

Chapter 2

EXPLICIT SOLUTIONS OF LINEAR QUADRATIC DIFFERENTIAL GAMES

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Abstract The theory of linear quadratic differential games is in principle known. An excellent reference for management and economics applications is Dockner et al. (2000). We review here the results, showing that in useful simple cases, explicit solutions are available. This treatment is not included in the previous reference and seems to be original. In non-stationary cases, explicit solutions are not available, we prove the existence of solutions of coupled Riccati equations, which provide a complete solution of the Nash equilibrium problem.

1. Introduction

Differential games is attracting a lot of interest in the management and economics literature. This is because many players appear in most situations, and traditional optimization techniques for a single decision maker are not sufficient. However the treatment of differential games is much more complex than that of control theory, especially as far as obtaining explicit solutions is concerned. In this article, we complete a presentation of Dockner et al. (2000), a main reference for management and economics applications of differential games, to derive explicit solutions of linear quadratic differential games in a fairly general context. However, when data depend on time, non-stationary situation, we do not have explicit solutions anymore. The problem reduces to solving coupled Riccati equations. We prove existence of the solution of this pair of equations, which provide control strategies for the players.

2. Open Loop Differential Games Finite Horizon

We consider the following model, [Dockner et al. (2000), Section 7.1]. We have two players, whose controls are denoted by v^1, v^2 . The state equation is described by

$$\dot{x} = ax + b^1v^1 + b^2v^2, \quad x(0) = x_0.$$

The payoffs of player $i = 1, 2$ are given by

$$J^i(x, \mathbf{v}) = \frac{1}{2} \int_0^T [\alpha^i x^2 + \beta^i (v^i)^2] dt.$$

We apply the theory of necessary conditions. We first define the Hamiltonians by the formulas

$$H^i(x, \mathbf{v}, q^i) = \frac{1}{2}(\alpha^i x^2 + \beta^i (v^i)^2) + q^i(ax + b^1v^1 + b^2v^2),$$

where $\mathbf{v} = (v^1, v^2)$. Writing the adjoint equations, we obtain

$$-\dot{p}^i = \alpha^i x + ap^i, \quad p^i(T) = 0.$$

Writing next the optimality conditions

$$H_{v^i}^i(y, \mathbf{u}, p^i) = 0,$$

we obtain

$$\beta^i u^i + p^i b^i = 0.$$

We then notice an important simplification, namely

$$\frac{p^i}{\alpha^i} = p,$$

with

$$-\dot{p} = y + ap, \quad p(T) = 0.$$

We next define

$$M = \sum_{i=1}^2 \frac{\alpha^i (b^i)^2}{\beta^i}.$$

Collecting results, we obtain the maximum principle conditions

$$\dot{y} = ay - Mp$$

$$-\dot{p} = y + pa$$

$$y(0) = x_0, \quad p(T) = 0,$$

and the optimal controls of both players are given by

$$u^i = -\frac{\alpha^i b^i p}{\beta^i}.$$

We can compute $y(t), p(t)$ following a decoupling argument. We postulate

$$p(t) = P(t)y(t).$$

It is easy to show that $P(t)$ is the solution of the Riccati equation

$$-\dot{P} - 2aP + MP^2 - 1 = 0, \quad P(T) = 0,$$

and that furthermore

$$\frac{1}{P(t)} = -a + s \frac{\exp 2s(T-t) + 1}{\exp 2s(T-t) - 1},$$

where

$$s = \sqrt{a^2 + M}.$$

We then obtain the explicit solution

$$y(t) = x_0 \frac{s(\exp s(T-t) + \exp -s(T-t)) - a(\exp s(T-t) - \exp -s(T-t))}{s(\exp sT + \exp -sT) - a(\exp sT - \exp -sT)}.$$

