

CHAPTER 1

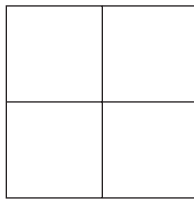
Asking the Right Questions

I'd like you to draw a square made from four unit squares.

A unit square is one where each of the sides is one unit long?

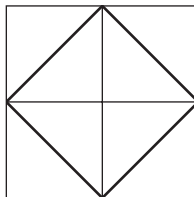
Yes.

Well, that shouldn't be too hard.



Will this do?

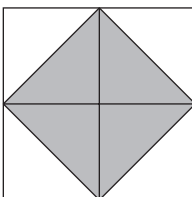
Perfect. Now let me add the following diagonals to your drawing.



You see that by doing this a new square is formed.

I do. One that uses a diagonal of each of the unit squares for its four sides.

Let's shade this square and call it the "internal" square.



Now, I want you to tell me the area of this internal square.

Let me think. The internal square contains exactly half of each unit square and so must have half the area of the large square. So it has an area of 2 square units.

Exactly. Now, what is the length of any one of those diagonals that forms a side of the internal square?

Off-hand I don't think I can say. I know that to get the area of a rectangular region you multiply its length by its breadth.

"Length by breadth," as you say, meaning multiply the length of one side by the length of a side at right angles to it.

So, for a square, this means that you multiply the length of one side by itself, since length and breadth are equal.

Yes.

But where does this get me? As I said, I don't know the length of the side.

As you say. But if we let s stand for the length of one of the sides, then what could you say about s ?

I suppose there is no way that we could have this little chat without bringing letters into it?

There is, but at the cost of the discussion being more longwinded than it need be. Incidentally, why did I chose the letter s ?

Because it is the initial of the word side?

Precisely. It is very common to use the initial of the word describing the quantity you're looking for.

So s stands for the length of the side of the internal square. I hope you are not going make me do algebra.

Just a very small amount—for the moment. So can you tell me something about the number s ?

When you multiply s by itself you get 2.

Exactly, because the area of the internal square is 2 (squared units). Do you recall that $s \times s$ is often written as s^2 ?

I do. My algebra isn't *that* rusty.

So you are saying that the number s “satisfies” the equation:

$$s^2 = 2$$

In words, “ s squared equals two.”

Okay, so the number s when multiplied by itself gives 2. Doesn't this mean that s is called the square root of 2?

Well, it would be more accurate to say that s is *a* square root of 2. A number is said to be a square root of another if, when multiplied by itself, it gives the other number.

So 3 is a square root of 9 because $3 \times 3 = 9$.

As is -3 , because $-3 \times -3 = 9$ also.

But most people would say that the square root of 9 is 3.

True. It is customary to call the positive square root of a number its square root. And since s is the length of the side of a square, it is obviously a positive quantity, so we may say . . .

. . . that s is the square root of 2.

Sometimes, we simply say “root two,” it being understood that it's a square root that is involved.

And not some other root like a cube root?

Yes. Now the fact that 3 is the square root of 9 is often expressed mathematically by writing $\sqrt{9} = 3$.

I've always liked this symbol for the square root.

It was first used by a certain Christoff Rudolff in 1525, in the book *Die Coss*, but I won't go into the reasons why he chose it.

Can we say goodbye to s and write $\sqrt{2}$ in its place from now on?

[See chapter note 1.]

If we want to, but we'll still use s if it serves our purposes.

So we have shown that the diagonal of a unit square is $\sqrt{2}$ in length.

Indeed we have. This wonderful way of establishing the existence of the square root of 2 originated in India thousands of years ago.

[See chapter note 2.]

You'd have to say that it is quite simple.

Which makes it all the more impressive.

So what number is $\sqrt{2}$?

As the equation $s^2 = 2$ says, it is the number that, when multiplied by itself, gives 2 exactly. This means no more or no less than what the equation

$$\sqrt{2} \times \sqrt{2} = 2$$

says it means: $\sqrt{2}$ is the number that when multiplied by itself gives 2.

I know, but what number does $\sqrt{2}$ actually stand for? I mean $\sqrt{16} = 4$, and 4 is what I would call a tangible number.

I understand. You have given me a concrete value for $\sqrt{16}$, namely the number 4. You want me to do the same for $\sqrt{2}$, that is, to show you some number of a type with which you are familiar, and that when squared, gives 2.

Exactly. I'm simply asking what the concrete value of s is, that makes $s^2 = 2$.

I can convince you quite easily that $\sqrt{2}$ is not a natural number.

The natural numbers are the ordinary counting numbers, 1, 2, 3, and so on.

Precisely.

Even though 2 itself is a natural number? The natural numbers 9 and 16 have square roots that are also natural numbers.

That's true, they do.

But you are saying that 2 doesn't.

I am. One way of seeing this is to write the first few natural numbers in order of increasing magnitude in a line, and beneath them on a second line write their corresponding squares:

1	2	3	4	5	6	7 . . .
1	4	9	16	25	36	49 . . .

The three dots, or ellipsis, at the end of a line means that the pattern continues without stopping.

Well, I can see straight away that the number 2 is missing from the second row.

As are

3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, . . .

I would say that there are a lot more numbers missing than are present.

Yes, in a sense "most" of the natural numbers are absent from this second line. The numbers 1, 4, 9, 16, . . . that appear on it are known as the *perfect squares*.

And those numbers that are missing from this line are not perfect squares?

Correct: 49 is a perfect square but 48 is not.

I think I see now why there is no natural number squaring to 2. The first natural number squares to 1 while the second natural number squared is 4, so 2 gets skipped over.

That's about it.

All right. It is fairly obvious, now at any rate, that there is no natural number that squares to 2, but surely there is some fraction whose square is 2?

By fraction, you mean a common fraction where one whole number is divided by another whole number?

That's what I mean, $\frac{7}{5}$, for example. Are there other types of fractions?

There are, but when we say "fraction" we mean one whole number divided by another one. The number being divided is the numerator and the one doing the dividing is called the denominator.

The number on the top is the numerator and the number on the bottom is the denominator.

That's it exactly. In your example, the whole number 7 is the numerator while the whole number 5 is the denominator.

Now mustn't there be some fraction close to this one that squares to give 2 exactly?

Why did you say *close* to this one?

Because my calculator tells me that $\frac{7}{5}$ is 1.4 in decimal form; and when I multiply this by itself I get 1.96, which is fairly close to 2.

Agreed. Let me show you how we can see this for ourselves without a calculator but using a little ingenuity instead.

Since

$$\begin{aligned}\left(\frac{7}{5}\right)^2 &= \frac{49}{25} \\ &\stackrel{!}{=} \frac{50-1}{25} \\ &= \frac{50}{25} - \frac{1}{25} \\ &= 2 - \frac{1}{25}\end{aligned}$$

we can say that the fraction $\frac{7}{5}$ when squared underestimates 2 by the amount $\frac{1}{25}$.

And according to my calculator $\frac{1}{25} = 0.04$, which is just $2 - 1.96$. By the way, why did you put the exclamation point over the second equals sign?

To indicate that the step being taken is quite a clever one.

It certainly wouldn't have occurred to me, which I know is not saying much.

Well, I don't lay any claim to originality for taking this step. I have seen many similar such tricks used by others in the past and, after all, I knew what it was I wanted to show.

