Relaxed Toll Sets for Congestion Pricing Problems

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Summary. Congestion or toll pricing problems in [HeR98] require a solution to the system problem (the traffic assignment problem that minimizes the total travel delay) to define the set of all valid tolls or the toll set. For practical problems, it may not be possible to obtain an exact solution to the system problem and the inaccuracy in an approximate system solution may render the toll set empty. When this occurs, this paper offers alternative toll sets based on relaxed optimality conditions. With carefully chosen parameters, tolls from the relaxed toll sets are shown theoretically and empirically (using four transportation networks in the literature) to induce route choices that are nearly system optimal.

Key words: Congestion Pricing, Traffic Equilibrium, Perturbation Analysis

1 Introduction

To encourage each traveller to choose a route in a transportation network that would collectively benefit all travellers, Hearn and Ramana [HeR98] proposed in 1998 a framework for determining the prices and locations at which to toll the network. This framework requires solving a congestion or toll pricing problem, an optimization problem with linear constraints that describe the set of all valid tolls or the toll set. Coefficients for the constraints depend on an optimal solution to the system problem, i.e., the traffic assignment problem (see, e.g., Florian and Hearn, [FlH95]) that minimizes the total travel delay among all travellers.

For small transportation networks, it is possible to compute an exact optimal solution to the system problem. However, obtaining such a solution for larger networks may be either impossible or impractical. When implemented on computers, algorithms for the system problem must perform all numerical computations using finite precision. This naturally induces small numerical

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inaccuracies because to perform some mathematical operations precisely requires infinite precision. Furthermore, the system problem is generally a nonlinear program for which most algorithms require in theory an infinite number of iterations to reach an exact optimal solution. In practice, it is common to terminate these algorithms when they find a solution with a small optimality gap, e.g., 10E-4.

On the other hand, using an approximate solution for the system problem (or an approximate system solution) to determine the coefficients for the constraints defining the toll set may cause the toll pricing problem to become infeasible, numerically (e.g., because of finite precision) or otherwise. To overcome this infeasibility, Hearn and Ramana [HeR98] employ a penalty function approach and Hearn et al. [HYR01] relax one of the constraints defining the toll set. For the latter, the relaxation is based on a parameter defined by an optimal solution to the penalty problem in [HeR98].

This paper studies the viability of using an approximate system solution in defining the toll set. Specifically, when an approximate system solution makes the toll set empty, this paper alleviates this inconsistency by relaxing one or more constraints, some of which are similar to those used in [HYR01]. However, our approach to relaxation does not require solving a penalty problem. Moreover, this paper also addresses three issues relating to the use of an approximate system solution. The first issue is whether an approximate system solution yields a consistent set of constraints defining the toll set. When it does not, the second issue is to find practical methods for relaxing the constraints in order to generate tolls that causes travellers to use the transportation network in nearly the most efficient manner. Finally, the last issue is to ascertain how well these methods work theoretically and empirically.

The remainder of the paper assumes that the travel demands are fixed. Results for the elastic demand case are similar and given in the Appendix. Section 2 defines two types of toll sets, system and non-system, and discusses their properties. Section 3 derives a relaxed toll set using an approximate system solution and shows that the tolls from this set have the desirable property. Section 4 gives an alternate representation of the relaxed toll set. Section 5 reports encouraging results for four transportation networks from the literature and Section 6 concludes the paper.

2 System and Non-System Toll Sets

To define toll sets, consider the two traffic assignment problems in transportation science literature, the system and user problems. (See, e.g., [FlH95].) Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a network with $\mathcal{N}$ and $\mathcal{A}$ denoting the set of nodes and arcs, respectively. Associated with $\mathcal{G}$, there are also a node-arc incidence matrix, A, and a set of commodities or origin-destination pairs, $\mathcal{K}$. For each commodity $k \in \mathcal{K}$, $b_k \in R^{|\mathcal{N}|}$ and $x^k \in R_+^{|\mathcal{A}|}$ denote the corresponding (fixed) demand and arc flow vectors, respectively. Hence, $v = \sum_k x^k$ is the vector of the total (or aggregate) flow on every arc and the set of feasible aggregate flow vectors can be described as follows:

$$V = \{v | v = \sum_k x^k, Ax^k = b_k, x^k \geq 0\},$$

where $x^k \geq 0$ means $x_a^k \geq 0, \forall a$. (More generally, $x \geq y$ means $x_a \geq y_a, \forall a$.) Without loss of generality, we can assume throughout the paper that V is bounded and, therefore, compact. (See, e.g., [FlH95].)

Let $s(v)$ be a travel cost vector in which each element, $s_a(v)$, is the cost to traverse arc a. This cost does not include any toll and can be measured in monetary or time units. For simplicity, we assume that $s_a(v)$ is differentiable for all a, i.e., $\nabla s(\overline{v})$, the Jacobian of $s(v)$, exists for all v. Then, the system optimal (SOPT) problem (or, more simply, system problem) is to find a feasible aggregate flow vector that minimizes the total travel cost or delay among all travellers. Mathematically, the system problem can be stated as follows:

$$\overline{v} = \operatorname{argmin}\{s(v)^T v : v \in V\}.$$

Instead of minimizing the total travel delay, an alternate traffic assignment problem, i.e., the user equilibrium problem (or, more simply, the user problem), assumes that each traveller tries to minimize his or her own travel time. The objective of the user problem is to find a solution for which no traveller can improve his or her travel time by unilaterally changing routes. In particular, v^* solves the user problem (or UOPT) if it satisfies the following variational inequality:

$$s(v^*)^T(v - v^*) \geq 0, \quad \forall v \in V.$$

Alternately, we say that v^* solves VI$[s(v), V]$.

The travel delay at the user solution, $s(v^*)^T v^*$, is generally larger than the one at the system solution, $s(\overline{v})^T \overline{v}$. In this sense, the user solution does not utilize the network in the most efficient manner. Mathematically, we can impose tolls on arcs in order to make travellers use the network more efficiently. For a given toll vector, β, let $v^*(\beta)$ solve VI$[s(v) + \beta, V]$, i.e., $v^*(\beta)$ satisfies the following tolled user equilibrium condition:

$$(s(v^*(\beta)) + \beta)^T(v - v^*(\beta)) \geq 0, \quad \forall v \in V.$$

We refer to $v^*(\beta)$ as the solution to the tolled user equilibrium problem and it is the user equilibrium flow resulting from imposing the toll β on the network.

Similar to [HeR98], we assume herein that $\overline{v}$ is a unique solution to SOPT and $v^*(\beta)$ is a unique solution to VI$[s(v) + \beta, V]$ for all $\beta \in R^{|\mathcal{A}|}$. Below, we refer to these two assumptions as [A] and [B], respectively. For example, both [A] and [B] hold when we use the Bureau of Public Road (BPR) function for travel costs, i.e., $s_a(v) = \tau_a(1.0 + \theta_a(v_a/\gamma_a)^4)$ and τ_a, θ_a, and γ_a are positive. More generally, both assumptions hold when $s_a(v)$ is a continuous convex function for each a and the cost vector $s(v)$ is strictly monotone on $\{v : v \geq 0\}$. These two assumptions allow us to define β as a valid or feasible toll vector if $v^*(\beta) = \overline{v}$, i.e., if the tolled user equilibrium solution associated with β equals the system solution. (See [HeR98] for a more general definition of a valid toll.) Then, the toll set is the set of all valid toll vectors, i.e., $\mathcal{T}(\overline{v}) = \{\beta | v^*(\beta) = \overline{v}\}$. The following result from [HeR98] describes $\mathcal{T}(\overline{v})$ algebraically.

Theorem 1. *The toll set, $\mathcal{T}(\overline{v})$, is the set consisting of the β component of every pair (β, ρ) that satisfies the following linear system*

$$s(\overline{v}) + \beta \geq A^T \rho^k, \quad \forall\ k \in \mathcal{K}, \tag{1}$$

$$(s(\overline{v}) + \beta)^T \overline{v} = \sum_k b_k{}^T \rho^k. \tag{2}$$

Observe that the above toll set is based on $\overline{v}$, the optimal solution to SOPT. To distinguish this toll set from others (to be defined later), we refer to $\mathcal{T}(\overline{v})$ as the "(unrestricted) system toll set." As defined above, β in the system toll set is unrestricted. (In practice, positive tolls represent payment for road usage and negative tolls represent subsidies for the same purpose.) Moreover, the system toll set is nonempty. In fact, $\beta = -s(\overline{v})$ belongs to the system toll set because $\beta = -s(\overline{v})$ and $\rho^k = 0$ for all k trivially satisfy (1) and (2). In addition, the optimality condition for the system problem also implies that $\beta_{\text{mscp}} = \nabla s(\overline{v})^T \overline{v} \in \mathcal{T}(\overline{v})$. (See, e.g., [HeR98].) Transportation economists (see, e.g., Arnott and Small [ArS94]), generally refer to β_{mscp} as the *marginal social cost vector*. Using $-s(\overline{v})$ and $\nabla s(\overline{v})^T \overline{v}$, Hearn and Ramana show in [HeR98] that $\mathcal{T}(\overline{v})$ is unbounded. Because an arbitrarily large toll is impractical, we assume that all toll vectors in $\mathcal{T}(\overline{v})$ are bounded and, when not explicitly stated, the constraint $\|\beta\| \leq B$, where B is a sufficiently large number, is included in all toll sets described herein.

