2

Solvability of systems of interval linear equations and inequalities

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2.1 Introduction and notations

This chapter deals with solvability and feasibility (i.e., nonnegative solvability) of systems of interval linear equations and inequalities. After a few preliminary sections, we delineate in Section 2.6 eight decision problems (weak solvability of equations through strong feasibility of inequalities) that are then solved in eight successive sections 2.7 to 2.14. It turns out that four problems are solvable in polynomial time and four are NP-hard. Some of the results are easy (Theorem 2.13), some difficult to prove (Theorem 2.14), and some are surprising (Theorem 2.24). Although solutions of several of them are already known, the complete classification of the eight problems given here is new. Some special cases (tolerance, control and algebraic solutions, systems with square matrices) are treated in Sections 2.16 to 2.19. The last, Section 2.21 contains additional notes and references to the material of this chapter. Some of the results find later applications in interval linear programming (Chapter 3).

We use the following notations. The ith row of a matrix A is denoted by $A_{i.}$ and the jth column by $A_{.j}$. For two matrices A, B of the same size, inequalities like $A \leq B$ or $A < B$ are understood componentwise. A is called nonnegative if $0 \leq A$; A^T is the transpose of A. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. We use the following easy-to-prove properties valid whenever the respective operations and inequalities are defined.

(i) $A \leq B$ and $0 \leq C$ imply $AC \leq BC$.
(ii) $A \leq |A|$.
(iii) $|A| \leq B$ if and only if $-B \leq A \leq B$.
(iv) $|A + B| \leq |A| + |B|$.
(v) $||A| - |B|| \leq |A - B|$.
(vi) $|AB| \leq |A||B|$.

The same notations and results also apply to vectors that are always considered one-column matrices. Hence, for $a = (a_i)$ and $b = (b_i)$, $a^T b = \sum_i a_i b_i$

is the scalar product whereas ab^T is the matrix $(a_i b_j)$. Maximum (or minimum) of two vectors a, b is understood componentwise: i.e., $(\max\{a,b\})_i = \max\{a_i, b_i\}$ for each i. In particular, for vectors a^+, a^- defined by $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$ we have $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+ \geq 0$, $a^- \geq 0$ and $(a^+)^T a^- = 0$. I denotes the unit matrix, e_j is the jth column of I and $e = (1, \dots, 1)^T$ is the vector of all ones (in these cases we do not designate explicitly the dimension which can always be inferred from the context). In our descriptions to follow, an important role is played by the set Y_m of all ± 1 vectors in $\mathbb{R}^m$; i.e.,

$$Y_m = \{y \in \mathbb{R}^m \mid |y| = e\}.$$

Obviously, the cardinality of Y_m is 2^m. For each $x \in \mathbb{R}^m$ we define its sign vector $\operatorname{sgn} x$ by

$$(\operatorname{sgn} x)_i = \begin{cases} 1 \text{ if } x_i \geq 0, \\ -1 \text{ if } x_i < 0 \end{cases} \qquad (i = 1, \dots, m),$$

so that $\operatorname{sgn} x \in Y_m$. For a given vector $y \in \mathbb{R}^m$ we denote

$$T_y = \operatorname{diag}(y_1, \dots, y_m) = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_m \end{pmatrix}. \tag{2.1}$$

With a few exceptions (mainly in the proof of Theorem 2.9), we use the notation T_y for vectors $y \in Y_m$ only, in which case we have $T_{-y} = -T_y$, $T_y^{-1} = T_y$ and $|T_y| = I$. For each $x \in \mathbb{R}^m$ we can write $|x| = T_z x$, where $z = \operatorname{sgn} x$; we often use this trick to remove the absolute value of a vector. Notice that $T_z x = (z_i x_i)_{i=1}^m$.

2.2 An algorithm for generating Y_m

It will prove helpful at a later stage to generate all the ± 1-vectors forming the set Y_m systematically one-by-one in such a way that any two successive vectors differ in exactly one entry. We describe here an algorithm for performing this task, formulated in terms of generating the whole set Y_m; in later applications the last-but-one line "$Y := Y \cup \{y\}$" is replaced by the respective action on the current vector y. The algorithm employs an auxiliary $(0,1)$-vector $z \in \mathbb{R}^m$ used for determining the index k for which the current value of y_k should be changed to $-y_k$, and its description is as follows.

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z := 0 ∈ R^m; select y ∈ Y_m; Y := {y};
while z ≠ e
    k := min{i | z_i = 0};
    for i := 1 to k − 1, z_i := 0; end
    z_k := 1; y_k := −y_k;
    Y := Y ∪ {y};
end
% Y = Y_m
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Theorem 2.1. *For each $m \geq 1$ the algorithm at the output yields the set $Y = Y_m$ independently of the choice of the initial vector y.*

Proof. We prove the assertion by induction on m. For $m = 1$ it is a matter of simple computation to verify that the algorithm, if started from $y = 1$, generates $Y = \{1, -1\}$, and if started from $y = -1$, generates $Y = \{-1, 1\}$; in both cases $Y = Y_1$. Thus let the assertion hold for some $m - 1 \geq 1$ and let the algorithm be run for m. To see what is being done in the course of the algorithm, let us notice that in the main loop the initial string of the form

$$(1, 1, \dots, 1, 0, \dots)^T$$

of the current vector z is being found, where 0 is at the kth position, and it is being changed to

$$(0, 0, \dots, 0, 1, \dots)^T$$

until the vector z of all ones is reached (the last vector preceding it is $(0, 1, \dots, 1, 1)^T$). Hence if we start the algorithm for m, then the sequence of vectors z and y, restricted to their first $m - 1$ entries, is the same as if the algorithm were run for $m - 1$, until vector z of the form

$$(1, 1, \dots, 1, 0)^T \tag{2.2}$$

is reached. By that time, according to the induction hypothesis, the algorithm has constructed all the vectors $y \in Y_m$ with y_m being fixed throughout at its initial value. In the next step the vector (2.2) is switched to

$$(0, 0, \dots, 0, 1)^T$$

and y_m is switched to $-y_m$. Now, from the point of view of the first $m - 1$ entries, the algorithm again starts from zero vector z and due to the induction hypothesis it again generates all the $(m - 1)$-dimensional ± 1-vectors in the first $m - 1$ entries, this time with the opposite value of y_m. This implies that at the end (when vector z of all ones is reached) the whole set Y_m is generated, which completes the proof by induction. $\square$

We have needed a description starting from an arbitrary $y \in Y_m$ for the purposes of the proof by induction only; in practice we usually start with

$y = e$. The performance of the algorithm for $m = 3$ is illustrated in the following table. The algorithm is started from $z = 0$, $y = e$ (the first row) and the current values of z, y at the end of each pass through the "**while** ... **end**" loop are given in the next seven rows of the table.

z^T	y^T
(0, 0, 0)	(1, 1, 1)
(1, 0, 0)	(-1, 1, 1)
(0, 1, 0)	(-1, -1, 1)
(1, 1, 0)	(1, -1, 1)
(0, 0, 1)	(1, -1, -1)
(1, 0, 1)	(-1, -1, -1)
(0, 1, 1)	(-1, 1, -1)
(1, 1, 1)	(1, 1, -1)

2.3 Auxiliary complexity result

Given two vector norms $\|x\|_\alpha$ and $\|x\|_\beta$ in $\mathbb{R}^n$, a subordinate matrix norm $\|A\|_{\alpha,\beta}$ for $A \in \mathbb{R}^{n\times n}$ is defined by

$$\|A\|_{\alpha,\beta} = \max_{\|x\|_\alpha = 1} \|Ax\|_\beta \tag{2.3}$$

(see Higham [51], p. 121). If we use the norms $\|x\|_1 = e^T|x| = \sum_i |x_i|$, $\|x\|_\infty = \max_i |x_i|$, then from (2.3) we obtain $\|A\|_{1,1} = \max_j \sum_i |a_{ij}|$, $\|A\|_{\infty,\infty} = \max_i \sum_j |a_{ij}|$, and $\|A\|_{1,\infty} = \max_{ij} |a_{ij}|$, so that all three norms are easy to compute. This, however, is no longer true for the fourth norm $\|A\|_{\infty,1}$. In [165] it is proved that

$$\|A\|_{\infty,1} = \max_{y\in Y_n} \|Ay\|_1 = \max_{z,y\in Y_n} z^T Ay, \tag{2.4}$$

where the set Y_n consists of 2^n vectors. One might hope to find an essentially better formula for $\|A\|_{\infty,1}$, but such an attempt is not likely to succeed due to the following complexity result proved again in [165].

Theorem 2.2. *The problem of checking whether*

$$\|A\|_{\infty,1} \geq 1$$

holds is NP-complete in the set of symmetric rational M-matrices.

A square matrix $A = (a_{ij})$ is called an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$ (see p. 29). For our purposes it is advantageous to reformulate the result in terms of systems of inequalities.

Theorem 2.3. *The problem of checking whether a system of inequalities*

$$-e \leq Ax \leq e, \tag{2.5}$$

$$e^T|x| \geq 1 \tag{2.6}$$

has a solution is NP-complete in the set of nonnegative positive definite rational matrices.

Comment. Clearly, $e^T|x| = \|x\|_1$, so that the inequality (2.6) could be equivalently written as

$$\|x\|_1 \geq 1.$$

We prefer, however, the formulation given because terms of the form $e^T|x|$ arise quite naturally in the analysis of complexity of interval linear systems.

Proof. Given a symmetric rational M-matrix $A \in \mathbb{R}^{n\times n}$, consider the system

$$-e \leq A^{-1}x \leq e, \tag{2.7}$$

$$e^T|x| \geq 1 \tag{2.8}$$

which can be constructed in polynomial time since the same is true for A^{-1} (see Bareiss [8]). Since A is positive definite ([54], p. 114, assertion 2.5.3.3), A^{-1} is rational nonnegative positive definite. Obviously, the system (2.7), (2.8) has a solution if and only if

$$\begin{aligned} 1 &\leq \max\{e^T|x| \mid -e \leq A^{-1}x \leq e\} = \max\{e^T|Ax'| \mid -e \leq x' \leq e\} \\ &= \max\{\|Ax'\|_1 \mid -e \leq x' \leq e\} = \max\{\|Ay\|_1 \mid y \in Y_n\} = \|A\|_{\infty,1} \end{aligned}$$

holds, since the function $\|Ax'\|_1$ is convex over the unit cube $\{x' \mid -e \leq x' \leq e\}$ and therefore its maximum is attained at one of its vertices which are just the vectors in Y_n. Summing up, we have shown that $\|A\|_{\infty,1} \geq 1$ holds if and only if the system (2.7), (2.8) has a solution. Since the former problem is NP-complete (Theorem 2.2), the latter one is NP-hard; hence also the problem (2.5), (2.6) is NP-hard. Moreover, if (2.5), (2.6) has a solution, then, as we have seen, it also has a rational solution of the form $x = Ay$ for some $y \in Y_n$, and verification whether x solves (2.5), (2.6) can be performed in polynomial time. Hence the problem of checking solvability of (2.5), (2.6) belongs to the class NP and therefore it is NP-complete. □

We later use this result to establish NP-hardness of several decision problems concerning systems of interval linear equations and inequalities. For a detailed introduction into complexity theory, see Garey and Johnson [41].

2.4 Solvability and feasibility

From this section on we consider systems of linear equations $Ax = b$ or systems of linear inequalities $Ax \le b$. Unless said otherwise, it is always assumed that $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, where m and n are arbitrary positive integers.

A system of linear equations $Ax = b$ is called *solvable* if it has a solution, and *feasible* if it has a nonnegative solution. Throughout this and the next chapter the reader is kindly asked to bear in mind that *feasibility means nonnegative solvability.* The basic result concerning feasibility of linear equations was proved by Farkas [34] in 1902. As it is used at some crucial points in the sequel, we give here an elementary, but somewhat lengthy proof of it. The ideas of the proof are not exploited later, so that the reader may skip the proof without loss of continuity.

Theorem 2.4 (Farkas). *A system*

$$Ax = b \tag{2.9}$$

is feasible if and only if each p with $A^T p \ge 0$ satisfies $b^T p \ge 0$.

Proof. (a) If the system (2.9) has a solution $x \ge 0$ and if $A^T p \ge 0$ holds for some $p \in \mathbb{R}^m$, then $b^T p = (Ax)^T p = x^T (A^T p) \ge 0$. This proves the "only if" part of the theorem.

(b) We prove the "if" part by contradiction, proving that if the system (2.9) does not possess a nonnegative solution, then there exists a $p \in \mathbb{R}^m$ satisfying $A^T p \ge 0$ and $b^T p < 0$; for the purposes of the proof it is advantageous to write down this system in the column form

$$p^T A_{\cdot j} \ge 0 \qquad (j = 1, \ldots, n), \tag{2.10}$$

$$p^T b < 0. \tag{2.11}$$

We prove this assertion by induction on n.

(b1) If $n = 1$, then A consists of a single column a. Let $W = \{\alpha a \mid \alpha \in \mathbb{R}\}$ be the subspace spanned by a. According to the orthogonal decomposition theorem (Meyer [88], p. 405), b can be written in the form

$$b = b_W + b_{W^\perp},$$

where $b_W \in W$ and $b_{W^\perp} \in W^\perp$, $W^\perp$ being the orthogonal complement of W. We consider two cases. If $b_{W^\perp} = 0$, then $b \in W$, so that $b = \alpha a$ for some $\alpha \in \mathbb{R}$. Since $Ax = b$ does not possess a nonnegative solution due to the assumption, it must be $\alpha < 0$ and $a \ne 0$, so that if we put $p = a$, then $p^T a = \|a\|_2^2 \ge 0$ and $p^T b = \alpha \|a\|_2^2 < 0$; hence p satisfies (2.10), (2.11). If $b_{W^\perp} \ne 0$, put $p = -b_{W^\perp}$; then $p^T a = 0$ and $p^T b = -\|b_{W^\perp}\|_2^2 < 0$, so that p again satisfies (2.10), (2.11).

