2

EXPECTED UTILITY THEORY

2.1 INTRODUCTION

Let us now turn to what, in the eyes of the investor, is probably the main *raison d'être* of investment, namely, profitability. In focusing on risk in our first chapter, by no means do we belittle this all-important function of investment. Our discussion of risk simply serves to emphasize that, in arriving at an investment decision, the risk of the investment has to be weighed against its profitability. Thus, both profitability and risk have to be incorporated in the decision making process. We devote this chapter to the expected utility criterion that takes into account the whole distribution of returns (risk and return).

Investors face many alternative investment choices . In order to compare the risk and return of alternative investments decision criteria are needed. Most of this book is devoted to investment decision rules that rely on the expected utility paradigm. The expected utility framework does not analyze risk and return separately; it considers the whole distribution of returns simultaneously. Moreover, in this framework, there is no need to define risk.

This chapter deals with the foundations of expected utility theory. We will first discuss a number of investment criteria and then we will analyze how these decision criteria are related to the expected utility framework.

2.2 INVESTMENT CRITERIA

a) The Maximum Return Criterion (MRC).

The Maximum Return Criterion (MRC) is employed when there is no risk at all. According to this rule, we simply choose the investment with the highest rate of return. By making the right choice, we ensure maximum return on the invested wealth at the end of the investment period. Textbooks on economics or price theory are replete with models aimed at maximizing profits, or maximizing return. Let us illustrate the MRC with an example of an optimal production decision by a firm. Let P be the price of the product per unit, Q the quantity of units produced by the firm, and C(Q) the production costs. The firm's objective is to decide on the optimal quantity, Q^* , to be produced such that the profit, $\pi(Q)$, is maximized.

Thus, the objective of the firm is:

$$\text{Max } \pi(Q) = P \cdot Q - C(Q).$$

Taking the first derivative and equating it to zero, we obtain the well-known result that at the optimum production level, the following will hold:

$$P = C'(Q^*)$$

which implies that the marginal revenue, P, will be equal to the marginal cost $C'(Q)$. The value Q^* is the optimal number of units to be produced because, by selecting Q^*, the firm maximizes its return $\pi(Q)$.

Can we apply this MRC rule to selection among uncertain investments and, in particular, to selection of a portfolio of securities that have uncertain returns? As we shall see, this rule is applicable only when the returns are certain (as in the case of selecting the optimum production); it is not applicable in the case of uncertain returns. Indeed, when MRC is recommended in most economics textbooks, it is assumed (implicitly or explicitly) that the price of the product P and the costs C(Q) are certain.

To demonstrate that MRC is applicable only in the case of certain cash flows, let us first explain what we mean by an applicable decision rule. A decision rule is said to be applicable if it can be employed in a non-arbitrary manner. It is not applicable if it can be employed in more than one way. For instance, it is not applicable if investment A is shown to be better than investment B when the rule is used in one way, and an opposite ranking is obtained if it is employed in a different way. Let us explain this notion via a numerical example. Suppose that you want to rank the following four investments in order to arrive at an investment decision:

Investment A		Investment B		Investment C		Investment D	
x	p(x)	x	p(x)	X	p(x)	x	p(x)
+4	1	+5	1	–5	1/4	–10	1/5
				0	1/2	+10	1/5
				+40	1/4	+20	2/5
						+30	1/5

where x is the return (in $s or percentages) and p(x) is the probability of obtaining x. The MRC rule tells us that investment B dominates investment A because it has a higher return. However, it is ambiguous regarding the other pairs of investments and, therefore, it is not applicable to these investments. For example, if we pick the –5 return of investment C, then investment B is better than investment C. However, if we compare the +40 return of investment C with investment B, the opposite ranking is obtained. With uncertain investments, we do not obtain by the MRC a clear-cut unique ranking of investments because the ranking is a function of the arbitrary pairs of returns chosen for comparison. Therefore, MRC is not applicable in the case of uncertainty. It is not a "bad" or a "good" rule for uncertain situations, it is simply not applicable. A modified version of the MRC whereby the highest possible of all returns for each investment is identified and the investment with the highest maximum is then selected, technically helps overcome this problem to some extent. In this case, the rule is applicable and, in our example, investment C with the highest return of 40% is selected. The modified MRC is applicable to uncertainty because its result is not a function of the way it is employed. However, it can be misleading . For instance, let us reduce the probability of the +40% return of investment C to 1/1000, increase the probability of the –5% return to, say, 999/1000, and reduce the probability of x=0 to zero. By the modified MRC, investment C is still the most desirable investment. This is an obvious drawback because very few investors would consider investment C with these new probabilities to be the best investment.

Finally, it should be emphasized that when MRC is employed in finance and economics (especially in price theory), it is assumed that certainty prevails, that is, that there is only one possible return on the investment under consideration. In such cases, the MRC has no drawbacks and it is applicable. However, the certainty assumption regarding future returns is very unrealistic. We, therefore, need to search for other investment criteria.

b) The Maximum Expected Return Criterion (MERC)

The Maximum Expected Return Criterion (MERC) identifies the investment with the highest *expected* return and thereby overcomes the problem of non-unique ranking.

To employ this rule we first calculate the expected return of each possible investment. For investment A, it is 4, for B it is 5, and for C and D it is as follows:

$$E_c(x) = 1/4\ (-5) + 1/2\ (0) + 1/4\ (40) = 8.75;$$

$$E_d(x) = 1/5\ (-10) + 1/5\ (10) + 2/5\ (20) + 1/5\ (30) = 14.$$

Thus, the MERC provides a clear and an unambiguous ranking: In our example, investment D has the highest expected return. Thus, by MERC, investment D is ranked as the best investment.