At least I can see why it's clever.

Good. Why?

By writing the numerator 49 as $50 - 1$, you were able to divide the 50 by 25 to get 2 exactly and the 1 by 25 to get $\frac{1}{25}$ as the measure of the underestimate.

Desert Island Math

A useful trick if you're stranded on a desert island without any calculating devices other than your own poor head.

Pure do-it-yourself mathematics! I suppose using a calculator to get the value of something you wouldn't be able to calculate for yourself is a form of cheating?

Do you mean like asking for the decimal expansion of $\sqrt{2}$, for example?

Well, something like that. I wouldn't have a clue how to get the decimal expansion of $\sqrt{2}$ using my own very limited powers.

I'm sure you do your mental abilities an injustice. If we know and understand how to get a decimal expansion of a number "by hand," then we don't contravene the DIY philosophy if we use a calculator to save labor.

Are you saying that because I know how to get the decimal expansion of $\frac{7}{5}$ or $\frac{3}{11}$ by long division, even though I wouldn't like to be pressed on why the procedure works, I may use a calculator to avoid the "donkey work" involved with such a task?

I think we'll let this be a policy. We'll assume that if we were put to it we could explain to ourselves and others the "ins and outs" of the long-division algorithm.

Of course, completely!

Decimal expansions, or "decimals" as we often say for short, have certain advantages, one being that they convey the magnitude of a number more readily than their equivalent fractions do. When a number is expressed in decimal form, it is easy to say geometrically where it is located on the number line. No matter how long the decimal expansion of a number may be, we still know between which two whole numbers it lies on this number line:



$$\begin{array}{r} 1.4 \\ 5 \overline{) 7.00} \\ \underline{5} \\ 20 \\ \underline{20} \\ 00 \end{array}$$

algorithm: step-by-step procedure

$$\begin{array}{r} 0.272 \dots \\ 11 \overline{) 3.000} \\ \underline{22} \\ 80 \\ \underline{77} \\ 30 \\ \underline{22} \\ 80 \\ \vdots \end{array}$$

So we can see quite easily from 1.4 that it is a number between 1 and 2, whereas it is not as easy to see this from the fraction $\frac{7}{5}$.

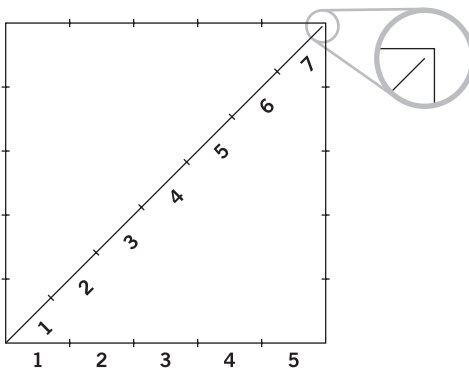
The fraction $\frac{7}{5}$ is perhaps too simple. It is not too difficult to mentally determine the two whole numbers between which it is located on the number line, but who can say without resorting to a calculation where the fraction $\frac{103993}{33102}$ is positioned on the same line?

I see the point, or should I say I do not see the (decimal) point!

Hmm! Speaking of the fraction $\frac{7}{5}$, you might like to get a box of matches and construct a square with five matches on each side.

Does this mean that the five matches between them make up the unit-length?

You can certainly think of it this way, if you like. Now you'll find that seven matches will fit along the diagonal:



These seven matches do not stretch the full length of the diagonal since $\frac{7}{5}$ underestimates $\sqrt{2}$.

That they don't is barely visible.

True, but the gap is there.

This is a rather neat way of visualising $\frac{7}{5}$ as an approximation to $\sqrt{2}$.

Yes it is, isn't it? Looked at another way it says that the ratio 7:5 is close to the ratio $\sqrt{2}:1$. Now, where were we?

Looking for a fraction that squares to 2.

Indeed, so let's continue the quest. Any further thoughts?

There must be some fraction a little bit bigger than $\frac{7}{5}$ that squares to give 2 exactly.

Well, there are lots of fractions just a little bit bigger than $\frac{7}{5}$.

I know. Isn't there an infinity of fractions between 1.4 and 1.5 alone?

Yes, but that this is so we can leave for another time. Why do you mention 1.5?

Simply because $(1.4)^2 = 1.96$ is less than 2 while $(1.5)^2 = 2.25$ is greater than 2.

So?

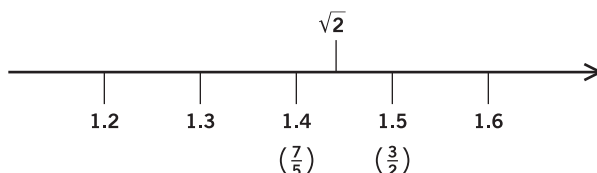
Doesn't this mean that the square root of 2 lies between these two values?

The symbol $<$ means "less than."

It does. In fact since $1.5 = \frac{3}{2}$ we may write that

$$\frac{7}{5} < \sqrt{2} < \frac{3}{2}$$

Let me display this arithmetic "inequality" on the number line:



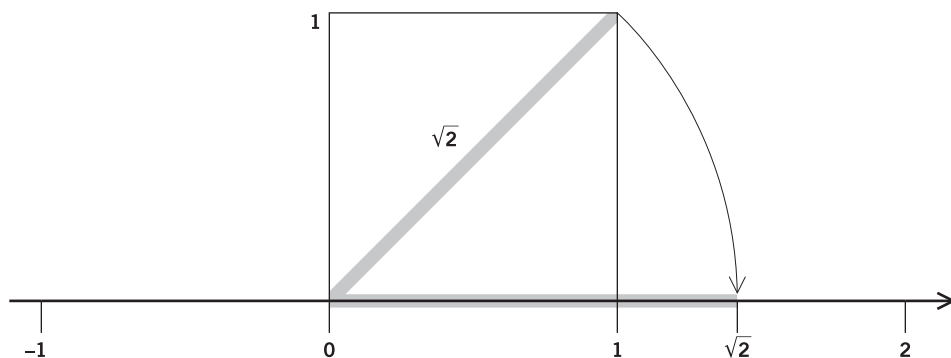
Notice that I have placed $\sqrt{2}$ to the right of 1.4 and closer to 1.4 than to 1.5 because $\frac{3}{2}$ squared overestimates 2 by $\frac{1}{4}$, which is much more than the $\frac{1}{25}$ by which $\frac{7}{5}$ squared underestimates $\sqrt{2}$.

But how do you locate $\sqrt{2}$ on the number line if you don't know what fraction it is?

A good question. The answer is that you do so geometrically.

I'd like to see how.

It's easy to construct a unit square geometrically on the interval that stretches between 0 and 1:



Now imagine the diagonal with one end at 0 and of length $\sqrt{2}$ being rotated clockwise about the point 0 until its other end lies on the number line.

At a point $\sqrt{2}$ from 0. Very smart.

Of course, this is an ideal construction where everything can be done to perfection.

I understand. It is the method that counts.

Yes.

An Exploration

But to return to the point I was making: surely among the infinity of fractions lying between 1.4 and 1.5 there is one that squares to give 2 exactly.

Well if there is, how do you propose finding it?

That's what is bothering me.

I'm sure you'll agree that it's not wise to begin checking fraction after fraction in this infinity of fractions without having some kind of plan.

