

Introduction

This book comes from the Internet. Browsing the Web, I stumbled on philosophers, cognitive scientists, sociologists, computer scientists, even mathematicians!—saying original, provocative things about mathematics. And many of these people had probably never heard of each other! So I have collected them here. This way, they can read each other's work. I also bring back a few provocative oldies that deserve publicity.

The authors are philosophers, mathematicians, a cognitive scientist, an anthropologist, a computer scientist, and a couple of sociologists. (Among the mathematicians are two Fields Prize winners and two Steele Prize winners.) None are historians, I regret to say, but there are two historically oriented articles. These essays don't share any common program or ideology. The standard for admission was: Nothing boring! Nothing trite, nothing trivial! Every essay is challenging, thought-provoking, and original.

Back in the 1970s when I started writing *about* mathematics (instead of just doing mathematics), I had to complain about the literature. Philosophy of science was already well into its modern revival (largely stimulated by the book of Thomas Kuhn). But philosophy of mathematics still seemed to be mostly foundationist ping-pong, in the ancient style of Rudolf Carnap or Willard Van Ormond Quine. The great exception was *Proofs and Refutations* by Imre Lakatos. But that exciting book was still virtually unknown and unread, by either mathematicians or philosophers. (I wrote an article entitled "Introducing Imre Lakatos" in the *Mathematical Intelligencer* in 1978.)

Since then, what a change! In the last few years newcomers—linguists, neuroscientists, cognitive scientists, computer scientists, sociologists—are bringing new ideas, studying mathematics with new tools. (George Lakoff-Rafael Núñez, Stanislas Dehaene, Brian Butterworth, Keith Devlin).

In previous centuries, old questions—"What Is Man?" "What Is Mind?" "What Is Language?"—were transformed from philosophical questions, free for speculation, into scientific problems. The subjects of linguistics, psychology and anthropology detached from philosophy to become autonomous disciplines. Maybe the question, "What is mathematics?" is coming into recognition as a scientific problem.

In 1981, in *The Mathematical Experience*, speaking about the prevailing alternative views of the nature of mathematics, Phil Davis and I asked, “Do we really have to choose between a formalism that is falsified by our everyday experience, and a Platonism that postulates a mythical fairyland where the uncountable and the inaccessible lie waiting to be observed by the mathematician whom God blessed with a good enough intuition? It is reasonable to propose a different task for mathematical philosophy, not to seek indubitable truth, but to give an account of mathematical knowledge as it really is—fallible, corrigible, tentative, and evolving, as is every other kind of human knowledge. Instead of continuing to look in vain for foundations, or feeling disoriented and illegitimate for lack of foundations, we have tried to look at what mathematics really is, and account for it as a part of human knowledge in general. We have tried to reflect honestly on what we do when we use, teach, invent, or discover mathematics.” (p. 406)

Before long, the historian Michael Crowe said these words were “a programme that I find extremely attractive.” In 1986–1987 Crowe visited Donald Gillies at King’s College in London. Gillies had been a student of Imre Lakatos. In 1992 Gillies published an anthology, *Revolutions in Mathematics*, where historians of mathematics like Crowe collaborated with philosophers of mathematics like Gillies.

Such collaboration developed further in Emily Grosholz and Herbert Breger’s anthology, *The Growth of Mathematical Knowledge* (Kluwer, 2000). Emily Grosholz wrote, “during the last decade, a growing number of younger philosophers of mathematics have turned their attention to the history of mathematics and tried to make use of it in their investigations. The most exciting of these concern how mathematical discovery takes place, how new discoveries are structured and integrated into existing knowledge, and what light these processes shed on the existence and applicability of mathematical objects.” She mentions books edited by Philip Kitcher and William Aspray, by Krueger, and by Javier Echeverria. She finds the Gillies volume “perhaps the most satisfactory synthesis.”

History of mathematics is today a lively and thriving enterprise. It is tempting to start listing my favorite historians, but I will limit myself to singling out the monumental work by Sanford L. Segal, *Mathematicians Under the Nazis* (Princeton, 2003).