Infinite Horizon

We consider the infinite horizon version of the basic model above. We introduce a discount factor r . The maximum principle leads to the following relations (usual changes with respect to the finite horizon case)

$$\dot{y} = ay - Mp,$$

$$-\dot{p} + rp = y + pa,$$

$$y(0) = x_0.$$

There is no final condition on $p(T)$, but we require the integration conditions

$$y \in L_r^2(0, \infty; R), p \in L_r^2(0, \infty; R).$$

We can check that the solution of the infinite horizon problem is obtained as follows

$$p = Py,$$

with

$$P = \frac{2a - r + \sqrt{(r - 2a)^2 + 4M}}{2M},$$

and y, p satisfy the integrability conditions.

Non-Zero Final Cost

We consider again the finite horizon case, with non-zero final cost. So the payoffs are given by

$$J^i(x, \mathbf{v}) = \frac{1}{2} \int_0^T [\alpha^i x^2 + \beta^i (v^i)^2] dt + \frac{1}{2} \gamma^i (x(T))^2.$$

The adjoint variables p^1, p^2 are then the solutions of

$$-\dot{p}^i = \alpha^i y + a p^i, \quad p^i(T) = \gamma^i y(T).$$

We do not have anymore the property

$$\frac{p^i}{\alpha^i} = p.$$

However we shall be able again to derive an explicit solution.

We can see indeed that

$$p^i = \alpha^i p + \gamma^i \pi,$$

where p, π satisfy

$$-\dot{p} = y + a p, \quad p(T) = 0,$$

$$-\dot{\pi} = a \pi, \quad \pi(T) = y(T).$$

We introduce a number analogous to M

$$N = \sum_{i=1}^2 \frac{\gamma^i (b^i)^2}{\beta^i},$$

and define

$$\varpi = M p + N \pi.$$

We deduce the equation for y

$$\dot{y} = a y - \varpi, \quad y(0) = x_0,$$

and we check that

$$-\dot{\varpi} = M y + a \varpi, \quad \varpi(T) = N y(T).$$

We can decouple the two-point boundary value problem in y, ϖ , by writing

$$\varpi(t) = Q(t) y(t),$$

and Q is the solution of the Riccati equation

$$-\dot{Q} - 2aQ + Q^2 = M, \quad Q(T) = N.$$

We deduce easily

$$p = Ry, \quad \pi = \rho y,$$

with

$$-\dot{R} = 1 + 2aR - QR, \quad R(T) = 0$$

$$-\dot{\rho} = 2a\rho - Q\rho, \quad \rho(T) = 1.$$

We next check that

$$Q(t) = a + s \frac{(N - a - s) + (N - a + s) \exp 2s(T - t)}{(N - a + s) \exp 2s(T - t) - (N - a - s)},$$

and that

$$p^1(t) = P^1(t)y(t), \quad p^2(t) = P^2(t)y(t),$$

where P^1, P^2 are solutions of the system

$$-\dot{P}^1 - 2aP^1 + \frac{(b^1)^2}{\beta^1} (P^1)^2 + \frac{(b^2)^2}{\beta^2} P^1 P^2 = \alpha^1$$

$$-\dot{P}^2 - 2aP^2 + \frac{(b^2)^2}{\beta^2} (P^2)^2 + \frac{(b^1)^2}{\beta^1} P^1 P^2 = \alpha^2$$

$$P^1(T) = \gamma^1, \quad P^2(T) = \gamma^2,$$

which is a system of Riccati equations. we then assert that

$$P^1(t) = \alpha^1 R(t) + \gamma^1 \rho(t),$$

$$P^2(t) = \alpha^2 R(t) + \gamma^2 \rho(t).$$

To complete the explicit solution, we check the following formulas

$$\rho(t) = \frac{2s \exp a(T - t)}{(N - a + s) \exp s(T - t) - (N - a - s) \exp -s(T - t)},$$

$$R(t) = \frac{1}{M} [-2sN \exp a(T - t) + (a + s)(N - a + s) \exp s(T - t) \\ + (N - a - s)(s - a) \exp -s(T - t)] \\ / [(N - a + s) \exp s(T - t) - (N - a - s) \exp -s(T - t)]$$