When β is required to be nonnegative, we refer to the set $\mathcal{T}^+(\overline{v}) = \{\beta \geq 0 | v^*(\beta) = \overline{v}\}$ as the "nonnegative system toll set." Algebraically, $\mathcal{T}^+(\overline{v})$ is the toll set described in Theorem 1 with an additional nonnegativity constraint on β. In practice, $\mathcal{T}^+(\overline{v})$ is nonempty. Practical traffic assignment models (see, e.g., [FGS87], [FlH95], [HLV87], and [LMP75]) typically use travel cost functions whose Jacobians, $\nabla s(\overline{v})$, are both nonnegative and diagonal. This makes β_{mscp} nonnegative and $\mathcal{T}^+(\overline{v})$ nonempty. Later, we provide a condition under which the latter holds without requiring the Jacobian to be nonnegative.

Consider the case when it is not practical to compute $\overline{v}$ exactly. Let $\widetilde{v}$ denote an approximate solution to SOPT. Without specifying the quality of the approximation, all that can be claimed is that $\widetilde{v}$ is a feasible aggregate flow vector and the toll set based on $\widetilde{v}$, or the non-system toll set, is $\mathcal{T}(\widetilde{v}) = \{\beta | v^*(\beta) = \widetilde{v}\}$. In words, this is the set of toll vectors whose tolled user equilibrium solution equals the aggregate flow vector $\widetilde{v}$. As shown below, the algebraic characterization of $\mathcal{T}(\widetilde{v})$ is essentially the same as that of $\mathcal{T}(\overline{v})$.

Theorem 2. *The nonsystem toll set, $\mathcal{T}(\widetilde{v})$, is the set consisting of the β component of every pair (β, ρ) that satisfies the following linear system:*

$$s(\widetilde{v}) + \beta \geq A^T \rho^k, \quad \forall\, k \in \mathcal{K}, \tag{3}$$

$$(s(\widetilde{v}) + \beta)^T \widetilde{v} = \sum_k {b_k}^T \rho^k. \tag{4}$$

Proof. Because of assumption [B], $\widetilde{v}$ must solve VI$[s(v) + \beta, V]$ uniquely for every $\beta \in \mathcal{T}(\widetilde{v})$. From Proposition 1.2.1 in Facchinei and Pang [FaP03], $\widetilde{v}$ solves VI$[s(v) + \beta, V]$ if and only if there exist ρ^k and σ^k that satisfy the following KKT conditions:

$$\begin{aligned} s(\widetilde{v}) + \beta - A^T\rho^k - \sigma^k &= 0, \quad \forall\, k \in \mathcal{K}, \\ (\widetilde{x}^k)^T \sigma^k &= 0, \quad \forall\, k \in \mathcal{K}, \\ \sigma^k &\geq 0, \quad \forall\, k \in \mathcal{K}. \end{aligned}$$

The pair (β, ρ), where ρ is determined by the above KKT conditions, satisfies (3) and (4). The first and third KKT conditions imply that (3) holds. Multiplying the first KKT conditions by $\widetilde{x}^k$ and summing the resulting equations together yield

$$(s(\widetilde{v}) + \beta)^T \sum_k \widetilde{x}^k = \sum_k (A\widetilde{x}^k)^T \rho^k + \sum_k (\widetilde{x}^k)^T \sigma^k.$$

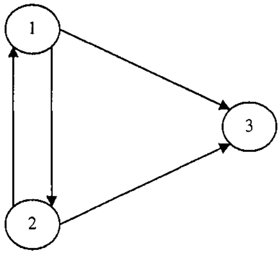
Because $\sum_k \widetilde{x}^k = \widetilde{v}$, $A\widetilde{x}^k = b_k$ and $(\widetilde{x}^k)^T\sigma^k = 0$, the above equation reduces to (4).

Conversely, let (β, ρ) satisfy (3) and (4) and set σ^k equal to $s(\widetilde{v}) + \beta - A^T\rho^k$ for all k. Then, it follows from (3) and (4) that $\sigma^k \geq 0$ and $\sum_k (\widetilde{x}^k)^T \sigma^k = 0$, respectively. The latter also implies that $(\widetilde{x}^k)^T\sigma^k$ must individually equal to zero because $\widetilde{x}^k \geq 0$ and $\sigma^k \geq 0$. Thus, the above KKT conditions are satisfied and, using the above result from [FaP03], $\widetilde{v}$ solves VI$[s(v) + \beta, V]$, i.e., $\beta \in \mathcal{T}(\widetilde{v})$. ■

In the above proof, assumption [B] is essential. Without uniqueness, there may be alternate tolled user equilibrium solutions not equal to $\widetilde{v}$. In addition, the non-system toll set described above is always nonempty because, as in the system toll set, $-s(\widetilde{v}) \in \mathcal{T}(\widetilde{v})$.

Consider the nonnegative and non-system toll set, i.e., $\mathcal{T}^+(\widetilde{v}) = \{\beta \geq 0 | v^*(\beta) = \widetilde{v}\}$. The algebraic representation of $\mathcal{T}^+(\widetilde{v})$ is the same as described in the above theorem with the addition of the nonnegativity constraint on β. However, the example below illustrates that $\mathcal{T}^+(\widetilde{v})$ can be empty.

The network in Figure 1 represents a transportation system with three nodes and four arcs where the travel cost function for every arc is constant and equals 1. There are two OD pairs, (1,3) and (2,3), each with a travel demand of 2 units. Table 1 displays a set of feasible flow vectors for the transportation system. For OD pair (1,3), the flow vector $\widetilde{x}^{(1,3)}$ corresponds to sending one unit of flow along each of the two possible paths, $1 \rightarrow 2 \rightarrow 3$



OD pairs	Demands
(1, 3)	2
(2, 3)	2

Travel Demands

Travel Cost function: $s_{ij}(v_{ij}) = 1$, $\forall$ arc (i, j)

Fig. 1. A Counterexample

and $1 \to 3$. Similarly, $\widetilde{x}^{(2,3)}$ corresponds to send one unit of flow along paths $2 \to 1 \to 3$ and $2 \to 3$. Clearly, the aggregate flow vector $\widetilde{v} = \widetilde{x}^{(1,3)} + \widetilde{x}^{(2,3)}$ is not system optimal because sending two units of flow along arcs (1,3) and (2,3) satisfies both travel demands and is less costly.

Table 1. Feasible Flow Vectors for the Network in Figure 1

Arc	$\widetilde{x}^{(1,3)}$	$\widetilde{x}^{(2,3)}$	$\widetilde{v}$
(1,2)	1	0	1
(1,3)	1	1	2
(2,1)	0	1	1
(2,3)	1	1	2

Because the two OD pairs can be treated as one commodity, the nonnegative and nonsystem toll set, $\mathcal{T}^{+}(\widetilde{v})$, reduces to the following linear system:

$$\begin{aligned} 1+\beta_{12} &\geq \rho_1 - \rho_2 \\ 1+\beta_{13} &\geq \rho_1 - \rho_3 \\ 1+\beta_{21} &\geq \rho_2 - \rho_1 \\ 1+\beta_{23} &\geq \rho_2 - \rho_3 \\ (1+\beta_{12}) + 2(1+\beta_{13}) + (1+\beta_{21}) + 2(1+\beta_{23}) &= 2\rho_1 + 2\rho_2 - 4\rho_3 \\ \beta_{ij} &\geq 0, \qquad\qquad \forall (i,j) \end{aligned}$$

The equality constraint in the above system can be equivalently written as

$$\begin{aligned} (1+\beta_{12} - [\rho_1 - \rho_2]) &+ 2(1+\beta_{13} - [\rho_1 - \rho_3]) \\ &+ (1+\beta_{21} - [\rho_2 - \rho_1]) + 2(1+\beta_{23} - [\rho_2 - \rho_3]) = 0. \end{aligned}$$

This equation implies that the four inequalities in the system must hold at equality, i.e.,

$$\begin{aligned} 1+\beta_{12} &= \rho_1 - \rho_2 \\ 1+\beta_{13} &= \rho_1 - \rho_3 \\ 1+\beta_{21} &= \rho_2 - \rho_1 \\ 1+\beta_{23} &= \rho_2 - \rho_3 \end{aligned}$$

Adding the first and third equations together yields

$$2+\beta_{12}+\beta_{21}=0.$$

However, this is impossible because $\beta_{ij} \geq 0, \forall (i,j)$. Thus, $\mathcal{T}^+(\widetilde{v}) = \emptyset$.

The following theorem provides a necessary and sufficient condition under which $\mathcal{T}^+(\widetilde{v})$ is nonempty. Independently, Fleischer et al. [FJM04] provide a different, but equivalent, condition for the nonemptiness of the nonnegative and non-system toll set. The condition in the theorem below is related to an earlier work on bounded traffic assignment problem in [Hea80] that was later continued in [Ber95] and [BHR97] under the setting of congestion pricing.

