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COMPLEMENTARITY PROBLEMS AND VARIATIONAL INEQUALITIES

We present in this chapter two classes of mathematical models, used in applied mathematics. The first class comprises *complementarity problems* and the second class *variational inequalities*. We present the necessary definitions and some important relations between complementarity problems, variational inequalities and the fixed-point problem.

2.1. Complementarity problems

The study of *complementarity problems* has developed sufficiently to call it *Complementarity Theory*. Now we consider it as a new domain of Applied Mathematics, having deep relations with several domains of fundamental mathematics and with numerical analysis. Complementarity problems represent a wide class of mathematical models related to optimization, economics, engineering, mechanics, elasticity, fluid mechanics and game theory.
It is important to note that the *complementarity condition* is a kind of *general equilibrium* concept that includes the equilibria of physics and economics. Equilibrium in physics has long been well known. Equilibrium in economics has become central to the understanding of competitive systems. One example is the general economic equilibrium problem in which all commodity prices are to be determined. A second example is the general financial equilibrium of markets in which firms compete to determine their profit-maximizing production outputs.

Many authors have studied equilibria of economic systems by several mathematical methods and from several points of view, but the recent development of Complementarity Theory helps us to understand better a number of more complex aspects of economic equilibrium. In this sense we cite the books (Isac, G. [20] and (Isac, G., Bulavsky, V. A. and Kalashnikov, V. V. [2]). A deep study of equilibrium in Economics help us to understand better the non-equilibrium state of particular economical systems.

There exist several kinds of complementarity problems [see books (Isac, G. [12], [20]), (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [2]), (Hyers, D. H., Isac, G. and Rassias, Th. [1])]. In this book we present only the most important kinds of complementarity problems, from the point of view of applications and related to the Leray–Schauder type alternatives. We must keep in mind the fact that Complementarity Theory stands at a point on the crossroads of *applied mathematics, fundamental mathematics and experimental mathematics* related to numerical solvability. The connection of Complementarity Theory with Variational Inequalities Theory, with Fixed Point Theory and with Nonlinear Analysis is an important factor in its development as a theory. The literature on complementarity problems is now huge [See the references cited in [(Cottle, R. W., Pang, J. S. and Stone, R. E. [1]), (Isac, G. [12], 20]), (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [2]), (Hyers, D. H., Isac, G. and Rassias, Th. [1]), (Murty, K. G. [1])].

A. The classical complementarity problem

First, we note that many problems arising in fields such as economics, game theory, mathematical programming, mechanics, elasticity theory and engineering, several equilibrium problems can be stated in the following unified form.

Consider the vector space $\mathbb{R}^n$ and the classical inner-product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i,\ x = (x_i),\ y = (y_i) \in \mathbb{R}^n$. Suppose that $\mathbb{R}^n$ is ordered by the closed pointed convex cone $\mathbb{R}^n_+$ and suppose given a function $f : \mathbb{R}^n_+ \to \mathbb{R}$. The classical complementarity problem defined by the function f and the convex cone $\mathbb{R}^n_+$ is

$$CP(f, \mathbb{R}^n_+) : \begin{cases} \mathit{find}\ x_0 \in \mathbb{R}^n_+ \mathit{such\ that} \\ f(x_0) \in \mathbb{R}^n_+\ \mathit{and}\ \langle x_0, f(x_0)\rangle = 0. \end{cases}$$

The origin of this problem is perhaps in the Kuhn–Tucker Theorem, known in nonlinear programming (which gives the necessary optimality conditions, under some differentiability assumptions), or perhaps in the old and neglected paper by Du Val published in 1940 (Du Val, P. [1]). We note also that the origin of the term "complementarity" is in the paper by Cottle (Cottle, R. W. [1]) published in 1964. Initially, this problem was called, in the linear case (i.e., when $f(x) = Ax + b$, where A is a matrix and b is a vector), the "copositive problem", the "fundamental problem of mathematical programming" and the "complementarity problem". It seems that the term "complementarity problem" was proposed by R. W. Cottle in 1965 and used in the papers of R. W. Cottle, G. J. Habetler and C. E. Lemke. From the mathematical point of view, the origin of the term "complementarity" is the following fact.

Let $x_* = \{x_{*_i}\}_{i=1}^n$ be a solution of $CP(f, \mathbb{R}_+^n)$. We say that x_* is *nondegenerate* if at most n components of a $2n$-components vector $(x_*, f(x_*))$ are equal to zero. Otherwise, it is a *degenerate solution.* Denote it by $N_n = \{1, 2, \ldots, n\}$. If x_* is a nondegenerate solution and $y_* = \{y_{*_i}\}_{i=1}^n$, where $y_* = f(x_*)$, then the sets $A = \{i : x_{*_i} > 0\}$ and $B = \{i : y_{*_i} > 0\}$ are complementary subsets of N_n, that is $A = C_{N_n} B$.

If the function f has the form $f(x) = Ax + b$, where A is an $n \times n$-*matrix* and $b \in \mathbb{R}^n$, then in this case $CP(f, \mathbb{R}_+^n)$ is called the *linear complementarity problem* defined by A, b and $\mathbb{R}_+^n$, and it is denoted by

$$LCP(A, b, \mathbb{R}_+^n): \begin{cases} \textit{find } x_* \in \mathbb{R}_+^n \textit{ such that} \\ Ax_* + b \in \mathbb{R}_+^n \textit{ and} \\ \langle x_*, Ax_* + b \rangle = 0. \end{cases}$$

We note that the linear complementarity problem was initially defined as a basic mathematical model that unified linear and quadratic programs, as well as the bimatrix game problem. Specifically, W. S. Dorn in 1961 proved that if A is a *positive-definite* (but not necessarily symmetric) *matrix* then the *minimum value* of the quadratic programming problem

$$(P): \begin{cases} \text{minimize } \langle x, Ax+b \rangle, \\ x \in \mathcal{F}, \\ \text{where } \mathcal{F} = \{x \in \mathbb{R}^n_+ : Ax+b \in \mathbb{R}^n_+\} \text{ and } b \in \mathbb{R}^n \end{cases}$$

is zero. [See (Dorn, W. S., [1])]. We note that Dorn's paper was the first step in treating the linear complementarity problem as an independent problem.

