

2 Closure Operators and Lattices

The collection of all varieties of a given type forms a complete lattice. These lattices play an important role in Universal Algebra and in applications, but their study is difficult. Thus we look for new approaches or tools to use in their study. A useful method is to try to study some smaller parts of the large and complex lattice. Such smaller parts should have the same algebraic structure, so we are interested in the study of complete sublattices of a complete lattice.

2.1 Closure Operators and Kernel Operators

In the previous chapter we have seen two examples of operators with closure properties. The generation of subalgebras of a given algebra \mathcal{A} from subsets X of the carrier set A of the algebra \mathcal{A} defines an operator

$$\langle \rangle_{\mathcal{A}} : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$$

on the power set of A , which is extensive, monotone and idempotent. The generation of congruence relations by a binary relation defines an operator

$$\langle \rangle_{\text{Con}\mathcal{A}} : \mathcal{P}(A^2) \rightarrow \mathcal{P}(A^2)$$

which also has these three properties. We define:

Definition 2.1.1 Let A be a set. A mapping $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is called a *closure operator* on A , if for all subsets $X, Y \subseteq A$ the following properties are satisfied:

- (i) $X \subseteq C(X)$ (extensivity),
- (ii) $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$ (monotonicity),

$$(iii) \ C(X) = C(C(X)) \quad (\text{idempotency}).$$

Subsets of A of the form $C(X)$ are called *closed* (with respect to the operator C) and $C(X)$ is said to be the closed set generated by X .

An operator $K : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is said to be a *kernel operator* on A , if it is monotone and idempotent and if instead of (i) the condition

$$(i') \ X \supseteq K(X) \quad (\text{intensity})$$

is satisfied.

Closure and kernel operators are closely related to complete lattices. Indeed, if $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is a closure operator, then the set

$$\mathcal{L}_C := \{X \mid X \subseteq A \text{ and } C(X) = X\} \quad (1)$$

is a complete lattice with respect to the operations

$$\wedge : \mathcal{L}_C \times \mathcal{L}_C \rightarrow \mathcal{L}_C$$

defined by

$$(X, Y) \mapsto X \wedge Y := X \cap Y \quad (2)$$

for arbitrary sets $X, Y \in \mathcal{L}_C$ since $X \cap Y \in \mathcal{L}_C$ whenever $X, Y \in \mathcal{L}_C$ and

$$\vee : \mathcal{L}_C \times \mathcal{L}_C \rightarrow \mathcal{L}_C$$

defined by

$$(X, Y) \mapsto X \vee Y := C(X \cup Y). \quad (3)$$

Actually, $\bigwedge_{j \in J} X_j$ and $\bigvee_{j \in J} X_j$ for arbitrary families $(X_j)_{j \in J}$ of elements from \mathcal{L}_C belong to \mathcal{L}_C . Therefore, every closure operator C on A defines a subset of $\mathcal{P}(A)$ which is a complete lattice. Conversely, if $\mathcal{L} \subseteq \mathcal{P}(A)$ is a complete lattice, then by

$$C_{\mathcal{L}}(Y) := \bigcap \{H \in \mathcal{L} \mid H \supseteq Y\} \quad (4)$$

for every $Y \subseteq A$ a closure operator $C_{\mathcal{L}} : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is defined. The closed sets with respect to $C_{\mathcal{L}}$ are exactly the elements of \mathcal{L} . There is a one-to-one correspondence between complete lattices in $\mathcal{P}(A)$ and closure operators on A . Altogether we have the following well-known theorem.

Theorem 2.1.2 *Let \mathcal{L} be a complete lattice in $\mathcal{P}(A)$. Let C be a closure operator on the set A . Then \mathcal{L}_C defined by (1) is a complete lattice where the operations are defined by (2) and (3) and $C_{\mathcal{L}}$, defined by (4) is a closure operator on A . Moreover, the closed sets with respect to $C_{\mathcal{L}}$ are exactly the elements of \mathcal{L} and the elements of \mathcal{L}_C are exactly the closed sets of C . Both equations*

$$C_{\mathcal{L}_C} = C \text{ and } \mathcal{L}_{C_{\mathcal{L}}} = \mathcal{L}$$

are satisfied.

There is a similar connection between kernel operators on A and complete lattices in $\mathcal{P}(A)$. In this case instead of (4) we have to define

$$K_{\mathcal{L}}(Y) := \bigvee \{H \in \mathcal{L} \mid H \subseteq Y\}. \quad (4')$$

The image $C_{\mathcal{L}}(Y)$ is defined as the infimum (with respect to \subseteq) of all elements in \mathcal{L} containing Y and $K_{\mathcal{L}}(Y)$ is the supremum of all elements from \mathcal{L} which are contained in Y . The partially ordered set $(\mathcal{P}(A); \subseteq)$ itself is a complete lattice with respect to intersection and union of arbitrary families of subsets of A . But as the examples $Sub(\mathcal{A})$ and $Con(\mathcal{A})$ show, the complete lattices \mathcal{L}_C are in general not complete sublattices of $(\mathcal{P}(A); \subseteq)$ since the join operation is not the set-theoretical union. In the next section we want to give a condition which characterizes complete sublattices of a complete lattice.

2.2 Complete Sublattices of a Complete Lattice

In this section we will describe a method to produce complete sublattices of a given complete lattice. We will do so by consideration of the fixed points of a certain kind of closure operators defined on the complete lattice.

In 2.1 we showed that the fixed points of a closure operator $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ on the power set lattice form a complete lattice in $\mathcal{P}(A)$ and that conversely every complete lattice in $\mathcal{P}(A)$ defines a closure operator on A . This can be generalized to arbitrary complete lattices \mathcal{L} instead of $\mathcal{P}(A)$. For any complete lattice \mathcal{L} and

any closure operator $\varphi : \mathcal{L} \rightarrow \mathcal{L}$, we get the complete lattice

$$\mathcal{S}_\varphi := \text{Fix}(\varphi) := \{T \mid \varphi(T) = T\}$$

of all fixed points of φ ; and for any complete lattice $\mathcal{S} \subseteq \mathcal{L}$ we have the closure operator $\varphi_{\mathcal{S}}$ defined by

$$\varphi_{\mathcal{S}}(T) := \bigwedge^{\mathcal{L}} \{T' \in \mathcal{S} \mid T \leq T'\} \text{ for } T \in \mathcal{L}.$$

(Here \leq denotes the partial order of \mathcal{L} , $\bigwedge^{\mathcal{L}}$ is the infimum and $\bigvee^{\mathcal{L}}$ the supremum with respect to \leq .) Moreover, for any complete lattice \mathcal{S} in \mathcal{L} and any closure operator φ , we have $\mathcal{S}_{\varphi_{\mathcal{S}}} = \mathcal{S}$ and $\varphi_{\mathcal{S}_{\varphi}} = \varphi$. The set \mathcal{S}_{φ} is also called a *closure system*.