(b2) Let the induction hypothesis hold for $n - 1 \ge 1$ and let a system (2.9), where $A \in \mathbb{R}^{m\times n}$, possess no nonnegative solution. Then neither does the system

$$\sum_{j=1}^{n-1} A_{\cdot j} x_j = b$$

(otherwise for $x_n = 0$ we would get a nonnegative solution of (2.9)); hence according to the induction hypothesis there exists a $\overline{p} \in \mathbb{R}^m$ satisfying

$$\overline{p}^T A_{\cdot j} \geq 0 \qquad (j = 1, \ldots, n-1), \tag{2.12}$$

$$\overline{p}^T b < 0. \tag{2.13}$$

If $\overline{p}^T A_{\cdot n} \geq 0$, then p satisfies (2.10), (2.11) and we are done. Thus assume that

$$\overline{p}^T A_{\cdot n} < 0. \tag{2.14}$$

Put

$$\alpha_j = \overline{p}^T A_{\cdot j} \qquad (j = 1, \ldots, n),$$
$$\beta = \overline{p}^T b;$$

then $\alpha_1 \geq 0, \ldots, \alpha_{n-1} \geq 0$, $\alpha_n < 0$ and $\beta < 0$. Consider the system

$$\sum_{j=1}^{n-1} (\alpha_n A_{\cdot j} - \alpha_j A_{\cdot n}) x_j = \alpha_n b - \beta A_{\cdot n}. \tag{2.15}$$

If it had a nonnegative solution $x_1, \ldots, x_{n-1}$, then we could rearrange it to the form

$$\sum_{j=1}^{n-1} A_{\cdot j} x_j + A_{\cdot n} x_n = b, \tag{2.16}$$

where

$$x_n = \frac{\beta - \sum_{j=1}^{n-1} \alpha_j x_j}{\alpha_n} > 0$$

due to (2.12), (2.13), (2.14), so that the system (2.16), and thus also (2.9), would have a nonnegative solution $x_1, \ldots, x_n$ contrary to the assumption. Therefore the system (2.15) does not possess a nonnegative solution and thus according to the induction hypothesis there exists a $\tilde{p}$ such that

$$\tilde{p}^T (\alpha_n A_{\cdot j} - \alpha_j A_{\cdot n}) \geq 0 \qquad (j = 1, \ldots, n-1), \tag{2.17}$$

$$\tilde{p}^T (\alpha_n b - \beta A_{\cdot n}) < 0. \tag{2.18}$$

Now we set

$$p = \alpha_n \tilde{p} - (\tilde{p}^T A_{\cdot n}) \overline{p}$$

and we show that p satisfies (2.10), (2.11). For $j = 1, \ldots, n-1$ we have according to (2.17),

$$p^T A_{\cdot j} = \alpha_n \tilde{p}^T A_{\cdot j} - (\tilde{p}^T A_{\cdot n}) \overline{p}^T A_{\cdot j} \geq \alpha_j \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n}) \alpha_j = 0; \tag{2.19}$$

for $j = n$ we get

$$p^T A_{\cdot n} = \alpha_n \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\overline{p}^T A_{\cdot n} = \alpha_n \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\alpha_n = 0, \qquad (2.20)$$

and finally from (2.18),

$$p^T b = \alpha_n \tilde{p}^T b - (\tilde{p}^T A_{\cdot n})\overline{p}^T b < \beta \tilde{p}^T A_{\cdot n} - (\tilde{p}^T A_{\cdot n})\beta = 0, \qquad (2.21)$$

so that (2.19), (2.20), (2.21) imply (2.10) and (2.11); hence p is a vector having the asserted properties, which completes the proof by induction. □

With the help of the Farkas theorem we can now characterize solvability of systems of linear equations.

Theorem 2.5. *A system $Ax = b$ is solvable if and only if each p with $A^T p = 0$ satisfies $b^T p = 0$.*

Proof. If x solves $Ax = b$ and $A^T p = 0$ holds for some p, then $b^T p = p^T b = p^T Ax = (A^T p)^T x = 0$. Conversely, let the condition hold. Then for each p such that $A^T p \geq 0$ and $A^T p \leq 0$ we have $b^T p \geq 0$. But this, according to the Farkas theorem, is just the sufficient condition for the system

$$Ax_1 - Ax_2 = b \qquad (2.22)$$

to be feasible. Hence (2.22) has a solution $x_1 \geq 0$, $x_2 \geq 0$; thus $A(x_1 - x_2) = b$ and $Ax = b$ is solvable. □

For systems of linear inequalities we introduce the notions of solvability and feasibility in the same way: a system $Ax \leq b$ is called *solvable* if it has a solution, and *feasible* if it has a nonnegative solution. Again, we can use the Farkas theorem for characterizing solvability and feasibility.

Theorem 2.6. *A system $Ax \leq b$ is solvable if and only if each $p \geq 0$ with $A^T p = 0$ satisfies $b^T p \geq 0$.*

Proof. If x solves $Ax \leq b$ and $A^T p = 0$ holds for some $p \geq 0$, then $b^T p = p^T b \geq p^T Ax = 0$. Conversely, let the condition hold, so that each $p \geq 0$ with $A^T p \geq 0$, $A^T p \leq 0$ satisfies $b^T p \geq 0$. This, however, in view of the Farkas theorem means that the system

$$Ax_1 - Ax_2 + x_3 = b$$

is feasible. Hence due to the nonnegativity of x_3 we have $A(x_1 - x_2) \leq b$, and the system $Ax \leq b$ is solvable. □

Theorem 2.7. *A system $Ax \leq b$ is feasible if and only if each $p \geq 0$ with $A^T p \geq 0$ satisfies $b^T p \geq 0$.*

Proof. If $x \geq 0$ solves $Ax \leq b$ and $A^T p \geq 0$ holds for some $p \geq 0$, then $b^T p = p^T b = p^T Ax = (A^T p)^T x \geq 0$. Conversely, let the condition hold; then it is exactly the Farkas condition for the system

$$Ax_1 + x_2 = b \tag{2.23}$$

to be feasible. Hence (2.23) has a solution $x_1 \geq 0$, $x_2 \geq 0$, which implies $Ax_1 \leq b$, so that the system $Ax \leq b$ is feasible. □

Finally, we sum up the results achieved in this section in the form of a table that reveals similarities and differences among the four necessary and sufficient conditions.

Problem	**Condition**
solvability of $Ax = b$	$(\forall p)(A^T p = 0 \Rightarrow b^T p = 0)$
feasibility of $Ax = b$	$(\forall p)(A^T p \geq 0 \Rightarrow b^T p \geq 0)$
solvability of $Ax \leq b$	$(\forall p \geq 0)(A^T p = 0 \Rightarrow b^T p \geq 0)$
feasibility of $Ax \leq b$	$(\forall p \geq 0)(A^T p \geq 0 \Rightarrow b^T p \geq 0)$

An important result published by Khachiyan [71] in 1979 says that feasibility of a system of linear equations can be checked (and a solution to it, if it exists, found) in polynomial time. Since all three other problems, as shown in the proofs, can be reduced to this one, it follows that all four problems can be solved in polynomial time.

2.5 Interval matrices and vectors

There are several ways to express inexactness of the data. One of them, which has particularly nice properties from the point of view of a user, employs the so-called interval matrices which we define in this section.

If $\underline{A}$, $\overline{A}$ are two matrices in $\mathbb{R}^{m \times n}$, $\underline{A} \leq \overline{A}$, then the set of matrices

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \mid \underline{A} \leq A \leq \overline{A}\}$$

is called an interval matrix, and the matrices $\underline{A}$, $\overline{A}$ are called its bounds. Hence, if $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$, then $\mathbf{A}$ is the set of all matrices $A = (a_{ij})$ satisfying

$$\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij} \tag{2.24}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$. It is worth noting that each coefficient may attain any value in its interval (2.24) independently of the values taken on by other coefficients. Introducing additional relations among different coefficients makes interval problems much more difficult to solve and we do not follow this line in this chapter.

As shown later, in many cases it is more advantageous to express the data in terms of the center matrix

$$A_c = \tfrac{1}{2}(\underline{A} + \overline{A}) \tag{2.25}$$

and of the radius matrix

$$\Delta = \tfrac{1}{2}(\overline{A} - \underline{A}), \tag{2.26}$$

which is always nonnegative. From (2.25), (2.26) we easily obtain that

$$\underline{A} = A_c - \Delta,$$

$$\overline{A} = A_c + \Delta,$$

so that $\mathbf{A}$ can be given either as $[\underline{A}, \overline{A}]$, or as $[A_c - \Delta, A_c + \Delta]$, and consequently we can also write

$$\mathbf{A} = \{A \mid |A - A_c| \le \Delta\}.$$

In the sequel *we employ both forms and we switch freely between them* according to which one is more useful in the current context. The following proposition is the first example of usefulness of the center-radius notation.

Proposition 2.8. *Let* $\tilde{\mathbf{A}} = [\tilde{A}_c - \tilde{\Delta}, \tilde{A}_c + \tilde{\Delta}]$ *and* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *be interval matrices of the same size. Then* $\tilde{\mathbf{A}} \subseteq \mathbf{A}$ *if and only if*

$$|A_c - \tilde{A}_c| \le \Delta - \tilde{\Delta}$$

holds.

Proof. If $\tilde{\mathbf{A}} \subseteq \mathbf{A}$, then from

$$A_c - \Delta \le \tilde{A}_c - \tilde{\Delta} \le \tilde{A}_c + \tilde{\Delta} \le A_c + \Delta \tag{2.27}$$

we obtain

$$-(\Delta - \tilde{\Delta}) \le A_c - \tilde{A}_c \le \Delta - \tilde{\Delta}, \tag{2.28}$$

which gives

$$|A_c - \tilde{A}_c| \le \Delta - \tilde{\Delta}. \tag{2.29}$$

Conversely, (2.29) implies (2.28) and (2.27); hence $\tilde{\mathbf{A}} \subseteq \mathbf{A}$. $\square$

For an interval matrix $\mathbf{A} = [\underline{A}, \overline{A}] = [A_c - \Delta, A_c + \Delta]$, its transpose is defined by $\mathbf{A}^T = \{A^T \mid A \in \mathbf{A}\}$. Obviously, $\mathbf{A}^T = [\underline{A}^T, \overline{A}^T] = [A_c^T - \Delta^T, A_c^T + \Delta^T]$.

A special case of an interval matrix is an interval vector which is a one-column interval matrix

$$\mathbf{b} = \{b \mid \underline{b} \le b \le \overline{b}\},$$

where $\underline{b}, \overline{b} \in \mathbb{R}^m$. We again use the center vector

$$b_c = \tfrac{1}{2}(\underline{b} + \overline{b})$$

and the nonnegative radius vector

$$\delta = \frac{1}{2}(\overline{b} - \underline{b}),$$

and we employ both forms $\mathbf{b} = [\underline{b}, \overline{b}] = [b_c - \delta, b_c + \delta]$. Notice that interval matrices and vectors are typeset in boldface letters.

Given an $m \times n$ interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$, we define matrices

$$A_{yz} = A_c - T_y \Delta T_z \tag{2.30}$$

for each $y \in Y_m$ and $z \in Y_n$ (T_y is given by (2.1)). The definition implies that

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i \Delta_{ij} z_j = \begin{cases} \overline{a}_{ij} \text{ if } y_i z_j = -1, \\ \underline{a}_{ij} \text{ if } y_i z_j = 1 \end{cases}$$

($i = 1, \ldots, m$, $j = 1, \ldots, n$), so that $A_{yz} \in \mathbf{A}$ for each $y \in Y_m$, $z \in Y_n$. This finite set of matrices from $\mathbf{A}$ (of cardinality at most 2^{m+n-1} because $A_{yz} = A_{-y,-z}$ for each $y \in Y_m$, $z \in Y_n$), introduced in [156], plays an important role because it turns out that many problems with interval-valued data can be characterized in terms of these matrices, thereby obtaining finite characterizations of problems involving infinitely many sets of data. In the theorems to follow we show several examples of this approach, the most striking one being Theorem 2.14. We write A_{-yz} instead of $A_{-y,z}$. In particular, we have $A_{ye} = A_c - T_y \Delta$, $A_{ez} = A_c - \Delta T_z$, $A_{ee} = \underline{A}$ and $A_{-ee} = \overline{A}$.

For an m-dimensional interval vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$, in analogy with matrices A_{yz} we define vectors

$$b_y = b_c + T_y \delta$$

for each $y \in Y_m$. Then for each such a y we have

$$(b_y)_i = (b_c)_i + y_i \delta_i = \begin{cases} \underline{b}_i \text{ if } y_i = -1, \\ \overline{b}_i \text{ if } y_i = 1 \end{cases}$$

($i = 1, \ldots, m$), so that $b_y \in \mathbf{b}$ for each $y \in Y_m$. In particular, $b_{-e} = \underline{b}$ and $b_e = \overline{b}$. Together with matrices A_{yz}, vectors b_y are used in finite characterizations of interval problems having right-hand sides.

