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## “Introduction” to *Filosofia e matematica*\*

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Mathematics has long been a preferential subject of reflection for philosophers, inspiring them since antiquity in developing their theories of knowledge and their metaphysical doctrines. Given the close connection between philosophy and mathematics, it is hardly surprising that some major philosophers, such as Descartes, Leibniz, Pascal and Lambert, have also been major mathematicians.

In the history of philosophy the reflection on mathematics has taken several forms. Since it is impossible to deal with all of them in a single volume, in this book I will present what seems to me the most satisfactory form today. My own view, however, differs considerably from the dominant view, and on a number of accounts.

1. According to the dominant view, the reflection on mathematics is the task of a specialized discipline, the philosophy of mathematics, starting with Frege, characterized by its own problems and methods, and in a sense “the easiest part of philosophy”<sup>1</sup>. In this view, the philosophy of mathematics “is a specialized area of philosophy, but not merely a specialized area. Many of the questions that arise within it, though by no means all, are particular cases of more general questions that arise elsewhere in philosophy, and occur within the philosophy of mathematics in an especially pure, or especially simplified, form”<sup>2</sup>. Thus, “if you cannot solve these problems, what philosophical problems can you hope to solve?”<sup>3</sup>.

The view expressed in this book is instead that entrusting reflection on mathematics to a specialized discipline poses serious limitations, because one cannot assume that philosophical problems occur in mathematics in an

\* Translation of a revised version of the “Introduction” to Cellucci 2002a.

<sup>1</sup> Dummett 1998, p. 190.

<sup>2</sup> *Ibid.*, p. 123.

<sup>3</sup> *Ibid.*, p. 191.

especially pure, or especially simplified, form. The reflection on mathematics entails dealing with such problems in all their impurity and complexity, and cannot be carried out adequately without dealing with them.

The idea that philosophical problems occur in mathematics in an especially pure or especially simplified form depends on the assumption that, whereas applied mathematics draws its “concepts from experience, observation, scientific theories, and even economics”, pure mathematics does not; on the contrary, “it is its purity that gives rise to many of the questions” on mathematics “we have been puzzling over”<sup>4</sup>. Pure mathematics “requires no input from experience: it is exclusively the product of thought”<sup>5</sup>.

This view is unjustified, however, since, like applied mathematics, pure mathematics draws its concepts from experience, observation, scientific theories and even economics. The questions considered by the reflection on mathematics have, therefore, all the impurity and complexity of which philosophical problems are capable.

2. According to the dominant view, the main problem in the philosophy of mathematics is the justification of mathematics. This problem arises because “our much-valued mathematics rests on two supports: inexorable deductive logic, the stuff of proof, and the set theoretic axioms”, which raises “the question of what grounds our faith in logical inference” and “what justifies the axioms of set theory”<sup>6</sup>. To answer such questions one must clarify the foundations of mathematics, providing a justification for them. On the other hand, the philosophy of mathematics does not concern itself with the problem of mathematical discovery, since it is only “concerned with the product of mathematical thought; the study of the process of production is the concern of psychology, not of philosophy”<sup>7</sup>.

The view expressed in this book is instead that the main problem in the reflection on mathematics is discovery. This includes the problem of justification, since discovery is not merely a part of mathematical activity but encompasses the whole of it, and therefore includes justification. Indeed, discovery requires making hypotheses capable of solving given problems, and in order to choose the hypotheses one must carefully evaluate the reasons for and against them. The evaluation process is intertwined with the process of hypothesis-formation, since one must compare alternative hypotheses in order to select one of them. This blurs the distinction between discovery and justification. In fact no such distinction is possible, since there are normally so many possible hypotheses to be formed and evaluated for any given problem that an exhaustive

<sup>4</sup> *Ibid.*, p. 190.

<sup>5</sup> Dummett 2001, p. 10.

<sup>6</sup> Maddy 1997, pp. 1-2.

<sup>7</sup> Dummett 1991, p. 305.

search cannot take place. Given that one cannot first make all possible hypotheses and then evaluate them, making hypotheses and evaluating them must be concurrent processes.

The idea that the main problem in the philosophy of mathematics is the justification of mathematics has made the philosophy of mathematics an increasingly less attractive subject, devoted to the study of questions – such as Frege’s: ‘What is the number one?’ – which seem irrelevant to mathematicians, neglecting those which are more important for understanding mathematics. No wonder, then, that there is widespread disregard and misunderstanding, and often outright antagonism, between philosophers of mathematics and mathematicians. The problem of the justification of mathematics seems unpalatable to the vast majority of mathematicians, who consider it irrelevant to their work.

Moreover, the idea that solving the problem of the justification of mathematics consists of clarifying the foundations of mathematics contradicts mathematical experience, which shows that mathematics is by no means a static structure, based on foundations given once and for all, but is a dynamic process, multifarious and articulated, whose ways of justification are also multifarious and articulated.