Absolutely, it could take forever. What would you suggest?

Thinking about the problem a little to see if we can find some systematic way of attacking it.

Sounds as if we are about to go into battle.

A mental battle. Let us begin our campaign by examining the implications of expressing the number $\sqrt{2}$ as a fraction.

This could get interesting. What are you going to call this fraction?

Well, since we don't know it, at least not yet, we must keep our options open. One way of doing this is to use distinct letters, one to stand for its numerator and the other for its denominator.

Here comes some more algebra.

Only a little, used as scaffolding as it were, just to get us started.

Well, I'll stop you if I think I'm losing the drift of the discussion.

Let's call the numerator of the fraction m and the denominator n .

So if the fraction were $\frac{7}{5}$, which I know it is not, then m would be 7 and n would equal 5.

Or put slightly differently, if $m = 7$ and $n = 5$ then

$$\frac{m}{n} = \frac{7}{5}$$

I'm with you.

Now if

$$\sqrt{2} = \frac{m}{n}$$

then

$$\sqrt{2} \times \sqrt{2} = \frac{m}{n} \times \frac{m}{n}$$

Agreed?

I think so. You are simply squaring both sides of the original equation.

I am, and I do so in this elaborate manner to highlight the presence of $\sqrt{2} \times \sqrt{2}$.

Which by definition is 2.

Yes, a simple but vital use of the defining property of $\sqrt{2}$, which allows us to write that

$$2 = \left(\frac{m}{n}\right)^2$$

We can turn this equation around and write

$$\left(\frac{m}{n}\right)^2 = 2$$

to put the emphasis on the fraction $\frac{m}{n}$. What is the equation saying about $\frac{m}{n}$?

That its square is 2.

Exactly. And since

$$\left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$$

we can say that

$$\frac{m^2}{n^2} = 2$$

or that

$$m^2 = 2n^2$$

So this equation is a consequence of writing $\sqrt{2}$ as $\frac{m}{n}$?

It is indeed. Now let us see what we can learn from it.

I'll leave this to you.

I'm sure it won't be long before you join in. For one thing, $m^2 = 2n^2$ tells us is that if we are to find a fraction that is equal to $\sqrt{2}$, then we must find two perfect squares, one of which is twice the other.

What are perfect squares again? Oh, I remember, 1, 4, 9, 16, . . .

That's right, a perfect number is one that is the square of a natural number.

Well, this is a task that I can definitely undertake.

Be my guest.

Why don't I make out a list of the first twenty squares along with their doubles and see if I can find a match between some square and the double of some other square.

An excellent plan. Nothing like a bit of "number crunching," as it's called, to really get one thinking.

Of course, I'm going to use a calculator just to speed things up.

Naturally. Nobody doubts that you can multiply one number by itself.

Here's the table I get:

Natural Number	Number Squared	Twice Number Squared
1	1	2
2	4	8
3	9	18
4	16	32
5	25	50
6	36	72
7	49	98
8	64	128
9	81	162
10	100	200
11	121	242
12	144	288
13	169	338
14	196	392
15	225	450
16	256	512
17	289	578
18	324	648
19	361	722
20	400	800

The three columns show, in turn, the first twenty natural numbers, their squares, and twice these squares.

Great. We can think of the second column as corresponding to m^2 numbers and the third column as corresponding to numbers of the form $2n^2$.

I'm not sure I understand what you are saying here.

I'll explain by example. We may think of the number 196 in the second column as being an m^2 number, where $m = 14$, while we may consider the number 450 in the third column as being a $2n^2$ number, where $n = 15$.

Let me test myself to see if I have got the idea. I can think of 16 in the second column as an m^2 number with $m = 4$, while I can think of the 648 in the third column as corresponding to $2n^2$, with $n = 18$, because $2(18)^2 = 648$. Do I pass?

With honors. Now if you can find an entry in the second column that matches an entry in the third column, you will have found values for m and an n which make $m^2 = 2n^2$ and so you'll have a fraction $\frac{m}{n}$ equal to $\sqrt{2}$.

As easy as that? So fingers crossed as I look at each entry of the second column of this table and then look upwards from its location along the third column for a possible match.

Of course! A time-saving observation. As you say, you need only look upwards because the corresponding entries in the third column are bigger than those in the second.

Unfortunately, I can't find a single entry in the second column that is equal to any entry in the third column.

So the second and third columns have no element in common.

Not that I can see. I'm going to experiment a little more by calculating the next ten perfect squares along with their doubles.

Good for you.

This time I get:

Natural Number	Number Squared	Twice Number Squared
21	441	882
22	484	968
23	529	1058
24	576	1152
25	625	1250
26	676	1352
27	729	1458

Natural Number	Number Squared	Twice Number Squared
28	784	1568
29	841	1682
30	900	1800

I realize that this is not much of an extension to the previous table.

Maybe, but perhaps you'll get a match this time.

I'm scanning the second column to see if any entry matches anything in the previous third column or the new third column.

Any luck?

I'm afraid not. However, I notice that there are some near misses in the first table.

What do you mean by "near misses"?

A discovery?

There are entries in the second column that are either just 1 less or 1 more than an entry in the third column.

I'm more than curious; please elaborate.

Well, take the number 9 in the second column. It is 1 more than the 8 in the third column.

True. Any others?

There's a 49 in the second column that is 1 less than the 50 appearing in the third column.

Again, true. Any more?

Yes. There's a 289 in the second column and a 288 in the third column.

Again, as you observed, with a difference of 1 between them.

Did you find any more examples?

Not that I can see in these two tables, except, of course, at the very beginning. There's a 1 in the second column and a 2 in the third column.

Indeed there is.

But I don't know what to make of these near misses.

However, you seem to have hit upon something interesting, exciting even, so let's take a little time out to mull over your observations.

Fine by me, but you'll have to do the thinking.

Why don't we look at the case of the 9 in the second column and the 8 in the third column. What is the m number corresponding to this 9 in the second column, and what is the n number corresponding to the 8 in the third column?

Let me think. I would say that $m = 3$ and that $n = 2$.

And you'd be right. Your observation tells us that

$$m^2 = 2n^2 + 1$$

where $m = 3$ and $n = 2$.

Because $3^2 = 2(2)^2 + 1$?

Exactly. Now let us move on to the case of the number 49 in the second column and the 50 in the third column.

Here the $m = 7$ and $n = 5$ since $2(5)^2 = 2(25) = 50$.

This time

$$m^2 = 2n^2 - 1$$

where $m = 7$ and $n = 5$.

Can I try the next case?

By all means.

The number 289 corresponds to $m = 17$ since in this case $m^2 = 289$. On the other hand, the number 288 corresponds to $n = 12$ since $2(12)^2 = 2(144) = 288$.

No argument there.

This time

$$m^2 = 2n^2 + 1$$

where $m = 17$ and $n = 12$.

So we're back to 1 over.

There seems to be an alternating pattern with these pairs of near misses.

There does indeed. For the sake of completeness, you should look at the first case.

You mean the case with 1 in the second column and 2 in the third column?

None other; the smallest case, so to speak.

Okay. Here $m = 1$ and $n = 1$.

And what is the value of $m^2 - 2n^2$ on this occasion?

This time

$$m^2 = 2n^2 - 1$$

Does this fit the alternating pattern?