2.6 Weak and strong solvability/feasibility

Let $\mathbf{A}$ be an $m \times n$ interval matrix and $\mathbf{b}$ an m-dimensional interval vector. Under a *system of interval linear equations*

$$\mathbf{A}x = \mathbf{b} \tag{2.31}$$

we understand the *family* of all systems of linear equations

$$Ax = b \tag{2.32}$$

with data satisfying

$$A \in \mathbf{A},\, b \in \mathbf{b}, \tag{2.33}$$

and similarly a system of interval linear inequalities

$$\mathbf{A}x \leq \mathbf{b} \tag{2.34}$$

is the *family* of all systems

$$Ax \leq b$$

whose data satisfy

$$A \in \mathbf{A},\, b \in \mathbf{b}.$$

We introduce the following definitions. A system (2.31) is said to be *weakly* solvable (feasible) if *some* system (2.32) with data (2.33) is solvable (feasible), and it is called *strongly* solvable (feasible) if *each* system (2.32) with data (2.33) is solvable (feasible). In the same way we define weak and strong solvability (feasibility) of a system of interval linear inequalities (2.34). Hence, the word "weakly" refers to validity of the respective property for some system in the family whereas the word "strongly" refers to its validity for all systems in the family.

Introduction of weak and strong properties has an obvious motivation. Assume we are to decide whether some system $A_0x = b_0$ is solvable, but the exact data of this system are not directly available to us (they come from some measurements, are afflicted with rounding errors, etc.); instead, we only know that they satisfy $A_0 \in \mathbf{A}$, $b_0 \in \mathbf{b}$. Then we can be sure that our system $A_0x = b_0$ is solvable only if we know that the system (2.31) is strongly solvable, and in a similar way we can be sure that the system $A_0x = b_0$ is not solvable only if we know that the system (2.31) is not weakly solvable. A similar reasoning also holds for feasibility and for interval linear inequalities.

In this way, combining weak and strong solvability or feasibility of systems of interval linear equations or inequalities, we arrive at eight decision problems:

- Weak solvability of equations,
- Weak feasibility of equations,
- Strong solvability of equations,
- Strong feasibility of equations,
- Weak solvability of inequalities,
- Weak feasibility of inequalities,
- Strong solvability of inequalities,
- Strong feasibility of inequalities.

We study these problems separately in the next eight sections. It is shown that all of them can be solved by finite means, however, in half of the cases the number of steps is exponential in matrix size and the respective problems are proved to be NP-hard.

2.7 Weak solvability of equations

In this section we study the first of the eight decision problems delineated in Section 2.6, namely weak solvability of systems of interval linear equations. As before, we assume that $\mathbf{A}$ is an $m \times n$ interval matrix and $\mathbf{b}$ is an m-dimensional interval vector, where m and n are arbitrary positive integers.

First we introduce a useful auxiliary term: a vector $x \in \mathbb{R}^n$ is called a *weak solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$. Oettli and Prager [112] proved in 1964 the following nice and far-reaching characterization of weak solutions.

Theorem 2.9 (Oettli–Prager). *A vector $x \in \mathbb{R}^n$ is a weak solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies*

$$|A_c x - b_c| \le \Delta|x| + \delta. \tag{2.35}$$

Proof. If x is a weak solution, then $Ax = b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$, which gives $|A_c x - b_c| = |(A_c - A)x + b - b_c| \le \Delta|x| + \delta$. Conversely, let $|A_c x - b_c| \le \Delta|x| + \delta$ hold for some x. Define $y \in \mathbb{R}^m$ by

$$y_i = \begin{cases} \frac{(A_c x - b_c)_i}{(\Delta|x|+\delta)_i} & \text{if } (\Delta|x| + \delta)_i > 0, \\ 1 & \text{if } (\Delta|x| + \delta)_i = 0 \end{cases} \qquad (i = 1, \dots, m), \tag{2.36}$$

then $|y| \le e$ and

$$A_c x - b_c = T_y(\Delta|x| + \delta). \tag{2.37}$$

Put $z = \operatorname{sgn} x$; then $|x| = T_z x$ and from (2.37) we get

$$(A_c - T_y \Delta T_z)x = b_c + T_y \delta. \tag{2.38}$$

Since $|y| \le e$ and $z \in Y_n$, we have $|T_y \Delta T_z| \le \Delta$ and $|T_y \delta| \le \delta$, so that $A_c - T_y \Delta T_z \in \mathbf{A}$ and $b_c + T_y \delta \in \mathbf{b}$, which implies that x is a weak solution of $\mathbf{A}x = \mathbf{b}$. □

The main merit of the Oettli–Prager theorem consists in the fact that it describes the set of all weak solutions by means of a single, but nonlinear, inequality (2.35). In the proof we have also established a constructive result that is worth stating independently.

Proposition 2.10. *If x solves (2.35), then it satisfies (2.38), where y is given by (2.36) and $z = \operatorname{sgn} x$.*

Weak solvability of a system $\mathbf{A}x = \mathbf{b}$, as it was defined in Section 2.6, is equivalent to existence of a weak solution to it. Hence we can employ the Oettli–Prager theorem to characterize weak solvability of interval linear equations. Let us recall that in accordance with the general definition (2.30) we have $A_{ez} = A_c - \Delta T_z$ and $A_{-ez} = A_c + \Delta T_z$.

Theorem 2.11. *A system* $\mathbf{A}x = \mathbf{b}$ *is weakly solvable if and only if the system*

$$A_{ez}x \le \overline{b}, \tag{2.39}$$

$$-A_{-ez}x \le -\underline{b} \tag{2.40}$$

is solvable for some $z \in Y_n$.

Proof. If $\mathbf{A}x = \mathbf{b}$ is weakly solvable, then it has a weak solution x that according to Theorem 2.9 satisfies (2.35) and thus also

$$-\Delta|x| - \delta \le A_c x - b_c \le \Delta|x| + \delta. \tag{2.41}$$

If we put $z = \operatorname{sgn} x$, then $|x| = T_z x$ and (2.41) turns into $A_{ez}x = (A_c - \Delta T_z)x \le b_c + \delta = \overline{b}$ and $A_{-ez}x = (A_c + \Delta T_z)x \ge b_c - \delta = \underline{b}$ which shows that x satisfies (2.39), (2.40). Conversely, let (2.39), (2.40) hold for some x and $z \in Y_n$. Then we have

$$-\Delta T_z x - \delta \le A_c x - b_c \le \Delta T_z x + \delta$$

and consequently

$$|A_c x - b_c| \le \Delta T_z x + \delta \le \Delta|x| + \delta;$$

hence x satisfies (2.35) and therefore it is a weak solution of $\mathbf{A}x = \mathbf{b}$. $\square$

This result shows that checking weak solvability of interval linear equations can be in principle performed by finite means by checking solvability of systems (2.39), (2.40), $z \in Y_n$ by some finite procedure (e.g., a linear programming technique). However, to verify that $\mathbf{A}x = \mathbf{b}$ is not weakly solvable, we have to check all the systems (2.39), (2.40), $z \in Y_n$, whose number in the worst case is 2^n. Clearly, this is nearly impossible even for relatively small values of n (say, $n = 30$). It turns out that the source of these difficulties does not lie with inadequateness of our description, but that it is inherently present in the problem itself which is NP-hard. In the proof of this statement we show an approach that is also used several times later, namely a polynomial-time reduction of our standard NP-complete problem from Theorem 2.3 to the current problem, which proves its NP-hardness.

Theorem 2.12. *Checking weak solvability of interval linear equations is NP-hard.*

Proof. Let A be a square matrix. We first prove that the system

$$-e \le Ax \le e, \tag{2.42}$$

$$e^T|x| \ge 1 \tag{2.43}$$

has a solution if and only if the system of interval linear equations

$$[A, A]x = [-e, e], \tag{2.44}$$
$$[-e^T, e^T]x = [1, 1] \tag{2.45}$$

is weakly solvable. If x solves (2.42), (2.43) and if we set $x' = \frac{x}{e^T|x|}$, then $|Ax'| = \frac{1}{e^T|x|}|Ax| \leq |Ax| \leq e$ and $e^T|x'| = 1$; hence x' satisfies $Ax' = b$, $z^T x' = 1$ for some $b \in [-e, e]$ and $z^T = (\operatorname{sgn} x')^T \in [-e^T, e^T]$, which means that (2.44), (2.45) is weakly solvable. Conversely, let (2.44), (2.45) have a weak solution x; then $Ax = b$ and $c^T x = 1$ for some $b \in [-e, e]$ and $c^T \in [-e^T, e^T]$; hence $|Ax| \leq e$ and $1 = c^T x \leq |c|^T |x| \leq e^T |x|$, so that x solves (2.42), (2.43). We have shown that the problem of checking solvability of (2.42), (2.43) can be reduced in polynomial time to that of checking weak solvability of (2.44), (2.45). Since the former problem is NP-complete by Theorem 2.3, the latter one is NP-hard. □

2.8 Weak feasibility of equations

Using the notion of a weak solution introduced in Section 2.7, we can say that a system $\mathbf{A}x = \mathbf{b}$ is weakly feasible (in the sense of the definition made in Section 2.6) if and only if it has a nonnegative weak solution. Hence we can again use the Oettli–Prager theorem to obtain a characterization of weak feasibility.

Theorem 2.13. *A system $\mathbf{A}x = \mathbf{b}$ is weakly feasible if and only if the system*

$$\underline{A}x \leq \overline{b}, \tag{2.46}$$
$$-\overline{A}x \leq -\underline{b} \tag{2.47}$$

is feasible.

Proof. If $\mathbf{A}x = \mathbf{b}$ is weakly feasible, then it possesses a nonnegative weak solution x that by Theorem 2.9 satisfies

$$|A_c x - b_c| \leq \Delta x + \delta \tag{2.48}$$

and thus also

$$-\Delta x - \delta \leq A_c x - b_c \leq \Delta x + \delta, \tag{2.49}$$

which is (2.46), (2.47). Conversely, if (2.46), (2.47) has a nonnegative solution x, then it satisfies (2.49) and (2.48) and by the same Theorem 2.9 it is a nonnegative weak solution to $\mathbf{A}x = \mathbf{b}$ which means that this system is weakly feasible. □

Hence, only one system of linear inequalities (2.46), (2.47) is to be checked in this case. Referring to the last paragraph of Section 2.4, we can conclude that checking weak feasibility of interval linear equations can be performed in polynomial time whereas checking weak solvability, as we have seen in Theorem 2.12, is NP-hard.

2.9 Strong solvability of equations

By definition (Section 2.6), $\mathbf{A}x = \mathbf{b}$ is strongly solvable if each system $Ax = b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ is solvable. If $\underline{A}_{ij} < \overline{A}_{ij}$ for some i, j or $\underline{b}_i < \overline{b}_i$ for some i, then the family $\mathbf{A}x = \mathbf{b}$ consists of infinitely many linear systems. Therefore the fact that solvability of these infinitely many systems can be characterized in terms of feasibility of finitely many systems is nontrivial, and so is the proof of the following theorem which also establishes a useful additional property. $\operatorname{Conv} X$ denotes the convex hull of X, i.e., the intersection of all convex subsets of $\mathbb{R}^n$ containing X.

Theorem 2.14. *A system $\mathbf{A}x = \mathbf{b}$ is strongly solvable if and only if for each $y \in Y_m$ the system*

$$A_{ye}x^1 - A_{-ye}x^2 = b_y, \tag{2.50}$$

$$x^1 \geq 0,\ x^2 \geq 0 \tag{2.51}$$

has a solution x_y^1, x_y^2. Moreover, if this is the case, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ the system $Ax = b$ has a solution in the set

$$\operatorname{Conv}\{x_y^1 - x_y^2 \mid y \in Y_m\}.$$

Proof. "Only if": Let $\mathbf{A}x = \mathbf{b}$ be strongly solvable. Assume to the contrary that (2.50), (2.51) does not have a solution for some $y \in Y_m$. Then the Farkas theorem implies existence of a $p \in \mathbb{R}^m$ satisfying

$$(A_c - T_y\Delta)^T p \geq 0, \tag{2.52}$$

$$(A_c + T_y\Delta)^T p \leq 0, \tag{2.53}$$

$$b_y^T p < 0. \tag{2.54}$$

Now (2.52) and (2.53) together give

$$\Delta^T T_y p \leq A_c^T p \leq -\Delta^T T_y p;$$

hence

$$|A_c^T p| \leq -\Delta^T T_y p = |-\Delta^T T_y p| \leq \Delta^T |p|,$$

and the Oettli–Prager theorem as applied to the system $[A_c^T - \Delta^T, A_c^T + \Delta^T]x = [0, 0]$ shows that there exists a matrix $A \in \mathbf{A}$ such that

$$A^T p = 0. \tag{2.55}$$

In the light of Theorem 2.5, (2.55) and (2.54) mean that the system

$$Ax = b_y$$

has no solution, which contradicts our assumption of strong solvability since $A \in \mathbf{A}$ and $b_y \in \mathbf{b}$.