Furthermore,

$$y(t) = \frac{2sx_0}{(N - a + s) \exp s(T - t) - (N - a - s) \exp -s(T - t)}.$$

3. Non-Stationary Mode Maximum Principle

We consider the same problem with non-constant parameters, namely

$$\dot{x}(t) = a(t)x(t) + b^1(t)v^1(t) + b^2(t)v^2(t), \quad x(0) = x_0,$$

and

$$J^i(v^1(\cdot), v^2(\cdot)) = \frac{1}{2} \int_0^T (\alpha^i(t)x^2(t) + \beta^i(t)(v^i)^2(t))dt + \frac{1}{2}\gamma^i x^2(T).$$

We can write the maximum principle in a way similar to the stationary case. To save notation, we shall not explicitly write the argument t . This leads to the system

$$\dot{y}(t) = ay - \frac{(b^1)^2}{\beta^1}p^1 - \frac{(b^2)^2}{\beta^2}p^2, \quad y(0) = x_0$$

$$-\dot{p}^i = \alpha^i y + ap^i, \quad p^i(T) = \gamma^i y(T).$$

Unfortunately the simplifications of the stationary case do not carry over. However, one can use the fact that the system obtained from the maximum principle arguments is linear. So if we set

$$A(t) = \begin{pmatrix} a(t) & -\frac{(b^1)^2}{\beta^1}(t) & -\frac{(b^2)^2}{\beta^2}(t) \\ -\alpha^1(t) & -a(t) & 0 \\ -\alpha^2(t) & 0 & -a(t) \end{pmatrix}$$

and

$$\mathbf{z}(t) = \begin{pmatrix} y(t) \\ p^1(t) \\ p^2(t) \end{pmatrix},$$

then the system of conditions from the maximum principle reads

$$\dot{\mathbf{z}}(t) = A(t)\mathbf{z}(t).$$

Fundamental Matrix

The solution of this non-stationary linear differential system is obtained as follows

$$\mathbf{z}(t) = \Phi(t, \tau)\mathbf{z}(\tau), \quad \forall t > \tau,$$

where $\Phi(t, \tau)$ is called the *fundamental matrix*. In the stationary case, where A does not depend on t we can find the eigenvalues and the

eigenvectors of A . We can check that these eigenvalues are $s, -s, -a$. We find next the corresponding eigenvectors, w^1, w^2, w^3 . We can show that

$$W = (w^1, w^2, w^3) = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{\alpha^1}{a+s} & -\frac{\alpha^1}{a-s} & \frac{(b^2)^2}{\beta^2} \\ -\frac{\alpha^2}{a+s} & -\frac{\alpha^2}{a-s} & -\frac{(b^1)^2}{\beta^1} \end{pmatrix}$$

Let Λ be the diagonal matrix with eigenvalues on the diagonal, we can show that the fundamental matrix is given by

$$\Phi(t, \tau) = W \exp \Lambda(t - \tau) W^{-1}.$$

The fundamental matrix satisfies the matrix differential equation

$$\frac{\partial}{\partial t} \Phi(t, \tau) = A(t) \Phi(t, \tau), \quad \Phi(\tau, \tau) = I.$$

Moreover, this matrix is invertible, with inverse $\Psi(t, \tau) = (\Phi(t, \tau))^{-1}$ given by

$$\frac{\partial}{\partial t} \Psi(t, \tau) = -\Psi(t, \tau) A(t), \quad \Psi(\tau, \tau) = I.$$