There is also a dual 1-1 correspondence between kernel operators and complete lattices in \mathcal{L} . For a kernel operator $\psi : \mathcal{L} \rightarrow \mathcal{L}$ and a complete lattice \mathcal{L} , we set

$$\begin{aligned} \mathcal{S}_{\psi} &:= \{T \in \mathcal{L} \mid \psi(T) = T\}; \\ \psi_{\mathcal{S}}(T) &:= \bigvee^{\mathcal{L}} \{T' \in \mathcal{S} \mid T' \leq T\} \text{ for } T \in \mathcal{L}. \end{aligned}$$

Then we have $\text{Fix}(\psi) = \mathcal{S}_{\psi}$ and $\psi_{\mathcal{S}_{\psi}} = \psi$, for any complete lattice \mathcal{S} and any kernel operator ψ on \mathcal{L} . These results date back to A. Tarski ([97]).

Theorem 2.2.1 *Let \mathcal{L} be a complete lattice.*

(i) *If φ is a closure operator on \mathcal{L} which satisfies*

$$\varphi\left(\bigvee^{\mathcal{L}} \{T_j \mid j \in J\}\right) = \bigvee^{\mathcal{L}} \{\varphi(T_j) \mid j \in J\} \quad (*)$$

for every index set J , then the set of all fixed points under φ ,

$$\mathcal{S}_{\varphi} = \{T \in \mathcal{L} \mid \varphi(T) = T\},$$

is a complete sublattice of \mathcal{L} and $\varphi(\mathcal{L}) = \{\varphi(T) \mid T \in \mathcal{L}\} = \mathcal{S}_{\varphi}$.

(ii) *Conversely, if \mathcal{S} is a complete sublattice of \mathcal{L} , then the function $\varphi_{\mathcal{S}}$ which is defined by*

$$\varphi_{\mathcal{S}}(T) := \bigwedge^{\mathcal{L}} \{T' \in \mathcal{S} \mid T \leq T'\}, \text{ for } T \in \mathcal{L},$$

is a closure operator on \mathcal{L} with $\varphi_{\mathcal{S}}(\mathcal{L}) = \mathcal{S}$, and $\varphi_{\mathcal{S}}$ satisfies condition (). Moreover, $\mathcal{S}_{\varphi_{\mathcal{S}}} = \mathcal{S}$ and $\varphi_{\mathcal{S}_{\varphi}} = \varphi$.*

(iii) If ψ is a kernel operator on \mathcal{L} which satisfies

$$\psi(\bigwedge^{\mathcal{L}}\{T_j \mid j \in J\}) = \bigwedge^{\mathcal{L}}\{\psi(T_j) \mid j \in J\} \quad (**)$$

for every index set J , then the set of all fixed points under ψ ,

$$\mathcal{S}_\psi = \{T \in \mathcal{L} \mid \psi(T) = T\},$$

is a complete sublattice of \mathcal{L} and $\psi(\mathcal{L}) = \mathcal{S}_\psi$.

(iv) Conversely, if \mathcal{S} is a complete sublattice of \mathcal{L} then the function $\psi_{\mathcal{S}}$ which is defined by

$$\psi_{\mathcal{S}}(T) := \bigvee^{\mathcal{L}}\{T' \in \mathcal{S} \mid T' \leq T\} \text{ for } T \in \mathcal{L},$$

is a kernel operator on \mathcal{L} with $\psi_{\mathcal{S}}(\mathcal{L}) = \mathcal{S}$, and ψ satisfies the condition (**). Moreover, $\mathcal{S}_{\psi_{\mathcal{S}}} = \mathcal{S}$ and $\psi_{\mathcal{S}_\psi} = \psi$.

Proof: (i) Let φ be a closure operator on \mathcal{L} which satisfies the condition (*). We have to prove that the set of all fixed points under φ is a complete sublattice of \mathcal{L} , that is, that for any index set J , both

$$\bigwedge^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\} \in \mathcal{S}_\varphi \quad \text{and} \quad \bigvee^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\} \in \mathcal{S}_\varphi.$$

It is clear that $\varphi(\bigwedge^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\}) \geq \bigwedge^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\}$. For each $j \in J$ we have $T_j = \varphi(T_j) \geq \varphi(\bigwedge^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\})$, and from this we obtain $\bigwedge^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\} \geq \varphi(\bigwedge^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\})$. Altogether this gives equality, and $\bigwedge^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\} \in \mathcal{S}_\varphi$.

The fact that φ satisfies the join condition (*) gives $\bigvee^{\mathcal{L}}\{T_j \in \mathcal{S}_\varphi \mid j \in J\} \in \mathcal{S}_\varphi$. Thus we have a sublattice of \mathcal{L} . Clearly, $\mathcal{S}_\varphi \subseteq \varphi(\mathcal{L})$. Since φ is idempotent, $\varphi(T)$ is in \mathcal{S}_φ for all $T \in \mathcal{L}$. This shows that $\varphi(\mathcal{L}) = \mathcal{S}_\varphi$.

(ii) It is easy to see that $\varphi_{\mathcal{S}}$ is a closure operator. We need only show that this closure operator satisfies condition (*) and that $\varphi_{\mathcal{S}}(\mathcal{L}) = \mathcal{S}$. We prove the latter fact first. Since \mathcal{S} is a complete sublattice of

\mathcal{L} , we have $\varphi_{\mathcal{S}}(\mathcal{L}) \subseteq \mathcal{S}$. For the opposite inclusion, we see that for any $T \in \mathcal{S}$

$$\varphi_{\mathcal{S}}(T) = \bigwedge^{\mathcal{L}} \{T' \in \mathcal{S} \mid T \leq T'\} = T.$$

Thus $\mathcal{S} \subseteq \varphi_{\mathcal{S}}(\mathcal{L})$, and altogether we have $\mathcal{S} = \varphi_{\mathcal{S}}(\mathcal{L})$. Since for each $j \in J$ we have $T_j \leq \varphi_{\mathcal{S}}(T_j)$ and $\varphi_{\mathcal{S}}(T_j) \in \mathcal{S}$, the set

$$\bigvee^{\mathcal{L}} \{\varphi_{\mathcal{S}}(T_j) \mid j \in J\}$$

is an upper bound of the set $\{T_j \mid j \in J\}$. Therefore

$$\bigvee^{\mathcal{L}} \{T_j \mid j \in J\} \leq \bigvee^{\mathcal{L}} \{\varphi_{\mathcal{S}}(T_j) \mid j \in J\}.$$

The right hand side is an element of \mathcal{S} . By application of $\varphi_{\mathcal{S}}$ on both sides we get

$$\varphi_{\mathcal{S}}(\bigvee^{\mathcal{L}} \{T_j \mid j \in J\}) \leq \bigvee^{\mathcal{L}} \{\varphi_{\mathcal{S}}(T_j) \mid j \in J\}.$$

Since $\bigvee^{\mathcal{L}} \{T_j \mid j \in J\} \geq T_j$, we have $\varphi_{\mathcal{S}}(\bigvee^{\mathcal{L}} \{T_j \mid j \in J\}) \geq \varphi_{\mathcal{S}}(T_j)$ for all $j \in J$. Thus also $\varphi_{\mathcal{S}}(\bigvee^{\mathcal{L}} \{T_j \mid j \in J\}) \geq \bigvee^{\mathcal{L}} \{\varphi_{\mathcal{S}}(T_j) \mid j \in J\}$, giving the required equality.

