On a Local Search for Reverse Convex Problems

Alexander Strekalovsky

Institute of System Dynamics and Control Theory SB of RAS, 134 Lermontov St., Irkutsk-33, 664033 Russia strekal@icc.ru

Summary. In this paper we propose two variants of Local Search Method for reverse convex problems. The first is based on well-known theorem of H. Tuy as well as on Linearization Principle. The second variant is due to an idea of J. Rosen. We demonstrate the practical effectiveness of the proposed methods computationally.

Key words: Nonconvex optimization, reverse convex problem, local search, computational testing.

1 Introduction

The present situation in Continuous Nonconvex Optimization may be viewed as dominated by methods transferred from other sciences [1]-[3], as Discrete Optimization (Branch&Bound, cuts methods, outside and inside approximations, vertex enumeration and so on), Physics, Chemistry (simulated annealing methods), Biology (genetic and ant colony algorithms) etc.

On the other hand the classical method [11] of convex optimization have been thrown aside because of its inefficiency [1]-[6]. As is well known, the conspicuous limitation of convex optimization methods applied to nonconvex problems is their inability of escape a local extremum or even a critical point depending on a starting point [1]-[3]. So, the classical apparatus shows itself inoperative for new problems arising from practice.

In such a situation it is desirable to create a global optimization approach aimed at nonconvex problems — in particular to Reverse Convex Problem (RCP) — connected on with Convex Optimization Theory and using the methods of Convex Optimization.

We ventured to propose such an approach [12] and to advance the following principles of Nonconvex Optimization.

I. The linearization of the basic (generic) nonconvexity of a problem of interest and consequently a reducing of the problem to a family of (partially) linearized problems.

II. The application of convex optimization methods for solving the linearized problems and, as a consequence, within special local search methods.

III. Construction of "good" (pertinent) approximations (resolving sets) of level surfaces and epigraph boundaries of convex functions.

Obviously, the first and the second are well known. The depth and effectiveness of the third may be observed in [12]-[23].

Developing these principles we get the solving methodology for nonconvex problems which can be represented as follows.

1. Exact classification of a problem under study.
2. Application of special (for a given class of problems, for instance, RCP) local search methods.
3. Applyication of special conceptual global search methods (strategies).
4. Using the experience of similar nonconvex problems solving to construct pertinent approximations of level surfaces of corresponding convex functions.
5. Application of convex optimization methods for solving linearized problems and within special local search methods.

This approach lifts Classical Convex Optimization to a higher level, where the effectiveness and the speed of the methods become of paramount importance not only for Convex Optimization, but for Nonconvex problems (in particular for RCP, which is discussed below).

Our computational experience suggests that if you follow the above methodology you have more chances to reach a global solution of a nonconvex problem of large size ($\geq$ 1000 variables) than applying the Branch-and-Bound or Cutting Plane methods.

In this paper we decided to focus only on the advantages of Principle I — Linearization applied for Local Search Problem. In order to do this we propose two variants of Local Search Method. The first is based on a well-known theorem of H. Tuy [1] as well as on the Linearization Principle. The second variant is due to an idea of J. Rosen [10], which was only slightly modified by adding the procedure of free descent on the constraint $g = 0$. Finally we demonstrate the practical effectiveness of these methods by a computational testing and propose to unify two methods.

Before this we recall a few facts from Reverse Convex Theory.

2 Some features of RCP

Let us consider the problem

$$h(x) \downarrow \min, \quad x \in S, \quad g(x) \geq 0, \tag{P}$$

where h is a continuous function and the function g is a convex function on $I\!R^n$, $S \in I\!R^n$.

Denote the feasible set of the problem (P) by D

$$D := \{x \in S \mid g(x) \geq 0\} \neq \emptyset. \tag{1}$$

Further, suppose

$$h_* := \inf(h, D) \triangleq \inf_x\{h(x) \mid x \in S,\ g(x) \geq 0\} > -\infty. \tag{2}$$

It can be easily seen that the nonconvexity of (P) is generated by the reverse convex constraints $g \geq 0$ defining the complement of convex open set $\{x \in I\!R^n \mid g(x) < 0\}$. That is why we suppose this constraint to be active at any solution of (P) $(Sol(P))$. Otherwise, by solving the relaxed problem

$$(PW): \qquad h(x) \downarrow \min, \quad x \in S, \tag{3}$$

(which is simpler than (P)) one can simultaneously find a solution to (P).

