Chapter 2

Functionals Bounded Below

If a functional has the same infimum on two balls $B_r \subset B_R$, we establish the existence result of a bounded (PS)-sequence. On the other hand, we give a new proof for classical results on functionals bounded below.

2.1 Pseudo-Gradients

The existence of a pseudo-gradient vector field is a foundation stone for some variational problems.

Lemma 2.1. *Let P be a continuous mapping from a Banach space E to its dual E', and let $\tilde{E} := \{u \in E : P(u) \neq 0\}$. For any $\alpha \in (0, 1)$, there exists a locally Lipschitz continuous mapping (i.e., pseudo-gradient vector field) $V : \tilde{E} \to E$ such that*

$$\|V(w)\| \leq 1, \quad \alpha\|P(w)\| \leq \big(P(w), V(w)\big), \quad \forall\, w \in \tilde{E}.$$

Proof. Take $\alpha_1 \in (\alpha, 1)$. For any $u \in \tilde{E}$, there exists an element $\phi = \phi(u) \in E$ such that

$$\|\phi(u)\| = 1, \quad \alpha_1\|P(u)\| \leq \big(P(u), \phi(u)\big), u \in \tilde{E}.$$

The continuity of $P(u)$ implies that there exists an open neighborhood $U(u)$ of u such that

$$\alpha\|P(w)\| \leq \big(P(w), \phi(u)\big), \quad w \in U(u).$$

Then we get an open covering $\{U(u)\}$ of $\tilde{E}$. By Proposition 1.3, there is a locally finite refinement $\{V_i\}_{i \in J}$ and a locally Lipschitz continuous partition of unity $\{\lambda_i\}_{i\in J}$ subordinate to this refinement. For each $i \in J$, $V_i \subset U(u_i)$ for some u_i. Define

$$V(w) = \sum_{i\in J} \lambda_i(w)\phi(u_i).$$

Then $V : \tilde{E} \to E$ is locally Lipschitz continuous. By Proposition 1.3,

$$\|V(w)\| \le \sum_{i\in J} \lambda_i(w) = 1.$$

Moreover,

$$\big(P(w), V(w)\big) \ge \alpha \sum_{i\in J} \lambda_i(w)\|P(w)\| = \alpha\|P(w)\|.$$

□

Notes and Comments. Many books and papers have addressed the existence of the pseudo-gradient vector field which was applied directly to prove miscellaneous deformation theorems. For examples, see P. Bartolo-V. Benci-D. Fortunato [28], V. Benci-P. H. Rabinowitz [55], K. C. Chang [95, 96] (on a Finsler manifold), Y. Du [143], M. R. Grossinho-S. A. Tersian [176], J. Mawhin-M. Willem [252], L. Nirenberg [265], P. Rabinowitz [293], M. Ramos-C. Rebelo [298], M. Schechter [310], M. Struwe [352] and M. Willem [376, 377].

2.2 Bounded Minimizing Sequences

Let $(E, \|\cdot\|)$ be a Banach space and $I \in \mathcal{C}^1(E, \mathbf{R})$.

Theorem 2.2. *Assume that there exist $R > r > 0$ such that*

$$m := \inf_{B_R} I = \inf_{B_r} I > -\infty,$$

where $B_R := \{u \in E : \|u\| \le R\}$. Then there exists $\{u_n\} \subset B_R$ such that

$$I(u_n) \to m, \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Proof. Let $D(R, \varepsilon) = \{u \in B_R : I(u) < m + \varepsilon\}$. Then $\inf_{u\in D(R,\varepsilon)} \|I'(u)\| = 0$ for all $\varepsilon > 0$. Otherwise, there would exist an $\varepsilon_0 > 0$ such that $\|I'(u)\| \ge \varepsilon_0/(R-r)$ when $u \in D(R, \varepsilon_0)$. Let $u \in D(r, \varepsilon_0/2)(\ne \emptyset)$. By Lemma 2.1, there is a $V(u) : \tilde{E} := \{u \in E : I'(u) \ne 0\} \to E$ such that

$$\|V(u)\| \le 1, \quad \big(V(u), I'(u)\big) \ge \frac{3}{4}\|I'(u)\|, \quad \forall u \in \tilde{E}.$$