Theorem 3. *For any $\widetilde{v} \in V$, the set $\mathcal{T}^+(\widetilde{v})$ is nonempty if and only if $\widetilde{v}$ solves VI[$s(v)$,$\mathcal{V}$], where $\mathcal{V} = \{v | v = \sum_k x^k, Ax^k = b_k, x^k \geq 0, v \leq \widetilde{v}\}$.*

Proof. Using Proposition 1.2.1 in [FaP03], $\widetilde{v}$ solve VI[$s(v)$, $\mathcal{V}$] if and only if there exist ρ^k, σ^k, and β that satisfies the following KKT conditions:

$$\begin{aligned} s(\widetilde{v}) - A^T\rho^k - \sigma^k + \beta &= 0, \forall k, \\ (\widetilde{x}^k)^T\sigma^k &= 0, \forall k, \\ \sigma^k &\geq 0, \forall k, \\ \beta &\geq 0. \end{aligned}$$

In the above system, β is the multiplier vector corresponding to the upper bounds $v \leq \widetilde{v}$ in $\mathcal{V}$ and the complementarity condition $\beta^T(v-\widetilde{v}) = 0$ is not required because $v = \widetilde{v}$ satisfies every upper bound in $\mathcal{V}$ exactly. By an argument similar to the one in Theorem 2, the above conditions are equivalent to those that describe $\mathcal{T}^+(\widetilde{v})$. Thus, the theorem holds. ■

The corollary below provides a similar condition for the nonnegative system toll set and follows immediately from the above theorem.

Corollary 1. *The nonnegative system toll set, $\mathcal{T}^+(\overline{v})$, is nonempty if and only if $\overline{v}$ solves VI[$s(v)$,$\overline{\mathcal{V}}$], where $\overline{\mathcal{V}} = \{v | v = \sum_k x^k, Ax^k = b_k, x^k \geq 0, v \leq \overline{v}\}$.*

3 Relaxed Toll Set

Consider the situation in which an algorithm terminates and produces $\widetilde{v}$ as an approximate solution to SOPT with some desired optimality gap. Using Theorem 3 from the previous section, it is possible to determine whether $\mathcal{T}^+(\widetilde{v})$ is nonempty. However, $\mathcal{T}^+(\widetilde{v})$ is often empty in practice. This section resolves this difficulty by finding nonnegative tolls that satisfy the conditions in Theorem 2 approximately. Moreover, the focus is on defining a nonnegative relaxed toll set based on $\widetilde{v}$ when $\nabla s(\widetilde{v})$ is nonnegative. (When β is unrestricted, the system and non-system toll sets are nonempty. As such, they require no relaxation. When $\nabla s(\overline{v})$ is nonnegative, the same holds for the nonnegative system toll set.)

The first condition in Theorem 2 is

$$s(\widetilde{v}) + \beta \geq A^T \rho^k, \quad \forall k \in \mathcal{K}.$$

When multiplied by $\widetilde{x}^k$ and summed together, the above implies that $(s(\widetilde{v}) + \beta)^T \widetilde{v} \geq \sum_k b_k^T \rho^k$ because $A\widetilde{x}^k = b_k$ and $\sum_k \widetilde{x}^k = \widetilde{v}$. Therefore, the equality in (4) can be replaced by an inequality '$\leq$.' This replacement motivates the definition of a relaxed toll set $\mathcal{T}^+(\widetilde{v}, \epsilon)$, for some $\epsilon > 0$, as the set of all β for which there exists a corresponding ρ satisfying the following conditions:

$$\begin{aligned} s(\widetilde{v}) + \beta &\geq A^T \rho^k, && \forall k \in \mathcal{K}, \\ (s(\widetilde{v}) + \beta)^T \widetilde{v} &\leq \sum_{k \in \mathcal{K}} b_k^T \rho^k + \epsilon, && \\ \beta &\geq 0. && \end{aligned}$$

Let $-\epsilon_{\text{mscp}} = \min\{(s(\widetilde{v}) + \nabla s(\widetilde{v})^T \widetilde{v})^T (u - \widetilde{v}) : u \in V\}$. In Hearn [Hea82], ϵ_{mscp} is the optimality gap for SOPT at $\widetilde{v}$ and the following theorem shows that $\mathcal{T}^+(\widetilde{v}, \epsilon_{\text{mscp}})$ is nonempty.

Theorem 4. *If $\nabla s(\widetilde{v})$ is nonnegative, then $\mathcal{T}^+(\widetilde{v}, \epsilon_{mscp}) \neq \emptyset$, where $\epsilon_{mscp} > 0$ is as defined above.*

Proof. Note that

$$\begin{aligned} \epsilon_{\text{mscp}} &= (s(\widetilde{v}) + \nabla s(\widetilde{v})^T \widetilde{v})^T \widetilde{v} - \min\{(s(\widetilde{v}) + \nabla s(\widetilde{v})^T \widetilde{v})^T u : u \in V\} \\ &= (s(\widetilde{v}) + \beta_{\text{mscp}})^T \widetilde{v} - \min\{(s(\widetilde{v}) + \beta_{\text{mscp}})^T u : u \in V\}. \end{aligned}$$

From linear programming duality, the following holds

$$\begin{aligned} \epsilon_{\text{mscp}} &= (s(\widetilde{v}) + \beta_{\text{mscp}})^T \widetilde{v} - \max_{\rho}\{\textstyle\sum_k b_k^T \rho^k : s(\widetilde{v}) + \beta_{\text{mscp}} \geq A^T \rho^k, \forall k\} \\ &= \min_{\rho}\{(s(\widetilde{v}) + \beta_{\text{mscp}})^T \widetilde{v} - \textstyle\sum_k b_k^T \rho^k : s(\widetilde{v}) + \beta_{\text{mscp}} \geq A^T \rho^k, \forall k\} \end{aligned}$$

Let $\widetilde{\rho}$ denote an optimal solution to the linear program in the last equation. Then, the pair $(\beta_{\text{mscp}}, \widetilde{\rho})$ satisfies the relaxed toll condition with $\epsilon = \epsilon_{\text{mscp}}$, i.e.,

$$\begin{aligned} s(\widetilde{v}) + \beta_{\text{mscp}} &\geq A^T \widetilde{\rho}^k, && \forall k \in \mathcal{K}, \\ (s(\widetilde{v}) + \beta_{\text{mscp}})^T \widetilde{v} &= \sum_{k \in \mathcal{K}} b_k^T \widetilde{\rho}^k + \epsilon_{\text{mscp}}. && \end{aligned}$$

Because $\nabla s(\widetilde{v})$ is nonnegative, $\beta_{\text{mscp}} \geq 0$. So, $\beta_{\text{mscp}} \in \mathcal{T}^+(\widetilde{v}, \epsilon_{\text{mscp}})$ and $\mathcal{T}^+(\widetilde{v}, \epsilon_{\text{mscp}}) \neq \emptyset$. ■

As shown above, ϵ_{mscp} can be computed with little effort because many algorithms (see, e.g., [FGS87], [LMP75], and [HLV87]) for SOPT compute ϵ_{mscp} and terminate when they find a $\widetilde{v} \in V$ such that the corresponding

$\epsilon_{\text{mscp}} \leq \epsilon$, for some small $\epsilon > 0$. Instead of ϵ_{mscp}, it is also possible to choose ϵ by solving the following linear program:

$$\begin{aligned} \epsilon^* = \min_{(\beta,\rho)} \ & (s(\widetilde{v}) + \beta)^T \widetilde{v} - \sum_k b_k^T \rho^k \\ \text{s.t.} \ & s(\widetilde{v}) + \beta \geq A^T \rho^k, \qquad \forall k, \\ & \beta \geq 0. \end{aligned}$$

Because $\mathcal{T}^+(\widetilde{v}, \epsilon_{\text{mscp}}) \neq \emptyset$, the above optimization is feasible. In addition, $\epsilon^* \leq \epsilon_{\text{mscp}}$.

One important property of the system toll sets (unrestricted or otherwise) is that, for any β in $\mathcal{T}(\overline{v})$ (or $\mathcal{T}^+(\overline{v})$), $\overline{v}$ solves VI$[s(v) + \beta, V]$, i.e., the system solution also solves the user equilibrium problem with the toll vector β. However, this property only holds approximately for the relaxed toll set. Assume that $\widetilde{v}$ solves SOPT approximately, i.e.,

$$\min\{(s(\widetilde{v}) + \nabla s(\widetilde{v})^T \widetilde{v})^T (u - \widetilde{v}) : u \in V\} = -\epsilon_{\text{mscp}} \geq -\epsilon.$$

Then, for any $\beta \in \mathcal{T}^+(\widetilde{v}, \epsilon_{\text{mscp}})$, the following must hold

$$\begin{aligned} s(\widetilde{v}) + \beta &\geq A^T \rho^k, \qquad \forall k \in \mathcal{K}, \\ (s(\widetilde{v}) + \beta)^T \widetilde{v} &\leq \sum_{k \in \mathcal{K}} b_k^T \rho^k + \epsilon_{\text{mscp}}. \end{aligned}$$

It follows from the above that

$$\begin{aligned} -\epsilon \leq -\epsilon_{\text{mscp}} &\leq \sum_{k \in \mathcal{K}} b_k^T \rho^k - (s(\widetilde{v}) + \beta)^T \widetilde{v}, \forall \rho \in \{\rho : A^T \rho^k \leq s(\widetilde{v}) + \beta, \forall k\} \\ &\leq \max_{\rho}\{\sum_{k \in \mathcal{K}} b_k^T \rho^k : A^T \rho^k \leq s(\widetilde{v}) + \beta, \forall k\} - (s(\widetilde{v}) + \beta)^T \widetilde{v}, \\ &= \min_u\{(s(\widetilde{v}) + \beta)^T u : u \in V\} - (s(\widetilde{v}) + \beta)^T \widetilde{v}, \\ &= \min_u\{(s(\widetilde{v}) + \beta)^T (u - \widetilde{v}) : u \in V\} \\ &\leq (s(\widetilde{v}) + \beta)^T (u - \widetilde{v}), \forall u \in V, \end{aligned}$$

where the first equality holds because of the strong duality theorem in linear programming. Observe that the last inequality implies that $\widetilde{v}$ solves VI$[s(v) + \beta, V]$ approximately. Thus, $\widetilde{v}$ approximately solves both SOPT and the tolled user equilibrium problem.