(iii) and (iv) can be proved similar to (i) and (ii).

The equations are also easy to realize. ■

2.3 Galois Connections and Complete Lattices

In chapter I we considered the Galois connection (Id, Mod) between identities and model classes. We generalize this example and consider Galois connections as pairs of mappings with special properties between the power sets of two sets.

Definition 2.3.1 A *Galois connection* between the sets A and B is a pair (μ, ι) of mappings between the power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$,

$$\mu : \mathcal{P}(A) \rightarrow \mathcal{P}(B) \text{ and } \iota : \mathcal{P}(B) \rightarrow \mathcal{P}(A),$$

such that for all $T, T' \subseteq A$ and all $S, S' \subseteq B$ the following conditions are satisfied:

- (i) $T \subseteq T' \Rightarrow \mu(T) \supseteq \mu(T')$ and $S \subseteq S' \Rightarrow \iota(S) \supseteq \iota(S')$;
- (ii) $T \subseteq \iota\mu(T)$ and $S \subseteq \mu\iota(S)$.

Galois connections are also related to closure operators, as the following proposition shows.

Theorem 2.3.2 *Let the pair (μ, ι) with*

$$\mu : \mathcal{P}(A) \rightarrow \mathcal{P}(B) \text{ and } \iota : \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

be a Galois connection between the sets A and B . Then:

- (i) $\mu\mu = \mu$ and $\iota\iota = \iota$;
- (ii) $\iota\mu$ and $\mu\iota$ are closure operators on A and B respectively;
- (iii) *The sets closed under $\iota\mu$ are precisely the sets of the form $\iota(S)$, for some $S \subseteq B$; the sets closed under $\mu\iota$ are precisely the sets of the form $\mu(T)$, for some $T \subseteq A$.*

Proof: (i) Let $T \subseteq A$. By the second Galois connection property, we have $T \subseteq \iota\mu(T)$. By the first property, applying μ to this, gives $\mu(T) \supseteq \mu\iota\mu(T)$. But we also have $\mu(T) \subseteq \mu\iota(\mu(T))$, by the second Galois connection property applied to the set $\mu(T)$. This gives us $\mu\iota\mu(T) = \mu(T)$. The second claim can be proved similarly.

(ii) The extensivity of $\iota\mu$ and $\mu\iota$ follows from the second Galois connection property. From the first property we see that

$$T \subseteq T' \Rightarrow \mu(T) \supseteq \mu(T') \Rightarrow \iota\mu(T) \subseteq \iota\mu(T'),$$

since $\mu(T)$ and $\mu(T')$ are subsets of B ; and in the analogous way we get from $S \subseteq S'$ the inclusion $\mu\iota(S) \subseteq \mu\iota(S')$. Applying μ to the equation $\iota\mu = \iota$ from (i) gives us the idempotency of $\mu\iota$, and similarly for $\iota\mu$.

(iii) This is straightforward to verify. ■

Galois connections are “induced” by relations. A relation between the sets A and B is simply a subset of $A \times B$. Any relation R

between A and B induces a Galois connection, as follows. We can define the mappings

$$\mu_R : \mathcal{P}(A) \rightarrow \mathcal{P}(B), \quad \iota_R : \mathcal{P}(B) \rightarrow \mathcal{P}(A),$$

by

$$\begin{aligned} \mu_R(T) &:= \{y \in B \mid \forall x \in T \ ((x, y) \in R)\}, \\ \iota_R(S) &:= \{x \in A \mid \forall y \in S \ ((x, y) \in R)\}. \end{aligned}$$

We verify that the pair (μ_R, ι_R) is a Galois connection. Indeed, if $T \subseteq T'$ and if $y \in \mu_R(T')$, that is, if for all $x \in T'$ we have $(x, y) \in R$, then also for all $x \in T$ we have $(x, y) \in R$ and this means, $y \in \mu_R(T)$. This shows that $\mu_R(T') \subseteq \mu_R(T)$. Similarly one proves the second condition from Definition 2.3.1 (i). Consider

$$\begin{aligned} \iota_R \mu_R(T) &= \{x \in A \mid \forall y \in \mu_R(T) ((x, y) \in R)\} \\ &= \{x \in A \mid \forall y \in \{z \in B \mid \forall x \in T ((x, z) \in R)\} ((x, y) \in R)\}. \end{aligned}$$

It is easy to see that $T \subseteq \iota_R \mu_R(T)$ and $S \subseteq \mu_R \iota_R(S)$.

If conversely (μ, ι) is a Galois connection between A and B then we construct the relation

$$R_{(\mu, \iota)} := \bigcup_{T \subseteq A} \{T \times \mu(T)\} \subseteq A \times B$$

and the Galois connection $(\mu_{R_{(\mu, \iota)}}, \iota_{R_{(\mu, \iota)}})$ defined by

$$\mu_{R_{(\mu, \iota)}}(T) := \{y \in B \mid \forall x \in T ((x, y) \in R_{(\mu, \iota)})\}$$

and

$$\iota_{R_{(\mu, \iota)}}(S) := \{x \in A \mid \forall y \in S ((x, y) \in R_{(\mu, \iota)})\}.$$

It turns out that $(\mu_{R_{(\mu, \iota)}}, \iota_{R_{(\mu, \iota)}}) = (\mu, \iota)$. Therefore, there is a one-to-one correspondence between relations between A and B and Galois connections between A and B .