Since the Maximum principle leads to a two-point boundary value problem, one must find the values

$$\varpi^1 = p^1(0), \quad \varpi^2 = p^2(0).$$

They are obtained from the conditions

$$p^1(T) = \gamma^1 y(T), \quad p^2(T) = \gamma^2 y(T),$$

which amounts to solving the linear system of algebraic equations

$$\begin{aligned} & (\Phi_{22}(T, 0) - \gamma^1 \Phi_{12}(T, 0)) \varpi^1 + (\Phi_{23}(T, 0) - \gamma^1 \Phi_{13}(T, 0)) \varpi^2 \\ & = (\gamma^1 \Phi_{11}(T, 0) - \Phi_{21}(T, 0)) x_0, \end{aligned}$$

$$\begin{aligned} & (\Phi_{32}(T, 0) - \gamma^2 \Phi_{12}(T, 0)) \varpi^1 + (\Phi_{33}(T, 0) - \gamma^2 \Phi_{13}(T, 0)) \varpi^2 \\ & = (\gamma^2 \Phi_{11}(T, 0) - \Phi_{31}(T, 0)) x_0, \end{aligned}$$

where $\Phi_{ij}(T, 0)$ represents the element of line i and column j of the matrix $\Phi(T, 0)$.

We can show the formulas

$$\varpi^1 = \frac{\Psi_{21}(T, 0) + \gamma^1 \Psi_{22}(T, 0) + \gamma^2 \Psi_{23}(T, 0)}{\Psi_{11}(T, 0) + \gamma^1 \Psi_{12}(T, 0) + \gamma^2 \Psi_{13}(T, 0)} x_0,$$

$$\varpi^2 = \frac{\Psi_{31}(T, 0) + \gamma^1 \Psi_{32}(T, 0) + \gamma^2 \Psi_{33}(T, 0)}{\Psi_{11}(T, 0) + \gamma^1 \Psi_{12}(T, 0) + \gamma^2 \Psi_{13}(T, 0)} x_0.$$

We can show more generally that

$$p^1(t) = P^1(t)y(t), \quad p^2(t) = P^2(t)y(t),$$

with the formulas

$$P^1(t) = \frac{\Psi_{21}(T, t) + \gamma^1 \Psi_{22}(T, t) + \gamma^2 \Psi_{23}(T, t)}{\Psi_{11}(T, t) + \gamma^1 \Psi_{12}(T, t) + \gamma^2 \Psi_{13}(T, t)},$$

$$P^2(t) = \frac{\Psi_{31}(T, t) + \gamma^1 \Psi_{32}(T, t) + \gamma^2 \Psi_{33}(T, t)}{\Psi_{11}(T, t) + \gamma^1 \Psi_{12}(T, t) + \gamma^2 \Psi_{13}(T, t)}.$$

We can check directly that $P^1(t), P^2(t)$ are solutions of the system of Riccati differential equations

$$-\dot{P}^1 - 2aP^1 + \frac{(b^1)^2}{\beta^1}(P^1)^2 + \frac{(b^2)^2}{\beta^2}P^1P^2 = \alpha^1,$$

$$-\dot{P}^2 - 2aP^2 + \frac{(b^2)^2}{\beta^2}(P^2)^2 + \frac{(b^1)^2}{\beta^1}P^1P^2 = \alpha^2,$$

$$P^1(T) = \gamma^1, \quad P^2(T) = \gamma^2,$$

already mentioned in the stationary case. This time the coefficients depend on time.

4. Closed-Loop Nash Equilibrium

System of PDE

We proceed with the Dynamic Programming formulation. The Hamiltonians are defined by

$$H^i(x, \mathbf{v}, q^i) = \frac{1}{2}(\alpha^i x^2 + \beta^i (v^i)^2) + q^i(ax + b^1 v^1 + b^2 v^2).$$