In 1963 G. B. Dantzig and R. W. Cottle generalized Dorn's result to the case when all the principal minors of the matrix A are positive (Dantzig, G. B. and Cottle, R. W. [1]). R. W. Cottle studied problem (P) in 1964, under the assumption that A is a *positive semi-definite* matrix and he remarked that, in this case it is not true that (P) must have an optimal solution. [See Cottle, R. W. [2]). Cottle proved that, if the matrix A is *positive semi-definite* and the set

$\mathcal{F} = \{x \in \mathbb{R}^n_+ : Ax+b \in \mathbb{R}^n_+\}$ *where* $b \in \mathbb{R}^n$ (called the *feasible set*) is non-empty, then an optimal solution for (P) exists and again $\min\limits_{x\in\mathcal{F}} \langle x, Ax+b \rangle = 0$.

After some time, G. B. Dantzig and R. W. Cottle constructively showed that if A is a square (not necessarily symmetric) matrix with all *the principal minors* positive, then problem (P) has an optimal solution x_* such that $\langle x_*, Ax_* + b \rangle = 0$. This result is in (Dantzig, G. B. and Cottle, R. W. [1]). In 1966 R. W. Cottle generalized this result. His generalization is the following:

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuously differentiable mapping. We say that* f *has a positively bounded Jacobian matrix* $J_f(x)$, *if there exists a real number* $0 < \delta < 1$ *such that for every* $x \in \mathbb{R}^n$ *each principal minor of* $J_f(x)$ *is an element of the interval* $[\delta, \delta^{-1}]$.

We recall that a solution (y, x) of the equation $y - f(x) = 0$ is said to be *nondegenerate* if at most n of the $2n$ components are zero.

THEOREM [Cottle]. *If* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous differentiable mapping such that the solutions of equation* $y - f(x) = 0$ *are nondegenerate, and if* f *has a positively bounded Jacobian matrix* $J_f(x)$, *then the problem*

$$NCP(f, \mathbb{R}_+): \begin{cases} \text{find } x_0 \in \mathbb{R}^n_+ \text{ such that} \\ f(x_0) \in \mathbb{R}^n_+ \text{ and } \langle x_0, f(x_0) \rangle = 0, \end{cases}$$

has a solution.

A proof of this theorem is in (Cottle, R. W., [3]) where he defined the *nonlinear complementarity problem* by f and the convex cone $\mathbb{R}^n_+$, and denoted it by $NCP\left(f, \mathbb{R}^n_+\right)$.

In studying the origin of the *Complementarity Theory* we must consider the papers (Lemke, C. E. [1]–[6]) and (Ingleton, A. W. [1]). Lemke proposed, in 1965, the complementarity problem as a method for solving matrix games (Lemke, C. E. [1]). His contribution to the development of complementarity theory was remarkable, because his algorithm for solving complementarity problems, known as *Lemke's algorithm*, has been widely used in many practical applications, (Lemke, C. E. [1]–[6]), (Lemke, C. E. and Howson, J, T. [1]).

In 1966, A. Ingleton showed the importance of complementarity problems in engineering (Ingleton, A. W. [1], [2]). Certainly, a strong influence on the development of complementarity theory is also found in (Eaves, B. C, [1]-[7]), (Eaves, B. C. and Lemke, C. E. [1], [2]), (Karamardian, S. [1]-[5]), (Kaneko, I, [1]-[13]) and (Kojima, M. [1]-[4]).

After 1970 the complementarity theory enjoyed a strong and ascending development from theoretical, numerical solvability and applicability points of view. Now, the literature on this subject is vast. To see this, the reader is referred to the books (Cottle, R. W., Pang, J. S. and Stone, R. E, [1]), (Isac, G. [12], [20]), (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [2]), (Murty, K. G. [1]) among others. Now, it is unanimously accepted that the study of complementarity problems is a necessary domain in applied mathematics and a stimulant for fundamental mathematics.

B. The general nonlinear complementarity problem

Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces and let $\mathbb{K} \subset E$ be a closed pointed convex cone. If $f : \mathbb{K} \to E^*$ is a given mapping, the (general) nonlinear complementarity problem defined by f and $\mathbb{K}$ is:

$$NCP(f, \mathbb{K}): \begin{cases} \text{find } x_* \in \mathbb{K} \text{ such that} \\ f(x_*) \in \mathbb{K}^* \text{ and } \langle x_*, f(x_*) \rangle = 0. \end{cases}$$

$NCP(f,\mathbb{K})$ contains as a particular case the classical complementarity problem $NCP(f,\mathbb{R}^n_+)$, where $f:\mathbb{R}^n_+ \to \mathbb{R}^n$. Also the (general) linear complementarity problem $LCP(T, b, \mathbb{K})$, where $T: E \to E^*$ is a linear operator and $b \in E^*$ can be considered as a particular case of the problem $NCP(f,\mathbb{K})$.

The problem $NCP(f,\mathbb{K})$ has many applications in optimization, game theory, economics, engineering, mechanics, etc. We will see in this chapter that the problem $NCP(f,\mathbb{K})$ is related to variational inequalities and in Hilbert spaces it is related to the *Fixed Point Problem.* The fixed-point problem represents an important chapter in nonlinear analysis. (Isac, G. [20]).

C. The multivalued complementarity problem

First, we note that the *multivalued complementarity problem* is necessary in the study of some problems in economics in the sensitivity analysis of classical complementarity problems and in numerical computation of solutions of practical complementarity problems, because of the accidental corruption of the problem data. Also, the *multivalued* complementarity problem is related with the theory of quasi-variational inequalities defined by set-valued mappings. Variational inequalities with set-valued mappings are used in the study of equilibrium in economics.

Let $\langle E,E^*\rangle$ be a dual system of locally convex spaces, $\mathbb{K} \subset E$ a pointed closed convex cone and $f:\mathbb{K}\to 2^{E^*}$ a set-valued mapping. The multivalued complementarity problem defined by f and $\mathbb{K}$ is:

$$MCP(f,\mathbb{K}):\ \begin{cases} \textit{find } x_0 \in \mathbb{K} \textit{ and } y_0 \in E^* \textit{ such that} \\ y_0 \in f(x_0) \cap \mathbb{K}^* \textit{ and } \langle x_0, y_0\rangle = 0. \end{cases}$$

This complementarity problem has been the subject of several papers as for example: (Chang, S. S. and Huang, N. J., [1]–[4]), (Gowda, M. S. and Pang, J. S., [1]), (Huang, N. J., [1]), (Isac, G. [12], [20]), (Isac, G. and Kostreva, M. M., [2]), (Isac, G. and Kalashnikov, V. V. [1]), (Luna, G. [1]), (Parida, J. and Sen, A., [1]), (Saigal, R., [1]).