The regularity conditions (when $g \geq 0$ is substantial) [1]-[6] may be given in different ways. For instance,

$$(G): \qquad \left.\begin{array}{c}\text{There is no any solution } x^* \in D \\ \text{to } (P) \text{ such that } g(x^*) > 0.\end{array}\right\} \tag{4}$$

The latter is equivalent to

$$(G'): \qquad Sol(P) \cap \{x \in I\!R^n \mid g(x) > 0\} = \emptyset, \tag{2.4'}$$

where $Sol(P)$ is the solution set of the problem (P) $Sol(P) = Argmin(P)$.

One can express the regularity condition with the help of the optimal value function for problem (P) and the relaxed problem (PW)–(3)

$$\mathcal{V}(PW) := \inf_x\{h(x) \mid x \in S\} < \mathcal{V}(P) \triangleq \inf_x\{h(x) \mid x \in S,\ g(x) \geq 0\}. \tag{5}$$

One of corollaries of the last condition is the fact that by solving the relaxed problem (PW)–(3), say, with convex h and S it is possible to perform a descent to the constraint $g = 0$ by means, for instance, one of the classical methods of convex optimization. As a consequence, one has

$$Sol(P) \subset \{x \in I\!R^n \mid g(x) = 0\}.$$

The following result is fundamental in RCP theory. Additionally, this theorem establishes a relation between Problem (P) and the problem of convex maximization [1]-[4], [12].

Theorem 1. *(H. Tuy [1, 2]) Let us suppose the assumption (G)–(4) to be fulfilled, and a point z to be a solution to (P). Then*

$$\max_x\{g(x) \mid x \in S,\ h(x) \leq h(z)\} = 0. \tag{6}$$

If the following assumption takes place

$$(H1): \qquad \left.\begin{array}{c} \forall y \in S: \ \ g(y) = 0, \ \exists \varepsilon > 0, \\ \exists u \in S \cap B(y, \varepsilon): \ g(u) > 0; \end{array}\right\} \tag{7}$$

then the condition (6) becomes sufficient for z to be a global solution to (P).

According to this result, instead of solving the Problem (P) one can consider the convex maximization problem

$$(Q_\beta): \qquad g(x) \uparrow \max, \ \ x \in S, \ h(x) \leq \beta. \tag{8}$$

If one got that the value of (Q_β)

$$V(\beta) := \max_x \{g(x) \mid x \in S, \ h(x) \leq \beta\}$$

with $\beta = h(z)$ is equal to zero, then $z \in Sol(P)$.

A theorem of H. Tuy generated a stream of interest in Solution Methods Theory for RCP leading to reducing Problem (P) to the dual problem (Q_β). Let us note two properties of such a reduction. First, the basic (generic) nonconvexity of the Problem (P) has not been dissipated. It stays in the goal function of (Q_β)–(8) so that even with convex h and S the problem (8) is nonconvex. Second, it is not clear how to choose the parameter β.

Finally, the question is: is it possible to apply convex optimization methods to solve (8)? It will be shown below that there exists another way to solve the Problem (P) [12], i.e. to employ an effective local search process.

3 Local search methods

Let us suppose the function h and the set S to be convex and, besides, the following regularity condition to be fulfilled (cf. (5))

$$(H_0): \qquad \exists v \in S, \ h(v) < h_* \triangleq \mathcal{V}(P), \ \ g(v) < 0. \tag{9}$$

Under this hypothesis we propose a special local search method consisting of two parts. The first procedure begins at a feasible point $y \in S$, $g(y) \geq 0$, and constructs a point $x(y) \in S$, such that

$$g(x(y)) = 0, \ \ h(x(y)) \leq h(y).$$

The second procedure consists in the consecutive solution of the Linearized problems:

$$(LQ(u,\beta)): \qquad \left.\begin{array}{c} \langle \nabla g(u), x \rangle \uparrow \max, \\ x \in S, \ \ h(x) \leq \beta; \end{array}\right\} \tag{10}$$

where the parameters u and β will be defined below.