The fact that the MERC provides an unambiguous ranking of risky investments does not imply that this rule should be employed in all instances. We are merely stating that, technically, the MERC is applicable to certainty and to uncertainty: Its theoretical justification has yet to be shown; hence, it is not necessarily the optimal rule. Actually, as we shall see below, this rule is not optimal and may lead to paradoxes (or irrational decisions) such as the famous St. Petersburg Paradox.

The St. Petersburg Paradox

The St. Petersburg Paradox first came to light in the 18th century and its solution paved the way to modern utility theory.

To illustrate this paradox, consider a game which requires a coin to be tossed until the first head shows up. The prize is $\$2^{x-1}$ where x is the number of tosses until the first head shows up. The game is over when the first head shows up. Theoretically, the game can be infinite. How much would you pay to participate in such a game? Or, specifically, what certain amount would you be willing to accept to be indifferent between playing the game for free or receiving this certain sum? This certain amount is called the *certainty equivalent* of the game. Note that the game can be seen as a risky investment. For example, if you pay, say \$100 for the game and the first head shows up after the first toss, you win in the game 2^{1-1} = \$1 and you lose \$99. How much would you be ready to pay for this risky investment? Experiments with this question reveal that most subjects report a very small certainty equivalent amount (\$2-\$3 in most cases). However, by the MERC, the certainty equivalent of this game is infinite; hence the paradox: Investors are ready to pay only a very small amount for an investment whose expected value is infinite. To see this, let us calculate the expected prize of this game. It is:

$$\sum_{x=1}^{\infty} \frac{1}{2^x} 2^{x-1} = \infty$$

where $\frac{1}{2^x}$ is the probability of an event $\underbrace{T,T,T,\ldots T}_{x-1}H$ which denotes the first head showing up on the x^{th} toss (T repeats x-1 times and then H occurs, x = 1, to 2,...where T stands for "tail shows" and H for "head shows up"). For example, if T appears 3 times in a row and then H appears, we obtain $2^{4-1} = 2^3$ = \$8 and the probability of such an event is $(1/2)^4 = 1/16$. As x can take on any number, we have a summation from x=1 up to infinity.

This paradox reveals the drawback of the MERC: it can lead to results that would be unacceptable to most investors because no investor would require an infinite amount, nor even a large amount, as the certainty equivalent. Indeed, way back in the 18th century, Nikolaus Bernoulli and Gabriel Cramer[1] suggested that investors, in making their decisions, aim at maximizing the expected utility of money like E(log (w)) or $E(w^{1/2})$ where log (w) or $w^{1/2}$ are possible utility functions, w stands for wealth, and E stands for expected value. Thus, according to Bernoulli and Cramer, what is
important to investors is the utility derived from the money received rather than the money itself. With a log(w) function, for example, we have log (10) =1, log(100) = 2 etc. Hence, the utility derived from the first \$10 is equivalent to the utility derived from the next \$90, showing a decreasing marginal utility of money (more details on the meaning of utility function will be provided later on in the chapter.)

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[1] For more details on the solution of Bernoulli and Cramer, see H.L. Levy and M. Sarnat, *Portfolio and Investment Selection: Theory and Practice*, Prentice Hall International, 1984.

Indeed, by calculating the expected utility, these two utility functions produce a reasonable solution. With the log (w) function (substituting the prize 2^{x-1}, for w) we obtain: [2]

$$E(\log w)=\sum_{x=1}^{\infty}\frac{1}{2^x}\log 2^{x-1}=\log 2\sum_{x=1}^{\infty}\frac{x-1}{2^x}=\log(2).$$

Thus, w=2 is the *certainty equivalent.* [3]

(Note that log (2) can also be considered to be the expected utility of the certainty equivalent because \$2 is received with a probability of 1.) In other words, the investor will be indifferent between receiving \$2 for sure or playing the St. Petersburg game because \$2 also yields the utility of log (2). If you offer the investor a higher sum, say, \$3 for sure or, alternatively, the chance to play the game for free by the expected utility criterion , the investor should choose the \$3 for sure because log (3) > E(log(w)) = log 2.

Similarly, with the function $U(w) = w^{1/2}$ suggested by Cramer, we obtain:

$$E(w^{\frac{1}{2}})=\sum_{x=1}^{\infty}\frac{1}{2^x}(2^{x-1})^{1/2}=1/2+\sqrt{2}/4+\sqrt{(2)^2}/8+\sqrt{(2)^3}/16+\cdots=1/2\cdot 1/(1-\sqrt{2}/2)$$

$$=1/(2-\sqrt{2})\cong\$1.707$$

hence, the investor will be indifferent between receiving $(\frac{1}{2-\sqrt{2}})^2=\$2.914$ for sure or, alternatively, playing the St. Petersburg game for free, because both alternatives yield the same expected utility of $U(w) = w^{1/2} = (2.914)^{1/2} =$ \$1.707.

The St. Petersburg paradox demonstrates why the MERC may be misleading and unacceptable. We have also seen above that by assuming that investors make

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[2] In the calculation of the expected utility we employ the following:

$$\sum_{x=1}^{\infty}\frac{x-1}{2^x}=1/4+2/8+3/16\cdots=(1/4+1/8+1/16\cdots)+(1/8+1/16+1/32\cdots)+$$
$$(1/16+1/32+1/64)\cdots=1$$

[3] Note that if we ask how much the player is ready to pay to play this game, the formula will be a little different. In such a case, we have to solve for the following equation, U(w) = EU(w+y-p) where w is the initial wealth, y is the prize received from the game, and p the price the player is willing to pay to participate in such a game.

investment decisions by the expected utility EU(w), and not by the expected return, E(w), we are able to solve the St. Petersburg Paradox. But is this solution to the paradox sufficient for the claim that investors should always select among the various investments according to the expected utility criterion (i.e., select the investment with the highest expected utility)?