Already in my 1979 article “Some Proposals for Reviving the Philosophy of Mathematics,” (reprinted in Thomas Tymoczko’s anthology, *New Directions in the Philosophy of Mathematics*, Birkhauser, 1986), and at greater length in my two subsequent books, I explained that, contrary to fictionalism, mathematical objects do exist—*really*! But, contrary to Platonism, their existence is not transcendental, or independent of humanity. It is created by human activity, and is part of human culture. I cited the 1947 essay by the famous anthropologist Leslie White, which is reprinted here. And at last, in 2003 and 2004, a few philosophers are also recognizing that mathematical objects are real and are our creations. (Jessica Carter, “Ontology and Math-

ematical Practice,” *Philosophia Mathematica* 12 (3), October 2004; M. Panza, “Mathematical Proofs,” *Synthese*, 134, 2003; M. Muntersbjorn, “Representational innovation and mathematical ontology,” *Synthese*, 134, 2003). I know of recent conferences in Mexico, Belgium, Denmark, Italy, Spain, Switzerland, and Hungary on philosophical issues of mathematical practice.

While others are starting to pay more attention to our ways, we mathematicians ourselves are having to look more deeply at what we are doing. A Special Interest Group on philosophy is now active in the Mathematical Association of America. A famous proposal by Arthur Jaffe and Frank Quinn in 1993 in the *Bulletin of the American Mathematical Society*, to accept not-so-rigorous mathematics by labeling it as such, provoked a flood of controversy. (The essay by William Thurston in this volume was his contribution to that controversy.)

After the rest of this book had gone to the editor at Springer, I found an article on the Web by Jonathan M. Borwein, the leader of the Centre for Experimental and Constructive Mathematics at Simon Fraser University in Vancouver. He quoted approvingly this five-point manifesto of mine:

- “1. *Mathematics is human.* It is part of and fits into human culture. It does not match Frege’s concept of an abstract, timeless, tenseless, objective reality.
2. *Mathematical knowledge is fallible.* As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The “fallibilism” of mathematics is brilliantly argued in Lakatos’ *Proofs and Refutations*.
3. *There are different versions of proof or rigor.* Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977, is just one example of an emerging nontraditional standard of rigor.
4. *Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics.* Aristotelian logic isn’t necessarily always the best way of deciding.
5. *Mathematical objects are a special variety of a social-cultural–historical object.* Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like *Moby Dick* in literature, or the Immaculate Conception in religion.”

(R. Hersh, “Fresh Breezes in the Philosophy of Mathematics,” *American Mathematical Monthly*, August–September 1995, 589–594; quoted by Jonathan Borwein in “The Experimental Mathematician: The Pleasure of Discovery and the Role of Proof,” prepared for the *International Journal of Computers for Mathematical Learning*, July 2004.)

As more and more important proofs approach and go beyond the limits of conventional verification, mathematicians are having to face honestly the

embarrassing ambiguity and temporal dependence of our central sacred icon—rigorous proof.

In his 1986 anthology Thomas Tymoczko called attention to the troublesome philosophical issues raised by the recent proof of the famous four-color theorem. This was the first time the solution of a major mathematical problem had relied essentially on machine computation.

Today, the status of several other famous problems raises even more prominent and severe difficulties. The story of Thomas Hales' "99% accepted" proof of the Kepler conjecture makes clear that something new and strange is happening in the very center of the mathematical enterprise (see George Szpiro, *Kepler's Conjecture*, Wiley, 2003). (Then there is also the on-going decades-long ups-and-downs, the many thousands of pages proof, of the classification of simple finite groups. See Ron Solomon, "On Finite Simple Groups and their Classification," *Notices of the AMS* 42 (2), February 1995, 231-239.)

Johannes Kepler in 1611 considered how spherical balls can be packed to fill space as densely as possible. There are three natural ways to pack spheres, and it's clear which of the three is best. Kepler guessed that this way is in fact the best possible. It turns out that this is fiendishly hard to prove. Wu-Yi Hsiang of the University of California, Berkeley, claimed to have a proof in 1993, but he failed to convince his colleagues and competitors. He has not relinquished his claim and continues to hold to it. Thomas Hales of the University of Michigan announced a proof by a different method in 1997. His proof follows suggestions made earlier by Laszlo Fejes-Toth, and it involves, like the famous computer proof of the four-color theorem, computer checking of thousands of separate cases, many of them individually very laborious. The *Annals of Mathematics* invited Hales to submit his manuscript. It is 250 pages long. A committee of 12 experts was appointed to referee the paper, coordinated by Gabor Fejes-Toth, Laszlo's son. After four years, the committee announced that they had found no errors, but still could not certify the correctness. They simply ran out of energy and gave up. Robert Macpherson, the editor of the *Annals*, wrote, "The news from the referees is bad, from my perspective. They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem. This is not what I had hoped for." He reluctantly acceded to their decision, and accepted the theoretical part of Hales' paper, leaving the computer part for publication elsewhere. Hales then announced that he was affiliating with a group of computer scientists known as the QED Project. This dormant project had as its original stated goal: to computerize all of mathematics! Hales' new project, the Flyspeck Project, proposes to do a computer coding and verification of his proof of the Kepler conjecture—a proof which, in ordinary mathematical form, was already too long and complicated to be completely checked, in four years, by a committee of 12 leading human experts. Project Flyspeck is expected to involve the work of hundreds of people and take 20 years. I do not know anyone who thinks either that this