It can be readily seen, that the linearized problems $(LQ(u,\beta))$ are convex. So a convex optimization method can be applied to get an approximate global solution to (10).

Now let us move to a more detailed description of the calculation process.

Procedure 1. [12, 20] Let us consider a point $y \in S,\quad g(y) \geq 0$. If $g(y) = 0$, we set $x(y) = y$. In the case $g(y) > 0$, there exists $\lambda \in]0,1[$ such that $g(x_\lambda) = 0$, where $x_\lambda = \lambda v + (1-\lambda)y$, since $g(y) > 0 > g(v)$ due to (H_0)–(9).

In addition, because of the convexity of $h(\cdot)$ one has

$$h(x_\lambda) \leq \lambda h(v) + (1-\lambda)h(y) < \lambda h_* + (1-\lambda)h(y) \leq h(y). \tag{11}$$

That is why we set $x(y) = x_\lambda$ and obtain

$$h(x(y)) < h(y), \tag{12}$$

which is what we wanted. Usually one calls Procedure 1 "free descent" (that is, free from the constraint $g(x) \geq 0$).

Procedure 2. This starts at a feasible point $\tilde{x} \in S,\quad g(\tilde{x}) = 0$, and constructs a sequence $\{u^r\}$ such that $(r = 0,1,2,\ldots)$

$$u^r \in S,\;\; g(u^r) \geq 0,\;\; h(u^r) \leq \beta, \tag{13}$$

where $\beta := h(\tilde{x}),\;\; u^0 := \tilde{x}$.

The sequence $\{u^r\}$ is constructed as follows. If a point u^r, $r \geq 0$ verifying (13) is given, then the next point u^{r+1} is constructed as an approximative solution to the linearized problem $(LQ(u^r,\beta))$, so that the following inequality holds:

$$\langle \nabla g(u^r), u^{r+1}\rangle + \delta_r \geq \sup_x\{\langle \nabla g(u^r), x\rangle \mid x \in S,\;\; h(x) \leq \beta\}, \tag{14}$$

where the sequence $\{\delta_r\}$ is such that

$$\delta_r > 0,\; r = 0,1,2,\ldots,\;\; \sum_{r=1}^{\infty} \delta_r < +\infty. \tag{15}$$

Theorem 2. *[12] Let us suppose that the optimal value of the dual problem:*

$$(Q_\gamma): \qquad g(x) \uparrow \max,\quad x \in S,\; h(x) \leq \gamma, \tag{16}$$

is finite for some $\gamma \geq \beta$:

$$\mathcal{V}(Q_\gamma) := \sup_x\{g(x) \mid x \in S,\; h(x) \leq \gamma\} < +\infty. \tag{17}$$

In addition, the function $g(\cdot)$ is convex and continuously differentiable on an open domain Ω containing the set

$$S \cup \{x \in I\!R^n \mid g(x) = 0\}. \tag{18}$$

Then:

i) the sequence $\{u^r\}$ generated by Procedure 2 verifies the condition

$$\lim_{r\to\infty}[\sup_x\{\langle\nabla g(u^r), x-u^r\rangle \mid x\in S,\ h(x)\le\beta\}]=0. \tag{19}$$

ii) For every cluster point u_* of the sequence $\{u^r\}$ the following conditions holds:

$$\langle\nabla g(u_*), x-u_*\rangle\le 0\ \ \forall x\in S:\ h(x)\le\beta, \tag{20}$$

$$g(u_*)\ge 0. \tag{21}$$

iii) If S is closed, then a cluster point u_* turns out to be normally critical (stationary) to the problem (Q_β).