D. Implicit complementarity problem

Another class of complementarity problems is *the class of implicit* *complementarity problems*. It seems that the origin of *implicit complementarity problems* is the dynamic programming approach of stochastic impulse and of continuous optimal control (Bensoussan, A., [1]), (Bensoussan, A. and Lions, J. L., [1]–[3]), (Bensoussan, A., Gourset, M. and Lions, J. L. [1]), (Capuzzo–Dolcetta. I. and Mosco, U., [1]), (Mosco, U. [1]), (Mosco, U. and Scarpini, F., [1].

The study of *implicit complementarity problems* has been stimulated by the applications of this class of mathematical models to the study of various free boundary problems associated to some particular differential operators. This class of complementarity problems has been studied by many authors as for example: (Pang, J. S. [1]–[2]), (Chan, D. and Pang J. S., [1]), (Noor, M. A., [1]), (Capuzzo–Dolcetta, I., Lorenzani, M. and Spizziachino, F. [1]), (Isac, G. and Nemeth, S. Z. [1]), (Kalashnikov, V. V. and Isac. G. [1]). We note that there exist deep and interesting relations between the *implicit complementarity problems and the quasivariational inequalities theory.*

Now, we give the most important kind of implicit complementarity problems. Let $E(\tau)$ be a locally convex space and let $\mathbb{K} \subset E$ be a closed convex cone. Suppose given an element $b \in E$ and two mappings A, $M : E \to E$. If $\langle \cdot, \cdot \rangle$ is a bilinear functional defined on $E \times E$ then the *implicit complementarity problem* is:

$$ICP(A,M,b,\mathbb{K}): \quad \begin{cases} \textit{find } x_0 \in E \textit{ such that} \\ M(x_0) - x_0 \in \mathbb{K}, b - A(x_0) \in \mathbb{K} \\ \textit{and } \langle A(x_0) - b, x_0 - M(x_0) \rangle = 0. \end{cases} \tag{2.1.1}$$

The *implicit complementarity problem* (2.1.1) has the following variant for a dual system. Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces, $\mathbb{K} \subset E$ a closed pointed convex cone, $M : E \to E$ and $A : E \to E^*$ arbitrary mappings and $b \in E^*$ an arbitrary element. In this case the problem (2.1.1) has the following form:

$$ICP(A,M,b,\mathbb{K}): \quad \begin{cases} \textit{find } x_0 \in E \textit{ such that} \\ M(x_0) - x_0 \in \mathbb{K}, b - A(x_0) \in \mathbb{K}^* \\ \textit{and } \langle A(x_0) - b, x_0 - M(x_0) \rangle = 0. \end{cases} \tag{2.1.2}$$

Obviously, if $E = H$, where $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space with respect to an inner-product $\langle\cdot,\cdot\rangle$, then the problem (2.1.2) is exactly the problem (2.1.1).

The most general form of the *implicit complementarity* problem is the following. Let $\langle E, E^*\rangle$ be a dual system of locally convex spaces, $\mathbb{K} \subset E$ a pointed closed convex cone and $D \subset E$ a non-empty subset. If $f : D \to E^*$ and $g : D \to E^*$ are arbitrary mappings, then the *generalized implicit complementarity problem* defined by *f*, *g*, *D* and $\mathbb{K}$ is:

$$GICP(f,g,D,\mathbb{K}): \begin{cases} \textit{find } x_0 \in D \textit{ such that} \\ g(x_0) \in \mathbb{K}, f(x_0) \in \mathbb{K}^* \textit{ and} \\ \langle g(x_0), f(x_0)\rangle = 0. \end{cases}$$

Finally, *the generalized implicit complementarity problem* has the following *multivalued* variant. Let $D \subset E$ be a non-empty subset, $\mathbb{K} \subset E$ a closed pointed convex cone and $f : D \to 2^{E^*}$, $g : D \to 2^E$ set-valued mappings. The *multivalued generalized implicit complementarity problem* is:

$$MGICP(f,g,D,\mathbb{K}): \begin{cases} \textit{find } x_0 \in D \textit{ such that} \\ \textit{there exist } x_* \in g(x_0) \cap \mathbb{K} \textit{ and} \\ y_* \in f(x_0) \cap \mathbb{K}^*, \textit{satisfying}^* \\ \langle x_*, y_*\rangle = 0. \end{cases}$$

E. Order complementarity problem

A new chapter in complementarity theory is the study of complementarity problems with respect to an ordering. The introduction of *order complementarity problems in complementarity theory* is justified by two reasons.

(i) In the study of some particular classical complementarity problems the essential fact is not the orthogonality in the sense of an inner-product, but the lattice orthogonality. Therefore, in some circumstances it is useful to represent the classical complementarity problem as an order complementarity problem.

(ii) In some practical problems, we must use the complementarity condition simultaneously with respect to several operators.

Denote by $E(\tau)$ [respectively by $(E,\|\cdot\|)$ or by $(E,\langle\cdot,\cdot\rangle)$] a locally convex space (respectively, a Banach space or a Hilbert space). Suppose that, E is ordered by a closed, pointed convex cone $\mathbb{K}$. Denote by "$\leq$" the ordering defined by $\mathbb{K}$, that is $x \leq y$ if and only if $y - x \in \mathbb{K}$. Assume that the ordered vector space $(E,\mathbb{K})$ is a *vector lattice*, i.e., for every pair $(x,y)\in E\times E$, *the supremum* $\vee(x,y)$ and the *infimum* $\wedge(x,y)$ with respect to the ordering $\leq$ exist in E. In this case, for every $x_1,x_2,x_3 \in E$ we have the following formulas:

(1) $\vee(x_1,x_2)+x_3 = \vee(x_1+x_3,x_2+x_3)$,

(2) $\wedge(x_1,x_2)+x_3 = \wedge(x_1+x_3,x_2+x_3)$,

(3) $\vee(x_1,x_2,x_3) = \vee(\vee(x_1,x_2),x_3) = \vee(\vee(x_1,x_2),\vee(x_2,x_3))$.