We need also the following well-known proposition (see e.g. [37], exercise 2.4.2):

Lemma 2.3.3 *Let $R \subseteq A \times B$ be a relation between the sets A and B and let (μ, ι) be the Galois connection between A and B induced by R . Then for any families $\{T_j \subseteq A \mid j \in J\}$ and $\{S_j \subseteq B \mid j \in J\}$, the following equalities hold:*

$$\text{a) } \mu\left(\bigcup_{j \in J} T_j\right) = \bigcap_{j \in J} \mu(T_j),$$

$$\text{b) } \iota\left(\bigcup_{j \in J} S_j\right) = \bigcap_{j \in J} \iota(S_j).$$

2.4 Galois Closed Subrelations

In section 2.2 we developed a method to produce complete sublattices of a given complete lattice. If $R \subseteq A \times B$ is a relation between two sets A and B and if (μ, ι) is the Galois connection induced by R , then the fixed points of the closure operators $\mu\iota$ and $\iota\mu$ form two complete lattices which are dually isomorphic. Now we consider a subrelation R' of the initial relation R , from which we obtain a new Galois connection and two new complete lattices. We describe a property of the subrelation R' which is sufficient to guarantee that the new complete lattices will be complete sublattices of the original lattices. This property is called the *Galois closed subrelation property*. Moreover, we show that any complete sublattices of our original lattices arise in this way.

Definition 2.4.1 Let R and R' be relations between sets A and B , and let (μ, ι) and (μ', ι') be the Galois connections between A and B induced by R and R' , respectively. The relation R' is called a *Galois closed subrelation of R* if:

- 1) $R' \subseteq R$ and
- 2) $\forall T \subseteq A, \forall S \subseteq B \quad (\mu'(T) = S \text{ and } \iota'(S) = T \Rightarrow \mu(T) = S \text{ and } \iota(S) = T).$

The following equivalent characterizations of Galois closed subrelations can easily be derived (see e.g. [45], [4])

Proposition 2.4.2 Let $R' \subseteq R$ be relations between sets A and B . Then the following are equivalent:

- (i) R' is a Galois closed subrelation of R ;
- (ii) For any $T \subseteq A$, if $\iota'\mu'(T) = T$ then $\mu(T) = \mu'(T)$, and for any $S \subseteq B$, if $\mu'\iota'(S) = S$ then $\iota(S) = \iota'(S)$;
- (iii) For all $T \subseteq A$ and for all $S \subseteq B$ the equations $\iota'\mu'(T) = \iota\mu'(T)$ and $\mu'\iota'(S) = \mu\iota'(S)$ are satisfied.

Proof: (i) \Rightarrow (ii) Define S to be the set $\mu'(T)$. Then from $\iota'\mu'(T) = T$, i.e. $\iota'(S) = T$ and $\mu'(T) = S$ by the definition of a Galois closed subrelation we obtain $\mu(T) = S$ and $\iota(S) = T$, i.e. $\mu(T) = \mu'(T)$

and $\iota(S) = \iota'(S)$. The claim for subsets S of B can be proved similarly.

(ii) \Rightarrow (iii) From $\mu'\iota'\mu'(T) = \mu'(T)$ and $\iota'\mu'\iota'(S) = \iota'(S)$ (Proposition 2.3.2) using (ii) we obtain $\mu\iota'(S) = \mu'\iota'(S)$ and $\iota\mu'(T) = \iota'\mu'(T)$.

(iii) \Rightarrow (i) If $\mu'(T) = S$ and $\iota'(S) = T$, then there follows:

$$\begin{aligned} \mu'\iota'(S) &= \mu'(T), & \iota'\mu'(T) &= \iota'(S) \quad \text{and} \\ \mu\iota'(S) &= \mu(T), & \iota\mu'(T) &= \iota(S). \end{aligned}$$

Then by condition (iii) we get

$$\mu(T) = \mu'(T) = S \quad \text{and} \quad \iota(S) = \iota'(S) = T.$$

This shows that R' is a Galois closed subrelation of R . ■

Let $\mathcal{H}_{\iota\mu}$ and $\mathcal{H}_{\mu\iota}$ be the complete lattices induced by the Galois connection (μ, ι) . We show that any Galois closed subrelation R' of the relation R yields a lattice of closed subsets of A which is a complete sublattice of the corresponding lattice $\mathcal{H}_{\iota\mu}$ for R . Conversely, we also show that any complete sublattice of the lattice $\mathcal{H}_{\iota\mu}$ occurs as the lattice of closed sets induced from some Galois closed subrelation of R . Dual results of course hold for the set B .

Theorem 2.4.3 *Let $R \subseteq A \times B$ be a relation between sets A and B , with induced Galois connection (μ, ι) . Let $\mathcal{H}_{\iota\mu}$ be the corresponding lattice of closed subsets of A .*

(i) *If $R' \subseteq A \times B$ is a Galois closed subrelation of R , then the class $\mathcal{U}_{R'} := \mathcal{H}_{\iota'\mu'}$ is a complete sublattice of $\mathcal{H}_{\iota\mu}$.*

(ii) *If \mathcal{U} is a complete sublattice of $\mathcal{H}_{\iota\mu}$, then the relation*

$$R_{\mathcal{U}} := \bigcup \{T \times \mu(T) \mid T \in \mathcal{U}\}$$

is a Galois closed subrelation of R .

(iii) *For any Galois closed subrelation R' of R and any complete sublattice \mathcal{U} of $\mathcal{H}_{\iota\mu}$, we have*

$$\mathcal{U}_{R_{\mathcal{U}}} = \mathcal{U} \quad \text{and} \quad R_{\mathcal{U}_{R'}} = R'.$$

Proof: (i) We begin by verifying that any subset of A which is closed under the operator $\iota'\mu'$ is also closed under $\iota\mu$, so that the lattice $\mathcal{H}_{\iota'\mu'}$ is at least a subset of $\mathcal{H}_{\iota\mu}$. Indeed, if $T \in \mathcal{H}_{\iota'\mu'}$, so that $\iota'\mu'(T) = T$, then by Proposition 2.4.2, we have

$$\iota\mu(T) = \iota\mu'(T) = \iota'\mu'(T) = T$$

and therefore, $\mathcal{H}_{\iota'\mu'} \subseteq \mathcal{H}_{\iota\mu}$. Since by 2.1 the infimum in a complete lattice which is included in the power set lattice of a set is the set-theoretical intersection we have $\bigwedge_{\mathcal{H}_{\iota\mu}} \{T_j \mid j \in J\} = \bigcap_{j \in J} T_j = \bigwedge_{\mathcal{H}_{\iota'\mu'}} \{T_j \mid j \in J\}$ and this means, $\mathcal{H}_{\iota'\mu'}$ is closed under the infimum operation of $\mathcal{H}_{\iota\mu}$.

Now we consider the join. By repeated use of Lemma 2.3.3 and Proposition 2.4.2, we see that $\bigvee_{\mathcal{H}_{\iota'\mu'}} \{T_j \mid j \in J\} = \iota'\mu'(\bigcup_{j \in J} T_j) = \iota\mu'(\bigcup_{j \in J} T_j) = \iota(\bigcap_{j \in J} \mu'(T_j)) = \iota(\bigcap_{j \in J} \mu(T_j)) = \iota\mu(\bigcup_{j \in J} T_j) = \bigvee_{\mathcal{H}_{\iota\mu}} \{T_j \mid j \in J\}$ and therefore $\mathcal{H}_{\iota'\mu'}$ is closed under the supremum operation of $\mathcal{H}_{\iota\mu}$.