For a slightly stronger statement, Theorem 5 below demonstrates that, for any $\eta > 0$, there exists a $\delta > 0$ such that $||v^*(\beta') - \overline{v}|| \leq \eta$ when $\beta' \in \mathcal{T}^+(\widetilde{v}, \epsilon_{\text{mscp}})$ and $||\widetilde{v} - \overline{v}|| \leq \delta$. (Here, $|| \cdot ||$ represents the Euclidean norm.) In words, the theorem states that a toll vector from the relaxed toll set yields a tolled user equilibrium solution that is approximately system optimal. To establish this theorem, the following lemmas are necessary.

Lemma 1. *Let P be a compact set, $c_1(\cdot)$ be continuous and strongly monotone with modulus α (i.e., $(c_1(v_1) - c_1(v_2))^T (v_1 - v_2) \geq \alpha ||v_1 - v_2||^2$), and $c_2(\cdot)$ be continuous. If p_1 and p_2 solve VI$[c_1(\cdot), P]$ and VI$[c_2(\cdot), P]$, respectively, then $||p_2 - p_1|| \leq \frac{1}{\alpha} ||c_2(p_2) - c_1(p_2)||$.*

Proof. See Dafermos and Nagurney [DaN84]. ■

Lemma 2. *For $i = 1$ and 2, let $F_i = \{x | U_i x \le r_i,\ W_i x = t_i\}$, where U_i and W_i are $(l \times n)$ and $(m \times n)$ matrices, respectively, and r_i and t_i are vectors in R^l and R^m, respectively. If $x_2 \in F_2$, then there exists a $x_1 \in F_1$ such that $\|x_1 - x_2\| \le \sigma(U_1, W_1) \left\| \begin{matrix} [(U_1 - U_2)x_2 - (r_1 - r_2)]^+ \\ [(W_1 - W_2)x_2 - (t_1 - t_2)] \end{matrix} \right\|_2$, where $\sigma(U_1, W_1)$ is a finite real number associated with U_1 and W_1.*

Proof. See Robinson [Rob73]. ■

Lemma 3. *Let $s(v)$ be continuously differentiable and $\|\widetilde{v} - \overline{v}\| \le \delta$ for some $\delta > 0$. For any $\beta' \in \mathcal{T}^+(\widetilde{v}, \epsilon)$, there must exist a $\beta \in \mathcal{T}^+(\overline{v})$ and constants K_1 and K_2 such that $\|\beta' - \beta\| \le K_1\delta + K_2\epsilon$.*

Proof. The conditions defining $\mathcal{T}^+(\overline{v})$ and $\mathcal{T}^+(\widetilde{v}, \epsilon)$ can be written more compactly as follows:

$$\begin{cases} -\beta \le 0, \\ \boldsymbol{A}^T \rho - \boldsymbol{I}\beta \le \boldsymbol{I} s(\overline{v}), \\ -\boldsymbol{b}^T \rho + \overline{v}^T \beta \le -s(\overline{v})^T \overline{v}, \end{cases} \tag{5}$$

and

$$\begin{cases} -\beta \le 0, \\ \boldsymbol{A}^T \rho' - \boldsymbol{I}\beta' \le \boldsymbol{I} s(\widetilde{v},) \\ -\boldsymbol{b}^T \rho' + \widetilde{v}^T \beta' \le -s(\widetilde{v})^T \widetilde{v} + \epsilon. \end{cases} \tag{6}$$

where $\boldsymbol{A} = \mathrm{diag}(A, A, \cdots, A)$, and $\boldsymbol{b}^T = ({b_1}^T, {b_2}^T, \cdots, {b_{|\mathcal{K}|}}^T)$. To further simplify our notation, let (U_1, r_1) and (U_2, r_2) denote the pairs of matrix and right-hand-side vector for (5) and (6), respectively. Because $\beta' \in \mathcal{T}^+(\widetilde{v}, \epsilon)$, there must exists a ρ' such that (β', ρ') solves (6). From Lemma 2, there must exist a pair (β, ρ) satisfying (5) for which the following hold

$$\begin{aligned}
&\left\| \begin{pmatrix} \rho' \\ \beta' \end{pmatrix} - \begin{pmatrix} \rho \\ \beta \end{pmatrix} \right\|_2 \\
&\le \sigma(U_1) \left\| \left[(U_1 - U_2) \begin{pmatrix} \rho' \\ \beta' \end{pmatrix} - (r_1 - r_2) \right]^+ \right\|_2 \\
&\le \sigma(U_1) \left\| (U_1 - U_2) \begin{pmatrix} \rho' \\ \beta' \end{pmatrix} - (r_1 - r_2) \right\|_2 \\
&\le \sigma(U_1) \left(\left\| \begin{pmatrix} 0 \\ 0 \\ (\overline{v} - \widetilde{v})^T \beta' \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} 0 \\ \boldsymbol{I}(s(\overline{v}) - s(\widetilde{v})) \\ -s(\overline{v})^T \overline{v} + s(\widetilde{v})^T \widetilde{v} - \epsilon \end{pmatrix} \right\|_2 \right) \\
&\le \sigma(U_1)(\|\beta'\|_2 \|\overline{v} - \widetilde{v}\|_2 + \|s(\overline{v}) - s(\widetilde{v})\|_2 + \|s(\overline{v})^T \overline{v} - s(\widetilde{v})^T \widetilde{v}\| + \epsilon) \\
&\le \sigma(U_1)(\|\beta'\|_2 \|\overline{v} - \widetilde{v}\|_2 + \| \bigtriangledown s(u_1)\|_2 \|\overline{v} - \widetilde{v}\|_2 \\
&\qquad + \|s(u_2) + \bigtriangledown s(u_2)^T s(u_2)\|_2 \|\overline{v} - \widetilde{v}\|_2 + \epsilon) \\
&\le \sigma(U_1)(B\|\overline{v} - \widetilde{v}\|_2 + L_1 \|\overline{v} - \widetilde{v}\|_2 + L_2 \|\overline{v} - \widetilde{v}\|_2 + \epsilon),
\end{aligned}$$

where the first inequality follows from Lemma 2, the second from the fact that $\|[x]^+\| \leq \|x\|$, the third from the definition of U_i and r_i and the triangle inequality and the fourth from Cauchy-Schwarz inequality. In the fifth inequality, u_1 and u_2 are some points between $\overline{v}$ and $\widetilde{v}$ and the gradient of $s(v)^T v$ is $s(v) + \nabla s(v)^T v$. Furthermore, the inequality holds because of Cauchy-Schwarz inequality, the differentiability of $s(v)$, and the mean value theorem. Finally, the last inequality is true because we assume earlier that $\|\beta\| \leq B$ and the continuous functions $\nabla s(v)$ and $s(v)+\nabla s(v)^T v$ are bounded on the compact set V by some constants L_1 and L_2, respectively. By letting $K_1 = (B + L_1 + L_2)\sigma(U_1)$ and $K_2 = \sigma(U_1)$, the above reduces to $\|\beta' - \beta\| \leq \left\| \begin{pmatrix} \rho' - \rho \\ \beta' - \beta \end{pmatrix} \right\|_2 \leq K_1\delta + K_2\epsilon.$ ■

Theorem 5. *Let $s(\cdot)$ be strongly monotone with modulus α. For any $\eta > 0$, there exists a $\delta > 0$ such that $\|v^*(\beta') - \overline{v}\| \leq \eta$ whenever $\beta' \in \mathcal{T}^+(\widetilde{v}, \epsilon_{mscp})$ and $\|\widetilde{v} - \overline{v}\| \leq \delta$.*

Proof. For any $\beta' \in \mathcal{T}^+(\widetilde{v}, \epsilon_{\text{mscp}})$, Lemma 3 implies that there exists a $\beta \in \mathcal{T}^+(\overline{v})$ such that $\|\beta' - \beta\| \leq K_1\delta + K_2\epsilon_{\text{mscp}}$. As defined earlier, ϵ_{mscp} depends on $\widetilde{v}$. In particular, $\epsilon_{\text{mscp}} \to 0$ as $\delta \to 0$. When combining the latter with the fact that α, K_1, and K_2 are constant and independent of η it must be possible to choose δ so that $(1/\alpha)(K_1\delta + K_2\epsilon_{\text{mscp}}) \leq \eta$.