We now show how to construct the point $y(\tilde{x})$ with the help of the sequence $\{u^r\}$. If we consider numbers $\varepsilon>0$ and $r\ge 0$ such that $\delta_r\le\varepsilon/2$ and

$$\langle\nabla g(u^r), u^{r+1}-u^r\rangle\le\varepsilon/2, \tag{22}$$

then we set $y=y(\tilde{x},\varepsilon):=u^r$. It can be shown [12] that the point y verifies the condition

$$\sup_x\{\langle\nabla g(y), x-y\rangle\ \mid\ x\in S,\ h(x)\le\beta\}\le\varepsilon, \tag{23}$$

i.e. y turns out to be an ε-solution to the linearized problem $(LQ(y,\beta))$–(10) where $\beta=h(\tilde{x})$.

Let us now unify the procedures 1 and 2 into one method. In what follows, we consider a feasible point $x_0\in S$, $g(x_0)\ge 0$, and number sequences $\{\delta_r\}$ and $\{\varepsilon_s\}$ verifying (15) and the condition:

$$\varepsilon_s>0,\ s=0,1,2,\dots,\ \ \varepsilon_s\downarrow 0\ \ (s\to+\infty).$$

Special Local Search Method (SLSM).
Step 0. Set $s:=0$, $x^s:=x_0$, $\beta_s:=h(x^s)$.
Step 1. (Procedure 2) Beginning at the point x^s, construct a point $y^s=y(x^s,\varepsilon_s)$:

$$y^s\in S,\ \ g(y^s)\ge 0,\ \ h(y^s)\le\beta_s,$$

which is ε_s-solution to linearized problem $(LQ(y^s,\beta_s))$, i.e.

$$\langle\nabla g(y^s), x-y^s\rangle\le\varepsilon_s\ \ \forall x\in S:\ h(x)\le\beta_s.$$

Step 2. (Stopping criterion) If $g(y^s)\le 0$, **STOP**.
Step 3. (Procedure 1) With the help of the point y^s construct $u:=x(y^s)$ such that

$$u\in S,\ \ g(u)=0,\ \ h(u)<h(y^s)\le\beta_s.$$

Step 4. Set $s:=s+1$, $x^s:=u$, $\beta_s:=h(u)$ and loop to Step 1.

It is easy to see [12] that SLSM described above

(a) either is finite with N iterations, $g(y^N) = 0$;
(b) or generates two sequences $\{x^s\}$ and $\{y^s\}$ with the properties:

$$x^s \in S, \;\; g(x^s) = 0, \;\; y^s \in S, \;\; g(y^s) > 0, \tag{24}$$

$$\beta_{s+1} := h(x^{s+1}) < h(y^s) \leq \beta_s := h(x^s). \tag{25}$$

Besides, the following equalities hold

$$\beta_* := \lim_{s \to \infty} \beta_s = \lim_{s \to \infty} h(y^s). \tag{26}$$

Theorem 3. *Let us consider a convex function $h(\cdot)$ and a convex set S. In addition, suppose the set $F_0 = \{x \in S \mid h(x) \leq h(x_0)\}$ to be bounded and the regularity condition (H_0)–(9) to be fulfilled. Then SLSM:*

(a) either (in the finite case) obtains a point $y^N \in S, \;\; g(y^N) = 0$, that is an ε_N-solution to linearized problem $(LQ(y_N, \beta_N))$ where N is the number of the stopping iteration;
(b) or (in the general case) in addition to the properties (24)–(26) the sequences $\{x^s\}$ and $\{y^s\}$ verify the conditions:

$$0 = g(x^s) = \lim_{s \to \infty} g(y^s), \tag{27}$$

$$x_* = \lim_{s \to \infty} x^s = \lim_{s \to \infty} y^s, \tag{28}$$

with a point $x_* \in I\!R^n, \;\; g(x_*) = 0$.

Furthermore, the point x^* is a solution to the linearized problem $(LQ(x_*, \beta_*))$:

$$\langle \nabla g(x_*), x - x_* \rangle \leq 0 \;\; \forall x \in S, \;\; h(x) \leq \beta_*, \tag{29}$$

and a normal stationary point with respect to the dual problem $(Q(\beta_*))$.