If $x_1,x_2,\ldots,x_n \in E$, then by induction $\vee(x_1,x_2,\ldots,x_n)$ and $\wedge(x_1,x_2,\ldots,x_n)$ are well defined for any $n \in N$, considering also the formula $\wedge(x,y) = -\vee(-x,-y)$.

Let $\langle\cdot,\cdot\rangle : E\times E \to \mathbb{R}$ be a bilinear form. We say that the bilinear form $\langle\cdot,\cdot\rangle$ is $\mathbb{K}$*-local*, if and only if $\langle x,y\rangle = 0$, whenever $x, y \in \mathbb{K}$ and $\wedge(x,y)=0$. (The term $\mathbb{K}$*-local* is used in the axiomatic potential theory). Let D be a non-empty subset of E. In particular the set D can be the cone $\mathbb{K}$. Given m, linear or nonlinear mappings $f_1,f_2,\ldots,f_m : E \to E$, the *order complementarity problem* defined by the family of mappings $\{f_i\}_{i=1}^m$ and the set D is:

$$OCP\left(\{f_i\}_{i=1}^m, D\right) = \begin{cases} find\ x_0 \in D\ such\ that \\ \wedge\left(f_1(x_0), f_2(x_0), \ldots, f_m(x_0)\right) = 0. \end{cases}$$

In (Isac, G. and Goeleven, D. [1]) this problem is called the *implicit general order complementarity problem.* We have several interesting particular cases:

(1) If $m = 2$, $D = E$, $f_1 = I$ (the identity mapping) and $f_2(x) = T(x)+q$, where $T : E \to E$ is a linear mapping and q is an element in E, we have *the linear order complementarity problem* denoted by

LOCP(T, q). This problem was studied systematically for the first time in 1989 in (Borwein, J. M. and Dempster, M. A. H., [1]), where several interesting new classes of linear operators were introduced. We find for example the operators of class (H^{1}), (S), (Z), ($\mathbb{K}$), (P) and (A).

(2) If m is an arbitrary natural number and $f_i, (i = 1, 2, \ldots, m)$ are affine mappings we have *the generalized linear order complementarity problem.* Several results about this problem are in (Gowda, M. S. and Sznajder, R., [1]), (Isac, G. and Goeleven, D., [1]), and (Sznajder, R. [1]).

(3) If $m = 2$, $D = \mathbb{K}$ and f_1, f_2 are nonlinear mappings, then in this case we have *the nonlinear order complementarity problem,* studied for the first time in 1986 (Isac, G. [5]).

(4) If $m = 3$, $D = E$, $f_1 = I$ (the identity mapping) and f_2, f_3 are nonlinear we have *an order complementarity problem.* In 1986 Oh, K. P introduced this notion in lubrication theory. (Oh, K. P., [1]). This interesting order complementary problem is the following. Consider the mixed lubrication in the context of a journal bearing with elastic support. The problem is to study the contact pressure X. In this case $E = H^1(\Omega)$ (defined over $L^2(\Omega)$) and the cone is $\mathbb{K} = \{u \in H^1(\Omega) | u \geq 0 \; a.e., on \; \Omega\}$. We have two operators, $T_1(X)$ and $T_2(X)$, where T_1 is the Reynolds partial differential operator. For the definition of these operators, the reader is referred to (Oh, K. P., [1]), (Isac, G. and Kostreva, M., [1], (Isac, G. and Goeleven, D., [1]). In this case, there are three distinct functions, which cause the decomposition of the spatial area into three disjoint regios: *the innermost region* (solid-to-solid contact), *the elasto-hydrodynamic lubrication region* (solid-to-fluid contact) and *the cavity region* (in which the pressure returns to the ambient value). The complementarity formulation is based on the observation that the contact pressure X satisfies the following equation specified for every region:

(i) $X \geq 0$, $T_1(X) = 0$, $T_2(X) \geq 0$ (solid-to-solid contact),

(ii) $X = 0$, $T_1(X) \geq 0$, $T_2(X) \geq 0$ (cavity point),

(iii) $X \geq 0$, $T_1(X) \geq 0$, $T_2(X) = 0$ (lubrication point).

The problem of finding the contact pressure X is equivalent to solvability of *OCP*(I, T_1, T_2; $\mathbb{K}$). This problem, defined in 1986 in (Oh, K. P., [1]) is theoretical not yet solved, but it has many interesting applications. In practical problems this mathematical model is implemented by simulation and by numerical approximations. Finally, we note that the *order complementarity problems* are used also in the study of the *global reproduction of economic systems working with several technologies,* in the study of *discrete dynamic complementarity problems.* (Isac, G. [20]), and in the study of the fold *complementarity problems* (Isac, G. [15]) and (Isac, G. and Kostreva, M. [3]).

If m is an arbitrary natural number, $D = \mathbb{K}$, $f_1 = I$ (the identity mapping) and $f_2, f_3, \ldots, f_m$ are nonlinear but having the form $f_i(x) = x - T_i(x), (i = 1, 2, 3, \ldots, m)$, with T_i nonlinear mappings, then we have the *generalized order complementarity problem* studied systematically in (Isac, G. and Kostreva, M. [1]) and for set valued mappings in (Isac, G. and Kostreva, M. [2]) and (Huang, N. J. and Fang, Y. P. [1]). Some numerical methods for the order complementarity problem can be found in (Isac, G. [11]) and in (Isac, G. and Goeleven, D. [2]).

2.2. Variational inequalities

Another important domain of applied mathematics is the study of variational inequalities, which is deeply related to complementarity theory. It seems that the notion of *variational inequality* was introduced in the papers of G. Stampacchia and G. Stampacchia and P. Hartman. For references the reader is referred to the books (Stampacchia, G. [1]), (Kinderlehrer, D, [1]), (Baiocchi, C. and Capelo, A, [1]), (Duvaut, G. and Lions, J. L., [1]) and (Lions, J. L. and Magenes, E., [1]).

The theory of variational inequalities had from the beginning a rapid development and a prolific growth of its applications. Initially, one of the attractions of the theory of variational inequalities was its applications to many questions of physical interest, as for example: the lubrication theory, the steady filtration of a liquid through a porous membrane, the motion of a fluid past a given profile and the small deflections of an elastic beam etc. Many remarkable mathematicians added their contributions to the development of the variational inequalities theory as for example: H. Brezis, C. Baiocchi, L. Caffarelly, D. Kinderlehrer, H. Lewy, J. L. Lions and E. Magenes, among others. Now, the literature on variational inequalities is huge and contains several variations. We consider in this book only the classical variational inequalities.