Let $\beta' \in \mathcal{T}^+(\widetilde{v}, \epsilon_{\text{mscp}})$ and $\beta \in \mathcal{T}^+(\overline{v})$. Then Lemma 1 implies that the solutions $v^*(\beta')$ and $\overline{v}$ to $\text{VI}[s(v) + \beta', V]$ and $\text{VI}[s(v) + \beta, V]$, respectively, must satisfy

$$\begin{aligned} \|\overline{v} - v^*(\beta')\| &\leq (1/\alpha)\|s(\overline{v}) + \beta - s(\overline{v}) - \beta'\| \\ &\leq (1/\alpha)\|\beta - \beta'\| \\ &\leq (1/\alpha)(K_1\delta + K_2\epsilon_{\text{mscp}}). \end{aligned}$$

Therefore, the above choice of δ implies the theorem holds. ■

Because $\epsilon^* \leq \epsilon_{\text{mscp}}$, the above theorem also holds when ϵ^* replaces ϵ_{mscp}.

4 Disaggregate Representation of Relaxed Toll Sets

The second condition (2) in Theorem 1 is an aggregation of a number of complementarity conditions as shown in the proof of Theorem 2. When (2) is relaxed, the resulting relaxed toll set $\mathcal{T}^+(\widetilde{v}, \epsilon)$ may be larger than necessary. To define smaller relaxed toll sets, (2) can be disaggregated into its original form.

Using the argument in, e.g., Theorem 2, it is possible to show that $\mathcal{T}^+(\overline{v})$ is equivalent to the set consisting of the β component of every vector (β, ρ, σ) that satisfies the following linear system

$$\begin{aligned} s(\overline{v}) + \beta - A^T\rho^k &= \sigma^k, \forall k \in \mathcal{K}, \\ (\overline{x})^T \sigma^k &= 0, \quad \forall k \in \mathcal{K}, \\ \sigma^k &\geq 0, \quad \forall k \in \mathcal{K}, \\ \beta &\geq 0. \end{aligned}$$

The second equation is an aggregation of the complementarity condition for each arc $(i,j) \in \mathcal{A}$, i.e., $x_{ij}^k \sigma_{ij}^k = 0$. Thus, the above system is equivalent to the following:

$$\begin{aligned} s_{ij}(\overline{v}) + \beta_{ij} &\geq \rho_i^k - \rho_j^k, \forall k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ s_{ij}(\overline{v}) + \beta_{ij} &\leq \rho_i^k - \rho_j^k, \forall k \in \mathcal{K}, (i,j) \in \mathcal{A} : \overline{x}_{ij}^k > 0, \\ \beta_{ij} &\geq 0 \qquad\quad \forall (i,j) \in \mathcal{A}. \end{aligned}$$

As before, let $\widetilde{v} = \sum_k \widetilde{x}^k$ denote an approximate SOPT solution. Then, a relaxed toll set in the disaggregate form, $\Pi^+(\widetilde{v}, \xi)$, is the set consisting of the β component of every vector (β, ρ) that satisfies the following linear system

$$\begin{aligned} s_{ij}(\widetilde{v}) + \beta_{ij} &\geq \rho_i^k - \rho_j^k, \qquad\quad \forall k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ s_{ij}(\widetilde{v}) + \beta_{ij} &\leq \rho_i^k - \rho_j^k + \xi_{ij}^k, \forall k \in \mathcal{K}, (i,j) \in \mathcal{A} : \widetilde{x}_{ij}^k > 0, \\ \beta_{ij} &\geq 0 \qquad\qquad\qquad \forall (i,j) \in \mathcal{A}. \end{aligned}$$

Unlike ϵ (a constant) in $\mathcal{T}^+(\widetilde{v}, \epsilon)$, ξ is a nonnegative vector in the relaxed toll set $\Pi^+(\widetilde{v}, \xi)$. Below are two properties of this (disaggregate) toll set.

Theorem 6. *For any $\widetilde{v} \in V$, let $\beta_{mscp} = \nabla s(\widetilde{v})^T \widetilde{v}$ and, for all $k \in \mathcal{K}$ and $(i,j) \in \mathcal{A}$ such that $\widetilde{x}_{ij}^k > 0$, let $\widetilde{\xi}_{ij}^k = s_{ij}(\widetilde{v}) + [\beta_{mscp}]_{ij} - \widetilde{\rho}_i^k + \widetilde{\rho}_j^k$, where $\widetilde{\rho}$ is an optimal solution to the linear program in Theorem 4. If $\nabla s(\widetilde{v})$ is nonnegative, then $\Pi^+(\widetilde{v}, \widetilde{\xi}) \neq \emptyset$.*

Proof. Recall from Theorem 4 that

$$\epsilon_{\text{mscp}} = \min_{\rho}\{(s(\widetilde{v}) + \beta_{\text{mscp}})^T \widetilde{v} - \sum\nolimits_k b_k^T \rho^k : s(\widetilde{v}) + \beta_{\text{mscp}} \geq A^T \rho^k, \forall k\}.$$

For all $k \in \mathcal{K}$ and $(i,j) \in \mathcal{A}$ such that $\widetilde{x}_{ij}^k > 0$, let $\widetilde{\xi}_{ij}^k = s_{ij}(\widetilde{v}) + [\beta_{\text{mscp}}]_{ij} - \widetilde{\rho}_j^k + \widetilde{\rho}_i^k$, where $\widetilde{\rho}$ is an optimal solution to the above linear program. Moreover, its constraints also ensure that $\widetilde{\xi}_{ij}^k \geq 0$ and, when combined with the definition of $\widetilde{\xi}_{ij}^k$, the following must hold

$$\begin{aligned} s_{ij}(\widetilde{v}) + [\beta_{\text{mscp}}]_{ij} &\geq \widetilde{\rho}_i^k - \widetilde{\rho}_j^k, \qquad\quad \forall k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ s_{ij}(\widetilde{v}) + [\beta_{\text{mscp}}]_{ij} &= \widetilde{\rho}_i^k - \widetilde{\rho}_j^k + \widetilde{\xi}_{ij}^k, \forall k \in \mathcal{K}, (i,j) \in \mathcal{A} : \widetilde{x}_{ij}^k > 0. \end{aligned}$$

Then, the nonnegativity of $\nabla s(\widetilde{v})^T \widetilde{v}$ implies that $[\beta_{\text{mscp}}]_{ij} \geq 0$ for all $(i,j) \in \mathcal{A}$. Therefore, $\beta_{\text{mscp}} \in \Pi^+(\widetilde{v}, \widetilde{\xi})$ and $\Pi^+(\widetilde{v}, \widetilde{\xi}) \neq \emptyset$. ∎

Theorem 7. *If $\Pi^+(\widetilde{v}, \xi) \neq \emptyset$, then $\Pi^+(\widetilde{v}, \xi) \subseteq \mathcal{T}^+(\widetilde{v}, \epsilon)$, where $\epsilon = \sum_k \sum_{(i,j) \in \mathcal{A}} \widetilde{x}_{ij}^k \xi_{ij}^k$.*

Proof. Multiplying the second equation in the definition of $\Pi^+(\widetilde{v},\xi)$ by $\widetilde{x}_{ij}^k$ yields

$$(s_{ij}(\widetilde{v}) + \beta_{ij})\widetilde{x}_{ij}^k \le (\rho_i^k - \rho_j^k)\widetilde{x}_{ij}^k + \xi_{ij}^k \widetilde{x}_{ij}^k,\ \forall (i,j,k) : \widetilde{x}_{ij}^k > 0.$$

Then, summing the above equations together and recognizing that $A\widetilde{x}^k = b_k$ yield that $(s(\widetilde{v}) + \beta)^T \widetilde{v} \le \sum_{k\in\mathcal{K}} b_k^T \rho^k + \epsilon$, where $\epsilon = \sum_k \sum_{(i,j)\in\mathcal{A}} \widetilde{x}_{ij}^k \xi_{ij}^k$. Thus, $\beta \in \Pi^+(\widetilde{v},\xi)$ implies that $\beta \in \mathcal{T}^+(\widetilde{v},\epsilon)$, i.e., $\Pi^+(\widetilde{v},\xi) \subseteq \mathcal{T}^+(\widetilde{v},\epsilon)$. ■

Instead of choosing ξ as in Theorem 6, it is also possible to choose ξ that solves one of the following two problems:

$$\begin{array}{lll} \min\limits_{(\beta,\rho,\xi)} & \sum_k \sum_{(i,j):\widetilde{x}_{ij}^k>0} \xi_{ij}^k & \\ s.t. & s_{ij}(\widetilde{v}) + \beta_{ij} = \rho_i^k - \rho_j^k + \xi_{ij}^k, & \forall k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ & \xi_{ij}^k \ge 0, & \forall k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ & \beta_{ij} \ge 0 & \forall (i,j) \in \mathcal{A}, \end{array}$$

or

$$\begin{array}{lll} \min\limits_{(\beta,\rho,\xi,z)} & z & \\ s.t. & s_{ij}(\widetilde{v}) + \beta_{ij} = \rho_i^k - \rho_j^k + \xi_{ij}^k, & \forall k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ & \xi_{ij}^k \le z, & \forall k \in \mathcal{K}, (i,j) \in \mathcal{A} : \widetilde{x}_{ij}^k > 0, \\ & \xi_{ij}^k \ge 0 & \forall k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ & \beta_{ij} \ge 0 & \forall (i,j) \in \mathcal{A}. \end{array}$$

Both problems yield a ξ that makes $\Pi^+(\widetilde{v},\xi)$ nonempty.