Although, solving a paradox indicates a good property of the maximum expected utility criterion (MEUC), it cannot serve as justification for employing the MEUC in all cases. Yet, as we shall prove below, the maximum expected utility criterion (MEUC) is the correct rule as long as certain axioms are fulfilled. We will show below that if certain axioms are accepted, then the MEUC is the optimal decision rule. We will first discuss the axioms and then prove that the MEUC is the optimal rule given these axioms. Finally, we will discuss the relationship between MEUC, MERC and MRC, and analyze a few properties of the expected utility criterion.

2.3 THE AXIOMS AND PROOF OF THE MAXIMUM EXPECTED UTILITY CRITERION (MEUC).

Although Bernoulli and Cramer succeeded in solving the St. Petersburg Paradox, they did not provide a theoretical foundation for their solution. This came only in the 20th century when Ramsey and later on, von-Neumann and Morgenstern developed the theory of expected utility which determines that alternative investments should be ranked by their expected utility. [4, 5] The expected utility proof can be formulated in various ways. Here, we adopt six axioms from which the maximum expected utility criterion easily follows. We first discuss the axioms and then provide the proof.

a) The Payoff of the Investments

Suppose that you have to make a choice between two investments called also lotteries, which are denoted, by L_1 and L_2. These two investments can be written as:

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[4] See F.P. Ramsey, "Truth and Probability," in *The Foundations of Mathematics and Other Logical Essays*, London: K. Paul, Trench, Trusner and Co., 1931. See also, J.M. Keynes, *Essays in Biography*, London: Rupert Hart-Davis, 1951.

[5] See J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton, N.J.: Princeton University Press, 3rd ed., 1953.

$$L_1 = \{p_1\, A_1, p_2\, A_2, \ldots, p_n\, A_n\}$$

$$L_2 = \{q_1\, A_1\,, q_2\, A_2, \ldots, q_n\, A_n\}$$

where A_i are the possible outcomes with probability p_i and q_i, respectively, and the outcomes are ranked from the smallest (A_1) to the largest (A_n). Thus, under L_1 we have probability p_1 to get A_1, probability p_2 to get A_2, etc. Similarly, under L_2 we have probability q_1 to get A_1, probability q_2 to get A_2, etc. These are mutually exclusive and comprehensive events, that is, only one outcome can be realized under each investment and $\Sigma p_i = \Sigma q_i = 1$. In practice, it is rare for the two investments to have an identical series of outcomes $A_1, A_2, \ldots A_n$, but this fact imposes no constraints on our analysis: Simply write down all the outcomes of the two options under consideration and assign probabilities of p_i or q_i equal to zero when relevant. For example, if $L_1 = \{\frac{1}{4}4, \frac{3}{4}5\}$ (which should be read as probability of 1/4 to get 4 and a probability of 3/4 to get 5), $L_2 = \{\frac{1}{2}1, \frac{1}{2}10\}$, then we can write these two investments as $L_1 = \{\frac{0}{1}1, \frac{1}{4}4, \frac{3}{4}5, \frac{0}{1}\ 10\}$ and $L_2 = \{\frac{1}{2}1, \frac{0}{1}4, \frac{0}{1}5, \frac{1}{2}\ 10\}$.

b) The Axioms

Axiom 1: Comparability. By this axiom, when faced by two monetary outcomes, say A_i and A_j, the investor must say whether he/she prefers A_i to A_j ($A_i \succ A_j$) (where the sign $\succ$ means "prefers " as distinct from the sign $>$meaning "greater than"), or A_j to A_i ($A_j \succ A_i$) or whether he/she is indifferent between the two ($A_i \sim A_j$) (where the sign $\sim$ means "indifferent"). By this axiom, the answer "I do not know which monetary outcome I prefer" is simply not accepted.

Axiom 2: Continuity. If A_3 is preferred to A_2 and A_2 is preferred to A_1 then there must be a probability $U(A_2)$ ($0 \le U(A_2) \le 1$) such that,

$$L = \{\,(1\text{-}U(A_2))\, A_1,\ (U(A_2))\, A_3\} \sim A_2\,.$$

Thus, the investor will be indifferent between two choices: to receive A_2 with certainty or to receive either A_1 with probability $1\text{-}U(A_2)$ or A_3 with probability $U(A_2)$. For a given A_1 and A_3, these probabilities are a function of A_2; hence, the notation $U(A_2)$. Why is this axiom called the *continuity* axiom? Simply choose $U(A_2) = 1$ to obtain $L = A_3 \succ A_2$ (because by assumption $A_3 \succ A_2$). Then choose $U(A_2) = 0$ to obtain $L = A_1 \prec A_2$ (because by assumption $A_1 \prec A_2$). Thus, if you increase *continuously* $U(A_2)$ from zero to 1, you will hit a value $U(A_2)$ such that $L \sim A_2$.

Previously, we used the notation p and q for probabilities. Why do we suddenly switch here to $U(A_i)$ (rather than, say, $p(A_i)$)? The reason is simply because $U(A_i)$ is also the investor's utility function; hence, the new notation. This will be demonstrated as we continue with the proof.

Axiom 3: Interchangeability Suppose that you have a lottery (investment) L_1 given by:

$$L_1 = \{ p_1 A_1, p_2 A_2, p_3 A_3 \} .$$

Assume, also, that you are indifferent between A_2 and another lottery B, where $B = \{q A_1, (1-q) A_3 \}$.

Then by the *Interchangeability* axiom, you will be indifferent between L_1 and L_2 where $L_2 = \{ p_1 A_1, p_2 B, p_3 A_3\}$.