Let $\langle E, E^* \rangle$ be a duality of locally convex spaces, i.e., E is a locally convex space, E^* is the topological dual of E and $\langle \cdot, \cdot \rangle$ is a bilinear form on $E \times E^*$ satisfying the following separation axioms:

(s_1) $\langle x_0, y \rangle = 0$ *for all* $y \in E^*$ *implies* $x_0 = 0$,

(s_2) $\langle x, y_0 \rangle = 0$ *for all* $x \in E$ *implies* $y_0 = 0$.

Let $f : E \rightarrow E^*$ be a mapping. We recall the following classical notions.

(a) We say that f is *monotone*, if for any $x, y \in E$ we have $\langle x - y, f(x) - f(y) \rangle \geq 0$.

(b) We say that f is *pseudomonotone* (in Karamardian's sense) if for any $x, y \in E$ we have that $\langle x - y, f(y) \rangle \geq 0$ implies $\langle x - y, f(x) \rangle \geq 0$.

We have similar definitions if $f : \Omega \rightarrow E^*$, where Ω is an arbitrary non-empty subset of E. The *Hartman–Stampacchia variational inequality* defined by f and Ω is:

$$HSVI(f, \Omega): \begin{cases} \textit{find } x_* \in \Omega \textit{ such that} \\ \langle x - x_*, f(x_*) \rangle \geq 0 \textit{ for all } x \in \Omega, \end{cases}$$

and the *Minty variational inequality* defined by f and Ω is:

$$MVI(f, \Omega): \begin{cases} \textit{find } x_* \in \Omega \textit{ such that} \\ \langle x - x_*, f(x) \rangle \geq 0 \textit{ for all } x \in \Omega. \end{cases}$$

For more information about Minty's variational inequality the reader is referred to (Minty, G. J., [1]). The Hartman–Stampacchia variational inequality has many applications in physics, engineering and in economics, while the Minty variational inequality is important in the study of solvability of $HSVI(f, \Omega)$.

About the solvability of problem $HSVI(f, \Omega)$, first we note the following classical result, which is a generalization to locally convex spaces of Hartman–Stampacchia's theorem (proved initially in Euclidean space).

THEOREM 2.2.1 [Hartman–Stampacchia]. *Let Ω be a compact convex subset of a locally convex space E and let $f : \Omega \rightarrow E^*$ be a continuous mapping, (with respect to the strong topology). Then, there exists an element $x_* \in \Omega$ such that $\langle x - x_*, f(x_*) \rangle \geq 0$ for all $x \in \Omega$.*

Proof. A proof of this result is in (Holmes, R. B., [1]). The proof is based on the Fan–Kakutani Fixed Point Theorem. □

Remark. The study of solvability of problem *HSVI*(f, Ω) in the case when Ω is unbounded, generally is based on special mathematical tools. In this book we develop a new method to study variational inequalities with respect to unbounded closed convex sets.

The following result establishes a relation between problems *HSVI*(f, Ω) and *MVI*(f, Ω). If $\Omega \subset E$ is a convex set and $f : \Omega \to E^*$ is a mapping, we say that f is *hemicontinuous* if it is continuous from the line segments of Ω to the *weak topology* of E^*.

THEOREM 2.2.2. *Let $E(\tau)$ be a locally convex space, $\Omega \subset E$ a closed convex set and $f : \Omega \to E^*$ a pseudomonotone, hemicontinuous mapping. Then, an element $u_0 \in \Omega$ is a solution to the problem HSVI(f, Ω), if and only if u_0 is a solution to the problem MVI(f, Ω).*

Proof. Suppose that $u_0 \in \Omega$ is a solution to the problem *HSVI*(f, Ω). Then, in this case we have,

$$\langle x - u_0, f(u_0) \rangle \geq 0 \text{, for all } x \in \Omega$$

and the pseudomonotonicity implies that

$$\langle x - u_0, f(x) \rangle \geq 0 \text{, for all } x \in \Omega,$$

i.e., u_0 is a solution to the problem *MVI*(f, Ω).

Conversely, suppose that an element $u_0 \in \Omega$ is a solution to the problem *MVI*(f, Ω). In this case, if $x \in \Omega$ is an arbitrary element, we denote it by

$$x_t = (1-t)u_0 + tx,\ t \in]0,1[\,.$$

If we put x_t in the definition of the problem *MVI*(f, Ω), then we have

$$\langle x_t - u_0, f(x_t) \rangle \geq 0,$$

which implies

$$\langle t(x_t - u_0), f(x_t) \rangle \geq 0,$$

and finally,

$$\langle x - u_0, f(x_t) \rangle \geq 0.$$

Supposing that $t \to 0$ and using the hemicontinuity of f we obtain that $f(x_t)$ is weakly convergent to $f(u_0)$, which implies that

$$\langle x - u_0, f(u_0) \rangle \geq 0 \text{, for any } x \in \Omega,$$

i.e., u_0 is a solution to the problem *HSVI*(f, Ω) and the proof is complete. □

Obviously, the variational inequalities *HSVI*(f, Ω) and *MVI*(f, Ω) can be defined for set-valued mappings. Indeed, let f be a set-valued mapping from Ω into E^*, i.e., $f:\Omega \to 2^{E^*}$. The *multivalued Hartman–Stampacchia variational inequality* defined by f and Ω is:

$$MHSVI(f,\Omega): \begin{cases} \textit{find } x_* \in \Omega \textit{ and } y_* \in E^* \\ \textit{such that } y_* \in f(x_*) \textit{ and} \\ \langle x - x_*, y_* \rangle \geq 0 \textit{ for all } x \in \Omega \end{cases}$$

and the *multivalued Minty variational inequality* defined by f and Ω is:

$$MMVI(f,\Omega): \begin{cases} \textit{find } x_* \in \Omega \textit{ such that} \\ \textit{for any } x \in \Omega \textit{ there exists} \\ y_x \in f(x) \textit{ satisfying} \\ \langle x - x_*, y_x \rangle \geq 0. \end{cases}$$

Finally, we consider in this book a special *implicit* variational inequality.