5 Numerical Results

To illustrate the effectiveness of the relaxed toll sets, $\Pi^+(\widetilde{v},\xi)$ and $\mathcal{T}^+(\widetilde{v},\epsilon)$, we solved the MINSYS problem originally introduced in [Ber95] and [BHR97] and later referred to as the minimum toll revenue problem in [Dia99]. Using the (aggregate) relaxed toll set $\mathcal{T}^+(\widetilde{v},\epsilon)$, the (aggregate) minimum toll revenue (AMR) problem can be stated as follows:

$$\min\{\widetilde{v}^T\beta : \beta \in \mathcal{T}^+(\widetilde{v},\epsilon)\}.$$

The objective function in AMR is simply the sum of the product of the flow and the toll amount on each arc, i.e., the toll revenue. Using $\Pi^+(\widetilde{v},\xi)$ instead of $\mathcal{T}^+(\widetilde{v},\epsilon)$, the disaggregate minimum toll revenue (DMR) problem can be defined as follows:

$$\min\{\widetilde{v}^T\beta : \beta \in \Pi^+(\widetilde{v},\xi)\}.$$

Data for our experiments are from four transportation networks whose attributes are listed in Table 2. For each network, we used the restricted simplicial decomposition or RSD (see, Hearn et al. [HLV87]) to obtain a solution

Table 2. Network Attributes

Network	Links	Nodes	Commodities
Sioux Falls[LMP75]	76	24	528
Hull [FGS87]	798	501	138
Stockholm [HeR98]	962	416	1,623
Winnipeg [FGS87]	2836	1052	4,344

to SOPT with a relative optimality gap of 10^{-4}. Because they are readily available from RSD, we set $\epsilon = \epsilon_{\text{mscp}}$ and $\xi_{ij}^k = s_{ij}(\widetilde{v}) + [\beta_{\text{mscp}}]_{ij} - \rho_i^k + \rho_j^k$ in AMR and DMR, respectively. Both problems, AMR and DMR, were implemented in GAMS [GAM80] and solved using CPLEX 8.1 [CPL96].

Table 3 reports the results for the four networks. For Sioux Falls, the SOPT solution from RSD provides a consistent toll set and ϵ can be set to zero. The same does not hold for the remaining three networks. The values of their ϵ_{mscp} are listed in the table along with the ratio $\epsilon_{\text{mscp}}/s(\widetilde{v})^T\widetilde{v}$ to provide the magnitude of ϵ_{mscp} relative to the total travel delay at the approximate SOPT solution. The last two sets of columns compare the tolled user equilibrium solutions, $v^*(\beta)$, using toll vectors from AMR and DMR against the approximate SOPT solution, $\widetilde{v}$, from RSD. The two columns under the heading "Total Delay Error" reports $(s(v^*(\beta))^T v^*(\beta) - s(\widetilde{v})^T\widetilde{v})/(s(\widetilde{v})^T\widetilde{v})$, i.e., the error in travel delay relative to the delay at the approximate system solution, $\widetilde{v}$. The remaining two columns (under the heading "Link Flow Error") reports the percentage of arcs with relatively large link flow errors. In calculating this percentage, we only consider arcs with a moderately large amount of flow, i.e., we consider arcs in the set $\mathcal{A}' = \{a | v_a^*(\beta) \geq 0.25C_a \text{ or } \widetilde{v}_a \geq 0.25C_a\}$, where C_a is the capacity of arc a. Then, the link flow error is the percentage of arcs in $\mathcal{A}'$ such that $\frac{|v_a^*(\beta) - \widetilde{v}_a|}{\widetilde{v}_a} > 0.10$. Observe that results in the last two columns indicate that the relaxation based on the marginal social costs produces good toll vectors for they yield tolled user equilibrium solutions that are approxi-

Table 3. Numerical Results

			Total Delay Error		Link Flow Error	
Networks	ϵ_{mscp}	$\frac{\epsilon_{\text{mscp}}}{s(\widetilde{v})^T\widetilde{v}}$	(AMR)	(DMR)	(AMR)	(DMR)
Sioux Falls	0	0	0%	0%	0%	0%
Hull	4.85	9.59E-5	0.07%	0.07%	2.6%	1.3%
Stockholm	1,134.12	9.74E-5	0.06%	0.01%	0%	0%
Winnipeg	107.82	9.22E-5	0.05%	0.04%	0.1%	0.3%

mately optimal to SOPT. However, DMR on average yields tolls with slightly less error.

6 Conclusions

Congestion or toll pricing problems in [HeR98] require a solution to the system problem (the traffic assignment problem that minimizes the total travel delay) to define a toll set, i.e., a set of all valid tolls. Instead of an exact solution, it is more practical to obtain an approximate solution to the system problem for large networks. In this paper, we provide necessary and sufficient conditions to determine whether the toll set constructed from an approximate solution is empty. When it is so, we derive alternative toll sets based on relaxed optimality conditions. With carefully chosen parameters, tolls from the relaxed toll sets possess the desirable property, i.e., they induce travellers to choose routes that are nearly system optimal. Numerical solutions from four transportation networks in the literature also verify empirically the previous statement.

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Appendix

Relaxed Toll Sets for the Elastic Demand Case

This appendix describes the results concerning the toll sets when demands are elastic. Many results for the fixed demand case naturally extend to the case with elastic demands. The presentation below follows the same outline as in the main part of the paper.

A Elastic Demand System and User Problems

To state the traffic assignment problems with elastic demands, let t_k and $w_k(t_k)$ denote the travel demand and the inverse demand function for commodity k, respectively. For each k, E_k is a vector in $R^{|\mathcal{N}|}$ with exactly two nonzero elements, one equals 1 at the origin node and the other equals -1 at the destination node. Then, the set of feasible flow-demand vectors is

$$V_{\text{ED}} = \{(v,t)|v = \sum_k x^k, Ax^k = E_k t_k, x^k \geq 0, t_k \geq 0\}.$$

Without loss of generality, we assume V_{ED} is bounded, thus, compact. (See, e.g., [FlH95].)

Among several alternatives (see, e.g., [Gar80], [YaB97] and [ZhG97]), one system problem with elastic demands maximizes the net user benefit, i.e., the difference between the user benefit as measured by $\sum_{k\in\mathcal{K}} \int_0^{t_k} w_k(z)dz$ and the total delay (or cost) $s(v)^T v$. In its minimization form, this system problem can be written as

$$(\overline{v},\overline{t}) = \text{argmin}\{s(v)^T v - \sum_{k\in\mathcal{K}} \int_0^{t_k} w_k(z)dz : (v,t) \in V_{\text{ED}}.\}$$

As in the fixed demand case, the corresponding user problem with elastic demand is a variational inequality. In particular, (v^*, t^*) is a solution to the user equilibrium problem if the pair satisfies the following:

$$s(v^*)^T(v - v^*) - w(t^*)^T(t - t^*) \geq 0, \quad \forall\ (v,t) \in V_{\text{ED}}.$$

For a given toll vector β, $(v^*(\beta), t^*(\beta))$ is a solution to the tolled user equilibrium problem if the pair satisfies the following:

$$(s(v^*(\beta)) + \beta)^T(v - v^*(\beta)) - w(t^*(\beta))^T(t - t^*(\beta)) \geq 0, \quad \forall\ (v,t) \in V_{\text{ED}}.$$

As in the fixed demand case, we assume throughout this appendix that the system, user, and tolled user equilibrium problems have unique solutions.

B System and Non-system Toll Sets

Analogous to the fixed demand case, the system toll set when demands are elastic is $\mathcal{T}(\overline{v},\overline{t}) = \{\beta | v^*(\beta) = \overline{v}, t^*(\beta) = \overline{t}\}$. Under the uniqueness assumptions stated earlier, Hearn and Yildirim [HeY01] prove that $\mathcal{T}(\overline{v},\overline{t})$ consists of the β component of every pair (β, ρ) that satisfies the following system:

$$\begin{aligned} s(\overline{v}) + \beta &\geq A^T \rho^k, \quad \forall\, k \in \mathcal{K}, \\ w_k(\overline{t}_k) &\leq E_k^T \rho^k, \quad \forall\, k \in \mathcal{K}, \\ (s(\overline{v}) + \beta)^T \overline{v} &= w(\overline{t})^T \overline{t}. \end{aligned}$$

In [HeY01], Hearn and Yildirim show that both $\mathcal{T}(\overline{v},\overline{t})$ and $\mathcal{T}^+(\overline{v},\overline{t})$ are nonempty. The latter assumes that $\bigtriangledown s(\overline{v})$ is nonnegative.