Axiom 4: Transitivity. Suppose that there are three lotteries, L_1, L_2 and L_3, where $L_1 \succ L_2$, $L_2 \succ L_3$. Then, by the *transitivity* axiom, $L_1 \succ L_3$. Similarly, if $L_1 \sim L_2$ and $L_2 \sim L_3$ then, by this axiom, $L_1 \sim L_3$.

Axiom 5: Decomposability. A *complex* lottery is one in which the prizes are lotteries themselves. A *simple* lottery has monetary values A_1, A_2 etc. as prizes. Suppose that there is a complex lottery L^* such that:

$$L^* = (q L_1, (1-q) L_2)$$

where L_1 and L_2 themselves are (simple) lotteries. L_1 and L_2 are given by:

$$L_1 = \{p_1 A_1, (1-p_1) A_2\}$$
$$L_2 = \{p_2 A_1, (1-p_2) A_2\}$$

Then, by this axiom, the complex lottery L^* can be decomposed into a simple lottery L having only A_1 and A_2 as prizes. To be more specific:

$$L^* \sim L_1 = \{p^* A_1, (1-p^*) A_2\}$$

where $$p^* = qp_1 + (1-q) p_2$$

Axiom 6: Monotonicity

If there is certainty, then the *monotonicity* axiom determines that if $A_2 > A_1$ then $A_2 \succ A_1$. If there is an uncertainty, the *monotonicity* axiom can be formulated in two alternate ways. First:

Let $L_1 = \{p\, A_1, (1\text{-}p)\, A_2\}$,
and $L_2 = \{p\, A_1, (1\text{-}p)\, A_3\}$. If $A_3 > A_2$, hence $A_3 \succ A_1$
then $L_2 \succ L_1$.

Second:

Let $L_1 = \{p\, A_1, (1\text{-}p)\, A_2\}$,
and $L_2 = \{q\, A_1, (1\text{-}q)\, A_2\}$, and $A_2 > A_1$ (hence $A_2 \succ A_1$).
If $p<q$ [or if $(1\text{-}p) > (1\text{-}q)$]

then $L_1 \succ L_2$.

Each of these six axioms can be accepted or rejected. However, if they are accepted, then we can prove that the MEUC should be used to choose among alternative investments. Any other investment criterion will simply be inappropriate and may lead to a wrong investment decision. We turn next to this proof.

c) Proof that the Maximum Expected Utility Criterion (MEUC) is Optimal Decision Rule

Theorem 2.1: The MEUC. The optimum criterion for ranking alternative investments is the expected utility of the various investments.

To prove Theorem 2.1, suppose that you have to make a choice between two investments L_1 and L_2 given by:

$$L_1 = \{p_1\, A_1\,,\, p_2\, A_2, \ldots, p_n\, A_n\}\,;$$
$$L_2 = \{q_1 A_1\,,\, q_2 A_2, \ldots, q_n\, A_n\},$$

and $A_1 < A_2 < \ldots A_n$, where A_i are the various monetary outcomes.

First , note that by the *comparability* axiom, we are able to compare the A_i. Moreover, because of the *monotonicity* axiom we can determine that:

$$A_1 < A_2 < \ldots\ldots A_n \text{ implies } A_1 \prec A_2 \prec \ldots\ldots A_n.$$

Define $A_i^* = \{ (1-U(A_i))\, A_1, U(A_i)\, A_n \}$ where $0 \le U(A_i) \le 1$. By the *continuity* axiom, for every A_i, there is a probability $U(A_i)$ such that:

$$A_i \sim A_i^*$$

Note that for A_1, we have $U(A_1) = 0$; hence $A_1 \sim A_1$ and for A_n, $U(A_n) = 1$, hence $A_n \sim A_n$. For all other values A_i, we have $0 < U(A_i) < 1$ and, due to the *monotonicity* and *transitivity* axioms, $U(A_i)$ increases from zero to 1 as A_i increases from A_1 to A_n.[6]

Substitute A_i by A_i^* in L_1 and, by the *interchangeability* axiom, we obtain:

$$L_1 \sim L_1^* \equiv \{p_1 A_1, p_2 A_2, \ldots p_i A_i^* \ldots . p_n A_n\}$$

where the superscript of L_1 indicates that one element A_i^* has been substituted in L. Then substitute one more element in L_1^* and use the *interchangeability* and *transitivity* axioms to obtain that $L_1 \sim L_1^* \sim L_1^{**}$ where L_1^{**} is the lottery when two elements are substituted. Continue this process and denote the lottery by $\tilde{L}_1$ where all its elements A_i $(i = 1, 2, \ldots, n)$ are substituted by A_i^* to obtain:

$$L_1 \sim \tilde{L} \equiv \{p_1 A_1^*, \ldots p_2 A_2^*, \ldots, p_n A_n^*\}$$

By the *decomposability* and *transitivity* axioms, we have:

$$L_1 \sim \tilde{L}_1 \sim \tilde{\tilde{L}}_1 \equiv \{A_1 \sum p_i (1-U(A_i)), A_n \sum p_i U(A_i)\} \; .$$

We repeat all these steps with lottery L_2 to obtain:

$$L_2 \sim \tilde{L}_2 \sim \tilde{\tilde{L}}_2 = \{A_1 \sum q_i (1-U(A_i)), A_n \sum q_i U(A_i)\} \, .$$

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[6] To see this, suppose that $6 \sim \{\frac{1}{4}1, \frac{3}{4}10\} \equiv A_6^*$. Then we claim that for a higher value, say 7, we have $7 \sim \{(1-\alpha)\ 1,\ \alpha\{10\} \equiv A_7^*$ where $\alpha > 3/4$. Due to the *monotonicity* 7 $\succ$ 6; due to the *transitivity*, $A_7^* \succ A_6^*$, and due to the *monotonicity*, $\alpha > 3/4$, which is exactly what is claimed above.