"If": Conversely, let for each $y \in Y_m$ the system (2.50), (2.51) have a solution x_y^1, x_y^2. Let $A \in \mathbf{A}$ and $b \in \mathbf{b}$. To prove that the system $Ax = b$ has a solution, take an arbitrary $y \in Y_m$ and put $x_y = x_y^1 - x_y^2$. Then we have

$$\begin{aligned} T_y(Ax_y - b) &= T_y(A_c x_y - b_c) + T_y(A - A_c)x_y + T_y(b_c - b) \\ &\geq T_y(A_c x_y - b_c) - \Delta|x_y| - \delta \end{aligned}$$

since $|T_y(A - A_c)x_y| \leq \Delta|x_y|$, which implies $T_y(A - A_c)x_y \geq -\Delta|x_y|$, and similarly $|T_y(b_c - b)| \leq \delta$ implies $T_y(b_c - b) \geq -\delta$; thus

$$\begin{aligned} T_y(Ax_y - b) &\geq T_y(A_c(x_y^1 - x_y^2) - b_c) - \Delta|x_y^1 - x_y^2| - \delta \\ &\geq T_y(A_c(x_y^1 - x_y^2) - b_c) - \Delta(x_y^1 + x_y^2) - \delta \\ &= T_y((A_c - T_y\Delta)x_y^1 - (A_c + T_y\Delta)x_y^2 - (b_c + T_y\delta)) \\ &= T_y(A_{ye}x_y^1 - A_{-ye}x_y^2 - b_y) \\ &= 0 \end{aligned}$$

since x_y^1, x_y^2 solve (2.50), (2.51). In this way we have proved that for each $y \in Y_m$, x_y satisfies

$$T_y A x_y \geq T_y b. \tag{2.56}$$

Using (2.56), we next prove that the system of linear equations

$$\sum_{y \in Y_m} \lambda_y A x_y = b, \tag{2.57}$$

$$\sum_{y \in Y_m} \lambda_y = 1 \tag{2.58}$$

has a solution $\lambda_y \geq 0$, $y \in Y_m$. In view of the Farkas theorem, it suffices to show that for each $p \in \mathbb{R}^m$ and each $p_0 \in \mathbb{R}$,

$$p^T A x_y + p_0 \geq 0 \text{ for each } y \in Y_m \tag{2.59}$$

implies

$$p^T b + p_0 \geq 0. \tag{2.60}$$

Thus let p and p_0 satisfy (2.59). Put $y = -\operatorname{sgn} p$; then $p = -T_y|p|$ and from (2.56), (2.59) we have

$$p^T b + p_0 = -|p|^T T_y b + p_0 \geq -|p|^T T_y A x_y + p_0 = p^T A x_y + p_0 \geq 0,$$

which proves (2.60). Hence the system (2.57), (2.58) has a solution $\lambda_y \geq 0$, $y \in Y_m$. Put $x = \sum_{y \in Y_m} \lambda_y x_y$; then $Ax = b$ by (2.57) and x belongs to the set $\operatorname{Conv}\{x_y \mid y \in Y_m\} = \operatorname{Conv}\{x_y^1 - x_y^2 \mid y \in Y_m\}$ by (2.58). This proves the "if" part, and also the additional assertion. □

Let us have a closer look at the form of the systems (2.50). If $y_k = 1$, then the kth rows of A_{ye} and A_{-ye} are equal to the kth rows of $\underline{A}$ and $\overline{A}$, respectively, and $(b_y)_k = \overline{b}_k$. This means that in this case the kth equation of (2.50) has the form

$$(\underline{A}x^1 - \overline{A}x^2)_k = \overline{b}_k, \tag{2.61}$$

and similarly in case $y_k = -1$ it is of the form

$$(\overline{A}x^1 - \underline{A}x^2)_k = \underline{b}_k. \tag{2.62}$$

Hence we can see that the family of systems (2.50) for all $y \in Y_m$ is just the family of all systems whose kth equations are either of the form (2.61), or of the form (2.62) for $k = 1, \ldots, m$. Now we can use the algorithm of Section 2.2 to generate the systems $A_{ye}x^1 - A_{-ye}x^2 = b_y$ in such a way that any pair of successive systems differs in exactly one equation. In this way, a feasible solution x^1, x^2 of the preceding system satisfies all but at most one of the equations of the next generated system, so that this solution x^1, x^2 can be used as the initial iteration for the procedure for checking feasibility of the next system (the procedure is not specified in the algorithm; e.g., phase I of the simplex method may be used for this purpose). The complete description of the algorithm is as follows.

$z := 0$; $y := e$; $strosolv := true$;
$A := \underline{A}$; $B := \overline{A}$; $b := \overline{b}$;
if $Ax^1 - Bx^2 = b$ is not feasible **then** $strosolv := false$; **end**
while $z \neq e$ & $strosolv$
    $k := \min\{i \mid z_i = 0\}$;
    for $i := 1$ **to** $k-1$, $z_i := 0$; **end**
    $z_k := 1$; $y_k := -y_k$;
    if $y_k = 1$ **then** $A_{k\cdot} := \underline{A}_{k\cdot}$; $B_{k\cdot} := \overline{A}_{k\cdot}$; $b_k := \overline{b}_k$;
                else $A_{k\cdot} := \overline{A}_{k\cdot}$; $B_{k\cdot} := \underline{A}_{k\cdot}$; $b_k := \underline{b}_k$;
    end
    if $Ax^1 - Bx^2 = b$ is not feasible **then** $strosolv := false$; **end**
end
% $\mathbf{A}x = \mathbf{b}$ is strongly solvable if and only if $strosolv = true$.

A small change can greatly improve the performance of the algorithm. Observe that if

$$\underline{A}_{k\cdot} = \overline{A}_{k\cdot} \quad \text{and} \quad \underline{b}_k = \overline{b}_k \tag{2.63}$$

hold for some k, then the equations (2.61) and (2.62) are the same and there is no need to solve the same system anew. Hence only rows satisfying

$$\underline{A}_{k\cdot} \neq \overline{A}_{k\cdot} \quad \text{or} \quad \underline{b}_k < \overline{b}_k \tag{2.64}$$

play any role. Let us reorder the equations of $\mathbf{A}x = \mathbf{b}$ so that those satisfying (2.64) go first, followed by those with (2.63). Hence, for the reordered system the matrix (Δ, δ) has first q nonzero rows, followed by $m-q$ zero rows ($0 \leq q \leq m$). Now we can employ the algorithm in literally the same formulation, but started with $z := 0 \in \mathbb{R}^q$, $y := e \in \mathbb{R}^q$ (instead of $z, y \in \mathbb{R}^m$ in the original version). In this way, in the case of strong solvability 2^q systems $A_{ye}x^1 - A_{-ye}x^2 = b_y$ are to be checked for feasibility. Clearly, the whole procedure can be considered acceptable for moderate values of q only.

Since the number of systems to be checked is in the worst case exponential in the matrix size, we may suspect the problem to be NP-hard. It turns out to be indeed the case, and the NP-complete problem of Theorem 2.3 can again be used for the purpose of the proof of this result.

Theorem 2.15. *Checking strong solvability of interval linear equations is NP-hard.*

Proof. Let A be square $n \times n$. We prove that the system

$$-e \leq Ax \leq e, \tag{2.65}$$

$$e^T|x| \geq 1 \tag{2.66}$$

has a solution if and only if the system of interval linear equations

$$[A - ee^T, A + ee^T]x = [0, e] \tag{2.67}$$

is *not* strongly solvable. "If": Assume that (2.67) is not strongly solvable, so that $A'x = b'$ does not have a solution for some $A' \in [A - ee^T, A + ee^T]$ and $b' \in [0, e]$. Then A' must be singular; hence $A'x' = 0$ for some $x' \neq 0$. Then x' is a weak solution of the system $[A - ee^T, A + ee^T]x = [0, 0]$; hence $|Ax'| \leq ee^T|x'|$ by the Oettli–Prager theorem. Now if we set $x = \frac{x'}{e^T|x'|}$, then $|Ax| \leq e$ and $e^T|x| = 1$, so that x solves (2.65), (2.66). "Only if" by contradiction: Assume that (2.67) is strongly solvable, and let A' be an arbitrary matrix in $[A - ee^T, A + ee^T]$. Then for each $j = 1, \ldots, n$ the system $A'x = e_j$ (where $e_j \in [0, e]$ is the jth column of the unit matrix I) has a solution x^j; hence the matrix X consisting of columns $x^1, \ldots, x^n$ satisfies $A'X = I$, so that A' is nonsingular. Hence, strong solvability of (2.67) implies nonsingularity of each $A' \in [A - ee^T, A + ee^T]$. Assume now that (2.65), (2.66) has a solution x. Then $|Ax| \leq e \leq ee^T|x|$, and the Oettli–Prager theorem implies that x solves $A'x = 0$ for some $A' \in [A - ee^T, A + ee^T]$; hence A' is singular which contradicts the above fact that each $A' \in [A - ee^T, A + ee^T]$ is nonsingular. This contradiction shows that strong solvability of (2.67) precludes existence of a solution to (2.65), (2.66), which proves the "only if" part of the assertion. In view of the established equivalence, we can see that the problem of checking solvability of (2.65), (2.66) can be reduced in polynomial time to that of checking strong solvability of (2.67). By Theorem 2.3, the former problem is NP-complete; hence the latter one is NP-hard. $\square$

In an analogy with weak solutions, we may also introduce strong solutions of systems of interval linear equations. A vector x is said to be a *strong solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for each $A \in \mathbf{A}$, $b \in \mathbf{b}$. We have this characterization of strong solutions:

Theorem 2.16. *A vector $x \in \mathbb{R}^n$ is a strong solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies*

$$A_c x = b_c, \tag{2.68}$$

$$\Delta|x| = \delta = 0. \tag{2.69}$$

Proof. Let x be a strong solution of $\mathbf{A}x = \mathbf{b}$. Put $z = \operatorname{sgn} x$; then $|x| = T_z x$, and x satisfies both

$$A_c x = b_c \tag{2.70}$$

and

$$(A_c + \Delta T_z)x = b_c - \delta. \tag{2.71}$$

Subtracting (2.70) from (2.71), we obtain

$$\Delta|x| = \Delta T_z x = -\delta,$$

where $\Delta|x| \geq 0$ and $-\delta \leq 0$; hence $\Delta|x| = \delta = 0$. Conversely, if (2.68) and (2.69) hold, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ we have

$$|Ax - b| = |A_c x - b_c + (A - A_c)x + b_c - b| \leq \Delta|x| + \delta = 0,$$

so that $Ax = b$; hence x is a strong solution of $\mathbf{A}x = \mathbf{b}$. $\square$

The condition $\Delta|x| = 0$ in (2.69) says that it must be $x_j = 0$ for each j with $\Delta_{\cdot j} \neq 0$. Hence, putting $J = \{j \mid \Delta_{\cdot j} \neq 0\}$, we may reformulate (2.68), (2.69) in the form

$$\sum_{j \notin J} (A_c)_{\cdot j} x_j = b_c, \tag{2.72}$$

$$x_j = 0 \qquad (j \in J), \tag{2.73}$$

$$\delta = 0, \tag{2.74}$$

which shows that checking existence of a strong solution (and, in the positive case, also computation of it) may be performed by solving a single system of linear equations (2.72). But on the whole the system (2.72)–(2.74) shows that strong solutions exist on rare occasions only, as could have been expected already from the definition.

2.10 Strong feasibility of equations

By definition in Section 2.6, a system $\mathbf{A}x = \mathbf{b}$ is strongly feasible if each system $Ax = b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ is feasible. It turns out that characterization of strong feasibility can be easily derived from that of strong solvability.

Theorem 2.17. *A system* $\mathbf{A}x = \mathbf{b}$ *is strongly feasible if and only if for each* $y \in Y_m$ *the system*

$$A_{ye}x = b_y \tag{2.75}$$

has a nonnegative solution x_y. *Moreover, if this is the case, then for each* $A \in \mathbf{A}$, $b \in \mathbf{b}$ *the system* $Ax = b$ *has a solution in the set*

$$\text{Conv}\{x_y \mid y \in Y_m\}.$$

Proof. If $\mathbf{A}x = \mathbf{b}$ is strongly feasible, then each system (2.75) has a nonnegative solution since $A_{ye} \in \mathbf{A}$ and $b_y \in \mathbf{b}$ for each $y \in Y_m$. Conversely, if for each $y \in Y_m$ the system (2.75) has a nonnegative solution x_y, then setting $x_y^1 = x_y$, $x_y^2 = 0$ for each $y \in Y_m$, we can see that x_y^1, x_y^2 solve (2.50), (2.51). This according to Theorem 2.14 means that each system $Ax = b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$ has a solution in the set $\text{Conv}\{x_y^1 - x_y^2 \mid y \in Y_m\} = \text{Conv}\{x_y \mid y \in Y_m\}$ which is a part of the nonnegative orthant; hence $\mathbf{A}x = \mathbf{b}$ is strongly feasible. $\square$

Repeating the argument following the proof of Theorem 2.14, we can say that the kth row of (2.75) is of the form

$$(\underline{A}x)_k = \overline{b}_k$$

if $y_k = 1$ and of the form

$$(\overline{A}x)_k = \underline{b}_k$$

if $y_k = -1$. Hence, the algorithm for checking strong solvability can be easily adapted for the present purpose.

$z := 0$; $y := e$; $strofeas := true$;
$A := \underline{A}$; $b := \overline{b}$;
if $Ax = b$ is not feasible **then** $strofeas := false$; **end**
while $z \neq e$ & $strofeas$
  $k := \min\{i \mid z_i = 0\}$;
  for $i := 1$ **to** $k - 1$, $z_i := 0$; **end**
  $z_k := 1$; $y_k := -y_k$;
  if $y_k = 1$ **then** $A_{k\cdot} := \underline{A}_{k\cdot}$; $b_k := \overline{b}_k$; **else** $A_{k\cdot} := \overline{A}_{k\cdot}$; $b_k := \underline{b}_k$; **end**
  if $Ax = b$ is not feasible **then** $strofeas := false$; **end**
end
% $\mathbf{A}x = \mathbf{b}$ is strongly feasible if and only if $strofeas = true$.

As in Section 2.9, the equations of $\mathbf{A}x = \mathbf{b}$ should be first reordered so that the first q of them satisfy (2.64) and the last $m - q$ of them are of the form (2.63). Then the algorithm remains in force if it is initialized with $z := 0 \in \mathbb{R}^q$, $y := e \in \mathbb{R}^q$.

In contrast to checking weak feasibility which is polynomial-time (Section 2.8), checking strong feasibility remains NP-hard. The proof, going along similar lines as before, is a little bit different since $n \times 2n$ matrices are needed here.

