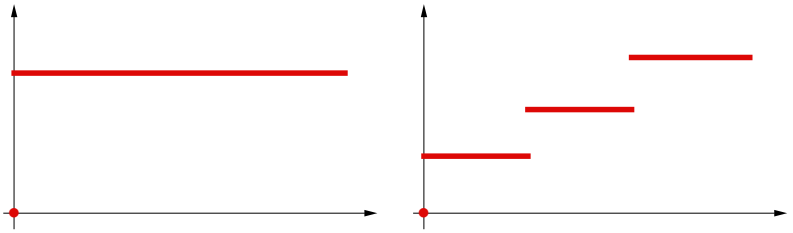

Figure 2.2. (a) flat rate policy (b) multiple-zones policy.

Once the existence of Σ_{opt} is established, several interesting questions arise:

- study the regularity properties of Σ_{opt}, under reasonable regularity assumptions on the data f^+ and f^-;
- study the geometrical necessary conditions of optimality that Σ_{opt} has to fulfill (nonexistence of closed loops, bifurcation points, distance from the boundary $\partial\Omega$, ...);
- perform an asymptotic analysis of the optimization problem (2.3) as $L \to 0$ and as $L \to +\infty$.

Most of the questions above are still open in the general framework covered by the existence Theorem 2.2 above. However, some partial results are available in particular situations; we refer the interested reader to the several recent papers on the subject, see for instance (Brancolini and Buttazzo, 2003, Buttazzo et al., 2002, Buttazzo and Stepanov, Mosconi and Tilli, 2003).

4. Optimal pricing policies

With the notation above, we consider the urban area Ω and the measures f^+, f^- as fixed, as well as the transportation network Σ. The unknown is in this case the pricing policy function B that the manager of the network has to choose among all lower semicontinuous monotone nondecreasing functions B, with $B(0) = 0$. The goal is to maximize the total income, which of course depends on the policy B chosen, so it can be seen as a functional $F(B)$.

The function B can be seen as a control variable and the corresponding Kantorovich transport plan γ_B as a state variable, which solves the minimum problem

$$\min\left\{\int_{\Omega\times\Omega} d_B^p(x,y)\,d\gamma(x,y) \;:\; \gamma \text{ admissible}\right\} \tag{2.5}$$

where p is the Wasserstein exponent and d_B is the cost function

$$d_B(x,y) = \inf\big\{A\big(\mathcal{H}^1(\phi \setminus \Sigma)\big) + B\big(\mathcal{H}^1(\phi \cap \Sigma)\big) \ : \ \phi \in \mathcal{C}_{x,y}\big\}. \qquad (2.6)$$

The quantity $d_B(x,y)$ can be seen as the total minimal cost a customer has to pay to go from a point x to a point y, using the best path ϕ. This cost is divided in two parts: a part $A\big(\mathcal{H}^1(\phi \setminus \Sigma)\big)$ due to the use of his own means, and a part $i_B(x,y) = B\big(\mathcal{H}^1(\phi \cap \Sigma)\big)$ due to the ticket to pay for using the transportation network. The only condition we assume to make the problem well posed is that, in case several paths ϕ realize the minimum in (2.6), the customer chooses the one with minimal own means cost (and so with maximal network cost). The total income is then

$$F(B) = \int_{\Omega\times\Omega} i_B(x,y)\, d\gamma_B(x,y), \qquad (2.7)$$

so that the optimization problem we consider is:

$$\max\big\{F(B) \ : \ B \text{ l.s.c., nondecreasing, } B(0) = 0\big\}. \qquad (2.8)$$

The following result has been proved in (Buttazzo et al.).

THEOREM 2.3 *There exists an optimal pricing policy B_{opt} solving the maximal income problem (2.8).*

Also in this case some necessary conditions of optimality can be obtained. It may happen that several functions B_{opt} solve the maximal income problem (2.8); in this case, as a canonical representative, we choose the smallest one, with respect to the usual order between functions. It is possible to show that it is still a solution of problem (2.8). In particular, the function B_{opt} turns out to be continuous, and its Lipschitz constant can be bounded by the one of A. We refer to (Buttazzo et al.) for all details as well as for the proofs above.

5. Optimal design of an urban area

We consider the following model for the optimal planning of an urban area, see (Buttazzo and Santambrogio, 2003).

- The domain Ω (the geographical region or urban area), a regular compact subset of $\mathbb{R}^N$, is prescribed;
- the probability measure f^+ on Ω (the density of residents) is unknown;
- the probability measure f^- on Ω (the density of services) is unknown.

Here the distance d in Ω is taken for simplicity as the Euclidean one, but with a similar procedure one could also study the cases in which the distance is induced by a transportation network Σ, as in the previous sections. The unknowns of the problem are f^+ and f^-, that have to be determined in an optimal way taking into account the following facts:

- residents have to pay a transportation cost for moving from the residential areas to the services poles;
- residents like to live in areas where the density of population is not too high;
- services need to be concentrated as much as possible, in order to increase efficiency and decrease management costs.

The transportation cost will be described through a Monge-Kantorovich mass transportation model; it is indeed given by a p-Wasserstein distance ($p \geq 1$) $W_p(f^+, f^-)$.

The *total unhappiness* of residents due to high density of population will be described by a penalization functional, of the form

$$H(f^+) = \begin{cases} \int_\Omega h(u)\,dx & \text{if } f^+ = u\,dx \\ +\infty & \text{otherwise,} \end{cases}$$

where h is assumed to be convex and superlinear (i.e. $h(t)/t \to +\infty$ as $t \to +\infty$). The increasing and diverging function $h(t)/t$ then represents the *unhappiness* to live in an area with population density t.

Finally, there is a third term $G(f^-)$ which penalizes sparse services. We force f^- to be a sum of Dirac masses and we consider $G(f^-)$ as a functional defined on measures, of the form studied by Bouchitté and Buttazzo in (Bouchitté and Buttazzo, 1990, Bouchitté and Buttazzo, 1992, Bouchitté and Buttazzo, 1992):

$$G(f^-) = \begin{cases} \sum_n g(a_n) & \text{if } f^- = \sum_n a_n \delta_{x_n} \\ +\infty & \text{otherwise,} \end{cases}$$

where g is concave and with infinite slope at the origin ((i.e. $g(t)/t \to +\infty$ as $t \to 0^+$). Every single term $g(a_n)$ in the sum above represents the cost for building and managing a service pole of dimension a_n, located at the point $x_n \in \Omega$.

We have then the optimization problem

$$\min\big\{W_p(f^+, f^-) + H(f^+) + G(f^-) \;:\; f^+, f^- \text{ probabilities on } \Omega\big\}. \quad (2.9)$$

THEOREM 2.4 *There exists an optimal pair (f^+, f^-) solving the problem above.*

Also in this case we obtain some necessary conditions of optimality. In particular, if Ω is sufficiently large, the optimal structure of the city consists of a finite number of disjoint subcities: circular residential areas with a service pole at their center.

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