It does.

Which is great.

But returning to the original reason for constructing the tables, I haven't found a single square among the first thirty perfect squares that is equal to twice another square.

True, and that means that, as of yet, you haven't found a fraction $\frac{m}{n}$ that squares to 2. But, on the other hand, you have found a number of very interesting fractions.

I have? I would have thought that I've only found pairs of natural numbers that are within 1 of each other.

In a sense, you could say that. But you actually have discovered fractions with the property that the square of their numerator is within 1 of double the square of their denominator.

I'm afraid you'll have to elaborate.

Of course. You remember we said, when you observed that 9 in the second column was 1 greater than the 8 in the third column, that the 9 corresponded to m^2 where $m = 3$, while the 8 corresponded to $2n^2$ where $n = 2$?

I do.

Furthermore, $m^2 - 2n^2 = 1$, in this case.

That's correct.

Suppose now that we form the fraction

$$\frac{m}{n} = \frac{3}{2}$$

Then can't we say that the equation $m^2 - 2n^2 = 1$ tells us that this fraction is such that the square of its numerator is 1 more than twice the square of its denominator?

It seems to. I'll have to think a little more about this. Yes: $3^2 = 9$ and $2(2)^2 = 8$.

Try another one. Ask yourself, "What fraction is associated with the observation that the 49 in the second column is 1 less than the 50 in the third column?"

Okay. Here $m = 7$ and $n = 5$, so the fraction is $\frac{7}{5}$, right?

Absolutely. Now what can you say about the numerator and denominator of this fraction?

That the square of the numerator is 1 less than twice the square of its denominator.

Exactly.

I think I understand now. You are saying that every time we observe the near miss phenomenon we actually find a special fraction.

Yes. You looked for a fraction whose numerator squared would match twice its denominator squared; you didn't find one, but instead you found fractions whose numerators squared are within 1 of twice their denominators squared.

That's a nice way of looking at it.

Often when you look for something specific you chance upon something else.

So I suppose you could say that I found the next best thing.

I think we can say this, and not a bad reward for your labors.

Actually, I'm really curious to know if there are any more than just these four misses and to see if the plus or minus pattern continues to hold.

Let's hope so. Why don't we do a little more exploring?

I'd be happy to but shouldn't we stick to our original mission of finding a difference of exactly 0?

Very nicely put. Finding an m and n such that $m^2 = 2n^2$ means that the difference $m^2 - 2n^2$ would be 0.

Thanks.

However, I think we'll indulge ourselves and investigate your observation about near misses a little further, particularly as it looks so promising.

Okay. I'll extend my tables and then go searching.

You could do that, but it might be an idea to look more carefully at what you have already found.

Like good scientists would.

As you say. Begin by cataloguing the specimens found to date and examine them carefully for any clues.

Will do.

Time to Reflect

Beginning with the smallest, and listing them in increasing order, the fractions are

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}$$

Not many as of yet, but tantalizing.

What secrets do they hold, if any?

Indeed. Can you spot some connection between them?

Just like one of those sequence puzzles, "What is the next number in the sequence?" except here it looks harder because these are fractions and not ordinary numbers.

A puzzle certainly, but one we have encountered quite naturally.

And not just made up for the sake of it.

Yes, something like that.

I hope this is an easy puzzle.

It is always best to be optimistic so I advise that you say to yourself, “This is sure to be easy,” and look for simple connections.

Optimism it is then, but where to start?

It is often a good idea to begin by examining a pair of terms some way out along a sequence rather than at the very beginning of it.

Right. On that advice I’ll see if I can spot a connection between

$$\frac{7}{5} \quad \text{and} \quad \frac{17}{12}$$

and if I think I have found one, I’ll check it on the earlier fractions.

Very sensible. Happy hunting!

I think I’ll begin by focusing on the denominator 12 of the fraction $\frac{17}{12}$.

Following a very definite line of inquiry, as they say.

I think I have spotted something already.

Which is?

That $12 = 7 + 5$, the next denominator looks as if it might be the sum of the numerator and denominator of the previous fraction.

If it’s true, it will be a big breakthrough. I must say that was pretty quick.

I must check the earlier terms of the sequence

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}$$

to see if this rule holds also for their denominators.

Fingers crossed, then.

I obviously cannot check the first fraction, $\frac{1}{1}$.

Why not?

Because there is no fraction before it.

A good point.

But the second fraction, $\frac{3}{2}$, has denominator 2, which is just $1 + 1$, the sum of the numerator 1 and denominator 1 of the first fraction $\frac{1}{1}$. This is getting exciting.

That's great. How about the third fraction $\frac{7}{5}$?

Right, Mr. $\frac{7}{5}$, let's see if you fit the theory. Your denominator is 5, is it not? Indeed it is, and the sum of the numerator and denominator of the previous fraction, $\frac{3}{2}$, is $3 + 2$, which I'm happy to say is none other than 5. This is fantastic! Who would have thought?

Great again! Now is there an equally simple rule for the numerators?

I hope so, because discovering that rule for the denominators gave me a great thrill.

We couldn't ask for more than that.

Right, back to the drawing board. So is there a connection between the numerator 17 of the fraction $\frac{17}{12}$ and the numbers 7 and 5 from the previous fraction $\frac{7}{5}$?

It would be marvelous if there were.

If I'm not mistaken, there is. It's simply that $17 = 7 + (2 \times 5)$.

Well spotted, though not quite as simple as the rule for the denominators.

No, but still easy enough.

Once you see it. How do you interpret this rule?

Doesn't it say that the next numerator is obtained by adding the numerator of the previous fraction to twice the denominator of the previous fraction?

Indisputable. You had better check this rule on the other fractions.

It works for the fraction $\frac{3}{2}$ since $3 = 1 + (2 \times 1)$, and it also works for $\frac{7}{5}$ since $7 = 3 + (2 \times 2)$.

This is wonderful. So how would you summarize the overall rule, which allows one to go from one fraction to the next?

Well, the general rule obtained by combining the denominator rule and the numerator rule seems to be:

To get the denominator of the next fraction, add the numerator and denominator of the previous fraction; to get the numerator of the next fraction, add the numerator of the previous fraction to twice its denominator.

Well done! And a fairly straightforward rule, at that.

Isn't it amazing?

Indeed it is. After all, there was no reason to believe that there had to be any rule whatever connecting the fractions, but to find that there is one and that it's simple is remarkable.

I must now apply this general rule to $\frac{17}{12}$ to see what fraction comes out and to see if it has the property that its top squared minus twice the bottom squared is either 1 or -1 .

Let's hope that the property holds.

According to the rule, the next fraction has a denominator of $17 + 12 = 29$ and a numerator of $17 + 2 \times 12 = 41$, and so is $\frac{41}{29}$.

Good. And now what are we hoping for?

Based on the pattern displayed by the previous four fractions, that $(41)^2 - 2(29)^2$ will work out to be -1 .

Perform the acid test.

Here goes:

$$41^2 - 2(29)^2 = 1681 - 2(841) = 1681 - 1682 = -1$$

This is fantastic!

So now you have found another example of a perfect square that is within 1 of twice another perfect square—the whole point of this investigative detour—*without* having to go to the bother of extending your original tables.

That's true. Our more thorough examination of the four cases we found seems to have paid off.