Let $(\widetilde{v},\widetilde{t})$ denote a flow-demand vector feasible to V_{ED}. Then, the non-system toll set is $\mathcal{T}(\widetilde{v},\widetilde{t}) = \{\beta | v^*(\beta) = \widetilde{v}, t^*(\beta) = \widetilde{t}\}$ and, using an argument similar to the one in Theorem 2, the following holds.

Theorem 8. *The toll set $\mathcal{T}(\widetilde{v},\widetilde{t})$, where $(\widetilde{v},\widetilde{t}) \in V_{ED}$, is the set consisting of the β component of every pair (β, ρ) that satisfies the following linear system:*

$$\begin{aligned} s(\widetilde{v}) + \beta &\geq A^T \rho^k, \quad \forall\, k \in \mathcal{K}, \\ w_k(\widetilde{t}_k) &\leq E_k^T \rho^k, \quad \forall\, k \in \mathcal{K}, \\ (s(\widetilde{v}) + \beta)^T \widetilde{v} &= w(\widetilde{t})^T \widetilde{t}. \end{aligned}$$

The theorem below shows that the non-system toll set is nonempty for any non-trivial $(\widetilde{v},\widetilde{t}) \in V_{\mathrm{ED}}$.

Theorem 9. *For any $(\widetilde{v},\widetilde{t}) \in V_{ED}$ such that $\widetilde{v} \neq 0$, $\widetilde{\beta} = \nabla s(\widetilde{v})^T \widetilde{v} - \alpha \widetilde{v} \in \mathcal{T}(\widetilde{v},\widetilde{t})$ when $\alpha = \left[\left(s(\widetilde{v}) + \nabla s(\widetilde{v})^T \widetilde{v}\right)^T \widetilde{v} - w(\widetilde{t})^T \widetilde{t}\right] / \widetilde{v}^T \widetilde{v}$.*

Proof. Consider the following direction finding problem associated with VI$(s(v) + \widetilde{\beta}, V_{\mathrm{ED}})$ at $(\widetilde{v},\widetilde{t})$:

$$\begin{aligned} \text{DIR-ED}(\widetilde{\beta}) : \min\ & \left(s(\widetilde{v}) + \widetilde{\beta}\right)^T \sum_{k \in K} x^k - w(\widetilde{t})^T d \\ s.t\ \ & Ax^k - E_k d_k = 0, && \forall k, \\ & x^k \geq 0, && \forall k, \\ & d_k \geq 0, && \forall k. \end{aligned}$$

The dual of DIR-ED$(\widetilde{\beta})$ is

$$\begin{aligned} & \max 0 \\ s.t.\ \ & A^T \rho^k \leq s(\widetilde{v}) + \widetilde{\beta}, \forall k, \\ & E_k^T \rho^k \geq w_k(\widetilde{t}_k), \quad \forall k, \\ & \rho^k \text{ unrestricted}, \quad \forall k. \end{aligned}$$

The relationships between the primal and dual problems in linear programming imply that the objective value of the direction finding problem is bounded below by zero. Thus, $(u, d) = (0, 0)$, where $u = \sum_k x^k$, is an optimal solution because its objective value equals the lower bound. Furthermore, the dual of DIR-ED has a feasible solution, say $\widetilde{\rho}$. Then the pair $(\widetilde{\beta}, \widetilde{\rho})$ satisfies the linear system in Theorem 1. The first two conditions of the linear system in Theorem 8 follow from the first two constraints of the dual problem and our choice of $\widetilde{\beta}$ ensures that the following holds

$$\left(s(\widetilde{v}) + \widetilde{\beta}\right)^T \widetilde{v} = \left(s(\widetilde{v}) + \nabla s(\widetilde{v})^T \widetilde{v}\right)^T \widetilde{v} - \alpha \widetilde{v}^T \widetilde{v} = w(\widetilde{t})^T \widetilde{t}.$$

Thus, the last condition in the linear system is also satisfied and $\widetilde{\beta} \in \mathcal{T}(\widetilde{v}, \widetilde{t})$. ■

As defined above, α is zero and $\widetilde{\beta} = \nabla s(\widetilde{v})^T \widetilde{v}$, when $(\widetilde{v}, \widetilde{t})$ solves the system problem. Moreover, other choices for α and $\widetilde{\beta}$ exist. For example, $\widetilde{\beta} = \nabla s(\widetilde{v})^T \widetilde{v} - \alpha s(\widetilde{v})$, where $\alpha = [\left(s(\widetilde{v}) + \nabla s(\widetilde{v})^T \widetilde{v}\right)^T \widetilde{v} - w(\widetilde{t})^T \widetilde{t}] / s(\widetilde{v})^T \widetilde{v}$, is also valid when $s(\widetilde{v})^T \widetilde{v} \neq 0$.

The following theorem provides a necessary and sufficient condition under which the nonnegative and non-system toll set is nonempty. The proof is omitted because it is similar to that of Theorem 3 in the main part of the paper.

Theorem 10. *For any $(\widetilde{v}, \widetilde{t}) \in V_{ED}$, $\mathcal{T}^+(\widetilde{v}, \widetilde{t})$ is nonempty if and only if $(\widetilde{v}, \widetilde{t})$ solves VI[$(s(v), -w(t)), \mathcal{V}_{ED}$], where $\mathcal{V}_{ED} = \{(v, t) | v = \sum_k x^k, Ax^k = E_k t_k, x^k \geq 0, t_k \geq 0, v \leq \widetilde{v}\}$.*

C Relaxed Toll Set

In this and the following sections, we focus on the relaxations of the (unrestricted) non-system toll set. However, similar results also hold for the non-system toll set requiring tolls to be nonnegative.

For a given $\epsilon > 0$, the relaxed toll set $\mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon)$ is the set consisting of the β component of the pair (β, ρ) that satisfies the following:

$$\begin{aligned} s(\widetilde{v}) + \beta &\geq A^T \rho^k, && \forall k \in \mathcal{K}, \\ w_k(\widetilde{t}_k) &\leq E_k^T \rho^k, && \forall k \in \mathcal{K}, \\ (s(\widetilde{v}) + \beta)^T \widetilde{v} &\leq w(\widetilde{t})^T \widetilde{t} + \epsilon. \end{aligned}$$

Then, the following results are analogous to those in Section 3.

Theorem 11. *For any $(\widetilde{v}, \widetilde{t}) \in V_{ED}$, let $\epsilon_{mscp} = (s(\widetilde{v}) + \nabla s(\widetilde{v})^T \widetilde{v})^T \widetilde{v} - w(\widetilde{t})^T \widetilde{t}$. Then, $\mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon_{mscp}) \neq \emptyset$.*

Proof. From the discussion in Section B, the optimal objective value of DIR-ED (β_{mscp}) is zero, where $\beta_{\text{mscp}} = \nabla s(\widetilde{v})^T \widetilde{v}$ as before. Thus ϵ_{mscp} can be equivalently expressed as follows:

$$\begin{aligned}\epsilon_{\text{mscp}} = &(s(\widetilde{v}) + \beta_{\text{mscp}})^T \widetilde{v} - w(\widetilde{t})^T \widetilde{t} \\ &- \min\{(s(\widetilde{v}) + \beta_{\text{mscp}})^T u - w(\widetilde{t})^T t : (u, t) \in V_{\text{ED}}\}.\end{aligned}$$

As in Theorem 4 in the main part of the paper, the vector β_{mscp} and an optimal solution, $\widetilde{\rho}$, to the dual of DIR-ED(β_{mscp}) form a pair of (β, ρ) that belongs to $\mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon_{\text{mscp}})$. ■

Theorem 12. *Let $s(v)$ and $w(t)$ be strongly monotone with modula α and γ, respectively. For any $\eta > 0$, there exists a $\delta > 0$ such that $\|(v^*(\beta) - \overline{v}, t^*(\beta) - \overline{t})\|_2 \leq \eta$ whenever $\beta \in \mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon_{mscp})$ and $\|(\widetilde{v} - \overline{v}, \widetilde{t} - \overline{t})\| \leq \delta$.*

Proof. Because both $s(v)$ and $w(t)$ are strong monotone, $(s(v), -w(t))$ is also strongly monotone with modulus $\min\{\alpha, \gamma\}$. The rest of the proof requires lemmas and uses an argument similar to the one in Theorem 5. ■

The following linear program also provides an ϵ for which $\mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon)$ is nonempty.

$$\begin{aligned}\epsilon^* = \min_{(\beta,\rho)} \ &(s(\widetilde{v}) + \beta)^T \widetilde{v} - w(\widetilde{t})^T \widetilde{t} \\ s.t. \ &s(\widetilde{v}) + \beta \geq A^T \rho^k, \quad \forall k \in \mathcal{K}, \\ &w_k(\widetilde{t}_k) \leq E_k^T \rho^k, \quad \forall k \in \mathcal{K}.\end{aligned}$$

D Disaggregate Representation of Relaxed Toll Sets

Let $(\widetilde{v}, \widetilde{t})$ be an approximate system solution. For a given pair of (ξ, μ) such that $\xi, \mu \geq 0$, the following are three possible disaggregate representations of a relaxed toll set, all of which are analogous to the one presented in Section 4.