3. According to the dominant view, another important problem in the philosophy of mathematics is the existence of mathematical objects. This problem arises because “the point of view of common sense is perhaps that, if a proposition is true, it is because there are entities existing independently of the proposition which have the properties or stand in the relations which the proposition asserts of them”<sup>8</sup>. This “suggests that since mathematical propositions are true, that there are entities in virtue of which the propositions are true. The ontological issue is whether there are such entities and if so what their nature is”<sup>9</sup>. This problem supplements “the epistemological question” of the justification of mathematics, namely, “how mathematical beliefs come to be completely justified”<sup>10</sup>.

The view expressed in this book is instead that the problem of the existence of mathematical objects is irrelevant to mathematics because, as Locke pointed out, “all the discourses of the mathematicians about the squaring of a circle” – or any other geometrical figure – “concern not the existence of any of those figures”, and their proofs “are the same whether there be any square or circle existing the world, or no”<sup>11</sup>. Indeed, it is compatible with mathematical practice that there are no mathematical objects of which its theorems are true. Mathematical objects are simply hypotheses introduced to solve specific problems. To speak of

<sup>8</sup> Lehman 1979, p. 1.

<sup>9</sup> *Ibid.*

<sup>10</sup> *Ibid.*

<sup>11</sup> Locke 1975, p. 566.

mathematical objects is misleading because the term ‘object’ seems to designate the things the investigation is about, whereas hypotheses are the tools for the investigation. The latter is intended not to study properties of mathematical objects but to solve problems.

4. According to the dominant view, the philosophy of mathematics does not add to mathematics. Since its main problem is the justification of mathematics, it aims at clarifying the foundations of mathematics, not at expanding mathematics. Thus, “as the philosophy of law does not legislate, or the philosophy of science devise or test scientific hypotheses, so – we must realize from the outset – the philosophy of mathematics does not add to the number of mathematical theorems and theories”<sup>12</sup>. Its “arguments should have no doctrinal or practical impact on mathematics at all”<sup>13</sup>. For “mathematics comes first, then philosophizing about it, not the other way around”<sup>14</sup>. This is simply a special case of the fact that “philosophy does not contribute to the progress of knowledge: it merely clarifies what we already know”<sup>15</sup>.

The view expressed in this book is instead that the reflection on mathematics is relevant to the progress of mathematics. Since its main problem is mathematical discovery, it aims at improving existing methods of discovery and at inventing new ones. In this way the reflection on mathematics may contribute to the progress of mathematics, because the improvement in existing methods of discovery and the invention of new ones are of the utmost importance to that aim. Even Frege acknowledges that “a development of method, too, furthers science. Bacon, after all, thought it better to invent a means by which everything could easily be discovered than to discover particular truths, and all steps of scientific progress in recent times have had their origin in an improvement of method”<sup>16</sup>.

That the reflection on mathematics can contribute to the progress of mathematics entails that mathematics does not come first, with philosophizing about it following. On the contrary, they proceed together, both contributing to the advancement of learning.

5. According to the dominant view, the philosophy of mathematics does not require any detailed knowledge of mathematics, because its main aim – the justification of mathematics through a clarification of its foundations – does not require any detailed knowledge of the edifice built up on such foundations. Thus, even “if you have little knowledge of mathematics, you do not need to remedy that defect before interesting

<sup>12</sup> Körner 1960, p. 9.

<sup>13</sup> Wagner 1982, p. 267.

<sup>14</sup> Hersh 1997, p. xi.

<sup>15</sup> Dummett 2001, p. 24.

<sup>16</sup> Frege 1967, p. 6.

yourself in the philosophy of mathematics”<sup>17</sup>. You can “very well understand a good deal of the debates on the subject and a good deal of the theories advanced concerning it without an extensive knowledge of its subject-matter”<sup>18</sup>. Similarly, the philosophy of mathematics does not require any detailed knowledge of the history of mathematics, because “the etiology of mathematical ideas, however interesting, is not something whose study promises to reveal much about the structure of thought: for the most part, the origin and development of mathematical ideas are simply far too determined by extraneous influences”<sup>19</sup>. On the other hand, the philosophy of mathematics requires detailed knowledge of mathematical logic, “not so much as part of the object of study as serving as a tool of inquiry”<sup>20</sup>.

The view expressed in this book is instead that the reflection on mathematics does in fact require detailed knowledge of mathematics. Neglecting this has led the philosophy of mathematics to deal with marginal issues, deliberately excluding the broader ones. The philosophy of mathematics has done so on the assumption that, although the broader questions are “more interesting, more pressing, more significant than the narrower logical questions that are properly foundational”, the latter are “amenable to solution, whereas solutions to the broader questions may depend upon further advances in mathematics itself, advances which we cannot as yet foresee”<sup>21</sup>. But this assumption overlooks the fact that, owing to Gödel’s incompleteness theorems and related results, no logical question that is properly foundational has been solved by the philosophy of mathematics. Moreover, neither is there any evidence that the solutions to the broader questions may depend upon further advances in mathematics itself. As Wittgenstein put it, “even 500 years ago a philosophy of mathematics was possible, a philosophy of what mathematics was then”<sup>22</sup>.