project can be completed, or that even if claimed to be complete it would be universally accepted as a convincing proof of Kepler's conjecture. Donald Mackenzie's article in this volume sheds some light on these issues.

Such a story undermines our faith that mathematical proof will remain as we have always thought of it—that after reasonable time and effort, its correctness must be definitely decidable by unanimous consensus of competent specialists.

In fact, even without regard to Hales' theorem, it is easy to see that in principle there must be an upper bound on the length and complexity of the longest proof that at any time can be completely checked and verified by the mathematical community. In principle it is possible for a recognized, established mathematician to submit a proof longer than this upper bound. What should be the status of such a proof? Should it be accepted for publication? What degree of conviction or credibility should we attribute to its conclusion? Should it depend on our estimation of the reliability of its creator? May we use it as a building block in our own research? What if it has "applications" in physics? Such judgments are made every day in the "real world" of ordinary life. But in mathematics????!!

As I explained in my book, *What Is Mathematics, Really?* (1997, Oxford) the words "mathematical proof" have two different meanings—and the difference is not usually acknowledged. One meaning, found in logic texts and philosophy journals, is "a sequence of formalized statements, starting with unproved statements about undefined terms, and proceeding by steps permitted in first-order predicate calculus." The other meaning, not found in a precise or formal statement anywhere, is "an argument accepted as conclusive by the present-day mathematical community." The problem is to clarify and understand—not justify!!—the second meaning. A first stab at clarification might be, "an argument accepted as conclusive by the highest levels of authority in the present-day mathematical community." Such a clarification rests on several implicit hypotheses:

- (1) that there *is* a "mathematical community."
- (2) that this "community" has accepted "high levels of authority."
- (3) that these "high authorities" have a legitimacy based on some generally accepted rationale.
- (4) that the highest level of authority can agree on what to accept.
- (5) that arguments accepted as proofs by the recognized highest levels of authority in the mathematical community will remain accepted, at least for a very long time, at least with very high probability.

What seems to be threatening is that increasing length and complexity of proposed proofs, whether involving heavy use of computers or not, may go beyond the capacity of recognized authorities to reach a convincingly informed consensus.

The first of our eighteen articles is Alfréd Rényi's "A Socratic Dialogue on Mathematics" (*Dialogues on Mathematics*, Holden-Day, 1967.) Rényi was a

famous probabilist and number theorist, co-creator with Pal Erdos of the subject random graphs, and for many years director of the Institute of Mathematics in Budapest. This is a most inviting, charming and thought-provoking tour de force and jeu d'esprit. It poses the basic problem, and answers it in a way that invites further questioning and deeper development. I have heard that Prof. Rényi used to give live performances of this work, assisted by his daughter Zsuzsanna, to whom he dedicated the book from which this excerpt is taken.

The next article, by the logician-philosopher Carlo Cellucci of Rome, is the introductory chapter to his book, *Filosofia e matematica*. He simply lists 13 standard assumptions about mathematics (what he calls "the dominant view") and demolishes all of them. A most impressive and stimulating performance. I eagerly await the translation of his whole book into English.

William Thurston's friendly, down-to-earth article, "On Proof and Progress in Mathematics," provides a rare, invaluable glimpse for outsiders at some aspects of mathematical creation at the highest level. Its frank, unpretentious look at what really is done, what really happens at that level is told in a style and language accessible to anyone. It was published in the *Bulletin of the AMS*, one of the responses to the Jaffe-Quinn proposal mentioned above.

The U.S.-based English philosopher Andrew Aberdein's article, "The Informal Logic of Mathematical Proof," draws on "informal logic," a subject that was revived by Stephen Toulmin. "Informal logic" is closely allied to "rhetoric". An old article by Phil Davis and myself called "Rhetoric and Mathematics" may be relevant to Aberdein's article. (It appeared in *The Rhetoric of the Human Sciences*, edited by John S. Nelson, Allan Megill and Donald N. McCloskey, University of Wisconsin Press, 1987 and also as a chapter in our book, *Descartes' Dream*).