Moreover, V is a locally Lipschitz continuous map. Let $\sigma(t,u)$ be the solution of the Cauchy initial value problem

$$\begin{cases} \sigma'(t,u) = -V\big(\sigma(t,u)\big), \\ \sigma(0,u) = u \in D(r,\, \varepsilon_0/2). \end{cases}$$

Then

$$\|\sigma(t,u) - u\| \le \int_0^t \|\sigma'(s,u)\| ds \le t.$$

Hence, $\|\sigma(t,u)\| \le \|u\| + t \le R$ for $u \in D(r,\, \varepsilon_0/2), t \in [0, R-r]$. Therefore,

$$\begin{aligned} &\frac{d}{dt} I\big(\sigma(t,u)\big) \\ &= -\big(I'(\sigma), V(\sigma)\big) \\ &\le -\frac{3}{4}\|I'(\sigma(t,u))\| \\ &\le -\frac{3}{4(R-r)}\varepsilon_0 \end{aligned}$$

for $u \in D(r,\, \varepsilon_0/2)$, and therefore, $\sigma(t,u) \in D(R,\, \varepsilon_0)$ for $t \in [0, R-r]$ and $u \in D(r,\, \varepsilon_0/2)$. Furthermore,

$$I\big(\sigma(R-r,u)\big) \le I(u) - \frac{3}{4}\varepsilon_0 \le m + \frac{1}{2}\varepsilon_0 - \frac{3}{4}\varepsilon_0 = m - \frac{1}{4}\varepsilon_0,$$

a contradiction. $\square$

Consider a family of $\mathcal{C}^1(E, \mathbf{R})$-functionals of the form

$$I_\lambda(u) := \frac{1}{2}\lambda H(u) - J(u), \qquad \lambda \in \Lambda, u \in E,$$

where $\Lambda \subset (0, \infty)$ is an open interval; $H(u) \ge 0$ for all $u \in E$. Assume that one of the following conditions holds.

(**A**) For any $\beta > 0$, $\sup\big\{\|u\| : u \in E \text{ with } H(u) \le \beta\big\} < +\infty$.

(**B**) For any $\beta > 0$, $\sup\big\{\|u\| : u \in E \text{ with } J(u) \le \beta\big\} < +\infty$.

Theorem 2.3. *Assume that either (A) or (B) holds and that I_λ is bounded below for each $\lambda \in \Lambda$. Then for each $\lambda \in \Lambda$, there exists a sequence $\{u_n\}$ such that*

$$\sup_n \|u_n\| < \infty, \quad I_\lambda(u_n) \to \mathcal{M}_\lambda := \inf_E I_\lambda, \quad I'_\lambda(u_n) \to 0, \qquad \text{as } n \to \infty.$$

Proof. We only prove the first case. Note that the mapping $\lambda \to \mathcal{M}_\lambda$ is concave with respect to $\lambda \in \Lambda$. Therefore, it is Lipschitz continuous on each closed subinterval of Λ. For $\lambda \in \Lambda$, we choose a closed subinterval $\Lambda_\lambda \subset \Lambda$ containing λ as an interior point. Then, there exists a constant $\mathcal{M}'_\lambda > 0$ depending on λ such that

$$|\mathcal{M}_\lambda - \mathcal{M}_{\lambda'}| \leq \mathcal{M}'_\lambda |\lambda - \lambda'|, \quad \forall \lambda' \in \Lambda_\lambda.$$

Choose $\lambda_n \in (\lambda, 2\lambda) \cap \Lambda_\lambda, \lambda_n \to \lambda$ as $n \to \infty$. Then $\dfrac{|\mathcal{M}_{\lambda_n} - \mathcal{M}_\lambda|}{\lambda_n - \lambda} \leq \mathcal{M}'_\lambda$ for all n. We claim that there exists a sequence $\{u_n\} \subset E$ such that $\|u_n\| \leq k_0(\lambda) := k_0$ and

$$\mathcal{M}_\lambda \leq I_\lambda(u_n) \leq I_{\lambda_n}(u_n) \leq \mathcal{M}_{\lambda_n} + (\lambda_n - \lambda) \leq \mathcal{M}_\lambda + (\mathcal{M}'_\lambda + 1)(\lambda_n - \lambda).$$