Recall that $A_n > A_1$. Therefore, by the *monotonicity* axiom, $\tilde{\tilde{L}}_1$ is preferred to $\tilde{\tilde{L}}_2$ if the following condition holds:

$$\Sigma\, p_i\, U(A_i) > \Sigma\, q_i\, U(A_i)\,.$$

But, because of the *transitivity* , this also implies the same inequality with the original investments; hence $L_1 \succ L_2$.

How is this result related to expected utility? Assume for a moment that $U(A_i)$ is the utility of A_i. Then, given the above set of axioms, the investment with the highest expected utility is preferred, namely:

$$L_1 \succ L_2 \Leftrightarrow \sum p_i\, U(A_i) \equiv E_{L_1} U(x) > \sum q_i\, U(A_i) \equiv E_{L_2} U(x)$$

where x denotes the possible monetary outcomes (the A_i in our proof) and the subscripts L_1 and L_2 denote the expected utility of L_1 and L_2, respectively. We shall see below that the probabilities $U(A_i)$ do, indeed, represent the investor's preference regarding the various outcomes, hence they will also represent the utility corresponding to outcome A_i. Thus, $U(A_i)$ will be shown to be the investor's utility function.

2.4 THE PROPERTIES OF UTILITY FUNCTION

a) Preference and Expected Utility

We proved above that if the expected utility of L_1 is larger than the expected utility of L_2 then L_1 will be preferred to L_2. Actually, preference is a fundamental property reflecting the investor's taste. Therefore, it is more logical to turn the argument around and assert that if L_1 f L_2 then there is a non-decreasing function U_1 such that $E_{L_1} U_1(x) > E_{L_2} U_1(x)$. Note that $L_1 \succ L_2$ is possible for one investor and $L_2 \succ L_1$, is possible for another investor. This implies that there is another non-decreasing function U_2 reflecting the second investor's preference such that $E_{L_2} U(x) > E_{L_1} U(x)$. This non-decreasing function is called *utility* function. Why does the function $U(A_i)$ reflect the investor's taste or the investor's utility from money? The reason is that by the *continuity* axiom for any two values

A_1 and A_n (where $A_n > A_1$) and $A_1 < A_i < A_n$, there is a function (probability) $U(A_i)$ such that:

$$\{(1-U(A_i)A_1, U(A_i)A_n\} \equiv A_i^* \sim A_i \ .$$

Not all investors would agree on the specific value of $U(A_i)$ but, for each investor, such a function $U(A_i)$, (with $0 \le U(A_i) \le 1$), must exist. Because $U(A_i)$ differs from one investor to another, it reflects the investor's preference; hence, it is called utility function and it reflects the investor's taste or indifference curve. The indifference curve is generally measured by a comparison between uncertain investment and a certain cash flow as we shall see in the next example.

Example:

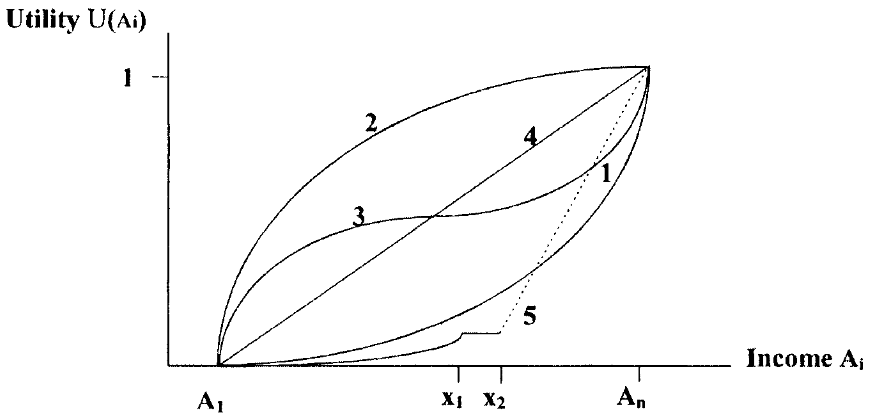
Suppose that A_1 = \$0, A_2 = \$10 and A_3 = \$20.

By the *continuity* axiom, then there is a function $0 < U(A_2) < 1$ such that:

$$L^* \equiv \{ (1-U(A_2))\ \$0\ ,\ U(A_2)\ \$20 \} \sim \$10\ .$$

If various investors were asked to determine the $U(A_2)$ which would make them indifferent between receiving \$10 for sure or L*, they would probably assign different values $U(A_2)$. One investor might decide on $U(A_2) = 1/2$. Another investor who dislikes uncertainty might decide on $U(A_2) = 3/4$. Because $U(A_2)$ varies from one investor to another according to his/her taste or preference, it is called the utility function, or the utility assigned to the value A_2. Therefore, $U(A_i)$ is called the utility of A_i , and the investment with the highest expected utility $\Sigma\ P_i\ U(A_i)$ is the optimal investment. The function $U(A_i)$ has only one property: It is $U(A_1) = 0$ for the lowest value A_1 and $U(A_n) = 1$ for the highest value (see Footnote 6) A_n and, due to the *monotonicity* axiom, it increases (in the weak sense) as A_i increases. Thus, the only constraint on $U(A_i)$ is that it is non-decreasing. Figure 2.1 illustrates various possible utility functions: all of them are possible and none of them contradict expected utility theory.