Consider again a dual system $\langle E, E^* \rangle$ of locally convex spaces. $\Omega \subset E$ a closed convex cone and two mappings $S : \Omega \to \Omega$ and $f : \Omega \to E^*$. The *implicit variational inequality* defined by S, f and Ω is:

$$IVI(f,S,\Omega): \begin{cases} \textit{find } x_0 \in \Omega \textit{ such that} \\ \langle x - S(x_0), f(x_0) \rangle \geq 0, \textit{ for all } x \in \Omega. \end{cases}$$

The problem *IVI*(f, S,Ω) is a special variational inequality. It is implicit in the sense of implicit variational inequalities presented in (Mosco, U., [1]). Obviously, if $S(x) = x$ for every $x \in \Omega$, the problem *IVI*(f, S,Ω) is exactly the problem *HIVI*(f, Ω). We note that the problem *IVI*(f, S,Ω) is related to the problem $GICP(f, g, D, \mathbb{K})$ when $g = S$ and $D = \mathbb{K}$.

2.3 Complementarity problems, variational inequalities, equivalences and equations

We present in this section some equivalences between complementarity problems and variational inequalities. We show also, how a complementarity problem or a variational inequality can be transformed in an equation. These equations are essential for the next chapters of this book.

Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces. Let $\mathbb{K} \subset E$ be a closed convex cone and $f: E \to E^*$ a mapping.

THEOREM 2.3.1. *The problems $NCP(f, \mathbb{K})$ and $HSVI(f, \mathbb{K})$ are equivalent.*

Proof. Indeed, if x_* is a solution to the problem $HSVI(f, \mathbb{K})$, then we have

$$\langle x - x_*, f(x_*) \rangle \geq 0, \textit{ for all } x \in \mathbb{K}. \tag{2.3.1}$$

Let $y \in \mathbb{K}$ be an arbitrary element. If we put $x = y + x_*$ in (2.3.1), then we obtain

$$\langle y, f(x_*) \rangle \geq 0, \textit{ for all } y \in \mathbb{K},$$

which implies that $f(x_*) \in \mathbb{K}^*$.

If we consider $x = 2x_*$ in (2.3.1) then we deduce that $\langle x_*, f(x_*) \rangle = 0$, i.e., x_* is a solution to the $NCP(f, \mathbb{K})$.

Conversely, if we suppose that $x_* \in \mathbb{K}$ is a solution to the problem $NCP(f, \mathbb{K})$, then we have $\langle x_*, f(x_*) \rangle = 0$ and $\langle x, f(x_*) \rangle \geq 0$ for every $x \in \mathbb{K}$, which obviously imply $\langle x - x_*, f(x_*) \rangle \geq 0$, *for all* $x \in \mathbb{K}$, that is, x_* is a solution to the problem $HSVI(f, \mathbb{K})$. □

Now, we consider the following problems. Let $\langle E, E^* \rangle$ be a duality of locally convex spaces and $\mathbb{K} \subset E$ a pointed closed convex cone. Suppose given two mappings, $f: E \to E^*$ and $g: E \to E$. The next theorem is related to the following two problems:

$$IVI(f, g, \mathbb{K}): \quad \begin{cases} \textit{find } x_* \in E \textit{ such that} \\ g(x_*) \in \mathbb{K} \textit{ and} \\ \langle x - g(x_*), f(x_*) \rangle \geq 0 \textit{ for all } x \in \mathbb{K}, \end{cases}$$

$$ICP(f,g,\mathbb{K}): \begin{cases} \text{find } x_* \in E \text{ such that} \\ g(x_*) \in \mathbb{K}, f(x_*) \in \mathbb{K}^* \text{ and} \\ \langle g(x_*), f(x_*) \rangle = 0. \end{cases}$$

THEOREM 2.3.2. *The problems $IVI(f,g,\mathbb{K})$ and $ICP(f,g,\mathbb{K})$ are equivalent.*

Proof. Indeed if $x_* \in E$ is a solution to the problem $ICP(f,g,\mathbb{K})$, then we have $g(x_*) \in \mathbb{K}$, $f(x_*) \in \mathbb{K}^*$ and $\langle g(x_*), f(x_*) \rangle = 0$ which imply

$$\langle x, f(x_*) \rangle \geq 0 \text{ for all } x \in \mathbb{K} \tag{2.3.2}$$

and

$$\langle g(x_*), f(x_*) \rangle = 0. \tag{2.3.3}$$

By using (2.3.2) and (2.3.3) we obtain $g(x_*) \in \mathbb{K}$, and $\langle x - g(x_*), f(x_*) \rangle \geq 0$ for any $x \in \mathbb{K}$, that is, x_* is a solution to the problem $IVI(f,g,\mathbb{K})$.

Conversely, we suppose that $x_* \in E$ is a solution to the problem $IVI(f,g,\mathbb{K})$. Then, we have $g(x_*) \in \mathbb{K}$, and $\langle x - g(x_*), f(x_*) \rangle \geq 0$ for all $x \in \mathbb{K}$. If we take $x = y + g(x_*)$, then we obtain that $\langle y, f(x_*) \rangle \geq 0$, which implies that $f(x_*) \in \mathbb{K}^*$. If we consider $x = 2g(x_*)$ in $ICP(f,g,\mathbb{K})$, then we obtain $\langle g(x_*), f(x_*) \rangle \geq 0$ and considering $x = 0$, we obtain $\langle g(x_*), f(x_*) \rangle \leq 0$. Therefore $\langle g(x_*), f(x_*) \rangle = 0$ and we have that x_* is a solution to the problem $ICP(f,g,\mathbb{K})$. □

For the method developed in this book, it is important to transform a complementarity problem or variational inequality in a fixed-point problem or in an equation. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\Omega \subset H$ a closed convex cone. Given f, g: $H \to H$ two arbitrary mappings, we consider the following implicit variational inequality:

$$IVI(f,g,\Omega): \begin{cases} \text{find } x_* \in H \text{ such that} \\ g(x_*) \in \Omega \text{ and} \\ \langle y - g(x_*), f(x_*) \rangle \geq 0 \text{ for all } y \in \Omega. \end{cases}$$

We have the following result.