In fact, by the definition of $\mathcal{M}_{\lambda_n}$, there exists a u_n such that $I_{\lambda_n}(u_n) \leq \mathcal{M}_{\lambda_n} + (\lambda_n - \lambda)$. Evidently, $\mathcal{M}_\lambda \leq I_\lambda(u_n) \leq I_{\lambda_n}(u_n)$ and

$$H(u_n) = \frac{2(I_{\lambda_n}(u_n) - I_\lambda(u_n))}{\lambda_n - \lambda} \leq 2\mathcal{M}'_\lambda + 4;$$

$$J(u_n) = \frac{\lambda I_{\lambda_n}(u_n) - \lambda_n I_\lambda(u_n)}{\lambda_n - \lambda} \leq -\mathcal{M}_\lambda + \mathcal{M}'_\lambda + 1.$$

Therefore, by (A), there exists a $k_0 = k_0(\lambda) > 0$ such that $\|u_n\| \leq k_0$. Define

$$D_\varepsilon(\lambda) := \{u \in E : \|u\| \leq k_0 + 3, \mathcal{M}_\lambda \leq I_\lambda(u) \leq \mathcal{M}_\lambda + \varepsilon\}.$$

Then, for any $\varepsilon > 0$, there exists an n large enough such that $u_n \in D_\varepsilon(\lambda)$. Now we claim that $\inf\{\|I'_\lambda(u)\| : u \in D_\varepsilon(\lambda)\} = 0$ for all $\varepsilon > 0$. If not, there exists an $\varepsilon_0 > 0$ such that $\|I'_\lambda(u)\| \geq \varepsilon_0$ for $u \in D_{\varepsilon_0}(\lambda)$. By Lemma 2.1, there is a locally Lipschitz continuous map $V_\lambda : \tilde{E} := \{u \in E : I'_\lambda(u) \neq 0\} \to E$ such that $\|V_\lambda(u)\| \leq 1$ and $\big(I'_\lambda(u), V_\lambda(u)\big) \geq \frac{1}{2}\|I'_\lambda(u)\|$ for all $u \in \tilde{E}$. Therefore, for any $u \in D_{\varepsilon_0}(\lambda)$, we have that $\|V_\lambda(u)\| \geq \dfrac{1}{2}$ and $\big(I'_\lambda(u), V_\lambda(u)\big) \geq \frac{1}{2}\varepsilon_0$. Define

$$A := \{u : \|u\| \geq k_0 + 2\} \cup \{u : I_\lambda(u) \geq \mathcal{M}_\lambda + \varepsilon_0/3\},$$

$$B := \{u : \|u\| \leq k_0 + 1, \quad \mathcal{M}_\lambda \leq I_\lambda(u) \leq \mathcal{M}_\lambda + \varepsilon_0/4\}.$$

Then, $A \cap B = \emptyset$. Moreover, if $u \notin A$ then $u \in D_{\varepsilon_0}(\lambda)$. In particular, $B \subset D_{\varepsilon_0}(\lambda)$. Define

$$\xi(u) := \frac{\operatorname{dist}(u, A)}{\operatorname{dist}(u, B) + \operatorname{dist}(u, A)}, \quad V^*_\lambda(u) := \xi(u) V_\lambda(u).$$

Then it is easy to check that $\big(I'_\lambda(u), V^*_\lambda(u)\big) \geq 0$ and $\|V^*_\lambda\| \leq 1$ for all $u \in \tilde{E}$. Furthermore, for $u \in B \subset D_{\varepsilon_0}(\lambda)$, $\xi(u) = 1$ and $\|I'_\lambda(u)\| \geq \varepsilon_0$, we have

$$\big(I'_\lambda(u), V^*_\lambda(u)\big) = \xi(u)\big(I'_\lambda(u), V_\lambda(u)\big) \geq \frac{1}{2}\|I'_\lambda(u)\| \geq \frac{1}{2}\varepsilon_0.$$

Now we consider the initial value problem $\dfrac{d\eta(t,u)}{dt} = -V^*_\lambda(\eta)$ with $\eta(0,u) = u$ for each $u \in E$ (note that ξ vanishes on an open set containing the points where $I'_\lambda = 0$). It is well known that there exists a unique solution $\eta(t,u)$ for $t \geq 0$. Moreover,

$$\mathcal{M}_\lambda \leq I_\lambda(\eta(t,u_n)) \leq I_\lambda(\eta(0,u_n)) \leq I_{\lambda_n}(u_n) \leq \mathcal{M}_{\lambda_n} + (\lambda - \lambda_n) \leq \mathcal{M}_\lambda + \frac{\varepsilon_0}{6}$$

for n large enough. Consequently,

$$\|\eta(t,u_n) - u_n\| = \|\int_0^t d\eta(s,u_n)\| \leq \int_0^t \|V^*_\lambda(\eta(s,u_n))\| ds \leq t.$$