(ii) Now let \mathcal{U} be any complete sublattice of $\mathcal{H}_{\iota\mu}$. We consider the relation

$$R_{\mathcal{U}} := \bigcup \{T \times \mu(T) \mid T \in \mathcal{U}\},$$

which we will prove is a Galois closed subrelation of R . First, for each non-empty $T \in \mathcal{U}$ we have $\mu(T) = \{s \in B \mid \forall t \in T((t, s) \in R)\}$, so that $T \times \mu(T) \subseteq R$. Therefore $R_{\mathcal{U}} \subseteq R$. To show that the second condition of the definition of a Galois closed subrelation is met, we let (μ', ι') be the Galois connection between sets A and B induced by $R_{\mathcal{U}}$, and assume that $\mu'(T) = S$ and $\iota'(S) = T$ for some $T \subseteq A$ and $S \subseteq B$. Our goal is to prove that

$$\mu(T) = S \quad \text{and} \quad \iota(S) = T.$$

Let $T \in \mathcal{U}$. By definition we have

$$\mu'(T) = \{s \in B \mid \forall t \in T((t, s) \in R_{\mathcal{U}})\}.$$

This means that $\mu'(T)$ is the greatest subset of B with $T \times \mu'(T) \subseteq R_{\mathcal{U}}$. Now from the definition of $R_{\mathcal{U}}$ we have $T \times \mu(T) \subseteq R_{\mathcal{U}}$.

Therefore, $\mu(T) \subseteq \mu'(T)$. The opposite inclusion also holds since $R_{\mathcal{U}} \subseteq R$. Altogether we have $\mu'(T) = \mu(T)$. If $\mu'(T) = S$, then $\iota(S) = \iota\mu'(T) = \iota\mu(T) = T$ since $T \in \mathcal{U} \subseteq \mathcal{H}_{\iota\mu}$.

(iii) Now we must show that for any complete sublattice \mathcal{U} of $\mathcal{H}_{\iota\mu}$, and any Galois closed subrelation R' of R , we have $\mathcal{U}_{R_{\mathcal{U}}} = \mathcal{U}$ and $R_{\mathcal{U}_{R'}} = R'$. We know that $\mathcal{U}_{R_{\mathcal{U}}} = \mathcal{H}_{\iota'\mu'}$, the lattice of subsets of A closed under the closure operator $\iota'\mu'$ induced from the relation $R_{\mathcal{U}}$. This means that $T \in \mathcal{U}_{R_{\mathcal{U}}}$ iff $\iota'\mu'(T) = T$. Let $T \in \mathcal{U}$. Then as before we have

$$\iota'\mu'(T) = \iota'\mu(T) = \iota\mu(T) = T$$

and this shows $T \in \mathcal{H}_{\iota'\mu'}$. Therefore $\mathcal{U} \subseteq \mathcal{U}_{R_{\mathcal{U}}}$. Now let $T \in \mathcal{U}_{R_{\mathcal{U}}}$ and let S be the set $\mu'(T)$. Then we have $\iota'(S) = T$, and since $R_{\mathcal{U}}$ is a Galois closed subrelation of R we conclude that

$$\mu(T) = S \quad \text{and} \quad \iota(S) = T.$$

If $T = \emptyset$ then $T \in \mathcal{U}$, and for $T \neq \emptyset$ for each $t \in T$ we define

$$D_t := \bigcap \{T' \in \mathcal{U} \mid t \in T' \text{ and } S \subseteq \mu(T')\}.$$

We can show that $T = \bigcup_{t \in T} D_t$. But now $T = \iota\mu(T) = \iota\mu(\bigcup_{t \in T} D_t) = \bigcup \{D_t \mid t \in T\} \in \mathcal{U}$. This shows the other direction $\mathcal{U}_{R_{\mathcal{U}}} \subseteq \mathcal{U}$.

Let R' be a Galois closed subrelation of R , and set

$$\begin{aligned} \mathcal{U}_{R'} &:= \mathcal{H}_{\iota'\mu'} = \{T \subseteq A \mid \iota'\mu'(T) = T\}, \quad \text{and} \\ R_{\mathcal{U}_{R'}} &:= \bigcup \{T \times \mu(T) \mid T \in \mathcal{U}_{R'}\}. \end{aligned}$$

We will show that $R_{\mathcal{U}_{R'}} = R'$. First, if $(t, s) \in R'$ then $s \in \mu'(\{t\})$. Setting $S := \mu'(\{t\})$, we have $s \in S$ and $\iota'(S) = \iota'\mu'(\{t\})$. Now taking $T := \iota'\mu'(\{t\})$, we have $\iota'\mu'(T) = T$, so $T \in \mathcal{U}_{R'}$, $\mu'(T) = S$ and $\iota'(S) = T$. Therefore $\mu(T) = S$ and $\iota(S) = T$. Since $t \in \iota'\mu'(\{t\}) = T$ and $s \in S = \mu(T)$, we get $(t, s) \in T \times \mu(T)$ and $T \in \mathcal{U}_{R'}$. Hence $(t, s) \in R_{\mathcal{U}_{R'}}$, and we have shown that $R' \subseteq R_{\mathcal{U}_{R'}}$.

To show the opposite inclusion, let $T \in \mathcal{U}_{R'}$, and let $S = \mu(T)$. Then we can prove

$$\mu'(T) = \mu(T) = S \quad \text{and} \quad \iota'(S) = \iota(S) = \iota\mu(T) = T.$$

Therefore $T \times \mu(T) \subseteq R'$, and $R_{\mathcal{U}_{R'}} \subseteq R'$. Altogether, we have $R_{\mathcal{U}_{R'}} = R'$. This completes the proof of (iii), and of our theorem. ■

2.5 Conjugate Pairs of Additive Closure Operators

In section 1 we saw that any closure operator γ defined on a set A gives us a complete lattice, the lattice \mathcal{H}_γ of all γ -closed subsets of A . In this lattice, the meet operation is the operation of intersection. The join operation however is not usually just the union; we have

$$\bigvee \mathcal{B} = \bigcap \{H \in \mathcal{H}_\gamma \mid H \supseteq \bigcup \mathcal{B}\}$$

for every $\mathcal{B} \subseteq \mathcal{H}_\gamma$. One situation when we do have the join operation equal to union is the following.

Definition 2.5.1 A closure operator γ defined on a set A is said to be *completely additive* if $\gamma(T) = \bigcup_{a \in T} \gamma(a)$ for all $T \subseteq A$. (Here we write $\gamma(a)$ for $\gamma(\{a\})$.)