Figure 2.1: Various Utility Functions



Curve 1 is convex, curve 2 is concave, curve 3 has convex as well as concave segments, and curve 4 is linear. Note that $U(A_i)$ do not have to strictly increase through the whole range. We allow also for a function that is constant for some ranges, say range $x_1 \leq x \leq x_2$ (see curve 5). Thus $U(A_i)$ is a non-decreasing function of A_i. All these functions reflect various preferences; all conform with the *monotonicity* axiom which determines that the higher the monetary outcome A_i, the higher (or equal) the utility $U(A_i)$. Thus, if $A_j > A_i$, then $U(A_j) \geq U(A_i)$. What is the intuitive explanation for the fact that the utility function is non-decreasing in income? The utility function cannot decrease because if $U(A_2) < U(A_1)$ and $(A_2 > A_1)$ your utility would increase by simply donating $A_2 - A_1$ to charity; your utility increases (and other people would also be able to enjoy your money!). As we shall see in the next chapter, we will develop stochastic dominance rules for various types of utility functions (e.g., all convex, all concave, etc.).

b) Is U(x) a Probability Function or a Utility Function?

In the proof of MEUC, we assume that U(x) is a probability with $0 \leq U(x) \leq 1$. However, we also called this function a utility function. Does this mean that the utility of any monetary outcome is bounded between 0 and 1? No, it doesn't: Utility can take on any value, even a negative one. We can start with $0 \leq U(x) \leq 1$ (as done in the proof of Theorem 2.1) and then conduct a transformation on U(x) and, therefore, it can take on any value, even a negative one, without changing the ranking of the investments. Therefore, we can switch from $0 \leq U(x) \leq 1$ to any other (unbounded) utility function. This is summarized in the following theorem.

Theorem 2.2: A utility function is determined up to a positive linear transformation , where "determined" means that the ranking of the projects by MEUC does not change.

Proof: First, we define a positive linear transformation as $U^*(x) = a+b \cdot U(x)$ where $b > 0$ and $a \gtreqless 0$. Suppose that there are two risky investments with returns x and y, respectively. Then by the Theorem claim $EU(x) > EU(y)$, if and only if $EU^*(x) > EU^*(x)$, where:

$$U^*(\cdot) = a + bU(\cdot) \text{ and } b > 0.$$

That is, $U^*(\cdot)$ is a positive linear transformation of $U(\cdot)$.

To see this recall that:

$$EU^*(x) = a + b \cdot EU(x);$$

$$EU^*(y) = a + b \cdot EU(y)$$

and it can be easily seen that for $b > 0$ and $a \lesseqgtr 0$, $EU(x) > EU(y)$ if and only if $EU^*(x) > EU^*(y)$.

Thus, the investments' ranking by U or U^* is identical; if x has a higher expected utility than y with U, it has a higher expected utility with U^* (and vice versa). Therefore, one can shift from U to U^* and vice versa without changing the ranking of the alternative choices under consideration.

Can we use the utility function as a probability function as in the proof of Theorem 2.1? Yes, we can. We use Theorem 2.2 to show this claim.

Suppose that there is a utility function reflecting the investor's preference. Then, one can conduct a linear transformation to obtain another utility function U^* such that U^* will be between zero and one; hence, U^* can be used as a probability in the proof of Theorem 2.1. To demonstrate how such a normalization is carried out, suppose that x reflects all possible values A_i , $x_1 = A_1$ is the lowest monetary value, and $x_n = A_n$ is the highest monetary value. $U(A_1)$ and $U(A_n)$ are unrestricted utilities which can even be negative. We can then conduct a linear positive transformation such that

$$U^*(A_1) = a + b \cdot U(A_1) = 0 ;$$

$$U^*(A_n) = a + b \cdot U(A_n) = 1 .$$

Thus, we have two equations with two unknowns (a and b), and we can solve for a and b as follows:

Subtract one equation from the others to obtain:

$$b(U(A_n) - U(A_1)) = 1 ,$$

or

$$b = 1/[U(A_n) - U(A_1)];$$ [7]

and, from the first equation:

$$a = -bU(A_1) = - U(A_1)/[U(A_n) - U(A_1)] .$$

Thus, for any utility function U, we can select a and b such that there will be a new function U^* with $U^*(a) = 0$ and $U^*(b) = 1$. Because such a transformation does not change the project ranking, U^* can be employed both as probability function as in the proof of Theorem 2.1 as well as a utility function for ranking investments.

Example:

Let $U(A_1) = -5$ and $U(A_n) = 95$. Then, by the above solution for a and b, we have:

$$b = 1/[95 - (-5)] = 1/100;$$

$$a = -(-5)/[95 - (-5)] = 5/100 .$$

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[7] Note that $U(A_n) > U(A_1)$, hence $b > 0$ which confirms that a positive linear transformation is employed.

Thus, the function U^* is given by $U^*(x) = 5/100 + 1/100\ U(x)$.

Indeed, for x = a, we have:

$$U^*(a) = 5/100 + 1/100 \cdot (-5) = 0$$

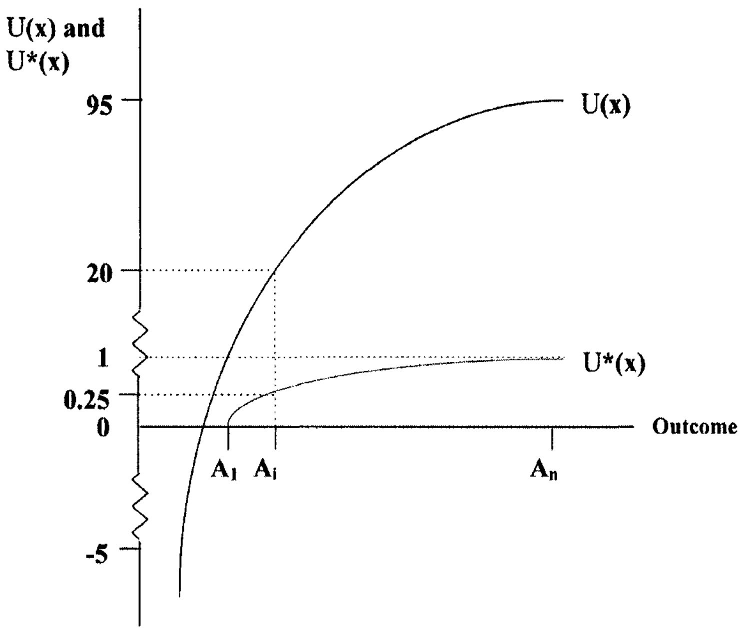
and, for x = b, we have:

$$U^*(b) = 5/100 + 1/100\ (95) = 1\ .$$

Hence U^* (a) = 0 and $U^*(b) = 1$ as in the proof of the MEUC.