The reflection on mathematics also requires a detailed knowledge of the history of mathematics. Neglecting this has led the philosophy of mathematics to consider mathematics as a static building, based on linear relations of logical dependence between *a priori* determined axioms and theorems. On the contrary, the history of mathematics shows that mathematics is a dynamic process, which often develops through tortuous and tormented paths not determined *a priori*, and proceeds through false starts and standstills, routine periods and sudden turnings. This has prevented the philosophy of mathematics from accounting not only

<sup>17</sup> Dummett 1998, p. 124.

<sup>18</sup> *Ibid.*

<sup>19</sup> George-Velleman 2002, p. 2.

<sup>20</sup> Dummett 1998, p. 124.

<sup>21</sup> Mayberry 1994, p. 19.

<sup>22</sup> Wittgenstein 1978, V, § 52.

for mathematical discovery but also for the real processes of mathematical justification.

As to the idea that the philosophy of mathematics requires detailed knowledge of mathematical logic, not so much as part of the object of study as serving as a tool of inquiry, it risks being empty. For, although the philosophy of mathematics has carried out an intense study of the foundations of mathematics using mathematical logic as a tool of inquiry, the hard core of mathematics has turned out to be impervious to what is found there. Thus the aim of clarifying the foundations of mathematics has lost momentum. Even supporters of mathematical logic, like Simpson, acknowledge that “foundations of mathematics is now out of fashion. Today, most of the leading mathematicians are ignorant of foundations”, and “foundations of mathematics is out of favor even among mathematical logicians”<sup>23</sup>. Indeed, the mainstream of mathematical logic has abandoned foundations to become a conventional albeit somewhat marginal branch of mathematics.

The idea that mathematical logic is the tool of inquiry of the philosophy of mathematics has its roots in the distrust towards the approach to philosophical problems of the philosophical tradition. This distrust has led to viewing the history of mathematical logic as a persistent struggle to free the subject from the grip of philosophy.

This distrust emerges among the first practitioners of the art, for example Russell, who maintains that “philosophy, from the earliest times, has made greater claims, and achieved fewer results, than any other branch of learning”<sup>24</sup>. Indeed, “so meagre was the logical apparatus that all the hypotheses philosophers could imagine were found to be inconsistent with facts”<sup>25</sup>. Nonetheless, “the time has now arrived when this unsatisfactory state of things can be brought to an end”<sup>26</sup>. This is made possible by mathematical logic, which has “introduced the same kind of advance into philosophy as Galilei introduced into physics”<sup>27</sup>. Mathematical logic “gives the method of research in philosophy, just as mathematics gives the method in physics. And as physics” finally “became a science through Galileo’s fresh observation of facts and subsequent mathematical manipulation, so philosophy, in our own day, is becoming scientific through the simultaneous acquisition of new facts and logical methods”<sup>28</sup>.

Statements of this kind are recurrent in the philosophy of mathematics. For example, Łukasiewicz claims that “philosophy must be reconstructed from its very foundations; it should take its inspiration from

<sup>23</sup> Simpson 1999, p. vii.

<sup>24</sup> Russell 1999, p. 13.

<sup>25</sup> *Ibid.*, p. 243.

<sup>26</sup> *Ibid.*, p. 13.

<sup>27</sup> *Ibid.*, pp. 68-69.

<sup>28</sup> *Ibid.*, p. 243.

scientific method and be based on the new logic”<sup>29</sup>. Beth maintains that “the lack of an adequate formal logic has strongly hampered the development of a systematic philosophy. Therefore”, although reflection on the philosophy of the past will remain one of the elements of future philosophy, “an adequate formal logic will be a second element of future philosophy”<sup>30</sup>. Kreisel claims that the approach to philosophical problems of the philosophical tradition is viable only “at an early stage, when we know too little about the phenomenon involved and about our knowledge of it in order to ask sensible specific questions”<sup>31</sup>. Such an approach must be replaced by one based on mathematical logic, which is “a tool in the philosophy of mathematics; just as other mathematics, for example the theory of partial differential equations, is a tool in what used to be called natural philosophy”<sup>32</sup>.

Distrust of the approach of the philosophical tradition and the urge to replace it by one based on mathematical logic are two basic features of the philosophy of mathematics which, on account of their very fruitlessness, have led to its progressive impoverishment and decline. This decline has become increasingly marked since the discovery of Gödel’s incompleteness theorems, so much so that Mac Lane claimed that the philosophy of mathematics is “a subject dormant since about 1931”<sup>33</sup>.

6. According to the dominant view, mathematics is theorem proving because it “is a collection of proofs. This is true no matter what standpoint one assumes about mathematics – platonism, anti-platonism, intuitionism, formalism, nominalism, etc.”<sup>34</sup>. Perhaps “in ‘doing mathematics’ proving theorems isn’t everything, but it’s way ahead of whatever is in second place”<sup>35</sup>. Of course, “the activity of mathematics is not just randomly writing down formal proofs for random theorems”, because “the choices of axioms, of problems, of research directions, are influenced by a variety of considerations – practical, artistic, mystical”, but the latter are “really non-mathematical”<sup>36</sup>. Therefore they are not a concern of the philosophy of mathematics.