We can show easily that when γ is an additive closure operator, the least upper bound operation on the lattice \mathcal{H}_γ agrees with $\bigcup \mathcal{B}$ (see D. Dikranjan and E. Giuli, [41] or M. Reichel, [91]). Indeed, we always have $\bigcup \mathcal{B} \subseteq \gamma(\bigcup \mathcal{B})$ because of the extensivity of γ . Conversely, if $a \in \bigcup \mathcal{B}$ then $a \in B$ for some set $B \in \mathcal{B}$, and since $B \in \mathcal{H}_\gamma$ we have $\gamma(a) \subseteq \bigcup \mathcal{B}$ and $\gamma(\bigcup \mathcal{B}) = \bigcup_{a \in \bigcup \mathcal{B}} \gamma(a) \subseteq \bigcup_{a \in \bigcup \mathcal{B}} \bigcup \mathcal{B} = \bigcup \mathcal{B}$.

This means that $\bigcup \mathcal{B}$ is γ -closed and $\bigvee \mathcal{B} = \bigcup \mathcal{B}$. In other words, when γ is an completely additive closure operator on A , the corresponding closure system forms a complete sublattice of the lattice $(\mathcal{P}(A); \cap, \cup)$ of all subsets of A . Now we study a situation where we have two completely additive closure operators which are closely connected. These results can be found also in [36] and in [19], respectively.

Definition 2.5.2 Let γ_1 be a closure operator defined on the set A and let γ_2 be a closure operator defined on the set B . Let $R \subseteq A \times B$ be a relation between A and B . Then γ_1 and γ_2 are called *conjugate* with respect to R if for all $t \in A$ and all $s \in B$ we have $\gamma_1(t) \times \{s\} \subseteq R$ iff $\{t\} \times \gamma_2(s) \subseteq R$.

When the two operators are completely additive, we can extend this definition given in terms of individual elements to sets of elements. Thus when (γ_1, γ_2) is a pair of additive closure operators,

γ_1 on A and γ_2 on B , and they are conjugate with respect to a relation $R \subseteq A \times B$, then for all $X \subseteq A$ and all $Y \subseteq B$ we have $X \times \gamma_2(Y) \subseteq R$ if and only if $\gamma_1(X) \times Y \subseteq R$.

Examples of conjugate pairs of additive closure operators will be given in the next chapter. In this section we develop the general theory of such operators. We assume that we have two sets A and B , and that R is a relation between A and B . This relation induces a Galois connection (μ, ι) between A and B , for which the two maps $\mu\iota$ and $\iota\mu$ are closure operators. Moreover, the pair $(\mu\iota, \iota\mu)$ is always conjugate with respect to the original relation R . But $\mu\iota$ and $\iota\mu$ need not be completely additive in general.

Definition 2.5.3 Let $\gamma := (\gamma_1, \gamma_2)$ be a conjugate pair of completely additive closure operators, with respect to a relation $R \subseteq A \times B$. Let R_γ be the following relation between A and B :

$$R_\gamma := \{(t, s) \in A \times B \mid \gamma_1(t) \times \{s\} \subseteq R\}.$$

Now we have two relations and Galois connections between A and B . The relation R induces a Galois connection (μ, ι) between A and B and the new relation R_γ induces a second Galois connection, which we shall denote by $(\mu_\gamma, \iota_\gamma)$. The following theorem gives some properties relating the two Galois connections.

Theorem 2.5.4 *Let $\gamma = (\gamma_1, \gamma_2)$ be a conjugate pair of completely additive closure operators with respect to $R \subseteq A \times B$. Then for all $T \subseteq A$ and $S \subseteq B$, the following properties hold:*

- (i) $\mu_\gamma(T) = \mu(\gamma_1(T))$,
- (ii) $\mu_\gamma(T) \subseteq \mu(T)$,
- (iii) $\gamma_2(\mu_\gamma(T)) = \mu_\gamma(T)$,
- (iv) $\gamma_1(\iota(\mu_\gamma(T))) = \iota(\mu_\gamma(T))$,
- (v) $\mu_\gamma(\iota_\gamma(S)) = \mu(\iota(\gamma_2(S)))$; *and dually,*
- (i') $\iota_\gamma(S) = \iota(\gamma_2(S))$,

$$(ii') \quad \iota_\gamma(S) \subseteq \iota(S),$$

$$(iii') \quad \gamma_1(\iota_\gamma(S)) = \iota_\gamma(S),$$

$$(iv') \quad \gamma_2(\mu(\iota_\gamma(S))) = \mu(\iota_\gamma(S)),$$

$$(v') \quad \iota_\gamma(\mu_\gamma(T)) = \iota(\mu(\gamma_1(T))).$$

Proof: We will prove only (i)-(v), the proofs of the other propositions are dual.

(i) By definition,

$$\begin{aligned} \mu_\gamma(T) &= \{b \in B \mid \forall a \in T ((a, b) \in R_\gamma)\} \\ &= \{b \in B \mid \forall a \in T (\gamma_1(a) \times \{b\} \subseteq R)\} \\ &= \{b \in B \mid \forall a \in \gamma_1(T) ((a, b) \in R)\} = \mu(\gamma_1(T)). \end{aligned}$$

(ii) Since γ_1 is a closure operator, we have $T \subseteq \gamma_1(T)$; and thus, since μ reverses inclusions, $\mu(T) \supseteq \mu(\gamma_1(T))$. Using (i) we obtain $\mu_\gamma(T) \subseteq \mu(T)$.

(iii) Extensivity of γ_2 implies $\mu_\gamma(T) \subseteq \gamma_2(\mu_\gamma(T))$. Now let $S \subseteq \mu_\gamma(T)$. Then for all $s \in S$ and for all $t \in T$, $(t, s) \in R_\gamma$, and by definition of R_γ we get $\{t\} \times \gamma_2(s) \subseteq R$. Idempotency of γ_2 gives $\{t\} \times \gamma_2(\gamma_2(s)) \subseteq R$ and thus $\gamma_2(s) \subseteq \mu_\gamma(T)$ for all $s \in S$. By additivity of γ_2 we get $\gamma_2(S) = \bigcup_{s \in S} \gamma_2(s) \subseteq \mu_\gamma(T)$; and taking $S = \mu_\gamma(T)$ we obtain $\gamma_2(\mu_\gamma(T)) \subseteq \mu_\gamma(T)$. Altogether we have the equality $\mu_\gamma(T) = \gamma_2(\mu_\gamma(T))$.

$$(iv) \quad \gamma_1(\iota(\mu_\gamma(T))) \stackrel{(iii)}{=} \gamma_1(\iota(\gamma_2(\mu_\gamma(T)))) \stackrel{(i')}{=} \gamma_1(\iota_\gamma(\mu_\gamma(T)))$$

$$\stackrel{(iii')}{=} \iota_\gamma(\mu_\gamma(T)) \stackrel{(i')}{=} \iota(\gamma_2(\mu_\gamma(T))) \stackrel{(iii)}{=} \iota(\mu_\gamma(T)).$$

$$(v) \quad \mu_\gamma(\iota_\gamma(S)) \stackrel{(i)}{=} \mu(\gamma_1(\iota_\gamma(S))) \stackrel{(iii')}{=} \mu(\iota_\gamma(S)) \stackrel{(i')}{=} \mu(\iota(\gamma_2(S))). \quad \blacksquare$$

The next theorem gives for sets $T \subseteq A$ which are closed under $\iota\mu$, i.e. with $\iota(\mu(T)) = T$ four equivalent characterizations.