Figure 2.2 illustrates the utility function U(x) and $U^*(x)$ corresponding to the above example. It should be noted that if there is another value, say A_i, for which $U(A_i) = 20$, then $U^*(A_i) = 5/100 + (1/100) \cdot 20 = 25/100 = 1/4$. All values $U(A_i)$ are determined by the same technique.

Figure 2.2: A linear positive transformation $U^*(x) = a + b \bullet U(x)$



This example illustrates that we can take any utility function $U(x)$ and conduct the linear transformation shifting to $U^*(x)$ without changing the investor's project ranking, and this $U^*(x)$ function can be employed as a probability function, as in the proof of Theorem 2.1.

2.5 THE MEANING OF THE UTILITY UNITS

The utility units, which are called *utiles*, have no meaning: If the utility of investment A is 100 and the utility of investment B is 150, we cannot claim that investment B is 50% better. The reason for this is that we can conduct a positive linear transformation and expand or suppress the difference between the utility of investment A and investment B arbitrarily. In the above example, we have the original utility function for the two values A_1 and A_n.

$$U(A_1) = -5\ , \qquad U(A_n) = 95\ , \text{ and } \ U(A_n) - U(A_1) = 100\ .$$

After the transformation, we have $U^*(A_1) = 0$ and $U^*(A_n) = 1$; hence the difference between the utility of A_n, and A_1 decreases from 100 to 1.

Because we can shift from U to U^* without changing the ranking of the various investments, we can say that the only important thing is the ranking of the investment by the expected utility and there is no meaning to the difference in the expected utility of the two projects under consideration. The "utilities" themselves and in particular, the magnitude of the difference of utilities, are meaningless. Moreover, a negative utility does not imply that the investment is unattractive. We demonstrate this in the following example:

Example:

Suppose that you are offered one of the following cash flows (denoted by x) for the same amount of money or free of charge corresponding to two distinct investments denoted by A and B:

Investment A		Investment B	
X	**p(x)**	**x**	**p(x)**
5	1/2	8	1
10	1/2		

Suppose that your preference is given by $U(x) = x^2$. Which cash flow would you select? By the MEUC, we have to select the one with the highest expected utility.

Simple calculation reveals:

$$E_AU(x) = (1/2)\ 5^2 + (1/2)\ 10^2 = 62.50\ ;$$

$$E_BU(x) = (1)\ 8^2 = 64.$$

Hence, B is preferred. Now use the utility function, $U^*(x) = 100\ x^2$, which is a positive linear transformation of U(x), with a = 0 and b = 100. With this new function we have:

$$E_A\, U_1^*(x) = 100 \cdot 62.5 = 6,250;$$

$$E_B\, U_1^*(x) = 100 \cdot 64 = 6{,}400\ .$$

Hence, $EU_B(x) > E_AU(x)$.

The difference, which was 64-62.5 = 1.5 between $E_BU(x)$ and $E_AU(x)$, increases to 6,400–6,250 = 150 with $E_B\, U_1^*(x)$ and $E_A\, U_1^*(x)$. This does not imply that B becomes much better because the only factor that is relevant is that $E_BU(x) > E_A$

U(x) (i.e., B is ranked above A), and the magnitude of the difference $E_AU(x)$ - $E_BU(x)$ is meaningless.

Now consider the following utility function $U_2^*(x) = -100 + x^2$ which is again a positive linear transformation of U(x) with a = −100 and b = 1.

With this function we have :

$E_A\ U_2^*(x) = -100 + 62.5 = -37.5;$

$E_B\ U_2^*(x) = -100 + 64 = -36.$

We obtain a negative expected utility. Does this mean that the investor should reject both cash flows? No, it doesn't. With no cash flow (x=0), the utility is:

$U_2^*(x) = -100 + 0^2 = -100$

and, because $-36 > -100$, the investor is better off selecting investment B. Thus, utility and expected utility can be negative. We cannot infer that the investment with a negative expected utility should be rejected. We simply rank all investments by their expected utility and select the one with the highest expected utility. Note, however, that the option not to invest at all may have the highest expected utility in which case all of the projects will be rejected.

One might be tempted to believe that any monotonic increasing transformation, not necessarily linear, also maintains the ranking of the project. This is not so. To see this, assume that we have U(x) for which $U(x) \geq 0$ (e.g., $U(x) = x^2$), the function employed earlier. Consider the following increasing monotonic transformation:

$$U^*(x) = [U(x)]^2 \text{ (for } U(x) > 0).$$

Because, by assumption, $U(x) = x^2$ (all $x > 0$), $U^*(x) = x^4$. This is an increasing monotonic transformation of U(x).

Recall that with the above example and with $U(x) = x^2$, we have $E_BU(x) = 64 > E_AU(x) = 62.5$. Let us show that the ranking of these two investments is reversed with $U^*(x)$. We have:

$E_A U^*(x) = (1/2)\ 5^4 + (1/2)\ 10^4 = (1/2)\ 625 + (1/2)\ 10{,}000 = 5{,}312.5;$

$E_B U^*(x) = (1)\ 8^4 = 4.096\ .$

Hence, $E_A U^*(x) > E_B\ U^*(x)$, which differs from the ranking by U(x) obtained before. Thus, a positive *linear* transformation is allowed (does not change the ranking), and a positive *monotonic* transformation is not allowed because it may change the ranking of the project under consideration.