It follows that $\|\eta(t,u_n)\| \leq \|u_n\| + t \leq k_0 + 1$ for $t \leq 1$. Therefore, $\eta(t,u_n) \in B$ for all $t \in [0,1]$. Moreover,

$$\begin{aligned}
&I_\lambda(\eta(1,u_n)) - I_\lambda(u_n)\\
&= \int_0^1 \frac{dI_\lambda(\eta(s,u_n))}{ds} ds\\
&= -\int_0^1 \Big(I'_\lambda(\eta(s,u_n)), V^*_\lambda(\eta(s,u_n))\Big) ds\\
&\leq -\frac{1}{2}\varepsilon_0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{M}_\lambda &\leq I_\lambda(\eta(1,u_n))\\
&\leq I_\lambda(u_n) - \frac{1}{2}\varepsilon_0\\
&\leq \mathcal{M}_\lambda + (\mathcal{M}'_\lambda + 2)(\lambda_n - \lambda) - \frac{1}{2}\varepsilon_0\\
&\leq \mathcal{M}_\lambda - \frac{1}{4}\varepsilon_0.
\end{aligned}$$

This is a contradiction. Thus, we know that there exists a sequence $\{u_n(\lambda)\}$ such that

$$\sup_n \|u_n(\lambda)\| < \infty, \quad I_\lambda(u_n(\lambda)) \to \mathcal{M}_\lambda, \quad I'_\lambda(u_n(\lambda)) \to 0 \text{ as } n \to \infty.$$

If H' is invertible and J' is compact, by standard arguments, there exists a u_λ such that $I_\lambda(u_\lambda) = \mathcal{M}_\lambda$ and $I'_\lambda(u_\lambda) = 0$.

$\square$

We therefore have

Theorem 2.4. *Assume that I_λ is bounded below for each $\lambda \in \Lambda$ and that any bounded (PS)-sequence of I_λ is precompact. Then for each $\lambda \in \Lambda$, $\mathcal{M}_\lambda = \inf_E I_\lambda$ is a critical value of I_λ.*

Theorem 2.3 can be easily generalized to a manifold (see R. Palais [269, 270, 271] for the definition of a Finsler manifold). Let M be a complete Finsler manifold with Finsler structure $\|\cdot\|$ and $I \in \mathcal{C}^1(M, \mathbf{R})$. Let $\partial I(u)$ denote the differential of I at u. Define

$$\|\partial I(u)\|_M := \sup\{|\partial I(u)\phi| : \phi \in T_u(M), \|\phi\| \le 1\},$$

where $T_u(M)$ denotes the tangent space of M at u. Let $T(M) = \cup_{u\in M} T_u(M)$ and $\tilde{M} = \{u \in M : \|\partial I(u)\|_M \neq 0\}$. Then we have the following theorem.

Theorem 2.5. *Let $M \neq \emptyset$ be a complete Finsler manifold with Finsler structure $\|\cdot\|$ and let $I_\lambda \in \mathcal{C}^1(M, \mathbf{R})$ be bounded below for all $\lambda > 0$. Assume that either (A) or (B) holds with E replaced by M. Then for each $\lambda \in \Lambda$, there exists a Palais-Smale sequence $\{u_n\} \subset M$ such that*

$$\sup_n \|u_n\| < \infty, \quad I_\lambda(u_n) \to \mathcal{M}_\lambda := \inf_M I_\lambda, \quad \|\partial I_\lambda(u_n)\|_M \to 0 \qquad \text{as } n \to \infty.$$

Proof. For any $\theta \in (0, 1)$, there exists a mapping $Y(u) : \tilde{M} \to T(S)$ which is locally Lipschitz continuous and satisfies

$$\|Y(u)\| \le 1, \quad \theta\|\partial I_\lambda(u)\|_M \le \big(\partial I_\lambda(u), Y(u)\big), \qquad u \in \tilde{M}.$$