Theorem 2.18. *Checking strong feasibility of interval linear equations is NP-hard.*

Proof. Let A be square $n \times n$. We prove that the system

$$-e \le Ax \le e, \tag{2.76}$$

$$e^T|x| \ge 1 \tag{2.77}$$

has a solution if and only if the system of interval linear equations

$$[(A^T - ee^T, -A^T - ee^T), (A^T + ee^T, -A^T + ee^T)]x = [-e, e] \tag{2.78}$$

(with an $n \times 2n$ interval matrix) is *not* strongly feasible. "If": Let (2.78) be not strongly feasible; then according to Theorem 2.17 there exists a $y \in Y_m$ such that the system $A_{ye}x = b_y$ is not feasible. In our case this system has the form

$$(A^T - ye^T)x^1 + (-A^T - ye^T)x^2 = y.$$

Since it is not feasible, the Farkas theorem assures existence of a vector x' satisfying

$$(A - ey^T)x' \ge 0, \tag{2.79}$$

$$(-A - ey^T)x' \ge 0, \tag{2.80}$$

$$y^T x' < 0; \tag{2.81}$$

then (2.79), (2.80) imply

$$|Ax'| \le -ey^T x' = |-ey^T x'| \le ee^T|x'|,$$

where $x' \ne 0$ by (2.81), hence the vector $x = \frac{x'}{e^T|x'|}$ satisfies $|Ax| \le e$ and $e^T|x| = 1$, so that it is a solution to (2.76), (2.77). "Only if" by contradiction: Assume that (2.78) is strongly feasible. Let $A' \in [A - ee^T, A + ee^T]$; then $A'^T \in [A^T - ee^T, A^T + ee^T]$ and $-A'^T \in [-A^T - ee^T, -A^T + ee^T]$, so that strong feasibility of (2.78) implies that for each $j = 1, \ldots, n$ the equation

$$A'^T x^1 - A'^T x^2 = e_j$$

is feasible; i.e., the equation $A'^T x = e_j$ has a solution x^j. Then the matrix X consisting of columns $x^1, \ldots, x^n$ satisfies $A'^T X = I$, which proves that A'^T, and thus also A', is nonsingular. We have proved that strong feasibility of (2.78) implies nonsingularity of each $A' \in [A - ee^T, A + ee^T]$. As we have seen in the proof of Theorem 2.15, solvability of (2.76), (2.77) would mean existence of a singular matrix $A' \in [A - ee^T, A + ee^T]$, a contradiction. Hence (2.76), (2.77) is not solvable, which concludes the proof of the "only if" part. In view of Theorem 2.3, the established equivalence shows that checking strong feasibility is NP-hard. $\square$

Of the four decision problems related to interval linear equations we have investigated so far, three were found to be NP-hard and only one to be solvable in polynomial time. In the next four sections we show that this ratio becomes exactly reciprocal for interval linear inequalities: only one problem is NP-hard, and three are solvable in polynomial time.

2.11 Weak solvability of inequalities

As in Section 2.7, we first define $x \in \mathbb{R}^n$ to be a *weak solution* of a system of interval linear inequalities $\mathbf{A}x \leq \mathbf{b}$ if it satisfies $Ax \leq b$ for some $A \in \mathbf{A}$, $b \in \mathbf{b}$. Gerlach [43] proved in 1981 an analogue of the Oettli–Prager theorem for the case of interval linear inequalities.

Theorem 2.19 (Gerlach). *A vector x is a weak solution of $\mathbf{A}x \leq \mathbf{b}$ if and only if it satisfies*

$$A_c x - \Delta|x| \leq \overline{b}. \tag{2.82}$$

Proof. If x solves $Ax \leq b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$, then

$$A_c x - b_c \leq (A_c - A)x + b - b_c \leq |(A_c - A)x + b - b_c| \leq \Delta|x| + \delta,$$

which is (2.82). Conversely, let (2.82) hold for some x. Put $z = \operatorname{sgn} x$, then substituting $|x| = T_z x$ into (2.82) leads to

$$A_{ez} x \leq \overline{b},$$

where $A_{ez} \in \mathbf{A}$ and $\overline{b} \in \mathbf{b}$; hence x is a weak solution of $\mathbf{A}x \leq \mathbf{b}$. $\square$

A system $\mathbf{A}x \leq \mathbf{b}$ is weakly solvable (Section 2.6) if some system $Ax \leq b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$ is solvable; in other words, weak solvability is equivalent to existence of a weak solution. Hence, Gerlach's theorem provides us with the following characterization.

Theorem 2.20. *A system $\mathbf{A}x \leq \mathbf{b}$ is weakly solvable if and only if the system*

$$A_{ez} x \leq \overline{b} \tag{2.83}$$

is solvable for some $z \in Y_n$.

Proof. If x is a weak solution of $\mathbf{A}x \leq \mathbf{b}$, then, as we have seen in the proof of the Gerlach theorem, it satisfies (2.83) for $z = \operatorname{sgn} x$. Conversely, if x satisfies (2.83) for some $z \in Y_n$, then it is a weak solution of the system $\mathbf{A}x \leq \mathbf{b}$ which is then weakly solvable. □

The description suggests that the problem might be NP-hard, and it turns out to be again the case.

Theorem 2.21. *Checking weak solvability of interval linear inequalities is NP-hard.*

Proof. Given a square matrix A, the system

$$-e \leq Ax \leq e, \tag{2.84}$$

$$e^T|x| \geq 1 \tag{2.85}$$

can be rewritten equivalently as

$$\begin{pmatrix} A \\ -A \\ 0^T \end{pmatrix} x - \begin{pmatrix} 0 \\ 0 \\ e^T \end{pmatrix} |x| \leq \begin{pmatrix} e \\ e \\ -1 \end{pmatrix},$$

which is just the Gerlach inequality (2.82) for the system

$$\mathbf{A}x \leq \mathbf{b}, \tag{2.86}$$

where

$$A_c = \begin{pmatrix} A \\ -A \\ 0^T \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 \\ 0 \\ e^T \end{pmatrix}, \quad \underline{b} = \overline{b} = \begin{pmatrix} e \\ e \\ -1 \end{pmatrix}. \tag{2.87}$$

Hence the system (2.84), (2.85) has a solution if and only if the system of interval linear inequalities (2.86), (2.87) is weakly solvable. Thus the NP-complete problem of checking solvability of (2.84), (2.85) (Theorem 2.3) can be reduced in polynomial time to the problem of checking weak solvability of interval linear inequalities, which is then NP-hard. □

2.12 Weak feasibility of inequalities

Weak feasibility of inequalities was defined in Section 2.6 as existence of a nonnegative weak solution. For nonnegative x we can replace the term $|x|$ in the Gerlach inequality simply by x, thereby obtaining this characterization:

Theorem 2.22. *A system $\mathbf{A}x \leq \mathbf{b}$ is weakly feasible if and only if the system*

$$\underline{A}x \leq \overline{b} \tag{2.88}$$

is feasible.

Proof. If $x \geq 0$ satisfies $Ax \leq b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$, then

$$\underline{A}x \leq Ax \leq b \leq \overline{b}$$

and x is a feasible solution to (2.88). Conversely, feasibility of (2.88) obviously implies weak feasibility of $\mathbf{A}x \leq \mathbf{b}$. □

Since feasibility of only one system of linear inequalities is to be checked, the problem is solvable in polynomial time (see the last paragraph of Section 2.4).

2.13 Strong solvability of inequalities

By definition, a system $\mathbf{A}x \leq \mathbf{b}$ is strongly solvable if each system $Ax \leq b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ is solvable. Since the problem of checking strong solvability of interval linear equations is NP-hard (Theorem 2.15), one might expect the same to be the case for interval linear inequalities. But this analogy is no longer true, and we have this rather surprising result:

Theorem 2.23. *A system $\mathbf{A}x \leq \mathbf{b}$ is strongly solvable if and only if the system*

$$\overline{A}x^1 - \underline{A}x^2 \leq \underline{b} \tag{2.89}$$

is feasible.

Proof. "Only if": Assume to the contrary that the system (2.89) is not feasible; then neither is the system

$$\overline{A}x^1 - \underline{A}x^2 + x^3 = \underline{b},$$

and the Farkas theorem implies existence of a vector $p \in \mathbb{R}^m$ satisfying

$$\overline{A}^T p \geq 0, \tag{2.90}$$

$$\underline{A}^T p \leq 0, \tag{2.91}$$

$$p \geq 0, \tag{2.92}$$

$$\underline{b}^T p < 0. \tag{2.93}$$

Then (2.90) and (2.91) give

$$-\Delta^T p \leq -A_c^T p \leq \Delta^T p;$$

hence

$$|A_c^T p| \leq \Delta^T p = \Delta^T |p|$$

because of (2.92), and the Oettli–Prager theorem as applied to the system

$$[A_c^T - \Delta^T, A_c^T + \Delta^T]x = [0, 0]$$

implies existence of a matrix $A \in \mathbf{A}$ satisfying

$$A^T p = 0,$$

which together with (2.92) and (2.93) shows in the light of Theorem 2.6 that the system

$$Ax \le \underline{b}$$

does not have a solution, a contradiction.

"If": Let $x^1 \ge 0$, $x^2 \ge 0$ solve (2.89). Then for each $A \in \mathbf{A}$ and each $b \in \mathbf{b}$ we have

$$A(x^1 - x^2) \le \overline{A}x^1 - \underline{A}x^2 \le \underline{b} \le b,$$

so that $x^1 - x^2$ solves $Ax \le b$. Hence $\mathbf{A}x \le \mathbf{b}$ is strongly solvable; even more, all the systems $Ax \le b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$ share a common solution $x^1 - x^2$. □

Hence checking strong solvability of inequalities can be performed in polynomial time. Let us call a vector x satisfying $Ax \le b$ for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ a *strong solution* of $\mathbf{A}x \le \mathbf{b}$. We have simultaneously proved the following result.

Theorem 2.24. *If a system $\mathbf{A}x \le \mathbf{b}$ is strongly solvable, then it has a strong solution.*

In other words, if each system $Ax \le b$ with data satisfying $A \in \mathbf{A}$, $b \in \mathbf{b}$ has a solution of its own (depending on A and b, say $x(A, b)$), then all these systems share a common solution. This fact is certainly not obvious.

We have this characterization of strong solutions:

Theorem 2.25. *The following assertions are equivalent.*

(i) *x is a strong solution of $\mathbf{A}x \le \mathbf{b}$.*
(ii) *x satisfies*

$$A_c x - b_c \le -\Delta|x| - \delta. \tag{2.94}$$

(iii) *$x = x^1 - x^2$, where x^1, x^2 satisfy*

$$\overline{A}x^1 - \underline{A}x^2 \le \underline{b}, \tag{2.95}$$

$$x^1 \ge 0,\ x^2 \ge 0. \tag{2.96}$$

Proof. We prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): If $Ax \le b$ for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, then also $A_{-ez}x \le \underline{b}$, where $z = \operatorname{sgn} x$; hence

$$A_c x + \Delta|x| = (A_c + \Delta T_z)x = A_{-ez}x \le \underline{b} = b_c - \delta,$$

which implies (2.94).

(ii)⇒(iii): If x satisfies (2.94), then for $x^1 = x^+ = \max\{x, 0\}$, $x^2 = x^- = \max\{-x, 0\}$ we have $x^1 \ge 0$, $x^2 \ge 0$ and

$$\overline{A}x^1 - \underline{A}x^2 = A_c(x^1 - x^2) + \Delta(x^1 + x^2) = A_c x + \Delta|x| \leq b_c - \delta = \underline{b};$$

hence x^1, x^2 solve (2.95), (2.96) and $x = x^1 - x^2$.

(iii)$\Rightarrow$(i) was proved in the "if" part of the proof of Theorem 2.23. □

We can sum up these results in the form of a simple algorithm:

> **if** (2.95), (2.96) has a solution x^1, x^2
> **then** set $x := x^1 - x^2$ and terminate:
> x is a strong solution of $\mathbf{A}x \leq \mathbf{b}$;
> **else** terminate: $\mathbf{A}x \leq \mathbf{b}$ is not strongly solvable;
> **end**

2.14 Strong feasibility of inequalities

Finally, checking strong feasibility of inequalities is easy to characterize and can be done in polynomial time.

Theorem 2.26. *A system $\mathbf{A}x \leq \mathbf{b}$ is strongly feasible if and only if the system*

$$\overline{A}x \leq \underline{b} \tag{2.97}$$

is feasible.

Proof. If $\mathbf{A}x \leq \mathbf{b}$ is strongly feasible, then (2.97) is feasible. Conversely, if (2.97) has a solution $x \geq 0$, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ we have

$$Ax \leq \overline{A}x \leq \underline{b} \leq b;$$

hence $\mathbf{A}x \leq \mathbf{b}$ is strongly feasible. □

2.15 Summary I: Complexity results

We can now summarize the results of the previous eight sections in the form of a table.

system of	equations	weakly	solvable	NP-hard
			feasible	polynomial-time
		strongly	solvable	NP-hard
			feasible	NP-hard
	inequalities	weakly	solvable	NP-hard
			feasible	polynomial-time
		strongly	solvable	polynomial-time
			feasible	polynomial-time

We can draw several conclusions from it. For interval problems, on the average:

(i) Properties of equations are more difficult to check than those of inequalities;
(ii) Checking solvability is more difficult than checking feasibility; and
(iii) There is no such distinction between weak and strong properties.

2.16 Tolerance solutions

So far we have investigated mainly decision problems and in that frame four types of solutions (weak and strong solutions of both equations and inequalities) were introduced as auxiliary tools only. In this and in the next two sections we define three additional types of solutions motivated by some practical considerations.

In the present section we study tolerance solutions. A vector $x \in \mathbb{R}^n$ is said to be a *tolerance solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax \in \mathbf{b}$ for each $A \in \mathbf{A}$. The name of this type of solution reflects the fact that vector Ax stays within the prescribed tolerance $[\underline{b}, \overline{b}]$ independently of the choice of $A \in \mathbf{A}$. Original motivations for introducing and studying tolerance solutions came from the problem of crane construction (Nuding and Wilhelm [110]) and from the problem of input–output planning with inexact data [146].