A little thought can save a lot of computing.

I know that I couldn't have spotted this example with my tables because they give only the first thirty perfect squares along with their doubles; but can we be sure that there is not an m value between 17 and 41 that gives a square that is within 1 of twice another perfect square?

An excellent question. At the moment we can't be sure without checking. However, if there is such an m value, then it corresponds to a fraction $\frac{m}{n}$ that doesn't fit in with the above rule. Of course, this doesn't exclude the possibility of there being such a value. However, if you check, you won't find any such value.

I must calculate the next fraction generated by the rule to see if it also satisfies the plus or minus 1 property, to give it a name. Applying the rule to $\frac{41}{29}$ gives $41 + 29 = 70$ as the next denominator and $47 + (2 \times 29) = 99$ as the numerator.

So $\frac{99}{70}$ is the next fraction to be tested.

I predict that $m^2 - 2n^2 = 1$ in this case. The calculation

$$99^2 - 2(70)^2 = 9801 - 2(4900) = 9801 - 9800 = 1$$

verifies this. Great!

Bravo! What now?

Obviously, we can apply the rule over and over again and so generate an infinite sequence beginning with

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \dots$$

True, you can generate an infinite number of fractions using the rule but . . .

How can we be sure that all the fractions of this sequence have the property that $m^2 - 2n^2$ is either plus or minus 1 without checking each, which is clearly out of the question.

Yes, this is a bit of a problem. It might be that answering such a question may prove difficult or even impossible.

And can we say that these are the only fractions having this plus or minus 1 property?

My, my, what truly mathematical questions! You need have no fear that you and mathematics are strangers if you can think up questions like this.

I don't know about that. Normally, I know I wouldn't even dream of asking questions such as these, but at the moment my mind seems to be totally engrossed by these particular fractions.

Ah, well, I've read somewhere that you really only see a person's true intelligence when his or her affections are fully engaged.

Perhaps tomorrow I won't care, but right now I really want to know if all the fractions generated by the rule actually obey the plus or minus 1 property; and I also want to know if these are the only fractions that do so.

Good for you. In mathematics it often seems that asking questions is the easy part, whereas it is the answering of them that is hard. But asking the right questions is a very important part of any investigation, whether it be mathematical or otherwise.

The good detectives always ask the right questions.

Well, by the end at any rate.

But can you tell me if my questions have answers; and if they do, what are their answers?

They do, but I am not going to tell you what they are. I don't want to spoil the fun you'll have in trying to answer them for yourself later.

Later could be an eternity away if it is left up to me on my own.

That remains to be seen. Anyway, you have opened up a rich vein for further exploration with your observation that there are squares whose doubles are within plus or minus 1 of other squares, and with your recent rule, both of which we'll come back to soon.

So, are we going to return to our original investigation? Not just yet.

Squeezing $\sqrt{2}$

Before leaving the fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \dots$$

let me show you how they connect with the number $\sqrt{2}$ itself.

Although none of them is $\sqrt{2}$?

Correct. But each of them can be thought of as an *approximation* to $\sqrt{2}$. In fact, each successive fraction provides a better approximation to $\sqrt{2}$ than its predecessor.

I hope you don't mind my saying so, but I would be much more interested in finding the exact fraction instead of approximations, however good they might be.

I appreciate that you are impatient to get on with your searching, but follow me for just a little longer so that I can show you how simply but cleverly we can use these fractions to close in on the location of $\sqrt{2}$ on the number line.

All right. Maybe I'll learn something that will help with my search.

Perhaps; we should look for clues anywhere we can. Now we know that

$$\begin{aligned} 1^2 &= 2(1)^2 - 1 \\ 3^2 &= 2(2)^2 + 1 \\ 7^2 &= 2(5)^2 - 1 \\ 17^2 &= 2(12)^2 + 1 \\ 41^2 &= 2(29)^2 - 1 \\ 99^2 &= 2(70)^2 + 1 \end{aligned}$$

Yes.

These equations are either of the form

$$m^2 = 2n^2 - 1$$

or

$$m^2 = 2n^2 + 1$$

alternating between one and the other.

Agreed. And I would bet that this jumping between -1 and 1 continues forever, though I have no idea how to prove it.

Now let us divide each of the equations

$$1^2 = 2(1)^2 - 1$$

$$3^2 = 2(2)^2 + 1$$

$$7^2 = 2(5)^2 - 1$$

$$17^2 = 2(12)^2 + 1$$

$$41^2 = 2(29)^2 - 1$$

$$99^2 = 2(70)^2 + 1$$

by their corresponding n^2 values to get

$$\left(\frac{1}{1}\right)^2 = 2 - \frac{1}{1^2}$$

$$\left(\frac{3}{2}\right)^2 = 2 + \frac{1}{2^2}$$

$$\left(\frac{7}{5}\right)^2 = 2 - \frac{1}{5^2}$$

$$\left(\frac{17}{12}\right)^2 = 2 + \frac{1}{12^2}$$

$$\left(\frac{41}{29}\right)^2 = 2 - \frac{1}{29^2}$$

$$\left(\frac{99}{70}\right)^2 = 2 + \frac{1}{70^2}$$

Are you with me?

Just about. I'm still mentally dividing across the second equation by 2^2 , putting it beneath the 3^2 and placing the combination $\frac{3}{2}$ under one umbrella with the power of 2 outside.

Takes practice, but it's all legal.

I'll accept this, since you did it, but I'm a little rusty when it comes to fractions and powers, so I can be slow. Anyway, I'm happy with this last set of equations now.

This simple but clever idea gives us equations that are very informative. They tell, in turn, how close the square of each fraction is to the number 2. Can you see why?

I'll have to take time on this. What is the equation

$$\left(\frac{17}{12}\right)^2 = 2 + \frac{1}{12^2}$$

saying? That when we square $\frac{17}{12}$ we get 2 plus the fraction $\frac{1}{144}$?

Yes. And?

Since $\frac{1}{144}$ is small, the fraction $\frac{17}{12}$ isn't a bad approximation of $\sqrt{2}$.

Not bad at all.

I think I see now why the approximations are getting better and better. As we move down through the set of equations, the fractions on the very right-hand side get smaller and smaller.

True. So?

So the corresponding fractions squared on the right-hand side are getting closer and closer to 2, which I take it means that the fractions themselves are better and better approximations of $\sqrt{2}$.

Excellent. We can say more.

We can?

We can say that the alternate fractions

$$\frac{1}{1}, \frac{7}{5}, \frac{41}{29}$$

are three underestimates of $\sqrt{2}$, each being a better approximation of $\sqrt{2}$ than its predecessor.

Because the minus sign before the last fraction in each equation tells us that these fractions squared are less than 2 by some amount.

That's it. The fraction $\frac{1}{1}$ is the smallest of these fractions, and $\frac{41}{29}$ is the largest:



You'll understand why I make this point in a moment.

But the fractions we skipped

$$\frac{3}{2}, \frac{17}{12}, \frac{99}{70}$$

on the other hand, are three overestimates of $\sqrt{2}$, which become progressively closer to $\sqrt{2}$.

Right again. When these fractions are squared, they give 2 plus something positive. Note that this time the first fraction is the largest and the last one the smallest.



This is the opposite of the previous case.