Theorem 2.5.5 (*Main Theorem for Conjugate Pairs of Additive Closure Operators*) *Let R be a relation between sets A and B , with corresponding Galois connection (μ, ι) . Let $\gamma = (\gamma_1, \gamma_2)$ be a conjugate pair of completely additive closure operators with respect to the relation R . Then for all sets $T \subseteq A$ with $\iota(\mu(T)) = T$ the following*

propositions (i) - (iv) are equivalent; and dually, for all sets $S \subseteq B$ with $\mu(\iota(S)) = S$, propositions (i') - (iv') are equivalent:

- (i) $T = \iota_\gamma(\mu_\gamma(T))$,
- (ii) $\gamma_1(T) = T$,
- (iii) $\mu(T) = \mu_\gamma(T)$,
- (iv) $\gamma_2(\mu(T)) = \mu(T)$; and dually,
- (i') $S = \mu_\gamma(\iota_\gamma(S))$,
- (ii') $\gamma_2(S) = S$,
- (iii') $\iota(S) = \iota_\gamma(S)$,
- (iv') $\gamma_1(\iota(S)) = \iota(S)$.

Proof: We prove the equivalence of (i), (ii), (iii) and (iv); the equivalence of the four dual statements can be proved dually.

(i) \Rightarrow (ii) We always have $T \subseteq \gamma_1(T)$, since γ_1 is a closure operator. Since $\iota\mu$ is a closure operator we also have $\gamma_1(T) \subseteq \iota\mu(\gamma_1(T)) = \iota_\gamma(\mu_\gamma(T)) = T$, by Theorem 2.5.4 (v').

(ii) \Rightarrow (iii) We have $\mu(T) = \mu(\gamma_1(T)) = \mu_\gamma(T)$ by (ii) and Theorem 2.5.4 (i).

(iii) \Rightarrow (iv) We have $\gamma_2(\mu(T)) = \gamma_2(\mu_\gamma(T)) = \mu_\gamma(T)$, using Theorem 2.5.4 (iii).

(iv) \Rightarrow (i) Since the $\iota_\gamma\mu_\gamma$ -closed sets are exactly the sets of the form $\iota_\gamma(S)$, we have to find a set $S \subseteq B$ with $T = \iota_\gamma(S)$. But we have $\iota_\gamma(\mu(T)) = \iota(\gamma_2(\mu(T))) = \iota(\mu(T)) = T$, by Theorem 2.5.4 (i') and our assumption that T is $\iota\mu$ -closed. ■

Before using this Main Theorem to produce our complete sublattices, we need the following additional properties.

Theorem 2.5.6 *Let R be a relation between sets A and B , with Galois connection (μ, ι) . Let $\gamma = (\gamma_1, \gamma_2)$ be a conjugate pair of*

completely additive closure operators with respect to R . Then for all sets $T \subseteq A$ and $S \subseteq B$, the following properties hold:

- (i) $\gamma_1(T) \subseteq \iota(\mu(T)) \Leftrightarrow \iota(\mu(T)) = \iota_\gamma(\mu_\gamma(T));$
- (ii) $\gamma_1(T) \subseteq \iota(\mu(T)) \Leftrightarrow \gamma_1(\iota(\mu(T))) = \iota(\mu(T));$
- (i') $\gamma_2(S) \subseteq \mu(\iota(S)) \Leftrightarrow \mu(\iota(S)) = \mu_\gamma(\iota_\gamma(S));$
- (ii') $\gamma_2(S) \subseteq \mu(\iota(S)) \Leftrightarrow \gamma_2(\mu(\iota(S))) = \mu(\iota(S)).$

Proof: We prove only (i') and (ii'); the others are dual.

(i') Suppose that $\gamma_2(S) \subseteq \mu(\iota(S))$. Since $\mu\iota$ is a closure operator we have $\mu(\iota(S)) = \mu(\iota(\mu(\iota(S)))) \supseteq \mu(\iota(\gamma_2(S))) = \mu_\gamma(\iota_\gamma(S))$, by our assumption and by Theorem 2.5.4 (v). Also $S \subseteq \gamma_2(S)$, and hence we have $\mu(\iota(S)) \subseteq \mu(\iota(\gamma_2(S))) = \mu_\gamma(\iota_\gamma(S))$, again by Theorem 2.5.4 (v). For the converse we have $\gamma_2(S) \subseteq \mu(\iota(\gamma_2(S))) = \mu_\gamma(\iota_\gamma(S)) = \mu(\iota(S))$, using the extensivity of $\mu\iota$, Theorem 2.5.4 (v) and our assumption.

(ii') Let $\gamma_2(S) \subseteq \mu(\iota(S))$. Then $S \subseteq \gamma_2(S)$ implies $\gamma_2(\mu(\iota(S))) \subseteq \gamma_2(\mu(\iota(\gamma_2(S))))$. We also have $\gamma_2(\mu(\iota(\gamma_2(S)))) = \gamma_2(\mu(\iota_\gamma(S)))$ by Theorem 2.5.4 (i'), and $\gamma_2(\mu(\iota(\gamma_2(S)))) = \mu(\iota_\gamma(S))$ by Theorem 2.5.4 (iv'). In addition, $\mu(\iota_\gamma(S)) = \mu(\iota(\gamma_2(S))) \subseteq \mu(\iota(\mu(\iota(S)))) = \mu(\iota(S))$. Altogether we obtain $\gamma_2(\mu(\iota(S))) \subseteq \mu(\iota(S))$. The opposite inclusion is always true, since γ_2 is a closure operator. Conversely, $S \subseteq \mu(\iota(S))$ implies $\gamma_2(S) \subseteq \gamma_2(\mu(\iota(S))) = \mu(\iota(S))$, by the extensivity of $\mu\iota$, the monotonicity of γ_2 and our assumption. ■

Now we are ready to produce our complete sublattices. We know that from the original relation R and Galois connection (μ, ι) we have two (dually isomorphic) complete lattices of closed sets, the lattices \mathcal{H}_μ and $\mathcal{H}_{\iota\mu}$. We also get two complete lattices of closed sets from the new Galois connection $(\mu_\gamma, \iota_\gamma)$ induced by R_γ . Our result is that each new complete lattice is in fact a complete sublattice of the corresponding original complete lattice.