The definition can also be recast by saying that x shall satisfy

$$\{Ax \mid A \in \mathbf{A}\} \subseteq \mathbf{b}. \tag{2.98}$$

We start therefore with a description of the left-hand-side set in (2.98).

Proposition 2.27. *Let* $\mathbf{A}$ *be an* $m \times n$ *interval matrix and let* $x \in \mathbb{R}^n$. *Then there holds*

$$\{Ax \mid A \in \mathbf{A}\} = [A_c x - \Delta|x|, A_c x + \Delta|x|]. \tag{2.99}$$

Proof. If $b \in \{Ax \mid A \in \mathbf{A}\}$, then $Ax = b$ for some $A \in \mathbf{A}$; hence x is a weak solution of

$$\mathbf{A}x = [b, b] \tag{2.100}$$

and by the Oettli–Prager theorem it satisfies

$$|A_c x - b| \leq \Delta|x|; \tag{2.101}$$

hence

$$-\Delta|x| \leq A_c x - b \leq \Delta|x| \tag{2.102}$$

and

$$A_c x - \Delta|x| \leq b \leq A_c x + \Delta|x|. \tag{2.103}$$

We have proved that $\{Ax \mid A \in \mathbf{A}\} \subseteq [A_c x - \Delta|x|, A_c x + \Delta|x|]$. Conversely, if $b \in [A_c x - \Delta|x|, A_c x + \Delta|x|]$, then b satisfies (2.103), (2.102), and (2.101); hence x is a weak solution of (2.100) which gives that $b \in \{Ax \mid A \in \mathbf{A}\}$. This proves the converse inclusion; hence (2.99) holds. $\square$

With the help of this auxiliary result we can give two equivalent descriptions of tolerance solutions.

Theorem 2.28. *The following assertions are equivalent.*

(i) x *is a tolerance solution of* $\mathbf{A}x = \mathbf{b}$.

(ii) x satisfies

$$|A_c x - b_c| \le -\Delta|x| + \delta. \tag{2.104}$$

(iii) $x = x_1 - x_2$, where x_1, x_2 satisfy

$$\overline{A}x_1 - \underline{A}x_2 \le \overline{b}, \tag{2.105}$$

$$\underline{A}x_1 - \overline{A}x_2 \ge \underline{b}, \tag{2.106}$$

$$x_1 \ge 0,\ x_2 \ge 0. \tag{2.107}$$

Proof. We prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): According to Proposition 2.27,

$$\{Ax \mid A \in \mathbf{A}\} = [A_c x - \Delta|x|, A_c x + \Delta|x|].$$

Hence, if x is a tolerance solution, then

$$[A_c x - \Delta|x|, A_c x + \Delta|x|] \subseteq [b_c - \delta, b_c + \delta],$$

which implies

$$b_c - \delta \le A_c x - \Delta|x| \le A_c x + \Delta|x| \le b_c + \delta$$

and thus also

$$-(-\Delta|x| + \delta) \le A_c x - b_c \le -\Delta|x| + \delta, \tag{2.108}$$

which is (2.104).

(ii)⇒(iii): If x satisfies (2.104), then for $x_1 = x^+$, $x_2 = x^-$ we have $x = x_1 - x_2$, $|x| = x_1 + x_2$ and the inequalities (2.108) turn into

$$\Delta(x_1 + x_2) - \delta \le A_c(x_1 - x_2) - b_c \le -\Delta(x_1 + x_2) + \delta,$$

which gives (2.105), (2.106), and (2.107) is satisfied because $x^+ \ge 0$, $x^- \ge 0$.

(iii)⇒(i): If $x_1 \ge 0$, $x_2 \ge 0$ solve (2.105), (2.106), then for $x = x_1 - x_2$ and for each $A \in \mathbf{A}$ we have

$$Ax = A(x_1 - x_2) \le \overline{A}x_1 - \underline{A}x_2 \le \overline{b}$$

and

$$Ax = A(x_1 - x_2) \ge \underline{A}x_1 - \overline{A}x_2 \ge \underline{b}$$

which shows that $Ax \in \mathbf{b}$ for each $A \in \mathbf{A}$, hence x is a tolerance solution. □

There is a remarkable similarity between the inequality (2.104) and the Oettli–Prager inequality (2.35): both descriptions differ in the sign preceding the matrix Δ only. Yet this seemingly small difference has an astounding impact: although checking the existence of solution of the Oettli–Prager inequality is NP-hard (Theorem 2.12), checking the existence of a tolerance solution can be performed in polynomial time simply by checking solvability of the system (2.105)–(2.107). The description (iii) also shows that the set of tolerance solutions is a convex polyhedron; it allows us to compute the range of components of tolerance solutions by solving the respective linear programming problems [154], etc.

2.17 Control solutions

A vector $x \in \mathbb{R}^n$ is called a *control solution* of $\mathbf{A}x = \mathbf{b}$ if for each $b \in \mathbf{b}$ there exists an $A \in \mathbf{A}$ such that $Ax = b$ holds, in other words, if

$$\mathbf{b} \subseteq \{Ax \mid A \in \mathbf{A}\}.$$

Control solutions were introduced by Shary [178] in 1992. The choice of the word "control" was probably motivated by the fact that each vector $b \in \mathbf{b}$ can be reached by Ax when properly controlling the coefficients of A within $\mathbf{A}$. We have this characterization.

Theorem 2.29. *The following assertions are equivalent.*

(i) x *is a control solution of* $\mathbf{A}x = \mathbf{b}$.
(ii) x *satisfies*

$$|A_c x - b_c| \le \Delta|x| - \delta. \tag{2.109}$$

(iii) x *solves*

$$A_{ez}x \le \underline{b}, \tag{2.110}$$
$$-A_{-ez}x \le -\overline{b} \tag{2.111}$$

for some $z \in Y_n$.

Proof. We prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): If x is a control solution, then by Proposition 2.27 it satisfies $[b_c - \delta, b_c + \delta] \subseteq \{Ax \mid A \in \mathbf{A}\} = [A_c x - \Delta|x|, A_c x + \Delta|x|]$, which implies

$$A_c x - \Delta|x| \le b_c - \delta \le b_c + \delta \le A_c x + \Delta|x|$$

and

$$-(\Delta|x| - \delta) \le A_c x - b_c \le \Delta|x| - \delta; \tag{2.112}$$

hence

$$|A_c x - b_c| \le \Delta|x| - \delta.$$

(ii)⇒(iii): If x satisfies (2.109), then (2.112) holds and with $z = \operatorname{sgn} x$ we can substitute $|x| = T_z x$ into (2.112) which leads to (2.110), (2.111).

(iii)⇒(i): If x solves (2.110), (2.111) for some $z \in Y_n$, then $|\Delta T_z x| \le \Delta|x|$, hence

$$\begin{aligned} A_c x - \Delta|x| &\le (A_c - \Delta T_z)x = A_{ez}x \le \underline{b} \le \overline{b} \le A_{-ez}x \\ &= (A_c + \Delta T_z)x \le A_c x + \Delta|x|, \end{aligned}$$

which implies

$$[\underline{b}, \overline{b}] \subseteq [A_c x - \Delta|x|, A_c x + \Delta|x|] = \{Ax \mid A \in \mathbf{A}\}$$

by Proposition 2.27; hence x is a control solution. □

Again, the inequality (2.109) differs from the Oettli–Prager inequality (2.35) in the sign preceding δ only. But this time the difference does not affect complexity of the problem.

Theorem 2.30. *Checking existence of control solutions is NP-hard.*

Proof. For a square matrix A, consider the system

$$-e \leq Ax \leq e, \tag{2.113}$$

$$e^T|x| \geq 1, \tag{2.114}$$

and the inequality

$$\left|\begin{pmatrix} A \\ 0^T \end{pmatrix} x - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right| \leq \begin{pmatrix} ee^T \\ e^T \end{pmatrix} |x| - \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.115}$$

If x solves (2.113), (2.114), then it also solves (2.115). Conversely, if x solves (2.115), then $x \neq 0$ and $x' = \frac{x}{e^T|x|}$ solves (2.113), (2.114). Hence, the system (2.113), (2.114) has a solution if and only if the inequality (2.115) has a solution. But (2.115) is exactly the inequality (2.109) for the system of interval linear equations

$$[A - ee^T, A + ee^T]x = [0, 0], \tag{2.116}$$

$$[-e^T, e^T]x = [1, 1], \tag{2.117}$$

which gives that (2.113), (2.114) has a solution if and only if (2.116), (2.117) has a control solution. Now an application of Theorem 2.3 concludes the proof. □

2.18 Algebraic solutions

A vector $x \in \mathbb{R}^n$ is called an *algebraic solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies

$$\{Ax \mid A \in \mathbf{A}\} = \mathbf{b}. \tag{2.118}$$

Algebraic solutions were first introduced by Ratschek and Sauer in [138]. This type of solution is easy to characterize.

Theorem 2.31. *x is an algebraic solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies*

$$A_c x = b_c, \tag{2.119}$$

$$\Delta|x| = \delta. \tag{2.120}$$

Proof. By Proposition 2.27, (2.118) is equivalent to

$$[A_c x - \Delta|x|, A_c x + \Delta|x|] = [b_c - \delta, b_c + \delta], \tag{2.121}$$

which implies (2.119), (2.120). On the other hand, (2.119) and (2.120) imply (2.121) and thus also (2.118). □

It follows from Theorems 2.28 and 2.29, inequalities (2.104) and (2.109), that x is an algebraic solution of $\mathbf{A}x = \mathbf{b}$ if and only if it is both the tolerance and control solution of it. If $m = n$ and A_c is nonsingular, then $\mathbf{A}x = \mathbf{b}$ has an algebraic solution if and only if the data satisfy

$$\Delta|A_c^{-1}b_c| = \delta, \tag{2.122}$$

in which case $x = A_c^{-1}b_c$ is the unique algebraic solution of it.

2.19 The square case

In this section we consider systems of interval linear equations $\mathbf{A}x = \mathbf{b}$ where $\mathbf{A}$ is square $n \times n$ and $\mathbf{b}$ is an n-dimensional interval vector. The square case, which has been a part of the mainstream of interval analysis for the last three decades, would have deserved a special chapter itself, if not a book. Here we confine ourselves to the most important theoretical result (Theorem 2.36), its prerequisites and some of its consequences. Let us repeat that throughout this section $\mathbf{A}$ is square $n \times n$.

As in the noninterval case, nonsingularity plays an important role here. A square interval matrix $\mathbf{A}$ is called *regular* if each $A \in \mathbf{A}$ is nonsingular, and *singular* in the opposite case (i.e., if $\mathbf{A}$ contains a singular matrix). Our previous results imply the following general characterization.

Theorem 2.32. $\mathbf{A}$ *is regular if and only if the system*

$$A_{ye}x^1 - A_{-ye}x^2 = y \tag{2.123}$$

is feasible for each $y \in Y_n$.

Proof. Consider the system of interval linear equations

$$\mathbf{A}x = [-e, e]. \tag{2.124}$$

If $\mathbf{A}$ is regular, then (2.124) is strongly solvable and Theorem 2.14 implies that the system (2.123) is feasible for each $y \in Y_n$ since in this case $b_y = T_y e = y$. Conversely, if (2.123) is feasible for each $y \in Y_n$, then (2.124) is strongly solvable by Theorem 2.14; hence for each $A \in \mathbf{A}$ the system $Ax = e_j$ has a solution for each j, where $e_j \in [-e, e]$ is the jth column of the unit matrix I, which shows that A is invertible and thus nonsingular. □

Theorem 2.33. *Checking regularity of interval matrices is NP-hard.*

Proof. Let A be square. From the proof of Theorem 2.30 we can infer that the system

$$-e \leq Ax \leq e,$$
$$e^T|x| \geq 1$$

has a solution if and only if the interval matrix

$$[A - ee^T, A + ee^T]$$

is singular. Now Theorem 2.3 provides for the rest. $\square$

Fortunately, there exists a verifiable sufficient regularity condition that covers most practical cases. $\varrho(A)$ denotes the spectral radius of A.

Proposition 2.34. *If A_c is nonsingular and*

$$\varrho(|A_c^{-1}|\Delta) < 1$$

holds, then $\mathbf{A}$ *is regular.*

Proof. For each $A \in \mathbf{A}$ we have

$$\varrho(A_c^{-1}(A_c - A)) \leq \varrho(|A_c^{-1}(A_c - A)|) \leq \varrho(|A_c^{-1}|\Delta) < 1.$$

Hence by Theorem 1.31 the matrix

$$I - A_c^{-1}(A_c - A) = A_c^{-1}A$$

is invertible and thus nonsingular. Then A is nonsingular, and $\mathbf{A}$ is regular. $\square$

A square matrix A is called a P-matrix if all its principal minors are positive. In 1962 Fiedler and Pták [37] proved this characterization: A is a P-matrix if and only if for each $x \neq 0$ there is an i such that $x_i(Ax)_i > 0$ (see Theorem 1.79). With the help of this fact we can prove the next assertion which forms a bridge towards the main result.

Theorem 2.35. *If* $\mathbf{A}$ *is regular, then $A_1^{-1}A_2$ is a P-matrix for each $A_1, A_2 \in$* $\mathbf{A}$.