THEOREM 2.3.3. *An element $x_* \in H$ is a solution to the problem IVI(f, g, Ω) if and only if, x_* is a solution to the coincidence problem*

$$CP(f,g,\Omega): \begin{cases} \text{find } x_* \in H \text{ such that} \\ g(x_*) = P_\Omega\left(g(x_*) - f(x_*)\right). \end{cases}$$

Proof. Indeed, if $x_* \in H$ and $g(x_*) = P_\Omega\left(g(x_*) - f(x_*)\right)$, then we have that $g(x_*) \in \Omega$ and $g(x_*) - f(x_*) \in g(x_*) + N_\Omega\left(g(x_*)\right)$. [We used *Theorem 1.9.4*]. Therefore $\langle -f(x_*), y - g(x_*)\rangle \le 0$ for all $y \in \Omega$ and $g(x_*) \in \Omega$, that is x_* is a solution to the problem *IVI(f, g, Ω)*. Conversely, if $x_* \in H$, $g(x_*) \in \Omega$ and $\langle f(x_*), y - g(x_*)\rangle \ge 0$ for all $y \in \Omega$, then we have

$$\langle -f(x_*), y - g(x_*)\rangle \le 0 \text{ for all } y \in \Omega,$$

or

$$g(x_*) - f(x_*) \in g(x_*) + N_\Omega\left(g(x_*)\right),$$

which implies that

$$g(x_*) = P_\Omega\left(g(x_*) - f(x_*)\right),$$

[using again Theorem 1.9.4]. □

COROLLARY 2.3.4. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f: H \to H$ a mapping. The problem $NCP(f, \mathbb{K})$ has a solution if and only if the mapping $\Psi : H \to H$ defined by $\Psi_{\mathbb{K}}(x) = P_{\mathbb{K}}\left(x - f(x)\right)$ has a fixed point, i.e., there exists an element $x_* \in H$ such that $x_* = P_{\mathbb{K}}\left(x_* - f(x_*)\right)$.*

Proof. We take in Theorem 2.3.2 and Theorem 2.3.3, $g(x) = x$, for any $x \in H$ and $\Omega = \mathbb{K}$. □

Also for problem *HSVI(f, Ω)* we have the following result.

COROLLARY 2.3.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\Omega \subset H$ a closed convex set and $f: H \to H$ a mapping. The problem HSVI(f, Ω) has a solution if and only if the mapping $\Psi_\Omega : H \to H$ defined by $\Psi_\Omega(x) = P_\Omega\left(x - f(x)\right)$*

has a fixed point, i.e., there exists an element $x_* \in H$ *such that* $x_* = P_\Omega(x_* - f(x_*))$.

Proof. We take in Theorem 2.3.3 $g(x) = x$, for any $x \in H$. □

Remark. We can prove Corollary 2.3.4 using Theorem 1.9.7.

The reader can extend Corollary 2.3.4 (resp. Corollary 2.3.5) to the case when f is a set-valued mapping, that is when $f: H \to 2^H$, but in this case the mapping $\Psi_{\mathbb{K}}$ (resp. Ψ_Ω) will be a set-valued mapping. Therefore, we have the following result, related to the problems:

$$MCP(f,\mathbb{K}): \begin{cases} \textit{find } x_0 \in \mathbb{K} \textit{ and} \\ y_0 \in f(x_0) \cap \mathbb{K}^* \textit{ such that} \\ \langle x_0, y_0 \rangle = 0, \end{cases}$$

and

$$MHSVI(f,\Omega): \begin{cases} \textit{find } x_0 \in \Omega \textit{ and} \\ y_0 \in f(x_0) \textit{ such that} \\ \langle x - x_0, y_0 \rangle \geq 0 \textit{ for all } x \in \Omega. \end{cases}$$

COROLLARY 2.3.6. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a closed convex set and* $f: H \to H$ *a set-valued mapping. The problem* $MCP(f,\mathbb{K})$ *(resp. the problem* $MHSVI(f, \Omega)$*) has a solution if and only if the set-valued mapping* $\Psi_{\mathbb{K}}(x) = P_{\mathbb{K}}(x - f(x))$ *(resp.* $\Psi_\Omega(x) = P_\Omega(x - f(x))$*) has a fixed point, i.e., there exists an element* $x_0 \in H$ *such that* $x_0 \in \Psi_{\mathbb{K}}(x_0) = P_{\mathbb{K}}(x_0 - f(x_0))$ *(resp.* $x_0 \in \Psi_\Omega(x_0) = P_\Omega(x_0 - f(x_0))$*).* □

Now, we introduce the normal operator and we will show that the solvability of a complementarity problem or a variational inequality is equivalent to the solvability of an equation. Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\Omega \subset H$ a closed convex set. Let $f: H \to H$ be an arbitrary mapping. Consider again the problem:

$$HSVI(f,\Omega): \begin{cases} \textit{find } x_0 \in \Omega \textit{ such that} \\ \langle x - x_0, f(x_0) \rangle \geq 0, \textit{ for all } x \in \Omega. \end{cases}$$

The operator $\mathcal{N}_f : H \to H$ defined by

$$\mathcal{N}_f(z) = f(P_\Omega(z)) + z - P_\Omega(z) \textit{ for all } z \in H$$

is called the *normal operator defined by f and* Ω.

Remark. In 1992, S. M. Robinson introduced the name *normal operator* [see (Robinson S. M., [1]–[4]), and this operator was used in several papers to transform a variational inequality in an equation of the form $\mathcal{N}_f(z) = 0$. In 1988, G. Isac used the same operator in complementarity theory, but in the form $z = P_{\mathbb{K}}(z) - f(P_{\mathbb{K}}(z))$. He used this operator to transform the solvability of the complementarity problem in a fixed-point problem. (Isac, G. [7]).

THEOREM 2.3.7. *An element* $z_* \in H$ *is a solution to the equation*

$$\mathcal{N}_f(z) = 0$$

if and only if, $x_* = P_\Omega(z_*)$ *is a solution to the problem HSVI(f,* Ω*).*

Proof. First, by Theorem 1.9.3 a, we have that $x_* = P_\Omega(z_*)$ if and only if

$$\langle z_* - x_*, x_* - x \rangle \geq 0, \textit{ for all } x \in \Omega. \tag{2.3.4}$$

If $\mathcal{N}_f(z_*) = 0$, then we have

$$f(P_\Omega(z_*)) + z_* - P_\Omega(z_*) = 0$$

or

$$-f(x_*) = z_* - x_*$$

which implies [using (2.3.4)]

$$\langle -f(x_*), x_* - x \rangle \geq 0, \textit{ for all } x \in \Omega,$$

and finally

$$\langle f(x_*), x_* - x \rangle \geq 0, \textit{ for all } x \in \Omega,$$

that is (using the commutativity of the inner-product, we have that x_* is a solution to the problem *HSVI(f,* Ω*)*.