This is a generalization of Lemma 2.1 (see also K. C. Chang [96, Lemma 3.1]). In fact, choose $\theta < \theta' < 1$. For $u \in \tilde{M}$, there exists an $h(u) \in T_u(M)$ such that

$$\|h(u)\| = 1, \quad \theta'\|\partial I_\lambda(u)\|_M \le \big(\partial I_\lambda(u), h(u)\big).$$

By the continuity of $\partial I_\lambda(u)$, for each $u \in \tilde{M}$, there is a neighborhood $N(u)$ such that $\theta\|\partial I_\lambda(v)\|_M \le \big(\partial I_\lambda(v), h(u)\big)$ for $v \in N(u)$. Then $\{N(u) : u \in \tilde{M}\}$ is an open covering of $\tilde{M}$. Since $\tilde{M}$ is a metric space, we may find a locally finite refinement $\{N_\tau\}$. Let $\{\psi_\tau\}$ be a locally Lipschitz continuous partition of unity subordinate to this refinement. For each τ, let u_τ be an element for which $N_\tau \subset N(u_\tau)$. Let $Y(v) = \sum \psi_\tau(v)h(u_\tau)$. Then $Y(v)$ is what we want. Similar to the proof of Theorem 2.3, we can prove that

$$\inf\{\|\partial I_\lambda(u)\|_M : \|u\| \le k_0 + 3, \mathcal{M}_\lambda \le I_\lambda(u) \le \mathcal{M}_\lambda + \varepsilon\} = 0$$

for all $\varepsilon > 0$ small enough. It is sufficient to know that the flow $\eta(t, u)$ still exists on the manifold. □

Notes and Comments. By I. Ekeland's variational principle (cf. [145]), if I is bounded from below, then there exists a minimizing sequence $\{u_n\}$ for I such that

$$I(u_n) \to \inf_E I, \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$

It is worth noting that if I satisfies the (PS) condition and maps bounded sets into bounded sets, then

$$\inf_E I > -\infty \quad \Leftrightarrow \quad I(u) \to +\infty \quad \text{as } \|u\| \to +\infty,$$

that is, I is coercive. This was proved in L. Caklovic-S. Li-W. Willem [77] using gradient flow (see also S. Li [211] and M. Willem [377]) and in D. G. Costa-E. A. B. Silva [114] by Ekeland's variational principle. Other properties for functionals bounded below (combining other assumptions) can be seen in H. Brézis-L. Nirenberg [70] (with local linking and without symmetry), in D. C. Clark [102] and in S. Li-Z. Q. Wang [217] (with locations of the critical points) for even functionals bounded below. Theorems 2.2-2.5 were obtained in M. Schechter-W. Zou [326].

2.3 An Application

We just give a simple example. We study the following elliptic eigenvalue problem:

$$(\mathbf{P}_\beta) \qquad -\Delta u = \beta g(x, u) \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$ and $g(x, t)$ is a Carathéodory function such that $g(x, 0) \in L^2(\Omega)$. Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be the sequence of eigenvalues of $-\Delta$ with Dirichlet zero boundary condition. Assume that there exists a constant α such that

$$\textbf{(C)} \quad \lambda_k \le \frac{g(x, t) - g(x, s)}{t - s} \le \alpha < \lambda_{k+1} \qquad \text{for all } x \in \Omega \text{ and } t \ne s.$$

Theorem 2.6. *Assume that (C) holds. Then for each $\beta \in (1, \frac{\lambda_{k+1}}{\alpha})$, problem (P_β) has a solution u_β. That is, the eigenvalue problem (P_β) has infinitely many solutions.*

However, we would like to show it by using Theorem 2.4. We need the following lemma.

Lemma 2.7. *Let E_i $(i = 1, 2)$ be two closed subspaces of a real Hilbert space E with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ such that $E = E_1 \oplus E_2$. Let $I \in \mathcal{C}^1(E, \mathbf{R})$. If there exists an increasing function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that $h(s) \to +\infty$ as $s \to +\infty$ and that*

$$\langle I'(u + v) - I'(u + w), v - w \rangle \geq \|v - w\| h(\|v - w\|) \tag{2.1}$$

for all $u \in E_1, v, w \in E_2$, then we have the following results.