Proof. Assume to the contrary that $A_1^{-1}A_2$ is not a P-matrix for some $A_1, A_2 \in \mathbf{A}$. Then according to the Fiedler–Pták theorem there exists an $x \neq 0$ such that $x_i(A_1^{-1}A_2x)_i \leq 0$ for each i. Take $x' = A_1^{-1}A_2x$; then $x_ix_i' \leq 0$ holds for each i which implies that

$$|x'| + |x| = |x' - x|. \tag{2.125}$$

Now we have

$$|A_c(x' - x)| = |(A_c - A_1)x' + (A_2 - A_c)x| \leq \Delta|x'| + \Delta|x| = \Delta|x' - x|$$

due to (2.125) which also gives that $x' \neq x$ since $x' = x$ would imply $x = 0$ contrary to $x \neq 0$. Hence by the Oettli–Prager theorem there exists an $A \in \mathbf{A}$ with $A(x' - x) = 0$ which means that A is singular, a contradiction. $\square$

When solving an interval linear system $\mathbf{A}x = \mathbf{b}$ with a square interval matrix $\mathbf{A}$, we are usually interested in the set X of weak solutions of it, i.e., in the set

$$X = \{x \mid Ax = b \text{ for some } A \in \mathbf{A}, b \in \mathbf{b}\}. \tag{2.126}$$

The main result of this section asserts that X contains some uniquely determined significant points ($\operatorname{Conv} X$ denotes the convex hull of X).

Theorem 2.36. *Let* $\mathbf{A}$ *be regular and let* $\mathbf{b}$ *be an n-dimensional interval vector. Then for each* $y \in Y_n$ *the equation*

$$A_c x - T_y \Delta |x| = b_y \tag{2.127}$$

has a unique solution x_y *that belongs to* X *and there holds*

$$\operatorname{Conv} X = \operatorname{Conv}\{x_y \mid y \in Y_n\}. \tag{2.128}$$

Proof. Consider the system

$$x^1 = A_{ye}^{-1} A_{-ye} x^2 + A_{ye}^{-1} b_y, \tag{2.129}$$

$$x^1 \geq 0,\ x^2 \geq 0, \tag{2.130}$$

$$(x^1)^T x^2 = 0. \tag{2.131}$$

We can see that (2.129)–(2.131) is a linear complementarity problem [97] whose matrix $A_{ye}^{-1} A_{-ye}$ is a P-matrix due to regularity of $\mathbf{A}$ (Theorem 2.35); hence by the result due to Samelson, Thrall and Wesler [175] (rediscovered independently by Ingleton [55] and Murty [97]), (2.129)–(2.131) has a unique solution x_y^1, x_y^2. Put $x_y = x_y^1 - x_y^2$. Then $A_{ye} x_y^1 - A_{-ye} x_y^2 = b_y$ and Theorem 2.14 gives that for each $A \in \mathbf{A}$ and each $b \in \mathbf{b}$ the unique (because of regularity) solution of $Ax = b$ belongs to $\operatorname{Conv}\{x_y \mid y \in Y_n\}$ which means that $X \subseteq \operatorname{Conv}\{x_y \mid y \in Y_n\}$ and thus also $\operatorname{Conv} X \subseteq \operatorname{Conv}\{x_y \mid y \in Y_n\}$. On the other hand, (2.129)–(2.131) imply

$$A_c x_y - b_c = A_c(x_y^1 - x_y^2) - b_c = T_y(\Delta(x_y^1 + x_y^2) + \delta) = T_y(\Delta |x_y| + \delta),$$

so that x_y solves (2.127) and

$$|A_c x_y - b_c| = \Delta |x_y| + \delta \tag{2.132}$$

holds, which in the light of the Oettli–Prager theorem means that $x_y \in X$ for each $y \in Y_n$. Hence $\operatorname{Conv}\{x_y \mid y \in Y_n\} \subseteq \operatorname{Conv} X$, which proves the converse inclusion. Finally, if x solves (2.127), then a simple rearrangement shows that $x^1 = x^+$, $x^2 = x^-$ solve (2.129)–(2.131) and in view of the above-stated uniqueness of solution of this linear complementarity problem we have

$$x = x^+ - x^- = x_y^1 - x_y^2 = x_y,$$

so that the solution of (2.127) is unique. $\square$

Let us emphasize that whereas in Theorem 2.17 x_y denoted an arbitrary of possibly infinitely many solutions of (2.75), in Theorem 2.36 x_y denotes the unique solution of (2.127). If $\mathbf{A}$ is regular, then for each $y \in Y_n$ the point x_y can be computed by the following finite algorithm, called the sign-accord algorithm because it works towards achieving a "sign accord" of vectors z and x (i.e., $z_j x_j \geq 0$ for each j).

$z := \operatorname{sgn}(A_c^{-1} b_y)$;
$x := A_{yz}^{-1} b_y$;
$C := A_{yz}^{-1} T_y \Delta$;
while $z_j x_j < 0$ for some j
  $k := \min\{j \mid z_j x_j < 0\}$;
  $z_k := -z_k$;
  $\alpha := 2z_k/(1 - 2z_k C_{kk})$;
  $x := x + \alpha x_k C_{\cdot k}$;
  $C := C + \alpha C_{\cdot k} C_{k \cdot}$;
end
$x_y := x$.

($C_{\cdot k}$ and $C_{k\cdot}$ denote the kth column and the kth row of C, respectively.) We refrain from including a proof here, which would lead us beyond the scope of this chapter. We refer an interested reader to [156, p. 48].

The narrowest interval vector $[\underline{x}, \overline{x}]$ containing the set X is called the *interval hull* of X. From (2.128) we immediately have that

$$\underline{x}_i = \min_{y \in Y_n} (x_y)_i, \tag{2.133}$$

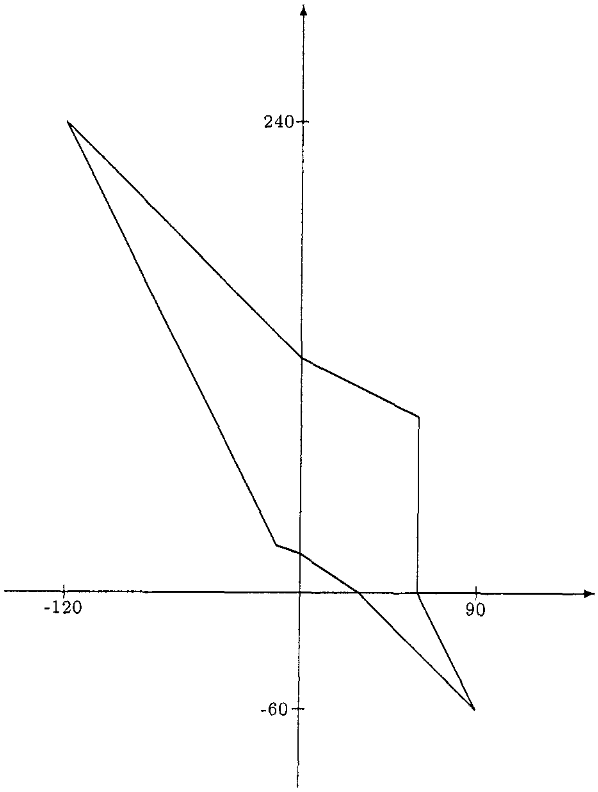
$$\overline{x}_i = \max_{y \in Y_n} (x_y)_i \tag{2.134}$$

($i = 1, \ldots, n$), which, when combined with the sign-accord algorithm, yields a finite procedure for computing the interval hull.

Example 2.37. Consider the example by Hansen [47]: $\mathbf{A} = [\underline{A}, \overline{A}]$, $\mathbf{b} = [\underline{b}, \overline{b}]$, where

$$\underline{A} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 0 \\ 60 \end{pmatrix}, \quad \overline{b} = \begin{pmatrix} 120 \\ 240 \end{pmatrix}.$$

Since for each $z \in Y_2$ the intersection of the set of weak solutions X with the orthant $\{x \in \mathbb{R}^2 \mid T_z x \geq 0\}$ is described by the system of linear inequalities (2.39), (2.40), considering separately all four orthants we arrive at this picture of the set X:



It can be seen that X is nonconvex and the four points x_y, $y \in Y_2$ are clearly visible since in view of (2.128) they must be exactly the four vertices of the convex hull of X. Using the sign-accord algorithm, we obtain

$$\begin{aligned} x_{(-1,-1)} &= (-12, 24)^T, \\ x_{(-1,1)} &= (-120, 240)^T, \\ x_{(1,-1)} &= (90, -60)^T, \\ x_{(1,1)} &= (60, 90)^T, \end{aligned}$$

and from (2.133), (2.134) we have that the interval hull of X is $[\underline{x}, \overline{x}]$, where

$$\begin{aligned} \underline{x} &= (-120, -60)^T, \\ \overline{x} &= (90, 240)^T. \end{aligned}$$

Unfortunately, the general problem is again NP-hard:

Theorem 2.38. *Computing the interval hull of the set X is NP-hard even for systems with interval matrices satisfying*

$$\varrho(|A_c^{-1}|\Delta) = 0.$$

Proof. Given a rational $n \times n$ matrix A, construct the $(n+1) \times (n+1)$ interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ with

$$A_c = \begin{pmatrix} 1 & 0^T \\ 0 & A \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & e^T \\ 0 & 0 \end{pmatrix},$$

and the $(n+1)$-dimensional interval vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$ with

$$b_c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 \\ e \end{pmatrix}$$

($e \in \mathbb{R}^n$). We have

$$|A_c^{-1}|\Delta = \begin{pmatrix} 0 & e^T \\ 0 & 0 \end{pmatrix};$$

hence

$$\varrho(|A_c^{-1}|\Delta) = 0.$$

Then each system $Ax = b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ has the form

$$\begin{aligned} x_1 + c^T x' &= 0, \\ Ax' &= d \end{aligned}$$

for some $c \in [-e, e]$ and $d \in [-e, e]$, where $x' = (x_2, \ldots, x_{n+1})^T$. If $[\underline{x}, \overline{x}]$ is the interval hull of (2.126), then for $\overline{x}_1$ we have

$$\overline{x}_1 = \max\{c^T x' \mid c \in [-e, e], \, -e \le Ax' \le e\} = \max\{e^T |x'| \mid -e \le Ax' \le e\};$$

hence

$$\overline{x}_1 \ge 1$$

holds if and only if the system

$$-e \le Ax' \le e,$$

$$e^T |x'| \ge 1$$

has a solution. Since the latter problem is NP-complete (Theorem 2.3), $\overline{x}_1$ is NP-hard to compute and the same holds for $[\underline{x}, \overline{x}]$. □

This result shows that we must set the goal differently: instead of trying to compute the exact interval hull $[\underline{x}, \overline{x}]$, we should be satisfied with computing a possibly narrow *enclosure* of X, i.e., an interval vector $[\underline{\underline{x}}, \overline{\overline{x}}]$ satisfying

$$X \subseteq [\underline{\underline{x}}, \overline{\overline{x}}].$$

There is a vast literature dedicated to this theme, comprising a number of ingenious enclosure methods, see, e.g., the monographs by Alefeld and Herzberger [2] or Neumaier [105]. We conclude this section with description of a nontrivial result that gives explicit formulae for computing an enclosure.

Theorem 2.39 (Hansen–Bliek–Rohn). *Let A_c be nonsingular and let*

$$\varrho(|A_c^{-1}|\Delta) < 1 \tag{2.135}$$

hold. Then we have

$$X \subseteq [\min\{\underset{\sim}{x}, T_\nu \underset{\sim}{x}\}, \max\{\tilde{x}, T_\nu \tilde{x}\}], \tag{2.136}$$

where

$$\begin{aligned} M &= (I - |A_c^{-1}|\Delta)^{-1}, \\ \mu &= (M_{11}, \dots, M_{nn})^T, \\ T_\nu &= (2T_\mu - I)^{-1}, \\ x_c &= A_c^{-1} b_c, \\ x^* &= M(|x_c| + |A_c^{-1}|\delta), \\ \underset{\sim}{x} &= -x^* + T_\mu(x_c + |x_c|), \\ \tilde{x} &= x^* + T_\mu(x_c - |x_c|). \end{aligned}$$

Proof. First we note that because of (2.135) we have

$$M = \sum_{j=0}^{\infty} (|A_c^{-1}|\Delta)^j \geq I \geq 0;$$

thus also $2T_\mu - I \geq I$, so that the diagonal matrix $T_\nu = (2T_\mu - I)^{-1}$ exists and $\nu_i = 1/(2M_{ii} - 1)$ for each i.

To prove (2.136), take an $x \in X$; then by the Oettli–Prager theorem it satisfies

$$|A_c x - b_c| \leq \Delta|x| + \delta;$$

hence

$$|x| - |x_c| \leq |x - x_c| = |A_c^{-1}(A_c x - b_c)| \leq |A_c^{-1}||A_c x - b_c| \leq |A_c^{-1}|(\Delta|x| + \delta). \tag{2.137}$$

Now, let us fix an $i \in \{1, \dots, n\}$. Then from (2.137) we have

$$x_i \leq (x_c)_i + (|A_c^{-1}|(\Delta|x| + \delta))_i \tag{2.138}$$

and

$$|x_j| \leq |x_c|_j + (|A_c^{-1}|(\Delta|x| + \delta))_j \tag{2.139}$$

for each $j \neq i$. Since $x_i = |x_i| + (x_i - |x_i|)$ and the same holds for $(x_c)_i$, we can put (2.138) and (2.139) together as

$$|x| + (x_i - |x_i|)e_i \leq |x_c| + ((x_c)_i - |x_c|_i)e_i + |A_c^{-1}|(\Delta|x| + \delta),$$

which implies

$$(I - |A_c^{-1}|\Delta)|x| + (x_i - |x_i|)e_i \le |x_c| + |A_c^{-1}|\delta + ((x_c)_i - |x_c|_i)e_i.$$

Premultiplying this inequality by the nonnegative vector $e_i^T M$, we finally obtain an inequality containing variable x_i only:

$$|x_i| + (x_i - |x_i|)M_{ii} \le x_i^* + ((x_c)_i - |x_c|_i)M_{ii} = \tilde{x}_i.$$

If $x_i \ge 0$, then this inequality becomes

$$x_i \le \tilde{x}_i,$$

and if $x_i < 0$, then it turns into

$$x_i \le \tilde{x}_i/(2M_{ii} - 1) = \nu_i \tilde{x}_i,$$

in both cases

$$x_i \le \max\{\tilde{x}_i, \nu_i \tilde{x}_i\}.$$

Since i was arbitrary, we conclude that

$$x \le \max\{\tilde{x}, T_\nu \tilde{x}\},$$

which is the upper bound in (2.136). To prove the lower bound, notice that if $Ax = b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$, then $A(-x) = -b$, hence $-x$ belongs to the solution set of the system $\mathbf{A}x = [-b_c - \delta, -b_c + \delta]$, and we can apply the previous result to this system by setting $b_c := -b_c$. In this way we obtain

$$-x \le \max\{x^* + T_\mu(-x_c - |x_c|), T_\nu(x^* + T_\mu(-x_c - |x_c|))\};$$

hence

$$x \ge \min\{-x^* + T_\mu(x_c + |x_c|), T_\nu(-x^* + T_\mu(x_c + |x_c|))\} = \min\{\underset{\sim}{x}, T_\nu \underset{\sim}{x}\},$$

which is the lower bound in (2.136). The theorem is proved. □

This theorem gives an enclosure (2.136) which is fairly good in practical cases, but generally not optimal (cf. Theorem 2.38). However, it is optimal (i.e., it yields the interval hull of X) in the case of $A_c = I$ (Hansen [48], Bliek [21], Rohn [160]).