Remark. If one changes the stopping criterion of SLSM $g(y^s) \leq 0$ to the simultaneous fulfilment of the three inequalities below

$$g(y^s) \leq \tau, \;\; \varepsilon_s \leq \tau, \;\; \beta_{s-1} - \beta_s \leq \tau, \tag{30}$$

where τ is a given tolerance, then it is easy to see that SLSM turns out to be finite. Besides, it yields the point y^N with the properties

$$\left.\begin{array}{c} g(y^N) \leq \tau, \;\; h(y^N) \leq \beta_N, \\ \langle \nabla g(y^N), x - y^N \rangle \leq \tau \;\; \forall x \in S, \;\; h(x) \leq \beta_N, \end{array}\right\} \tag{31}$$

which is suitable for a local search.

Note also that SLSM yields an approximate stationary point to the dual problem (Q_β) (for some β), but not for Problem (P), what completely corresponds to the duality Theorem 1 of H. Tuy [1].

Further, in addition to SLSM we consider a variant of a well-known method proposed by Rosen J.B. in 1966 [10]. This method consists in a consecutive solution of linearized problems of type different from $(LQ(u,\beta))$:

$$(PLR_r): \qquad \left.\begin{array}{c} h(x) \downarrow \min, \ \ x \in S, \\ \langle \nabla g(u^r), x - u^r \rangle + g(u^r) \geq 0, \end{array}\right\} \tag{32}$$

where $u^r \in S$ is a given point. The next point u^{r+1} is defined as an exact solution to (32).

In [9, 10], the convergence of the method was investigated. We proposed [12] a modification of Rosen method (MRM) which consists of two procedures.

The first procedure is an approximate solution of problem (PLR_r)–(32) which is obviously convex, if $h(\cdot)$ and S are convex. Then it becomes possible to apply a suitable convex optimization method to find a global (approximate) solution of (32).

The second procedure coincides with Procedure 1 of free descent on the constraint $g = 0$ (see description of SLSM). We were able to prove convergence of the proposed method [12].

In the following section we shall show the extent of the effectiveness of the local search theory proposed so far.

4 Computational testing

In this paragraph we present the results of computational solving by two local search methods presented above of a series of RCP of the following type:

$$\left.\begin{array}{c} \langle c, x \rangle \downarrow min, \\ x \in S \triangleq \{x \in I\!R^n \mid Ax \leq b, \ x \geq 0\}, \\ g(x) \geq 0, \end{array}\right\} \tag{33}$$

with the function g of two forms:

$$g_1(x) \triangleq \| x \|^2 - \langle d, x \rangle - \gamma, \tag{34}$$

$$g_2(x) \triangleq \langle x, Qx \rangle - \langle d, x \rangle - \gamma, \tag{35}$$

Here Q is an $n \times n$ symmetric ($Q = Q^T$) positive definite ($Q > 0$) matrix, with $d \in I\!R^n$, $\gamma \in I\!R$.

The result of computational experiments are presented in Table 1. Some comments on the particularities of the computational implementation are in order.

Note that the Linearized problems (LQ)–(10) and (PLR_r)–(32) are in fact Linear Programming (LP) problems and have been solved by one of the standard methods of LP [11].