The article by the Israeli-French mathematician Yehuda Rav is "Philosophical Problems of Mathematics in the Light of Evolutionary Epistemology." He shows that the human ability and inclination to mathematize can be understood as the result of natural selection. It is necessary and advantageous for our survival as a species. It was first published in the journal *Philosophica*, and reprinted in the anthology *Math Worlds: Philosophical and Social Studies of Mathematics and Mathematics Education*, edited by Sal Restivo, Jean Paul van Bendegem, and Roland Fischer (Albany; State University of New York Press, 1993.)

The English-American mathematician-turned cognitive scientist, Brian Rotman, provides a surprising insight into mathematics in the language of semiotics. His article, "Toward a Semiotics of Mathematics," clarifies what you do when you write mathematics. Three different personae participate: first of all, there is the disembodied pure thinker, the impersonal voice who calls himself "we." Secondly, there is also an imaginary automaton, who in imagination ("in principle") carries out any calculations or algorithms that "we" mention. And yes, there is also an actual live flesh-and-blood human being, who is sitting in your chair. This article first appeared in *Semiotica* 72-1/2 (1988).

Donald Mackenzie's article, "Computers and the Sociology of Mathematical Proof," gives a detailed history of the computer scientist's search for program correctness, and thereby shines a searchlight on the notion of mathematical certainty. It was presented at a conference at the University of Roskilde, Denmark, in 1998, whose proceedings were published as *New Trends in the History and Philosophy of Mathematics*, University Press of Southern Denmark, 2004.

"From G.H.H. and Littlewood to XML and Maple: Changing Needs and Expectations in Mathematical Knowledge Management," by computer scientist Terry Stanway of Vancouver, BC, looks at mathematics from a new vantage point, as a problem of information storage and retrieval. Stanway is connected to the Centre for Experimental and Constructive Mathematics at Simon Fraser University, led by Jonathan and Peter Borwein. Their remarkable work integrates theory and computation in surprising and fruitful ways.

"Do Real Numbers Really Move? Language, Thought, and Gesture: The Embodied Cognitive Foundations of Mathematics" is by the Chilean-Swiss cognitive scientist Rafael Núñez, now at the University of California, San Diego. Building on his previous work with Berkeley linguist George Lakoff (*Where Mathematics Comes from*, Basic Books, 2000), he uses rigorous study of how we unconsciously produce millisecond-precise hand gestures as we talk mathematics—*literal* "hand-waving"!—to prove that mathematics is indeed built from embodied metaphor. It appears in a collection, *Embodied Artificial Intelligence*, edited by F. Iida et al., Springer, 2004.

Timothy Gowers of Cambridge university contributes "Does Mathematics Need a Philosophy?" He writes as a dedicated teacher who is an outstanding creator of mathematics. Does a mathematician and a teacher of mathematics need a philosophy of mathematics? The answer, of course, turns out to be: "Yes and No."

The philosopher Jody Azzouni of Boston and Brooklyn analyzes "How and Why Mathematics Is Unique as a Social Practice." Addressing "mavericks" such as myself, he finds some agreement, and some unanswered questions.

Gian-Carlo Rota held the unheard of title, "Professor of Applied Mathematics and Philosophy" at M.I.T. Gianco was at once a leading combinatorialist and a deep phenomenologist. "The Pernicious Influence of Mathematics upon Philosophy" sticks some pins into academic analytic philosophy. This essay was originally a talk at a session on philosophy of mathematics, organized by me in 1990 at the national meeting of the American Association for Advancement of Science in New Orleans. It was previously published in *Synthese*, 88 (2), August 1991; and in Rota's book, *Indiscrete Thoughts*, Birkhauser, 1997.

Jack Schwartz was Rota's mentor at Yale, as recounted in Gian-Carlo's book. Jack's mischievous shocker, "The Pernicious Influence of Mathematics on Science" must have stimulated Gian-Carlo's parallel demolition piece. (It appeared in *Discrete Thoughts*, Birkhauser, 1986, which was coedited by

Rota, Schwartz, and Mark Kac.) Jack is co-author, with Nelson Dunford, of the prize-winning three-volume “bible” of functional analysis, *Linear Operators*. Having completed that gigantic task, he later transformed himself into a leading authority in computer science.