To sum up, a linear positive utility transformation does not affect the project's ranking and, as this is what is important for the investor, such a transformation is allowed. However, a monotonic transformation which is not linear may, or may not, keep the project's ranking unchanged and, therefore, it is not permissible.

Secondly, we can choose a transformation, which makes the utility function, intercept any two points a_1 and a_2. To do this, select $U(x_1) = a_1$ and $U(x_2) = a_2$. In our specific case, we selected $a_1 = 0$ and $a_2 = 1$ such that the utility can be used as a probability function. However, any pair (a_1, a_2), which we want the utility function to intercept, can be selected.

2.6 MRC, MERC AS SPECIAL CASES OF MEUC

Proving that MEUC is optimal does not imply that MRC and MERC should never be employed. On the contrary, these two criteria are special cases of MEUC.

Let us first show that if the utility function is linear of the type,

$$U(x) = a + bx \qquad (b > 0)$$

then MEUC and MERC coincide.

To see this, let us compare two investments denoted by x and y. By the MEUC we have that $x \succ y$ if and only if $EU(x) > EU(y)$. But with linear utility function we have:

$$EU(x) > EU(y) \Leftrightarrow a + bEx > a + b\,Ey \quad (\text{for } b > 0);\ \text{hence,}\ Ex > Ey.$$

Thus, the project ranking by EU(•) is the same as the ranking by the expected value; hence, for linear utility function, the MERC coincides with the MEUC.

Finally, if returns are certain, MRC is obtained as a special case of MEUC. To see this, consider two certain investments (1,x) and (1,y) where 1 means that the probability of obtaining x and y, respectively, is equal to one. Then, by the MRC, $x \succ y$ if $x > y$. But, because of the *monotonicity* of the utility function, $x > y \Rightarrow U(x) > U(y)$ and $1 \cdot U(x) > 1 \cdot U(y)$. However, the last inequity implies that $EU(x) > EU(y)$ for the degenerated case where the probability is 1.

2.7 UTILITY, WEALTH AND CHANGE OF WEALTH

Denote by w the investor's initial wealth and by x the change of wealth due to an investment under consideration. While w is constant, x is a random variable. The utility function is defined on total wealth w+x. The inclusion of w in the utility is crucial as the additional utility due to the possession of a risky asset, e.g., a stock or a bond, depends on w. For
example, a poor person with w = \$10,000 may appreciate an addition of x = \$1,000 (in terms of utility) more than a millionaire with w = \$10 million.

Despite the importance of w in the expected utility paradigm, there is evidence that investors in the decision making processes tend to ignore w and focus on change of wealth. Thus, instead of looking at U(w+x) the investors make decisions based on U(x). To the best of our knowledge, the first one to suggest that in practice investors make decisions based on change of wealth was Markowitz as early as 1952.[8] However, only in 1979[9] when Kahneman and Tversky published their famous Prospect Theory study, has this issue received more attention by economists. Indeed, one of the important components of Prospect Theory is that in practice (based on experimental findings) decisions are made based on change of wealth, x rather than total wealth w + x.

Making decisions based on change of wealth contradicts expected utility paradigm, but it does not contradict Stochastic Dominance (SD) framework despite the fact that SD is derived within expected utility paradigm. As we shall see in this book (see next chapter), if distribution F dominates distribution G for all U(w+x) in a given set of preferences (e.g., risk-aversion), the same dominance is intact also for all U(x) in the same set of preferences. In other words, the *partial ordering* of SD which determines the efficient and inefficient sets of investments

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[8] Markowitz, H., "The utility of wealth," *Journal of Political Economy*, *60*, 1952, pp. 151-156.
[9] Kahaneman, D.K., and Tversky, A., "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, *47*, 1979, pp. 263–291.

(as well as dominance by Markowitz's Mean-Variance rule) is invariant to w, while the selection of the optimal choice from the efficient set does depend on w. As we discuss in this book, in employing SD rules which provide the various efficient sets (i.e., partial ordering), we can safely ignore w.

2.8 SUMMARY

If a certain set of axioms is accepted , then the optimum investment criterion is the maximum expected utility criterion (MEUC) and all investments should be ranked by their expected utility. The ranking of the project is important but the magnitude of the difference of expected utility of two investments under consideration is meaningless because one can expand or shrink this difference by conducting a positive linear transformation. Thus, although utility function is cardinal (each project is not only ranked but also assigned a number), actually it is ordinal because what really matters in investment decisions is only the ranking of the projects.

The maximum return criterion (MRC) and the maximum expected return criterion (MERC) are special cases of MEUC where we have certainty or linear utility function, respectively.

In the proof of MEUC, nothing is assumed regarding the shape of the utility function apart from monotonicity (i.e., it is non-decreasing). In the next chapter we develop the stochastic dominance rules, which are optimal decision rules for various possible utility functions (i.e., subsets of all possible utility functions).

Key Terms

The Maximum Return Criterion (MRC).

The Maximum Expected Return Criterion (MERC).

The St. Petersburg Paradox.

Certainty Equivalent.

Comparability Axiom.

Continuity Axiom.

Interchangeability Axiom.

Transitivity Axiom.

Decomposability Axiom.

Monotonicity Axiom.

Optimal Decision Rule.

Utility Function.

Utiles.

Positive Linear Transformation.

Positive Monotonic Transformation.

Partial Ordering

Efficient Set

Inefficient Set

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