1. $\Pi^1(\widetilde{v}, \widetilde{t}, \xi, \mu) =$ the set of the β component of every pair (β, ρ) that satisfies the following:

$$\begin{aligned} &s_{ij}(\widetilde{v}) + \beta_{ij} \geq \rho_i^k - \rho_j^k, && \forall\ k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ &s_{ij}(\widetilde{v}) + \beta_{ij} \leq \rho_i^k - \rho_j^k + \xi_{ij}^k, && \forall\ k \in \mathcal{K}, (i,j) \in \mathcal{A} : \widetilde{x}_{ij}^k > 0, \\ &w_k(\widetilde{t}_k) \leq E_k^T \rho^k, && \forall\ k \in \mathcal{K}, \\ &w_k(\widetilde{t}_k) \geq E_k^T \rho^k - \mu_k, && \forall\ k \in \mathcal{K} : \widetilde{t}_k > 0. \end{aligned}$$

2. $\Pi^2(\widetilde{v}, \widetilde{t}, \xi) =$ the set of the β component of every pair (β, ρ) that satisfies the following:

$$\begin{aligned} &s_{ij}(\widetilde{v}) + \beta_{ij} \geq \rho_i^k - \rho_j^k, && \forall\ k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ &s_{ij}(\widetilde{v}) + \beta_{ij} \leq \rho_i^k - \rho_j^k + \xi_{ij}^k, && \forall\ k \in \mathcal{K}, (i,j) \in \mathcal{A} : \widetilde{x}_{ij}^k > 0, \\ &w_k(\widetilde{t}_k) \leq E_k^T \rho^k, && \forall\ k \in \mathcal{K}. \end{aligned}$$

3. $\Pi^3(\widetilde{v}, \widetilde{t}, \mu)$ = the set of the β component of every pair (β, ρ) that satisfies the following:

$$\begin{array}{ll} s_{ij}(\widetilde{v}) + \beta_{ij} \geq \rho_i^k - \rho_j^k, & \forall\ k \in \mathcal{K}, (i,j) \in \mathcal{A}, \\ w_k(\widetilde{t}_k) \leq E_k^T \rho^k, & \forall\ k \in \mathcal{K}, \\ w_k(\widetilde{t}_k) \geq E_k^T \rho^k - \mu_k, & \forall\ k \in \mathcal{K} : \widetilde{t}_k > 0. \end{array}$$

Because the last two sets contain subsets of the constraints appeared in the first, $\Pi^2(\widetilde{v}, \widetilde{t}, \xi)$ and $\Pi^3(\widetilde{v}, \widetilde{t}, \mu)$ are relaxations of $\Pi^1(\widetilde{v}, \widetilde{t}, \xi, \mu)$. Thus, $\Pi^1(\widetilde{v}, \widetilde{t}, \xi, \mu)$ must be a subset of both $\Pi^2(\widetilde{v}, \widetilde{t}, \xi)$ and $\Pi^3(\widetilde{v}, \widetilde{t}, \mu)$.

The following theorem shows that $\Pi^1(\widetilde{v}, \widetilde{t}, \xi, \mu)$ is nonempty. This in turn implies that both $\Pi^2(\widetilde{v}, \widetilde{t}, \xi)$ and $\Pi^3(\widetilde{v}, \widetilde{t}, \mu)$ are also nonempty.

Theorem 13. *For any* $(\widetilde{v}, \widetilde{t}) \in V_{ED}$, *let*

1. $\widetilde{\xi}_{ij}^k = s_{ij}(\widetilde{v}) + [\beta_{mscp}]_{ij} - \widetilde{\rho}_i^k + \widetilde{\rho}_j^k$, *for all* k *and arc* (i,j) *such that* $\widetilde{x}_{ij}^k > 0$, *and*
2. $\widetilde{\mu}_k = E_k^T \widetilde{\rho}^k - w_k(\widetilde{t}_k)$, *for all* k *such that* $\widetilde{t}_k > 0$,

where $\widetilde{\rho}$ *is an optimal solution to the dual of GAP-ED*(β_{mscp}). *Then,* $\Pi^1(\widetilde{v}, \widetilde{t}, \widetilde{\xi}, \widetilde{\mu}) \neq \emptyset$.

Proof. Because $\widetilde{\rho}$ solves the dual of GAP-ED(β_{mscp}), it satisfies

$$\begin{array}{ll} s(\widetilde{v}) + \beta_{\text{mscp}} \geq A^T \widetilde{\rho}^k, & \forall k \in \mathcal{K}, \\ w_k(\widetilde{t}_k) \leq E_k^T \widetilde{\rho}^k, & \forall k \in \mathcal{K}. \end{array}$$

The above implies that both $\widetilde{\xi}_{ij}^k$ and $\widetilde{\mu}_k$ defined above are nonegative and satisfy the following:

$$\begin{array}{lll} s_{ij}(\widetilde{v}) + [\beta_{\text{mscp}}]_{ij} & \geq \widetilde{\rho}_i^k - \widetilde{\rho}_j^k, & \forall k, (i,j) \in \mathcal{A}, \\ s_{ij}(\widetilde{v}) + [\beta_{\text{mscp}}]_{ij} & = \widetilde{\rho}_i^k - \widetilde{\rho}_j^k + \widetilde{\xi}_{ij}^k, & \forall k, (i,j) : \widetilde{x}_{ij}^k > 0, \\ w_k(\widetilde{t}_k) & \leq E_k^T \widetilde{\rho}^k, & \forall k, \\ w_k(\widetilde{t}_k) & = E_k^T \widetilde{\rho}^k - \widetilde{\mu}_k, & \forall k : \widetilde{t}_k > 0. \end{array}$$

Thus, $(\beta_{\text{mscp}}, \widetilde{\xi}_{ij}^k, \widetilde{\mu}^k)$ satisfies the conditions defining $\Pi^1(\widetilde{x}, \widetilde{t}, \widetilde{\xi}, \widetilde{\mu})$, i.e., $\Pi^1(\widetilde{v}, \widetilde{t}, \widetilde{\xi}, \widetilde{\mu}) \neq \emptyset$. ∎

Corollary 2. *For any* $(\widetilde{v}, \widetilde{t}) \in V_{ED}$, *let* $\widetilde{\xi}_{ij}^k = s_{ij}(\widetilde{v}) + [\beta_{mscp}]_{ij} - \widetilde{\rho}_i^k + \widetilde{\rho}_j^k$, *where* $\widetilde{\rho}$ *is an optimal solution to the dual of GAP-ED*(β_{mscp}). *Then,* $\Pi^2(\widetilde{v}, \widetilde{t}, \widetilde{\xi}) \neq \emptyset$.

Corollary 3. *For any* $(\widetilde{v}, \widetilde{t}) \in V_{ED}$, *let* $\widetilde{\mu}_k = E_k^T \widetilde{\rho}^k - w_k(\widetilde{t}_k)$, *where* $\widetilde{\rho}$ *is an optimal solution to the dual of GAP-ED*(β_{mscp}). *Then,* $\Pi^3(\widetilde{v}, \widetilde{t}, \widetilde{\mu}) \neq \emptyset$.

Furthermore, if $\Pi^1(\widetilde{v}, \widetilde{t}, \xi, \mu)$ is nonempty, then multiplying the second and fourth conditions in the relaxed toll set by $\widetilde{x}_{ij}^k$ and $\widetilde{t}_k$ yields the following:

$$
\begin{aligned}
(s_{ij}(\widetilde{v}) + \beta_{ij})\widetilde{x}_{ij}^k &\le (\rho_i^k - \rho_j^k)\widetilde{x}_{ij}^k + \xi_{ij}^k \widetilde{x}_{ij}^k, \ \forall\, k \in \mathcal{K}, (i,j) : \widetilde{x}_{ij}^k > 0, \\
(E_k^T \rho^k)\widetilde{t}_k &\le w_k(\widetilde{t}_k)\widetilde{t}_k + \mu_k \widetilde{t}_k, \qquad \forall\, k : \widetilde{t}^k > 0.
\end{aligned}
$$

Because $\widetilde{v} = \sum_k \widetilde{x}^k$ and $A\widetilde{x}^k = E_k\widetilde{t}_k$, the above equations imply that $(s(\widetilde{v}) + \beta)^T\widetilde{v} \le w(\widetilde{t})^T\widetilde{t} + \epsilon$, where $\epsilon = \sum_k \sum_{(i,j)\in\mathcal{A}} \widetilde{x}_{ij}^k \xi_{ij}^k + \sum_k \widetilde{t}_k \mu_k$. Thus, if $\beta \in \Pi^1(\widetilde{v}, \widetilde{t}, \xi, \mu)$, β must be in $\mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon)$ as well, i.e., $\Pi^1(\widetilde{v}, \widetilde{t}, \xi, \mu) \subseteq \mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon)$. Similarly, $\Pi^2(\widetilde{v}, \widetilde{t}, \xi) \subseteq \mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon)$ and $\Pi^3(\widetilde{v}, \widetilde{t}, \mu) \subseteq \mathcal{T}(\widetilde{v}, \widetilde{t}, \epsilon)$ when ϵ is chosen in a similar manner.

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