I think I see what you're driving at. The underestimates are creeping up on $\sqrt{2}$ from the left while the overestimates are creeping back toward $\sqrt{2}$ from the right.

That's right, as we can see when we show all six fractions together on the number line:



Here is one way of summarizing this information:

$$\frac{1}{1} < \frac{7}{5} < \frac{41}{29} < \sqrt{2} < \frac{99}{70} < \frac{17}{12} < \frac{3}{2}$$

I know we haven't proved anything yet about the fractions in the sequence that follow the first six:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$$

but it looks, then, as if the very first fraction is the smallest of all the fractions in the sequence, while the second of the fractions is the largest of them all.

Another interesting observation.

If this the case, all the fractions, except for $\frac{1}{1}$, are greater than or equal to $1.4 = \frac{7}{5}$ and less than or equal to $1.5 = \frac{3}{2}$.

It would appear that way.

So the fractions alternate between being underestimates and overestimates of $\sqrt{2}$ simply because of the plus and minus property.

Yes. Actually, it is very handy to have the fractions alternate between being underestimates and overestimates of $\sqrt{2}$ because we can use them to place $\sqrt{2}$ into narrower and narrower intervals of the number line.

As if you were squeezing $\sqrt{2}$.

You could say that. For example, taking only the fraction $\frac{7}{5}$ on the left of $\sqrt{2}$ and the fraction $\frac{3}{2}$ to the right of $\sqrt{2}$ we get the inequality

$$\frac{7}{5} < \sqrt{2} < \frac{3}{2}$$

which you may recognize.

Something I mentioned earlier?

Yes, you said that 1.4 squared is less than 2 but that 1.5 squared is greater than 2.

A pure accident.

Maybe, or a sign of deep mathematical intuition.

Without doubt! So now we can improve on this and say that

$$\frac{41}{29} < \sqrt{2} < \frac{99}{70}$$

Correct. We cannot say, at least not yet, exactly how close $\frac{99}{70}$ is to $\sqrt{2}$ in terms of fractions or in decimal terms because we don't know how to calculate

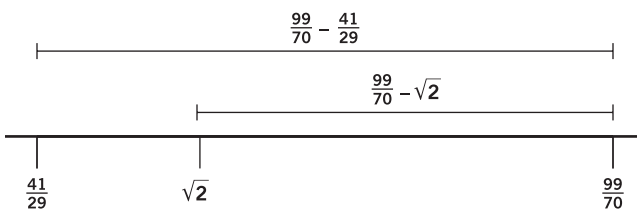
$$\frac{99}{70} - \sqrt{2}$$

But we could, if we could only find the fraction that is the same as $\sqrt{2}$.

Certainly, but this fraction is eluding us at the moment. Still we can *estimate* how close the fraction $\frac{99}{70}$ is to $\sqrt{2}$.

How?

Let us look at the interval between $\frac{41}{29}$ and $\frac{99}{70}$ under the microscope, as it were.



It may not strike you as a remarkable observation, but we can now at least say that the distance between $\sqrt{2}$ and $\frac{99}{70}$ is less than the length of the interval from $\frac{41}{29}$ to $\frac{99}{70}$, in which $\sqrt{2}$ resides.

This seems obvious from the picture you have just drawn. In fact, since we know that $\frac{99}{70}$ is greater than $\sqrt{2}$ but closer to it than the fraction $\frac{41}{29}$, we may say that $\frac{99}{70}$ is within *half* the length of the interval between $\frac{41}{29}$ and $\frac{99}{70}$.

Of course; simple but clever.

The length of the interval between $\frac{99}{70}$ and $\frac{41}{29}$ is

$$\frac{99}{70} - \frac{41}{29} = \frac{(99 \times 29) - (70 \times 41)}{70 \times 29} = \frac{1}{2030}$$

Pretty narrow.

Since 2030 is bigger than 2000, we can say that the fraction $\frac{1}{2030}$ is smaller than $\frac{1}{2000}$. So the length of the interval is less than

$$\frac{1}{2000} = 0.0005$$

Less than 5 ten-thousandths of a unit.

Yes. This means that

$$\left(\frac{99}{70} - \sqrt{2} \right) < \frac{0.0005}{2} = 0.00025$$

an estimate that shows with minimum computation that $\frac{99}{70}$ is within 0.00025 of $\sqrt{2}$.

Very smart.

Why don't you use your rule to show that the next two terms in the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \dots$$

are the fractions

$$\frac{239}{169} \quad \text{and} \quad \frac{577}{408}$$

respectively, and verify that $239^2 - 2(169)^2 = -1$, with $577^2 - 2(408)^2 = 1$?

So the next two fractions also follow the plus or minus 1 pattern.

Yes, but these two facts prove nothing about the remaining fractions.

I realise this.

However, you might like to use these two new arrivals to show that

$$\frac{1}{1} < \frac{7}{5} < \frac{41}{29} < \frac{239}{169} < \sqrt{2} < \frac{577}{408} < \frac{99}{70} < \frac{17}{12} < \frac{3}{2}$$

A further homing in on where $\sqrt{2}$ lives. It's very impressive how much can be said with just simple mathematics.

True, but it does help to have good observations to work on.

A lesson I've learned from all of this in relation to the search for a fraction exactly matching $\sqrt{2}$ is that it could be an awfully long search.

Why?

Well, we have just shown that the leading six fractions of the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \dots$$

provide successively improving approximations of $\sqrt{2}$, and I suspect that the fractions further out this sequence do even better.

For the sake of argument we'll grant for the moment that they do.

Judging from the numerators and denominators of the first eight terms, I'm guessing that the numerators and denominators grow longer and longer as we move further out the sequence.

Another interesting observation that we might discuss in more detail later. But for the moment, where is this line of reasoning taking you?

Well, it suggests that the actual fraction exactly matching $\sqrt{2}$ may also have a very large numerator and denominator and, if so, searching for it could take a very long time.

You have a point.

For example, even if $\sqrt{2}$ were the fraction $\frac{99}{70}$ with its very modest numerator and denominator, I would have to search as far as the ninety-ninth perfect square before hearing a click.

And if $\sqrt{2}$ were the fraction

$$\frac{351504323792998568782913107692171764446862638841}{248551090970421189729469473372814871029093002629}$$

... which it isn't, by the way, although it is very very close, you could be ...

... searching for the rest of my life.

What the Ancients Knew

So are you going to give up on the search idea?

Maybe, but I'd still like to test just a few more squares in the hope of getting lucky, even though I now realize that it is a most impractical method.

And one that would not produce a positive result no matter how far you, or countless millions of others armed with all the computing power in the world, were prepared to search.

What did you say?

That you would never succeed. Your search would be in vain.

Are you telling me that, of the infinity of fractions lying between 1.4 and 1.5 there is not one that squares to give 2 exactly?

That's correct. There *isn't* a fraction between these two numbers that squares to give 2 exactly.

But if there isn't such a fraction—and how on earth could you be convinced that there isn't—what kind of number is it that, when squared, gives 2? Or are you going to say that there is no such number?

Ah! A moment of truth has arrived. These crucial questions, which our opening geometrical demonstration has forced upon us, are ones we must attempt to answer.

Am I to understand that $\sqrt{2}$ is definitely not a fraction?

Yes, there is no *rational* number that, when squared, gives 2. Integers and fractions are known collectively as rational numbers. Put another way, there is no rational number that measures the length of the diagonal of a unit square.