Conversely, suppose that $z_* = x_* - f(x_*)$ and $\langle -f(x_*), x - x_* \rangle \geq 0$, for any $x \in \Omega$. We have $z_* - x_* = -f(x_*)$ or $f(x_*) = x_* - z_*$, which implies

$$\langle x_* - z_*, x - x_* \rangle \geq 0, \textit{ for all } x \in \Omega,$$

or

$$\langle z_* - x_*, x_* - x \rangle \geq 0, \textit{ for all } x \in \Omega,$$

which implies that $x_* = P_\Omega(z_*)$. Therefore, we have

$$f\left(P_\Omega(z_*)\right) + z_* - P_\Omega(z_*) = 0,$$

that is, $\mathcal{N}_f(z_*) = 0$, and the proof is complete. □

Now, we consider the case when the Hilbert space is replaced by a Banach space. In this case we must replace the projection operator defined in a Hilbert space by the projection operator in Alber's sense (See Chapter 1 of this book).

Let $(E, \|\cdot\|)$ be a *uniformly convex and uniformly smooth* Banach space. Let E^* be the topological dual of *E*. Denote by $\langle\cdot,\cdot\rangle$ the natural duality between E^* and *E*, that is, $\langle y, x\rangle = y(x)$, for all $y \in E^*$ and all $x \in E$. Let $\Omega \subset E$ be a closed convex set. Denote by $\|\cdot\|_*$ the norm on E^*. Let $J : E \to E^*$ be the duality mapping (See Chapter 1). We consider the mapping $V : E^* \times E \to \mathbb{R}$, defined by:

$$V(x, y) = \|y\|_*^2 - 2\langle y, x\rangle + \|x\|^2, \textit{ for any } (y, x) \in E^* \times E.$$

We know that the minimization problem:

$$\begin{cases} \textit{given } y \in E^*, \textit{ find } x_y \in \Omega \subset E \textit{ such that} \\ V(y, x_y) = \inf_{x \in \Omega} V(y, x) \end{cases}$$

has a unique solution. (see Chapter 1). The mapping $\Pi_\Omega : E^* \to \Omega \subset E$, defined by $\Pi_\Omega(y) = x_y$, is called the *generalized projection operator* (or the *Alber projection*). We need to use the following properties of mapping *V*.

(i) *$V(y, x)$ is convex with respect to y, when x is fixed and with respect to x, when y is fixed.*

(ii) $grad_x V(y, x) = 2(J(x) - y)$, *(because E is a smooth Banach space).*

For the proof of properties (i) and (ii) the reader is referred to (Alber, Y. I. [1]). We recall also the following property, well known in convex analysis.

(iii) *A differentiable mapping $\varphi : E \to \mathbb{R}$, is convex if and only if, for any x and x_0 in E we have*

$$\varphi(x) - \varphi(x_0) \geq \langle grad\varphi(x_0), x - x_0\rangle.$$

THEOREM 2.3.8. *An element $y_* \in \Omega$ is the generalized projection of an element $y \in E^*$ (i.e., $y_* = \Pi_\Omega(y)$, if and only if,*

$$\langle y - J(y_*), y_* - u \rangle \geq 0, \textit{ for all } u \in \Omega. \tag{2.3.5}$$

Proof. Considering the definition of $y_* = \Pi_\Omega(y)$ we have

$$V(y, y_*) \leq V(y, y_* + t(u - y_*)),$$

where $t \in]0, 1]$ and $y_* + t(u - y_*) \in \Omega$, because of the convexity of Ω. Using the properties (i), (ii) and (iii) we have

$$0 \geq V(y, y_*) - V(y, y_* + t(u - y_*))$$
$$\geq 2\langle J(y_* + t(u - y_*)) - y, y_* - y_* - t(u - y_*)\rangle,$$

which implies

$$\langle J(y_* + t(u - y_*)) - y, u - y_* \rangle \geq 0.$$

Letting $t \to 0$, we have

$$\langle J(y_*) - y, u - y_* \rangle \geq 0, \textit{ for all } u \in \Omega,$$

or

$$\langle y - J(y_*), y_* - u \rangle \geq 0, \textit{ for all } u \in \Omega,$$

that is condition (2.3.5) is satisfied.

Conversely, if condition (2.3.5) is satisfied, then we have (using properties (i), (ii) and (iii)),

$$V(y, u) - V(y, y_*) \geq 2\langle J(y_*) - y, u - y_* \rangle \geq 0, \text{ for any } x \in \Omega,$$

which implies

$$V(y, u) \geq V(y, y_*), \text{ for any } u \in \Omega.$$

Therefore $y_* = \Pi_\Omega(y)$ and the proof is complete. □

Let $f: E \to E^*$ be an arbitrary mapping. Consider again the problem

$$HSVI(f, \Omega): \begin{cases} \textit{find } x_* \in \Omega \textit{ such that} \\ \langle f(x_*), u - x_* \rangle \geq 0, \textit{ for all } u \in \Omega. \end{cases}$$

THEOREM 2.3.9. *Let f be a mapping from E to E^*, $\Omega \subset E$ a closed convex set and α an arbitrary fixed positive real number. Then an element $x_* \in \Omega$ is a solution to the problem HSVI(f, Ω) if and only if x_* is a fixed point of the mapping $\Psi_\Omega(x) = \Pi_\Omega[J(x) - \alpha f(x)]$, i.e., $x_* = \Pi_\Omega[J(x_*) - \alpha f(x_*)]$.*

Proof. Indeed, we observe that the problem *HSVI*(f, Ω) has the following representation:

$$\langle J(x_*) - \alpha f(x_*) - J(x_*), x_* - u \rangle \geq 0, \textit{ for all } u \in \Omega.$$

Considering this representation, and taking into account Theorem 2.3.8, $y = J(x_*) - \alpha f(x_*) \in E^*$ and $y_* = x_* \in \Omega \subset E$, we obtain the conclusion of the theorem. □

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