The view expressed in this book is instead that mathematics is problem solving. That does not mean that mathematics is only problem solving. First one must pose problems, then one can refine them, exhibit them, dismiss them or even dissolve them. But problem solving is the core of mathematical activity, so it seems justified to maintain that it is an essen-

<sup>29</sup> Lukasiewicz 1970, p. 112.

<sup>30</sup> Beth 1957, pp. 8-9.

<sup>31</sup> Kreisel 1984, p. 82.

<sup>32</sup> Kreisel 1967, p. 201.

<sup>33</sup> Mac Lane 1981, p. 462.

<sup>34</sup> Takeuti 1987, p. 1.

<sup>35</sup> Franks 1989b, p. 12.

<sup>36</sup> Monk 1976, p. 4.

tial feature of mathematics. Problem solving, however, does not concern single separated problems nor leads to a final solution. For each solution generates new problems, and depends on the solutions found for these new problems. Thus, no solution is final but is always subject to further reconsideration.

7. According to the dominant view, the method of mathematics is the axiomatic method. For “proof must begin from axioms that are not themselves proved”<sup>37</sup>. In fact, “to prove a proposition, you start from some first principles, derive some results from those axioms, then, using those axioms and results, push on to prove other results”<sup>38</sup>. The axiomatic method is available for the whole of mathematics, because “all mathematical theories, when sufficiently developed, are capable of axiomatization”<sup>39</sup>. Moreover, the axiomatic method provides “a strategy both for finding and remembering proofs”, because “relatively few properties, Bourbaki’s few, so-called basic structures, have been found adequate for similar strategies in a very broad domain of mathematics”, although “the use of axiomatic analysis as a proof strategy does not seem to be well known to people writing on heuristics, like Polya”<sup>40</sup>. Since the method of mathematics is the axiomatic method, “mathematics and science are intellectual undertakings which are complementary but opposed, distinguished by the direction of their view”<sup>41</sup>. For “the former proceeds forwards, from hypotheses to conclusions: i.e., from axioms to the theorems derivable from them. The latter proceeds backwards, from conclusions to premisses: i.e., from experimental data to physical laws from which they can be drawn”<sup>42</sup>.

The view expressed in this book is instead that the method of mathematics is the analytic method, a method which, unlike the axiomatic method, does not start from axioms which are given once and for all and are used to prove any theorem, nor does it proceed forwards from axioms to theorems, but proceeds backwards from problems to hypotheses. Thus proof does not begin from axioms that are not themselves proved. Unlike axioms, hypotheses are not given from the start, but are the very goal of the investigation. They are never definitive, but liable to be replaced by other hypotheses, and are introduced to solve specific problems<sup>43</sup>.

<sup>37</sup> Maddy 1990, p. 144.

<sup>38</sup> Leary 2000, p. 48.

<sup>39</sup> Dummett 1991, p. 305.

<sup>40</sup> Kreisel-MacIntyre 1982, pp. 232-233.

<sup>41</sup> Odifreddi 2001, p. 233.

<sup>42</sup> *Ibid.*

<sup>43</sup> The analytic method is meant here not in the sense of Aristotle or Pappus but in the sense of Hippocrates of Chios and Plato. On this distinction see Cellucci 1998a.



The idea that, to prove a proposition, you start from some first principles, derive some results from those axioms, then, using those axioms and results, push on to prove other results, contrasts with mathematical experience which shows that in mathematics one first formulates problems, then looks for hypotheses to solve them. Thus one does not proceed, as in the axiomatic method, from axioms to theorems but proceeds, as in the analytic method, from problems to hypotheses. As Hamming points out, in mathematics deriving theorems from axioms “does not correspond to simple observation. If the Pythagorean theorem were found to not follow from postulates, we would again search for a way to alter the postulates until it was true. Euclid’s postulates came from the Pythagorean theorem, not the other way”<sup>44</sup>.

Similarly, the idea that the axiomatic method is available for the whole of mathematics because all mathematical theories, when sufficiently developed, are capable of axiomatization, contrasts with mathematical experience, which shows that axiomatization does not naturally apply to all parts of mathematics. Some of them are not suitable for axiomatization, and exist as collections of solved or unsolved problems of a certain kind. This is true, for example, of number theory and of much of the theory of partial differential equations.

The idea that the axiomatic method provides a strategy both for finding and remembering proofs also contrasts with mathematical experience, which shows that proofs based on the axiomatic method often appear to be found only by a stroke of luck, and seem artificial and difficult to understand. Showing only the final outcome of the investigation, established in a way that is completely different from how it was first obtained, such proofs hide the actual mathematical process, thus contributing to make mathematics a difficult subject.

Similarly, the idea that, since the method of mathematics is the axiomatic method, mathematics and science are intellectual undertakings which are complementary but opposed, distinguished by the direction of their view, contrasts with mathematical experience, which shows that mathematics, like other sciences, proceeds backwards from conclusions to premises, i.e. from problems to hypotheses which provide conditions for their solution. This is adequately accounted for by the analytic method, which assimilates mathematics to other sciences, and in particular assimilates the concept of mathematical proof to the concepts of proof of other sciences.