Alfonso C. Avila del Palacio, of the University of Durango, Mexico, asks, “What Is Philosophy of Mathematics Looking for?” He finds that mathematicians, philosophers and historians can’t agree because they are asking different questions. His article is updated and translated from one that he published in *Syntesis* No. 3 (1997), *Revista de la Universidad Autonoma de Chihuahua*, Mexico.

Andrew Pickering of Urbana, a physicist who went to Edinburgh to become a sociologist, tells the story of Hamilton’s invention of the quaternion as an exemplary tale. His article, “Concepts and the Mangle of Practice Constructing Quaternions,” uses this historical material to bring mathematical practice in line with scientific practice, making the almost unprecedented move of aligning philosophy of mathematics with philosophy of science! It appeared in the collection, *Mathematics, Science, and Postclassical Theory*, edited by Barbara Herrnstein Smith and Arkady Plotnitsky, Duke University Press, 1997.

The philosopher Eduard Glas of Delft, in the Netherlands, in his article “Mathematics as Objective Knowledge and as Human Practice,” shows that the famous philosopher of science Karl Popper should also be considered a philosopher of mathematics. Indeed, Popper’s concept of science and mathematics as problem-solving avoids the traps of Platonism and formalism. Glas backs up his case with the illuminating history of descriptive and projective geometry in revolutionary and post-revolutionary France.

“The Locus of Mathematical Reality: An Anthropological Footnote” is by the famous anthropologist Leslie White, and is the oldest piece in our collection. White gave a clear, simple answer to the basic question about the nature of mathematical existence. He was a close friend to the leading topologist Raymond Wilder, and inspired Wilder’s writings on a cultural approach to the nature of mathematics. His article first appeared in the journal, *Philosophy of Science*, in October, 1947, and was reprinted in James R. Newman’s huge four-volume anthology, *The World of Mathematics* (Simon and Schuster, 1956).

Perhaps the most recalcitrant issue in philosophy of mathematics is what Eugene Wigner famously called the “unreasonable effectiveness” of mathematics. Mark Steiner has written a challenging book, *The Applicability of Mathematics as a Philosophical Problem*, that claims to refute naturalism on the basis of that effectiveness. My attempt to study this question, “Inner Vision, Outer Truth,” is my contribution to this collection. It is reprinted from *Mathematics and Science*, edited by Ronald Mickens, World Scientific, 1990.

The three articles by Aberdein, Azzouni and Glas all come from a conference I attended in Brussels in 2002, organized by Jean Paul von Bendegem

(*Theories of Mind, Social Science and Mathematical Practice*, Kluwer, 2005). Mackenzie's is from a conference in Denmark. The journal *Philosophia Mathematica* publishes a wide variety of perspectives. Recent books, from several different directions, have explored the nature of live mathematics. In English, I know of: *Where Mathematics Comes from* by George Lakoff and Rafael E. Núñez, *The Number Sense* by Stanislas Dehaene, *The Math Gene* by Keith Devlin, *What Counts* by Brian Butterworth, *Towards a Philosophy of Real Mathematics* by David Corfield, *Doing Mathematics* by Martin H. Krieger, *Social Constructivism as a Philosophy of Mathematics* by Paul Ernest, *Indiscrete Thoughts* by Gian-Carlo Rota, *Converging Realities* by Roland Omnès, and a forthcoming book by Alexandre Borovik. There is also my own *What Is Mathematics, Really?* (and, with Phil Davis, *Descartes' Dream* and *The Mathematical Experience*.) All in different ways go beyond traditional philosophizing, try to grapple with mathematical knowledge and activity as actual phenomena, as part of the real world. (We must mention forerunners—George Polya, Imre Lakatos, Karl Popper, Ludwig Wittgenstein, Raymond Wilder, Hans Freudenthal, Hao Wang, Philip Kitcher.) I know that there are also recent relevant books in several other languages. But I must leave it to others to compile those lists.

The articles here are largely limited to the cognitive aspect of mathematics. Of course the emotional, social and political aspects are also vitally important. The forthcoming book, *Loving and Hating Mathematics*, coauthored with Vera John-Steiner, turns to those aspects of mathematical life.

Although these articles were written independently, without contact between the authors, it's not surprising that there are several mutual references. Rav quotes Rényi and White; Aberdein quotes Thurston, Rav, and Rota; Azzouni, Núñez, Avila and Cellucci quote Hersh. But I have made no effort to weave these articles together, or to summarize what they add up to, or to announce what needs to be done next. The goal is to show the possibilities of thinking fresh, sticking close to actual practice, fearlessly letting go of standard shibboleths.

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