Incredible! Of the infinity of fractions between 1.4 or $\frac{7}{5}$ and 1.5 or $\frac{3}{2}$ you are absolutely certain that there isn't a single one of them that squares to give 2 exactly?

Absolutely.

But how do you know for certain that such a fraction doesn't exist?

I know because the Ancient Greeks proved that it is impossible. I will show you one beautiful numerical proof.

It must be a very deep proof that shows that there isn't a number that squares to 2 exactly.

No, that would be going too far! I'm definitely not saying that there isn't a "number" whose square is exactly 2. All I am saying is that there isn't an integer or a fraction which when squared gives 2 exactly. There is a difference.

But what other numbers are there besides the rational numbers, as you have just called them?

This is the mystifying point about the length of the diagonal of a unit square. It presents us with a paradox—an apparent contradiction—about the nature of numbers.

So all along you have known that my search was futile.

Futile in its ambition but not otherwise. I didn't want to give the game away. You are not the first to believe with complete conviction that there must be a fraction, however hard it might be to find, that squares to give 2 exactly. Besides, I wanted you to enjoy exploring and discovering, to experience the pleasure of finding things out for yourself.

I must gather my thoughts. I would not deny that the diagonal of the unit square has a length. In fact, this length is obviously greater than 1 unit, and as we know, less than 1.5 units. Yet you tell me that the length of this diagonal cannot be expressed as a unit plus a certain fraction of a unit.

That's right. While the rational numbers are perfectly adequate for the world of commerce, they are not up to the task of measuring the *exact* length of a diagonal of a unit square. No matter how close a rational number may come to measuring the length, there will always be an error, microscopically small perhaps, but nevertheless an error. Always. The ancient way of putting this was to say that the diagonal and side of a square are *incommensurable*.

So if we were to insist on thinking that *all* numbers are the ones with which we are familiar, namely the rationals, then we'd be forced to say that there is no number of units that measure this diagonal, or that there is no number whose square is 2.

Yes, but why restrict ourselves to such a viewpoint?

It seems natural.

Maybe, but perhaps it seems this way simply because most people's experience is limited to dealing with rational numbers. However, as you have said, if we were to insist on maintaining that rational numbers are the only type of number, then we'd have to be prepared to live in a world where there are lengths which are not measurable and where certain numbers have no square roots.

So we must accept that there are other types of numbers. For mathematicians, the proof that no unit plus a fraction of a unit can hope to exactly measure the diagonal forces us to broaden our notions of what constitutes a number. When we do this, the paradox surrounding $\sqrt{2}$ simply dissolves.

So what "number" measures the diagonal of a unit square?

The one whose square is 2 and that we denote by $\sqrt{2}$. We admit the existence of this number because it makes its presence necessary by being the length of a legitimate quantity—the diagonal of a unit-square.

So the length of any side of the internal square we talked about at the beginning is simply $\sqrt{2}$, with no need for further elaboration.

Yes. $\sqrt{2}$ is a number between 1.4 and 1.5 that is not a rational number but that, when squared, gives 2. As we have already said, $\sqrt{2}$ is defined by the equation $\sqrt{2} \times \sqrt{2} = 2$, which is the mathematical way of saying that $\sqrt{2}$ is the positive number that squares to give 2.

So $\sqrt{2}$ is a new type of number.

Yes, new or different, but we have not proved this yet. Because it is not a rational number, it is called an *irrational* number. Not that there is anything unreasonable about it. It is so named because it cannot be expressed as the *ratio* of two integers in the way that the fractions are.

So the word rational in “rational number” is used because of the word ratio, while the term “irrational” in connection with $\sqrt{2}$ is used because it cannot be so expressed.

Quite so. The number $\sqrt{2}$ is as real as any fraction. In fact, $\sqrt{2}$ is just one of an infinite number of irrational real numbers that exist “outside” the realm of the rational numbers.

Can you show me some other irrational numbers?

Yes, the positive square roots of each of the other numbers missing from the list of perfect squares we made out some time ago can also be shown to be irrational numbers.

This means that

$$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \dots$$

are all irrational numbers.

Yes.

This is why there is an infinity of these numbers.

Certainly, but the collection of irrational numbers contains not just all these *surds*, as they are sometimes called, but a whole galaxy of other weird and wonderful numbers, the most famous being π .

Ah, π ! The ratio of the length of the circumference of any circle to the length of its diagonal. I thought that π was the fraction $\frac{22}{7}$.

This is only an approximation of its true value, just like $\frac{7}{5}$ is an approximation of $\sqrt{2}$.

Reality is a lot more complicated than I naïvely thought.

Perhaps we should say that the world of mathematics is a lot more complicated than one might think at first. However, speaking of reality, the collection of rational numbers and the collection of irrational numbers between them constitute the set of *real* numbers.

The idea that there are new specimens of numbers other than the “usual ones” used in arithmetic takes a little getting used to.

You’re not the first person who was more than a little perplexed by these new numbers. The minds of the Ancient Greeks were bewildered when these irrational numbers thrust their existence upon the Greeks’ consciousness. Legend has it that they were positively perturbed by the intrusion of these new quantities into their reality. They experienced an intellectual and philosophical crisis.

They did? Why?

Well, there was a brotherhood of Pythagoreans, followers of the famous philosopher and mathematician Pythagoras, which was devoted to the pursuit of higher learning, in particular mathematics. They were very well respected and considered to know all that there was to know. They believed that everything could be quantified by the familiar rational numbers.

A reasonable enough belief, or should I say a rational belief?

Yes, a very justifiable one. After all, these numbers are the only ones needed for commercial transactions, and they are equally adequate at describing various other physical phenomena. They also suffice for most measuring purposes that occur in practice.

Even though they cannot be used to give the measure of the diagonal of a unit square.

Yes, the issue about the new nature of $\sqrt{2}$ and its cousins, $\sqrt{3}$, $\sqrt{5}$, . . . was a theoretical one rather than a concern with “practical” measurement. The Greeks were fully aware that even if fractions could not measure the diagonal of a unit square exactly, they could measure it to any desired degree of accuracy. For example, a length of

$$\frac{577}{408} = 1 + \frac{169}{408}$$

units measures the “true” length of the diagonal to well within a hundred-thousandth of a unit.

Which is less than one-hundredth of a millimeter if the unit is a meter.

[See chapter note 3.]

Tablet 7289 Yale
Collection

There is evidence that this approximation to $\sqrt{2}$ was known to the Babylonians around 1600 B.C. This is many centuries before the Ancient Greeks whom I mentioned, because a Babylonian tablet from that time gives 1; 24, 51, 10 as an approximation to $\sqrt{2}$.

What does 1; 24, 51, 10 stand for?

It's shorthand for

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$$

The Babylonians used a sexagesimal system.

What is this when it is simplified?

The fraction

$$\frac{30547}{21600}$$

which, as you can see, is not $\frac{577}{408}$ exactly.

Why, then, is it thought that they knew of "our" $\frac{577}{408}$?

Because in base 60,

$$\frac{577}{408} = 1; 24, 51, 10, 35 \dots$$

it is suspected that they just truncated (shortened) the sexagesimal expansion of this fraction.

To three places, as we'd say.

Yes.