The limitations of the axiomatic method are acknowledged by several mathematicians. For example, Lang stresses that “axiomatization is what one does last, it’s rubbish”, it is merely “the hygiene of mathematics”<sup>45</sup>. Giusti states that “setting out axioms is never the starting point, but is

<sup>44</sup> Hamming 1980, p. 87.

<sup>45</sup> Lang 1985, p. 19.

rather the arrival point of a theory”, and “the occasions where one started from the axioms are rather the exception than the rule”<sup>46</sup>. Hersh points out that, “in developing and understanding a subject, axioms come late”, and even if “sometimes someone tries to invent a new branch of mathematics by making up some axioms and going from there”, still “such efforts rarely achieve recognition or permanence. Examples, problems, and solutions come first. Later come axiom sets on which the existing theory can be ‘based’. The view that mathematics is in essence derivations from axioms is backward. In fact, it’s wrong”<sup>47</sup>.

The limitations of the axiomatic method are also acknowledged by some supporters of the dominant view, like Mayberry, who recognizes that “no axiomatic theory, formal or informal, of first or of higher order can logically play a foundational role in mathematics”<sup>48</sup>. For “it is obvious that you cannot use the axiomatic method to explain what the axiomatic method is”<sup>49</sup>. Since any theory put forward “as the foundation of mathematics must supply a convincing account of axiomatic definition, it cannot, on pain of circularity, itself be presented by means of an axiomatic definition”<sup>50</sup>.

8. According to the dominant view, the logic of mathematics is deductive logic. For theorems “are justified by deductive inference”<sup>51</sup>. In fact, “deductive inference patently plays a salient part in mathematics. The correct observation that the discovery of a theorem does not usually proceed in accordance with the strict rules of deduction has no force: a proof has to be set out in sufficient detail to convince readers, and, indeed, its author, of its deductive cogency”<sup>52</sup>. Admittedly, “deduction is only one component in mathematical reasoning understood in the broad sense of all the intellectual work that goes on when solving a mathematical problem. But this does not mean that the notion of deduction is not the key concept for understanding validity in mathematics, or that the distinction between discovery and justification loses its theoretical importance”<sup>53</sup>. For, “when it comes to explaining the remarkable phenomenon that work on a mathematical problem may end in a result that everyone finds definitive and conclusive, the notion of deduction is a central one”<sup>54</sup>. Indeed, “mathematics has a methodology unique among all the sciences. It is the only discipline in which

<sup>46</sup> Giusti 1999, p. 20.

<sup>47</sup> Hersh 1997, p. 6.

<sup>48</sup> Mayberry 1994, p. 34.

<sup>49</sup> *Ibid.*, p. 35.

<sup>50</sup> *Ibid.*

<sup>51</sup> Maddy 1984, p. 49.

<sup>52</sup> Dummett 1991, p. 305.

<sup>53</sup> Prawitz 1998, p. 332.

<sup>54</sup> *Ibid.*

deductive logic is the sole arbiter of truth. As a result mathematical truths established at the time of Euclid are still held valid today and are still taught. No other science can make this claim”<sup>55</sup>.

The view expressed in this book is instead that the logic of mathematics is not deductive logic but a broader logic, dealing with non-deductive (inductive, analogical, metaphorical, metonymical, etc.) inferences in addition to deductive inferences. It is by non-deductive inferences that one finds the hypotheses by which mathematical problems are solved. The logic of mathematics is not, therefore, that studied by mathematical logic, which is simply a branch of mathematics, but consists of a set of non-deductive methods and techniques in addition to deductive methods and techniques, and hence is not a theory but a set of tools.

To claim that the logic of mathematics is deductive logic because theorems are justified by deductive inference, restricts mathematical experience to ways of reasoning found only in textbooks of mathematical logic, and neglects those that are really used in mathematical activity. Moreover, it does not account for the real nature of mathematics, because mathematical reasoning is based mainly on non-deductive inferences, not on deductive inferences, which play a somewhat restricted role within it. Contrary to widespread misunderstanding, mathematics is never deductive in the making, since mathematicians first state problems, then find hypotheses for their solution by non-deductive inferences. As even some supporters of the dominant view, like Halmos, acknowledge, mathematics “is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof”<sup>56</sup>. The “deductive stage, writing the result down, and writing down its rigorous proof are relatively trivial once the real insight arrives; it is more like the draftsman’s work, not the architect’s”<sup>57</sup>. Furthermore, to claim that the logic of mathematics is deductive logic clashes with the results of the neurosciences, which show that the human brain is very inefficient even in moderately long chains of deductive inferences.

Similarly, to claim that, when it comes to explaining the remarkable phenomenon that work on a mathematical problem may end in a result that everyone finds definitive and conclusive, the notion of deduction is a central one, overlooks the fact that, according to the dominant view, several Euclid’s proofs are flawed. Thus, in this view, the fact that everyone finds Euclid’s results definitive and conclusive cannot depend on Euclid’s proofs. The same applies to contemporary mathematics, where

<sup>55</sup> Franks 1989a, p. 68.

<sup>56</sup> Halmos 1968, p. 380.

<sup>57</sup> *Ibid.*

several published proofs are flawed – somewhat surprising if mathematics is the rigorous deduction of theorems from axioms.