(1) There exists a continuous function $\phi : E_1 \to E_2$ such that

$$I(u + \phi(u)) = \min_{v \in E_2} I(u + v).$$

Moreover, $\phi(u)$ is the unique member of E_2 such that

$$\langle I'(u + \phi(u)), v \rangle = 0, \qquad \forall v \in E_2.$$

(2) The functional $J : E_1 \to \mathbf{R}$ defined by $J(u) = I(u + \phi(u))$ is of class $\mathcal{C}^1$ and

$$\langle J'(u), v \rangle = \langle I'(u + \phi(u)), v \rangle, \quad \forall u, v \in E_1.$$

(3) An element $u \in E_1$ is a critical point of J if and only if $u + \phi(u)$ is a critical point of I.

Proof. (1) For each $u \in E_1$, define $H_u : E_2 \to \mathbf{R}$ by $H_u(v) = I(u + v)$. By the assumption (2.1), H_u is of $\mathcal{C}^1$ and has at most one critical point. We claim that H_u is coercive. Note that

$$\begin{aligned} & H_u(v) \\ & = H_u(0) + \int_0^1 \langle H_u(sv), v \rangle ds \\ & \geq H_u(0) - \|H_u'(0)\| \|v\| + \int_0^1 s\|v\| h(\|sv\|) ds. \end{aligned}$$

By the hypotheses on h, we may choose R large enough such that

$$h(\|sv\|) \geq 8\|H_u'(0)\| \text{ uniformly for } \|v\| \geq R, s \in [1/2, 1].$$

Hence,

$$H_u(v) \geq H_u(0) + \|v\|,$$

which implies that $H_u(v) \to \infty$ as $\|v\| \to \infty$. Next, we show that H_u is convex. For given $v, w \in E_2$, define

$$\xi(s) = H_u(v + s(w - v)).$$

For $0 < \alpha < \beta < 1$, by (2.1), it is easy to show that

$$\xi'(\beta) - \xi'(\alpha) > 0.$$

This means that ξ is convex in s, and consequently, H_u is convex in v. Combining the above arguments, we see that H_u has a unique minimizer $\phi(u) \in E_2$ with $H_u(\phi(u)) = \min\{I(u+v) : v \in E_2\}$. Therefore, we have

$$\langle I'(u + \phi(u)), w\rangle = 0, \quad \forall w \in E_2. \tag{2.2}$$

To show that $\phi(u)$ is continuous in u, we assume on the contrary, that there are $\varepsilon_0 > 0$ and $u_k \to u$ as $k \to \infty$ such that

$$\|\phi(u_k) - \phi(u)\| \geq \varepsilon_0.$$

Let P be the projection from E to E_2. By (2.2), we see that $\|PI'(u_k + \phi(u))\| \leq h(\varepsilon_0/2)$ if k large enough. Therefore,

$$\begin{aligned}
&h(\varepsilon_0)\|\phi(u_k) - \phi(u)\| \\
&\leq \langle I'(u_k + \phi(u_k)) - I'(u_k + \phi(u)), \phi(u_k) - \phi(u)\rangle \\
&\leq \langle -I'(u_k + \phi(u)), \phi(u_k) - \phi(u)\rangle \\
&\leq \|PI'(u_k + \phi(u))\|\|\phi(u_k) - \phi(u)\| \\
&\leq h(\varepsilon_0/2)\|\phi(u_k) - \phi(u)\|,
\end{aligned}$$

a contradiction. Hence, $\phi(u)$ is continuous in u.

To show (2)-(3), it suffices to prove that

$$\langle J'(u), w\rangle = \langle I'(u + \phi(u)), w\rangle, \quad w \in E_1. \tag{2.3}$$

Indeed, for $s > 0$,

$$\begin{aligned}
&\frac{J(u + sw) - J(u)}{s} \\
&= \frac{I(u + sw + \phi(u + sw)) - J(u + \phi(u))}{s} \\
&\leq \frac{I(u + sw + \phi(u)) - J(u + \phi(u))}{s} \\
&= \int_0^1 \langle I'(u + \phi(u) + \tau sw), w\rangle d\tau.
\end{aligned} \tag{2.4}$$

Similarly, we have

$$\frac{J(u+sw)-J(u)}{s} \geq \int_0^1 \langle I'(u+\phi(u+sw)+\tau sw), w\rangle d\tau. \tag{2.5}$$