How did the Babylonians find such approximations?

It is not exactly known, but there is speculation that they knew of a method of approximation.

Was it a different method from the one using the sequence of fractions we have discovered?

It is related to this but faster.

Faster sounds interesting.

Accelerated, we might even say. This method also gave them 1; 25 as an approximation of $\sqrt{2}$. Convert 1; 25 to base 10 to see what it is.

I'll try. Since they used a sexagesimal system

What is 1; 24 as a
fraction?

$$1; 25 = 1 + \frac{25}{60} = \frac{85}{60} = \frac{17}{12}$$

this fraction, just like $\frac{577}{408}$, is in our sequence.

It is indeed, the fourth in the sequence. It is not as good an approximation of $\sqrt{2}$ as $\frac{577}{408}$, which is the eighth entry in the same sequence. As we said before, it doesn't do a bad job of approximating $\sqrt{2}$.

So these Mesopotamians must have known their mathematics.

And quite a bit more, by all accounts.

I should be able to verify for myself that $\frac{577}{408}$ approximates $\sqrt{2}$ as closely as you say.

You should and, what is significant, without knowing anything about the decimal expansion of $\sqrt{2}$.

Hmm. I didn't appreciate this point before.

I didn't emphasize it prior to this.

Please remind me of how exactly I would begin to go about this verification.

Recall that the fraction $\frac{239}{169}$ is the one before $\frac{577}{408}$ in our short sequence and that it underestimates $\sqrt{2}$, whereas $\frac{577}{408}$ overestimates it.

I think I remember now. Since $\frac{577}{408}$ is closer to $\sqrt{2}$ than $\frac{239}{169}$ is, its distance from $\sqrt{2}$ is less than half the distance between these two fractions.

Yes. This distance is $\frac{1}{68952}$, as you can check.

And what do we say now?

Since $50000 < 68952$, we say that $\frac{1}{68952}$ is less than $\frac{1}{50000}$. Hence the length of the interval is less than one fifty-thousandth of a unit, and so $\frac{577}{408}$ is within a hundred-thousandth of a unit of $\sqrt{2}$.

So we're done.

Yes. Maybe now is a good time to use what we know to get some idea of the leading digits in the decimal expansion of $\sqrt{2}$.

How are you going to do this?

Convert the fractions in the inequality

$$\frac{239}{169} < \sqrt{2} < \frac{577}{408}$$

to decimal form.

Using a calculator I hope.

Yes, because in theory this is something we can do ourselves by hand.

And so we are free to use a calculator to save time.

We get

$$1.\underline{4142}011834319526627 \dots < \sqrt{2} < 1.\underline{4142}156862745098039 \dots$$

working to twenty decimal places.

Some calculator! I see that both expansions agree to four decimal places. Is it safe so to say that

$$\sqrt{2} = 1.4142 \dots$$

which, I think, is pretty good?

It is. And because we did everything ourselves I think we can take a bow. We'll have fun improving on all of this at a later stage.

But to get back to the Ancient Greeks. You were saying that it is the *nature* of numbers that was of primary interest to these learned men.

Indeed. Such was their conviction that the rational numbers described all of nature exactly that their motto was, "All is number."

By which they meant the rational numbers.

Yes.

I'm glad to see I'm in good company.

You could certainly say that.

So they had their colors well and truly nailed to the mast.

This proclamation took on the status of an incontrovertible truth. It became a creed.

Oh! The discovery of the existence of $\sqrt{2}$ must have come as a shock.

A most unwelcome one we are told, because it challenged their cherished belief about the nature of numbers.

They took this whole business about numbers really seriously then?

I don't know how true much of the early lore surrounding the discovery of $\sqrt{2}$ is, but one story has it one of the brotherhood leaked the news that all was not well with the accepted dogma. For this breach of faith he was taken on a sea trip and cast overboard.

[See chapter note 4.]

You're kidding me!

Well, if it is true, it goes toward answering your question as to how seriously they took their mathematics.

So his number was up!

As fate would have it, the number $\sqrt{2}$ is referred to in some quarters as Pythagoras's constant.

I wonder what the Pythagorean brotherhood would have to say about that. But how did they come to know for certain that $\sqrt{2}$

is not a rational number? Surely they must have thought that $\sqrt{2}$ is actually a rational number but that they simply lacked the means to find it?

Perhaps they came upon numerical evidence similar to what you found in your search, but I don't know. I do know that their main mathematical focus was geometry.

Of course, the famous theorem of Pythagoras.

Actually, it may have been this very theorem that first brought $\sqrt{2}$ to the attention of the Ancient Greeks.

So they were the cause of the downfall of their own philosophy that "all is number."

You could say that. Coming back to what we were saying about searching, these clever Greeks would have known that the search method is one that, no matter how many perfect squares may have been checked, still leaves an infinite number of possibilities to be tested.

I hope they figured that out faster than I did!

I'm sure they were fully aware that any finite quantity, no matter what its size, is as nothing against the backdrop of infinity.

Still, they must have suspected right from the very moment the $\sqrt{2}$ problem reared its head that their doctrine of number was in trouble.

I'd be inclined to agree: they must have known that the doctrine wasn't as all-embracing as they originally proclaimed. It may be that some were really intrigued that $\sqrt{2}$ does not dwell in the infinite realm of rational numbers but is something that is "outside of it," as it were. Certainly minds over the centuries have been charmed by this aspect of numbers.

Did the Ancient Greeks find the proof you mentioned fairly soon after observing that there was more to $\sqrt{2}$ than meets the eye?

As far as I know, quite a stretch of time elapsed, about 300 years or so, before someone found an argument that turned suspicion into fact and established the irrationality of $\sqrt{2}$ once and for all. However, I don't know if the argument described by Euclid, which I am about to show you, was the first because there are many ingenious proofs of the irrationality of $\sqrt{2}$.

Elements X, §115a

But they must all be very difficult. It cannot be easy to be convinced that of the infinity of rational numbers, not one squares to give 2.

Not at all. The proof we are about to discuss is a magnificent *reductio ad absurdum* argument.

Which means?

In this case, you assume that there is a rational number that, when squared, gives 2, and then you show that this assumption leads to a contradiction or, put another way, reduces to something absurd. This form of logic—bequeathed to us by these Greek scholars—has been used ever since throughout mathematics.

If you arrive at a contradiction, you say that the assumption you made at the start is the cause of the trouble.

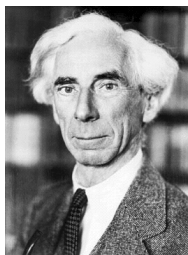
Yes, and you conclude that it must be false and must be abandoned.

So if the assumption is false then its opposite is true?

Precisely.

After thinking that $\sqrt{2}$ *must* be a fraction and having been frustrated in a vain search, I cannot wait to see this proof of irrationality. It must be a wonderful mathematical work of art.

A work of art indeed. Bertrand Russell once said, “Mathematics, rightly viewed, possesses not only truth but supreme beauty . . . sublimely pure, capable of a stern perfection as only the greatest art can show.” Judge for yourself whether the proof merits this accolade.



(1872–1970)

The Square Root of 2

A Dialogue Concerning a Number and a Sequence

Flannery, D.

2006, XII, 260 p. 31 illus., Hardcover

ISBN: 978-0-387-20220-4