Moreover, to claim that mathematics has a methodology unique among all the sciences because it is the only discipline in which deductive logic is the sole arbiter of truth, begs the question since it assumes that the logic of mathematics is deductive logic. On the contrary, the broader logic on which mathematics is based does not distinguish but rather assimilates the methodology of mathematics to that of the other sciences, which is based on inferences of the very same kind.

9. According to the dominant view, mathematical discovery is an irrational process based on intuition, not on logic. For “some intervention of intuition issuing from the unconscious is necessary at least to initiate the logical work”<sup>58</sup>. The activity “of a creating brain has never had any rational explanation, neither in mathematics nor in other fields”<sup>59</sup>. In particular, the discovery of axioms has nothing to do with logic, because “there is no hope, there is, as it were, a leap in the dark, a bet at any new axiom”, so “we are no longer in the domain of science but in that of poetry”<sup>60</sup>. Generally, “the creative and intuitive aspects of mathematical work evade logical encapsulation”<sup>61</sup>. The “mathematician at work relies on surprisingly vague intuitions and proceeds by fumbling fits and starts with all too frequent reversals. In this picture the actual historical and individual processes of mathematical discovery appear haphazard and illogical”<sup>62</sup>. The role of intuition in mathematical discovery is decisive “in most researchers, who are often put on the track that will lead them to their goal by an albeit confused intuition of the mathematical phenomena studied by them”<sup>63</sup>.

The view expressed in this book is instead that mathematics is a rational activity at any stage, including the most important one: discovery. Intuition does not provide an adequate explanation as to how we reach new hypotheses, so, either we must give up any explanation thus withdrawing into irrationalism, or we must provide an explanation, but then cannot appeal to intuition.

In fact, there is no need to appeal to intuition. Since ancient times, many have recognised not only that mathematical discovery is a rational process, but also that a method exists for it, namely the analytic method. This method gave great heuristic power to ancient mathematicians in solving geometrical problems, and has had a decisive role in the new

<sup>58</sup> Hadamard 1945, p. 112.

<sup>59</sup> Dieudonné 1988, p. 38.

<sup>60</sup> Girard 1989, p. 169.

<sup>61</sup> Feferman 1998, p. 178.

<sup>62</sup> *Ibid.*, p. 77.

<sup>63</sup> Dieudonné 1948, pp. 544-545.

developments of mathematics and physics since the beginning of the modern era. Within the analytic method, logic plays an essential role in the discovery of hypotheses, provided of course that logic is taken to include non-deductive inferences, unlike in the limited and somewhat parochial dominant view.

Not only is there no need to appeal to intuition, but Pascal even sets the mathematical mind against the intuitive mind. For he claims that “there are two kinds of mind, one mathematical, and the other what one might call the intuitive. The first takes a slow, firm and inflexible view, but the latter has flexibility of thought which it applies simultaneously to the diverse lovable parts of that which it loves”<sup>64</sup>.

10. According to the dominant view, in addition to mathematical discovery, mathematical justification too is based on intuition. For, if one assumes that the method of mathematics is the axiomatic method, then justifying mathematics amounts to justifying the certainty of its axioms, and their certainty is directly or indirectly based on intuition. Directly, when through intuition “the axioms force themselves upon us as being true”<sup>65</sup>. Indirectly, when “we apply contentual” – and hence intuitive – “inference, in particular, to the proof of the consistency of the axioms”<sup>66</sup>. Thus, “accounting for intuitive ‘knowledge’ in mathematics is the basic problem of mathematical epistemology”<sup>67</sup>.

The view expressed in this book is instead that justification is not based on intuition but on the fact that the hypotheses used in mathematics are plausible, i.e., compatible with the existing knowledge, in the sense that, if one compares the reasons for and against the hypotheses, the reasons for prevail. It is often claimed that ‘plausible’ has a subjective, psychological connotation, so that it is almost equivalent to ‘rhetorically persuasive’, hence plausible arguments are of little interest in mathematics. But ‘plausible’, in the sense explained above, has nothing subjective or psychological about it.

To assess whether a given hypothesis is plausible, one examines the reasons for and against it. This examination is carried out using facts which confirm the hypothesis or refute it, where these facts belong to the existing knowledge. Admittedly, such an assessment is fallible, because one’s choice of facts may be inadequate, and moreover the existing knowledge is not static but develops continuously, each new development providing further elements for assessing the hypothesis, which may lead to its rejection. But this procedure is neither subjective nor psychological. On the

<sup>64</sup> Pascal 1913, p. 50.

<sup>65</sup> Gödel 1986-, II, p. 268.

<sup>66</sup> Hilbert 1996, p. 1132.

<sup>67</sup> Hersh 1997, p. 65.

contrary, a justification of mathematics based on intuition is subjective and psychological.