		SLSM			MRM		
name	h_0	h_f	PL	$T(10)$	h_f	PL	$T(10)$
jm11x5h1	13,78	-12,000000	2	00.00	-12,000000	3	00.00
	6,00	-11,733224	2	00.00	-11,733224	2	00.00
	388,00	-11,725533	4	00.00	-11,725533	3	00.00
jm30x10h1	21,54	-25,998666	4	00.00	-25,998666	2	00.00
	-3,00	-25,998666	4	00.00	-25,998666	2	00.00
	-19,00	-26,000000	1	00.00	-26,000000	2	00.00
jm11x15h1	802,45	-502,000000	17	00.23	-502,000000	3	00.05
	-18,00	-498,348422	17	00.23	-498,348422	3	00.05
	90,79	-502,000000	16	00.19	-502,000000	4	00.05
jm10x20h1	1156,52	-723,996423	8	00.07	-723,996423	3	00.00
	23,00	-723,996423	8	00.07	-723,996423	3	00.00
	-701,00	-724,000000	7	00.06	-724,000000	2	00.00
sr15x10	17,03	-117,309715	29	00.22	-117,309716	5	00.05
	-29,78	-109,218649	9	00.11	-109,218649	4	00.00
	-5,61	-116,691999	23	00.16	-116,692001	5	00.06
sr20x15	22,86	-156,013872	72	01.26	-156,013874	6	00.17
	-25,41	-146,763729	19	00.38	-146,763729	6	00.16
	-69,45	-146,632051	8	00.16	-146,632052	3	00.11
sr25x15	35,31	*-37,911422*	16	00.33	-37,833231	5	00.11
	3,93	-62,031119	176	03.90	-62,031121	8	00.16
	-10,43	-51,415461	52	01.37	-51,415462	5	00.17
sr30x15	40,24	-13,725476	44	01.31	-13,725477	11	00.33
	4,96	-12,712342	31	00.93	-12,712343	5	00.22
	-24,47	*-10,876227*	2	00.00	-10,335201	9	00.55
sr25x18	40,63	-30,511466	184	07.96	-30,511469	8	00.39
	-10,31	-27,049467	59	02.14	-27,049468	5	00.22
	8,02	-7,707654	35	01.16	-7,707655	5	00.22
sr30x18	37,24	-101,765161	16	00.88	-101,765161	5	00.28
	-21,32	-102,354811	17	01.10	-102,354811	5	00.22
	-25,72	-122,043711	102	09.62	-122,043711	6	00.25
sr25x20	84,75	-262,645181	95	07.19	-262,645181	6	00.39
	-15,31	-276,171787	429	29.17	-276,171789	8	00.60
	-126,16	-240,721359	18	01.16	-240,721361	4	00.28
sr10x40	6197,17	-11644,52285	25	05.50	-11644,52285	4	00.77
	-5,23	-11634,37336	24	06.87	-11634,37336	7	00.88
	1736,43	-11634,37336	24	06.87	-11634,37336	8	00.88

Table 1. Computational results.

In Table 1 we use the following notation: *name* is the test problem name, which expresses the size $(m \times n)$ of the problem and the type of the function g, so that "$jm \ldots h1$" notes the problems with the function $g_1(\cdot)$ while "$sr \ldots$" marks the problems with the function $g_2(\cdot)$). Further, h_0 stands for the goal function value at an initial point. For each method (SLSM or MRM) the goal function value at obtained critical points have been denoted by h_f. Besides, PL means the number of solved linearized problems, and $T(10)$ is the solving time for 10 problems (since the solving time for one problem has turned out to be too small).

It can be easily seen from the Table 1 that SLSM and MRM have found the same τ-critical points in almost all test problems. At the same time the number of solved linearized problems is smaller for MRM in some cases (cf. problems $sr25 \times 15$, $sr25 \times 18$, $sr25 \times 20$). On the other hand, in some of the test problems SLSM found critical points which are better than the for MRM (cf. problems $sr25 \times 15$ and $sr30 \times 15$).

In summary, MRM works faster but SLSM sometimes finds a better critical point. Therefore according the results of computational experiments it would be practical, in order to solve similar problems, to apply a combination of SLSM and MRM. For instance, from the beginning MRM can be applied to get a critical point, and afterwards that point is improved on by SLSM.

5 Conclusion

In this paper, after presenting the principles and the methodology of Nonconvex Optimization:

- we discussed some features of RCP;
- further, we proposed two Local Search Methods for RCP and gave a convergence theorem for one of them;
- finally, we presented a computational testing of the Special Local Search Method (SLSM) and Modified Rosen Method (MRM) on a series of special RCPs.

The analysis of computational testing results led the author to propose a combination of SLSM and MRM.

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