Summing up (2.4)-(2.5), we get (2.3). □

Let $E := H_0^1(\Omega)$ be with the usual norm induced by the inner product $\langle u, v\rangle = \displaystyle\int_\Omega \nabla u \cdot \nabla v dx$. We denote by $E(\lambda_i)$ the eigenspace corresponding to λ_i. Set $E^- = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus E(\lambda_{k-1})$, $E^0 = E(\lambda_k)$ and $E^+ = E(\lambda_{k+1}) \oplus E(\lambda_{k+2}) \oplus \cdots$. Then $E = E^- \oplus E^0 \oplus E^+$. Solutions of (P_β) correspond to the critical points of the C^1-functional

$$I_\lambda(u) = \frac{\lambda}{2}\|u\|^2 - \int_\Omega G(x,u)dx, \quad \forall\, u \in E,$$

where $\lambda = \dfrac{1}{\beta}$, $G(x,u) = \displaystyle\int_0^u g(x,s)ds$.

Proof of Theorem 2.6. Choose $\Lambda = (\frac{\alpha}{\lambda_{k+1}}, 1)$. Then for $\lambda \in \Lambda$ and $u \in E^0 \oplus E^-, w, v \in E^+$, by (C), we have that

$$\begin{aligned}
&\langle I'_\lambda(u+v) - I'_\lambda(u+w), v-w\rangle \\
&= \lambda\|v-w\|^2 - \int_\Omega \Big(g(x,u+v) - g(x,u+w)\Big)(v-w)dx \\
&\geq \lambda\|v-w\|^2 - \int_\Omega \alpha(v-w)^2 dx \\
&\geq (\lambda - \frac{\alpha}{\lambda_{k+1}})\|v-w\|^2.
\end{aligned}$$

By Lemma 2.7, there exists a mapping $\phi_\lambda : E^0 \oplus E^- \to E^+$ such that

$$I_\lambda(u^0 + u^- + \phi_\lambda(u^0+u^-)) = \min_{u^+\in E^+} I_\lambda(u^0+u^-+u^+).$$

Moreover, $\phi_\lambda(u^0+u^-)$ is the unique member of E^+ such that

$$\langle I'_\lambda(u^0+u^-+\phi_\lambda(u^0+u^-), v\rangle = 0$$

for all $v \in E^+$. Define a functional $J_\lambda : E^0 \oplus E^- \to \mathbf{R}$ by

$$J_\lambda(u^0+u^-) = I_\lambda(u^0+u^-+\phi_\lambda(u^0+u^-)).$$

Then J is of class $\mathcal{C}^1$ and

$$\langle J'_\lambda(u^0+u^-),z\rangle = \langle I'_\lambda(u^0+u^-+\phi_\lambda(u^0+u^-)),z\rangle$$

for all $u^0+u^-, z \in E^0 \oplus E^-$. Moreover, u^0+u^- is a critical point of J if and only if $u^0+u^-+\phi_\lambda(u^0+u^-)$ is a critical point of I_λ. Next, we claim that $-I_\lambda$ is bounded below on $E^- \oplus E^0$. In fact, by condition (C) we see that $\lambda_k t^2 \le t(g(x,t)-g(x,0)) \le \alpha t^2$. Then

$$G(x,t) = \int_0^t sg(x,s)\frac{ds}{s} \ge \frac{1}{2}\lambda_k t^2 + tg(x,0).$$

Therefore,

$$I_\lambda(u) \le \frac{1}{2}(\lambda-1)\|u\|^2 - \int_\Omega ug(x,0)dx \to -\infty$$

as $\|u\| \to \infty$. Hence, $-I_\lambda$ and $-J_\lambda$ are bounded below. Evidently, $-J_\lambda$ satisfies the other assumptions of Theorem 2.3. Therefore, for all $\lambda \in \Lambda$, there exists a u_λ such that $-J'_\lambda(u_\lambda)=0$. This completes the proof of the theorem. $\square$

Notes and Comments. Lemma 2.7 was established in A. Castro [81]. Some applications of it can be found in A. Castro-J. Cossio [82] and M. Schechter [315]. Theorem 2.6 was given in M. Schechter-W. Zou [326]. Possibly, it can be proved by other methods such as the degree theory or the contraction mapping principle. We believe that Theorem 2.4 has far more extended applications. We would like to leave them to the readers.

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