11. According to the dominant view, mathematics is a body of truths – indeed a body of absolutely certain and hence irrefutable truths. For mathematics is “the paradigm of certain and final knowledge: not fixed, to be sure, but a steadily accumulating coherent body of truths obtained by successive deduction from the most evident truths. By the intricate combination and recombination of elementary steps one is led incontrovertibly from what is trivial and unremarkable to what can be nontrivial and surprising”<sup>68</sup>. This derives from the fact that, “while a physical hypothesis can only be verified to the accuracy and the interpretation of the best work in the laboratory”, a mathematical truth is established by a proof based on the axiomatic method, which “has the highest degree of certainty possible for man”<sup>69</sup>. Indeed, “there is at present no viable alternative to axiomatic presentation if the truth of a mathematical statement is to be established beyond reasonable doubt”<sup>70</sup>. While physical hypotheses come and go, none is definitive, and so “in physics nothing is completely certain”, mathematics, as based on the axiomatic method, “lasts an eternity”<sup>71</sup>.

The view expressed in this book is instead that mathematics is a body of knowledge but contains no truths. Speaking of truth is not necessary in mathematics, just as it is not necessary in the natural sciences, and is not necessary anywhere except perhaps in theology and in lovers’ quarrels. Assuming that mathematics is a body of truths leads to an inextricable muddle, which results in self-defeating statements such as: It is legitimate “to argue from ‘this theory has properties we like’ to ‘this theory is true’”<sup>72</sup>. On this basis Frege could have argued that, since his ideography had the property he liked of reducing arithmetic to logic, his ideography was true, only to be belied by Russell’s paradox.

That mathematics is not a body of truths does not mean that it has no objective content. It only means that, as with any other science, mathematics does not consist of truths but only of plausible statements, i.e., statements compatible with existing knowledge. The objectivity of mathematics does not depend on its being a body of truths but on its being a body of plausible statements.

Moreover, the idea that mathematics is a body of absolutely certain and hence irrefutable truths, overlooks the fact that we cannot be sure of the current proofs of our theorems. For, by Gödel’s incompleteness theorems and related results, we cannot be sure of the hypotheses on which they are based.

<sup>68</sup> Feferman 1998, p. 77.

<sup>69</sup> Jaffe 1997, p. 135.

<sup>70</sup> Rota 1997, p. 142.

<sup>71</sup> Jaffe 1997, pp. 139-140.

<sup>72</sup> Maddy 1997, p. 163.

As regards certainty, mathematics has no privilege and is as risky as any other human creation. Mathematical knowledge is not absolutely certain, but only plausible, i.e. compatible with the existing knowledge, and plausibility does not grant certainty, because the existing knowledge is not absolutely certain, but only plausible. For centuries mathematics was considered a body of absolutely certain truths, but now this is increasingly perceived as an illusion. Uncertainty and doubt have replaced the self-complacent certainty of the past. As some supporters of the dominant view, like Leary, also acknowledge, by Gödel's incompleteness theorems and related results, “mathematics, which had reigned for centuries as the embodiment of certainty, had lost that role”<sup>73</sup>.

12. According to the dominant view, the question of the applicability of mathematics to the physical sciences is inessential for the philosophy of mathematics. Mathematics is “a unified undertaking which we have reason to study as it is, and the study of the actual methods of mathematics, which includes pure mathematics, quickly reveals that modern mathematics also has goals of its own, apart from its role in science”<sup>74</sup>. Admittedly, “it is a wonderful thing when a branch of mathematics suddenly becomes relevant to new discoveries in another science; both fields benefit enormously. But many (maybe most) areas of mathematics will never be so fortunate. Yet most mathematicians feel their work is no less valid and no less important than mathematics that has found utility in other sciences. For them it is enough to experience and share the beauty of a new theorem. New mathematical knowledge” is “an end in itself”<sup>75</sup>.

The view expressed in this book is instead that the question of the applicability of mathematics to the physical sciences is important for the reflection on mathematics. While, on the one hand, mathematics is continuous with philosophy, on the other hand it is also continuous with the physical sciences, and many of its developments, even in pure mathematics, are inextricably linked to the physical sciences.

13. According to the dominant view, mathematics is based only on conceptual thought. For mathematics “is the purest product of conceptual thought, which is a feature of human life that both pervasively structures it and sets it apart from all else”<sup>76</sup>. Mathematics is “unconstrained by experience”, enters “the world touched only by the hand of reflection”, and is “justified by pure ratiocination, perceptual observation via any of our five sensory modalities being neither necessary nor even relevant”<sup>77</sup>.

<sup>73</sup> Leary 2000, p. 3.

<sup>74</sup> Maddy 1997, p. 205, footnote 15.

<sup>75</sup> Franks 1989b, p. 13.

<sup>76</sup> George-Velleman 2002, p. 1.

<sup>77</sup> *Ibid*



Thus, from the standpoint of the philosophy of mathematics, it is inessential to study questions concerning perception, or, more generally, “such questions as ‘What brain, or neural activity, or cognitive architecture makes mathematical thought possible?’”<sup>78</sup>. These questions focus on “phenomena that are really extraneous to the nature of mathematical thought itself”, i.e., “the neural states that somehow carry thought”, whereas “philosophers, by contrast, are interested in the nature of those thoughts themselves, in the content carried by the neural vehicles”<sup>79</sup>.