Theorem 2.5.7 *Let R be a relation between A and B , with induced Galois connection (μ, ι) . Let $\gamma = (\gamma_1, \gamma_2)$ be a conjugate pair of completely additive closure operators with respect to R . Then the*

lattice $\mathcal{H}_{\mu_\gamma \iota_\gamma}$ of sets closed under $\mu_\gamma \iota_\gamma$ is a complete sublattice of the lattice \mathcal{H}_μ , and dually the lattice $\mathcal{H}_{\iota_\gamma \mu_\gamma}$ is a complete sublattice of the lattice \mathcal{H}_{ι_μ} .

Proof: As a closure system $\mathcal{H}_{\mu_\gamma \iota_\gamma}$ is a complete lattice, and we have to prove that it is a complete sublattice of the complete lattice \mathcal{H}_μ . We begin by showing that it is a subset. Let $S \in \mathcal{H}_{\mu_\gamma \iota_\gamma}$, so that $\mu_\gamma(\iota_\gamma(S)) = S$. Then $\mu(\iota(S)) = \mu(\iota(\mu_\gamma(\iota_\gamma(S)))) = \mu(\iota(\mu(\iota(\gamma_2(S)))))) = \mu(\iota(\gamma_2(S))) = \mu_\gamma(\iota_\gamma(S)) = S$ by Theorem 2.5.4 (v), and thus $S \in \mathcal{H}_\mu$. This shows $\mathcal{H}_{\mu_\gamma \iota_\gamma} \subseteq \mathcal{H}_\mu$. Since every S in $\mathcal{H}_{\mu_\gamma \iota_\gamma}$ satisfies $\mu(\iota(S)) = S$, we can apply Theorem 2.5.5, to get

$$S \in \mathcal{H}_{\mu_\gamma \iota_\gamma} \iff S = \mu_\gamma(\iota_\gamma(S)) \iff S = \gamma_2(S) \iff S \in \mathcal{H}_{\gamma_2}.$$

As we remarked after Definition 2.5.1, the fact that γ_2 is an additive closure operator means that the corresponding closure system is a complete sublattice of the lattice $(\mathcal{P}(B); \cap, \cup)$ of all subsets of B ; that is, on our lattice $\mathcal{H}_{\mu_\gamma \iota_\gamma}$ the meet operation agrees with ordinary set-intersection and the join agrees with union. We already know that the meet operation in \mathcal{H}_μ also agrees with intersection, so we only need to show that $\mathcal{H}_{\mu_\gamma \iota_\gamma}$ is closed under the join operation of \mathcal{H}_μ . Let $(S_k)_{k \in J}$ be an indexed family of subsets of B . Then

$$\begin{aligned} \mu_\gamma(\iota_\gamma(\bigvee_{k \in J} S_k)) &= \gamma_2(\bigvee_{k \in J} S_k) = \gamma_2(\mu(\iota(\bigcup_{k \in J} S_k))) \\ &= \gamma_2(\mu(\iota_\gamma(\bigcup_{k \in J} S_k))) = \mu(\iota_\gamma(\bigcup_{k \in J} S_k)) = \mu(\iota(\bigcup_{k \in J} S_k)) = \bigvee_{k \in J} S_k, \end{aligned}$$

by Theorem 2.5.4 (iv'); and then using Lemma 2.3.3 we have

$$\iota(\bigcup_{k \in J} S_k) = \bigcap_{k \in J} \iota(S_k) = \bigcap_{k \in J} \iota_\gamma(S_k) = \iota_\gamma(\bigcup_{k \in J} S_k).$$

■

Thus conjugate pairs of additive closure operators give us a way to construct complete sublattices of a given closure lattice. We may also define an order relation on the set of all conjugate pairs of additive closure operators: for $\alpha = (\gamma_1, \gamma'_1)$ and $\beta = (\gamma_2, \gamma'_2)$ we set

$$\alpha \leq \beta :\Leftrightarrow (\forall T \subseteq A)(\forall S \subseteq B)(\gamma_1(T) \subseteq \gamma_2(T) \text{ and } \gamma'_1(S) \subseteq \gamma'_2(S)).$$

When $\alpha \leq \beta$, it can be shown that the lattice $\mathcal{H}_{\mu_\beta \iota_\beta}$ is a sublattice of $\mathcal{H}_{\mu_\alpha \iota_\alpha}$, and dually that $\mathcal{H}_{\iota_\beta \mu_\beta}$ is a sublattice of $\mathcal{H}_{\iota_\alpha \mu_\alpha}$.

The following additional properties may also be verified:

$$(i) \quad \gamma_1(\iota(\mu(T))) = \iota(\mu(T)) \iff T = \iota(\mu(\gamma_1(T))) \quad \text{and}$$

$$(i') \quad \gamma_2(\mu(\iota(S))) = \mu(\iota(S)) \iff S = \mu(\iota(\gamma_2(T))).$$

Sr. Arworn in [2] has generalized the theory of conjugate pairs of additive closure operators to the situation of conjugate pairs of extensive, additive operators. An alternative proof of Theorem 2.5.7 can be given by showing that R_γ is a Galois closed subrelation of R and then by using of Theorem 2.4.3.

Proposition 2.5.8 *Let $R \subseteq A \times B$ be a relation and let (μ) be the Galois connection induced by R . Let $\gamma := (\gamma_1, \gamma_2)$ be a conjugate pair of completely additive closure operators defined on A and on B , respectively. Let R_γ be the relation defined in Definition 2.5.3 and assume that $(\mu_\gamma, \iota_\gamma)$ is the Galois connection induced by R_γ . Then R_γ is a Galois closed subrelation of R .*

Proof: By Definition 2.5.3 the relation R_γ is a subrelation of R . We apply Proposition 2.4.2 (iii). Let $T \subseteq A$ and $S \subseteq B$. Then by Theorem 2.5.4 (iii) we have

$$\iota(\mu_\gamma(T)) = \iota(\gamma_2(\mu_\gamma(T))) = \iota_\gamma(\mu_\gamma(T)).$$

In a similar way by Theorem 2.5.4 (iii') and (i) we have

$$\mu(\iota_\gamma(S)) = \mu(\gamma_1(\iota_\gamma(S))) = \mu_\gamma(\iota_\gamma(S)).$$

This proves that R_γ is a Galois closed subrelation of R . ■



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