The result can be easily applied to bound the inverse of an interval matrix. In the next theorem, the minimum or maximum of two matrices is understood componentwise.

Theorem 2.40. *Let (2.135) hold. Then for each $A \in \mathbf{A}$ we have*

$$\min\{\underset{\sim}{B}, T_\nu \underset{\sim}{B}\} \le A^{-1} \le \max\{\widetilde{B}, T_\nu \widetilde{B}\},$$

where M, μ, and T_ν are as in Theorem 2.39 and

$$\underset{\sim}{B} = -M|A_c^{-1}| + T_\mu(A_c^{-1} + |A_c^{-1}|),$$

$$\widetilde{B} = \quad M|A_c^{-1}| + T_\mu(A_c^{-1} - |A_c^{-1}|).$$

Proof. Since $(A^{-1})_{\cdot j}$ is the solution of the system $Ax = e_j$, we obtain the result simply by applying Theorem 2.39 to interval linear systems $\mathbf{A}x = [e_j, e_j]$ for $j = 1, \ldots, n$. □

2.20 Summary II: Solution types

We have introduced altogether eight types of solutions. We summarize the results in the following table which clearly illustrates the tiny differences in their descriptions.

Solution	Description	Reference
weak solution of $\mathbf{A}x = \mathbf{b}$	$\lvert A_c x - b_c\rvert \le \Delta\lvert x\rvert + \delta$	(2.35)
strong solution of $\mathbf{A}x = \mathbf{b}$	$A_c x - b_c = \Delta\lvert x\rvert = \delta = 0$	(2.68), (2.69)
weak solution of $\mathbf{A}x \le \mathbf{b}$	$A_c x - b_c \le \Delta\lvert x\rvert + \delta$	(2.82)
strong solution of $\mathbf{A}x \le \mathbf{b}$	$A_c x - b_c \le -\Delta\lvert x\rvert - \delta$	(2.94)
tolerance solution	$\lvert A_c x - b_c\rvert \le -\Delta\lvert x\rvert + \delta$	(2.104)
control solution	$\lvert A_c x - b_c\rvert \le \Delta\lvert x\rvert - \delta$	(2.109)
algebraic solution	$A_c x - b_c = \Delta\lvert x\rvert - \delta = 0$	(2.119), (2.120)
x_y	$\lvert A_c x - b_c\rvert = \Delta\lvert x\rvert + \delta$	(2.132)

2.21 Notes and references

In this section we give some additional notes and references to the material contained in this chapter.

Section 2.1. We use standard linear algebraic notations except for Y_m, T_y and $\operatorname{sgn} x$ (introduced in [156]).

Section 2.2. The algorithm is a variant of the binary reflected Gray code (Gray [46]), see, e.g., [194].

Section 2.3. The first NP-hardness result for problems with interval-valued data was published by Poljak and Rohn as a report [116] in 1988 and as a journal paper [117] in 1993. They showed that for an $n \times n$ matrix A the value

$$\max_{z,y \in Y_n} z^T A y \tag{2.140}$$

is NP-hard to compute, and they used the result to prove that checking regularity of interval matrices is NP-hard (Theorem 2.33 here). Only in 1995 was it realized [162] that the value of (2.140) is equal to $\|A\|_{\infty,1}$ (see (2.4)) which led to the formulation of Theorem 2.2 ([162], in journal form [165]). Theorem 2.3, which is more useful in the context of interval linear systems, was also proved in [162]. Notice that all the NP-hardness results of this chapter were proved with the help of this theorem.

Section 2.4. The word "feasibility", which is a one-word substitute for nonnegative solvability, was inspired by linear programming terminology. Theorem 2.4, also known as Farkas' lemma, was proved by Farkas [34] in 1902. It is an important theoretical result (as evidenced throughout this chapter), but it does not give a constructive way of checking feasibility which must be done by another means (usually by a linear programming technique).

Section 2.5. Matrices A_{yz} and vectors b_y were introduced in [156]. The importance of the finite set of matrices A_{yz} becomes more apparent with problems involving square interval matrices only (as regularity, positive definiteness etc.). For example, an interval matrix $\mathbf{A}$ is regular (see Section 2.19) if and only if $\det(A_{yz})$ is of the same sign for each $z, y \in Y_n$ (Baumann [10]); for further results of this type see the monograph by Kreinovich, Lakeyev, Rohn, and Kahl [80, Chapters 21 and 22]. As we have seen, in the context of rectangular interval systems typically only matrices of the form A_{ye} or A_{ez} arise.

Section 2.6. The definition of an interval linear system $\mathbf{A}x = \mathbf{b}$ as a family of systems $Ax = b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$ makes it possible to define various types of solutions. The notion of strong feasibility of interval linear equations was introduced in [149], and weak solvability as a counterpart of strong solvability was first studied by Rohn and Kreslová in [168]. Formulation and study of the complete set of the eight decision problems is new and forms the bulk of this chapter.

Section 2.7. The Oettli–Prager theorem is formulated here in the form (2.35) which has become standard, although not explicitly present in the original paper [112] where the authors preferred an entrywise formulation. The theorem is now considered a basic tool for both backward error analysis (Golub and van Loan [44], Higham [51]) and interval analysis (Neumaier [105]) of systems of linear equations. Another form of Proposition 2.10 (perhaps more attractive, but less useful) is described in [153, Theorem 1.2]. NP-hardness of checking weak solvability of equations was proved by Lakeyev and Noskov [84] (preliminary announcement without proof in [83]) by another means. The proof given here employs polynomial reduction of our standard problem of Theorem 2.3 to the current problem, an approach adhered to throughout the chapter.

Section 2.8. Theorem 2.13 is a simple consequence of the Oettli–Prager theorem. It was discovered independently in [145].

Section 2.9. The proof of Theorem 2.14 is not straightforward and neither is its history. The "if" part was formulated and proved in technical reports [152], [151] in 1984, but the author refrained from further journal publication because he considered the sufficient condition too strong. In 1996 he discovered by chance that it was also necessary (paradoxically, it was the easier part of the proof), which gave rise to Theorem 2.14 published in [166]. The second part of the proof of the "if" part (starting from (2.56)) relies in fact on a new existence theorem for systems of linear equations which was published in [157] (existence proof, as given here) and in [159] (constructive proof). NP-hardness of checking strong solvability (Theorem 2.15) is an easy consequence of the same complexity result for the problem of checking regularity of interval matrices (Theorem 2.33), but because of the layout of this chapter it had to be proved independently.

Section 2.10. Characterization of strong feasibility of equations (Theorem 2.17) was published in [149] as part of a study of the interval linear programming problem. Many unsuccessful attempts by the author through the following years to find a characterization of strong feasibility that would not be inherently exponential finally led to the NP-hardness conjecture and to the proof of it in [164] (part 2 of the proof).

Section 2.11. Gerlach [43] initiated the study of systems of interval linear inequalities by proving Theorem 2.19 as a follow-up of the Oettli–Prager theorem. NP-hardness of checking weak solvability of inequalities was proved in technical report [162] and has not been published in journal form.

Section 2.12. The result of Theorem 2.22 is obvious and is included here for completeness.

Section 2.13. Both Theorems 2.23 and 2.24 are due to Rohn and Kreslová [168]. The contrast between the complexity results for strong solvability of interval linear equations (Theorem 2.15) and inequalities (Theorem 2.23) is striking and reveals that classical solvability-preserving reductions between linear equations and linear inequalities are no longer in force when inexact data are present. In fact, a system of linear equations $Ax = b$ can be equivalently written as a system if linear inequalities $Ax \leq b$, $-Ax \leq -b$ and solved as such. But in the case of interval data, the sets of weak solutions of $\mathbf{A}x = \mathbf{b}$ and of $\mathbf{A}x \leq \mathbf{b}$, $-\mathbf{A}x \leq -\mathbf{b}$ are generally not identical since the latter family contains systems of inequalities of type $Ax \leq b$, $-\tilde{A}x \leq -\tilde{b}$ ($A, \tilde{A} \in \mathbf{A}$, $b, \tilde{b} \in \mathbf{b}$) that may possess solutions which do not satisfy $Ax = b$ for any $A \in \mathbf{A}$, $b \in \mathbf{b}$. Existence of strong solutions in the case of strong solvability (Theorem 2.24) is a nontrivial fact that can be expected to find some applications, although none of them have been known to date.

Section 2.14. Theorem 2.26 is again obvious.

Section 2.16. Introduction of the notion of tolerance solutions was motivated by considerations concerning crane construction (Nuding and Wilhelm [110]) and input–output planning with inexact data of the socialist economy of the former Czechoslovakia [146]. Descriptions (ii), (iii) of tolerance solutions in Theorem 2.28 were proved in [154]. Tolerance solutions have been studied since by Neumaier [104], Deif [32], Kelling and Oelschlägel [70], Kelling [68], [69], Shaydurov and Shary [185], Shary [176], [179], [180], [181], and Lakeyev and Noskov [84].

Section 2.17. Control solutions were introduced by Shary [178] and further studied by him in [181], [183]. The description (2.109) in Theorem 2.29 is due to Lakeyev and Noskov [84] who in the same paper also proved NP-hardness of checking the existence of control solutions, as well as of algebraic solutions. For other possible types of solutions see the survey paper by Shary [184].

Section 2.18. Algebraic solutions were introduced by Ratschek and Sauer [138], although for the case $m = 1$ only. The condition (2.122) was proved in [153]. The topic makes more sense when the problem is formulated as $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is an interval vector and multiplication is performed in interval arithmetic. A solution of this problem in full generality is not known so far; for a partial solution see [158].

Section 2.19. NP-hardness of checking regularity of square matrices (Theorem 2.33) was proved by Poljak and Rohn [116], [117] whose work was motivated by the existence at that time of more than ten necessary and sufficient regularity conditions all of which exhibited exponential complexity (Theorem 5.1 in [156]; one of them is our Theorem 2.32). The sufficient regularity condition of Proposition 2.34 is usually attributed to Beeck [11], although allegedly (Neumaier [103]) it was derived earlier by Ris in his unpublished Ph.D. thesis [143]. The "convex-hull" Theorem 2.36, as well as finiteness of the sign accord algorithm, were proved in [156]. The NP-hardness result of Theorem 2.38 on complexity of computing the interval hull of the set X of weak solutions is due to Rohn and Kreinovich [167]. In 1992 Hansen [48] and Bliek [21] showed almost simultaneously that in the case $A_c = I$ the interval hull can be described by closed-form formulae, but their result lacked a rigorous proof which was supplied in [160]. The idea can be applied to a preconditioned system, as was done in the proof of Theorem 2.39, but in this way only an enclosure, not the interval hull, is obtained (Theorem 2.38 explains why it is so). Computation of the enclosure requires evaluation of two inverses, A_c^{-1} and $(I - |A_c^{-1}|\Delta)^{-1}$; the main result of [144] shows that we can also do with approximate inverses $R \approx A_c^{-1}$ and $M \approx (I - |A_c^{-1}|\Delta)^{-1}$ provided they satisfy certain additional inequality. The topic was later studied by Ning and Kearfott [108] and Neumaier [106]. *We refrain here from listing papers dedicated to computing enclosures since they are simply too many.* As for the latest developments,[1] we mention the method of Jansson [64], characterization of feasibility of preconditioned interval Gaussian algorithm by Mayer and Rohn [87], the techniques by Shary [177], [182], and a series of papers by Alefeld, Kreinovich, and Mayer [6], [3], [4], [5] which handle the complicated problem of solving interval systems with dependent data. An earlier version of this problem (with prescribed bounds on column sums of $A \in \mathbf{A}$) was studied in [148].

Works related to the material of this chapter include (but are not limited to) Albrecht [1], Coxson [28], Garloff [42], Heindl [49], Herzberger and Bethke [50], Jahn [60], Moore [89], [90], Nedoma [99], [100], [101], [102], Nickel [107], Nuding [109], Oettli [111], Rex [140], Rex and Rohn [141], [142], Rump [171], [172], [173] and Shokin [186], [187].

[1] Written in Spring 2002.

http://www.springer.com/978-0-387-32697-9

Linear Optimization Problems with Inexact Data
Fiedler, M.; Nedoma, J.; Ramik, J.; Rohn, J.; Zimmermann,
K.
2006, XVI, 214 p. 5 illus., Hardcover
ISBN: 978-0-387-32697-9