The view expressed in this book is instead that mathematics is based not only on conceptual thought but also on perception, which plays an important role in it, for example, in diagrams. Thus, from the viewpoint of the reflection on mathematics, it is important to study questions concerning perception and more generally the brain, the neural activity or the cognitive architecture which make mathematical thought possible. Mathematics, after all, is a human activity, and the only mathematics humans can do is what their brain, neural activity and cognitive architecture enable them to do. Therefore, what mathematics is essentially depends on what the human brain, neural activity or cognitive architectures are.

The idea that mathematics is based only on conceptual thought, and indeed is the purest product of conceptual thought, neglects the fact that the ability to distinguish shape, position and number is not restricted to humankind but is shared by several other forms of animal life. This ability is vital to these forms of life: they could not have survived without it. Mathematics, therefore, is not a feature of human life that distinguishes it from all the rest, but has its roots in certain basic abilities belonging both to humans and to several other forms of animal life, and is part of the process of adapting to the environment.

This brings to an end our examination of the main differences between the view expressed in this book and the dominant view. Not that there are no further differences, but those considered above will suffice to show to what extent the two views differ.

The arguments sketched above provide reasons for rejecting the dominant view. In short, the rejection is motivated by the fact that the dominant view does not explain how mathematical problems arise and are solved. Rather, it presents mathematics as an artificial construction, which does not reflect its important aspects, and omits those features which make mathematics a vital discipline. Thus the dominant view does not account for the richness, multifariousness, dynamism and flexibility of mathematical experience.

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<sup>78</sup> George-Velleman 2002, p. 2.

<sup>79</sup> *Ibid.*



In showing the limitations of the dominant view, in this book I do not describe it in all its historical and conceptual articulations, which would require considerably more space than is available in a single book. I only present as much as is necessary to show that it is untenable.

For statements of the dominant view, the interested reader may wish to consult, in addition to introductory texts<sup>80</sup>, the primary sources, many of which are readily available<sup>81</sup>. As to the different ways in which the reflection on mathematics has been carried out in the history of philosophy, he may wish to consult, in addition to introductory texts<sup>82</sup>, the primary sources, from Plato to Mill, most of which are also readily available<sup>83</sup>.

Partial challenges to the dominant view have been put forward by Pólya, Lakatos, Hersh and others<sup>84</sup>. The position stated in this book is, however, somewhat more radical, and perhaps more consequential.

For instance, unlike Pólya, I do not claim that “the first rule of discovery is to have brains and good luck”, nor that the “the second rule of discovery is to sit tight and wait till you get a bright idea”<sup>85</sup>. Nor do I distinguish between, on the one hand, mathematics in a finished form, viewed as “purely demonstrative, consisting of proofs only”, and, on the other hand, “mathematics in the making”, which “resembles any other human knowledge in the making”<sup>86</sup>. Moreover, I do not claim that axiomatic reasoning, characteristic of mathematics in finished form, “is safe, beyond controversy, and final”, unlike conjectural reasoning, characteristic of mathematics in the making, which “is hazardous, controversial, and provisional”<sup>87</sup>. Nor do I maintain that axiomatic reasoning is for the mathematician “his profession and the distinctive mark of his science”<sup>88</sup>. These views have prevented Pólya from developing a full alternative to the dominant view.

The view expounded in this book is a development of that presented in my earlier publications<sup>89</sup>. The reader might wish to consult them for matters which are not discussed or are discussed only too briefly here.

In this book I do not consider all philosophical questions concerning mathematics, even less all philosophical questions concerning knowledge,

<sup>80</sup> See, for example, Giaquinto 2002, Shapiro 2000.

<sup>81</sup> See Benacerraf-Putnam 1983, Ewald 1996, Hart 1996, Jacquette 2002, Mancosu 1998, van Heijenoort 1977.

<sup>82</sup> See, for example, Barbin-Caveing 1996.

<sup>83</sup> A basic choice can be found in Baum 1973.

<sup>84</sup> See, for example, Tymoczko 1998.

<sup>85</sup> Pólya 1948, p. 158.

<sup>86</sup> Pólya 1954, I, p. vi.

<sup>87</sup> *Ibid.*, I, p. v.

<sup>88</sup> *Ibid.*, I, p. vi.

<sup>89</sup> See, for example, Cellucci 1998a, 1998b, 2000, 2002b.

because that would require far more space than is available. To my mind, however, the questions discussed here should be dealt with in any investigation concerning the nature of mathematics.

The book consists of a number of short chapters, each of which can be read independently of the others, although its full meaning will emerge only within the context of the whole book. To illustrate my view, I often use fairly simple mathematical examples, which can be presented briefly and do not require elaborate preliminary explanations. Nonetheless, their simplicity does not detract from their exemplarity.

Since my view differs radically from the dominant view, which has exerted its supremacy for so long as to be now mistaken for common sense, I do not expect readers to agree with me immediately. I only ask that you to try and find counterarguments, and carefully assess whether they would stand up to the objections which could be raised against them from the viewpoint of this book.

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Mathematics

Hersh, R. (Ed.)

2006, XXII, 326 p. 10 illus., Softcover

ISBN: 978-0-387-25717-4