We look for Nash point equilibriums of the Hamiltonians $H^i(x, \mathbf{v}, q^i)$. We obtain easily

$$u^i(\mathbf{q}) = -q^i \frac{b^i}{\beta^i}.$$

We next write

$$H^1(x, \mathbf{q}) = \frac{1}{2}\alpha^1 x^2 + q^1 ax - \frac{1}{2}(q^1)^2 \frac{(b^1)^2}{\beta^1} - q^1 q^2 \frac{(b^2)^2}{\beta^2},$$

$$H^2(x, \mathbf{q}) = \frac{1}{2}\alpha^1 x^2 + q^2 ax - \frac{1}{2}q^2 \frac{(b^2)^2}{\beta^2} - q^1 q^2 \frac{(b^1)^2}{\beta^1}.$$

Dynamic Programming leads to the following system of partial differential equations

$$\begin{aligned}
 & \frac{\partial \Psi^1}{\partial t} + \frac{1}{2} \alpha^1 x^2 + \frac{\partial \Psi^1}{\partial x} a x - \frac{1}{2} \left(\frac{\partial \Psi^1}{\partial x} \right)^2 \frac{(b^1)^2}{\beta^1} - \frac{\partial \Psi^1}{\partial x} \frac{\partial \Psi^2}{\partial x} \frac{(b^2)^2}{\beta^2} = 0, \\
 (2.1) \quad & \frac{\partial \Psi^2}{\partial t} + \frac{1}{2} \alpha^1 x^2 + \frac{\partial \Psi^2}{\partial x} a x - \frac{1}{2} \left(\frac{\partial \Psi^2}{\partial x} \right)^2 \frac{(b^2)^2}{\beta^2} - \frac{\partial \Psi^1}{\partial x} \frac{\partial \Psi^2}{\partial x} \frac{(b^1)^2}{\beta^1} = 0, \\
 & \Psi^1(x, T) = \frac{1}{2} \gamma^1 x^2, \quad \Psi^2(x, T) = \frac{1}{2} \gamma^2 x^2, \quad \text{a.e.}
 \end{aligned}$$

System of Riccati Equations

The solutions are given by

$$\Psi^i(x, t) = \frac{1}{2} Q^i(t) x^2,$$

where $Q^i(t)$ are solutions of the system of Riccati equations

$$\begin{aligned}
 & -\dot{Q}^1 - 2aQ^1 + \frac{(b^1)^2}{\beta^1} (Q^1)^2 + 2 \frac{(b^2)^2}{\beta^2} Q^1 Q^2 = \alpha^1, \\
 (2.2) \quad & -\dot{Q}^2 - 2aQ^2 + \frac{(b^2)^2}{\beta^2} (Q^2)^2 + 2 \frac{(b^1)^2}{\beta^1} Q^1 Q^2 = \alpha^2 Q^1(T) = \gamma^1, \\
 & Q^2(T) = \gamma^2.
 \end{aligned}$$

These equations are different from those of open-loop control. The coupling term is different, reflecting the coupling through the state.

Stationary Case

The above Riccati equations cannot be solved as easily as in the open loop case. To simplify we consider the stationary case (it corresponds to an infinite horizon problem with no discount). The Riccati equations reduce to the algebraic equations

$$\begin{aligned}
 & -2aQ^1 + \frac{(b^1)^2}{\beta^1} (Q^1)^2 + 2 \frac{(b^2)^2}{\beta^2} Q^1 Q^2 = \alpha^1, \\
 (2.3) \quad & -2aQ^2 + \frac{(b^2)^2}{\beta^2} (Q^2)^2 + 2 \frac{(b^1)^2}{\beta^1} Q^1 Q^2 = \alpha^2.
 \end{aligned}$$

To simplify notation, we set

$$\nu^i = \frac{(b^i)^2}{\beta^i}.$$

We can write these equations as

$$(2.4) \quad \begin{aligned} (\nu^1 Q^1 + \nu^2 Q^2 - a)^2 &= \nu^1 \alpha^1 + (\nu^2 Q^2 - a)^2, \\ (\nu^1 Q^1 + \nu^2 Q^2 - a)^2 &= \nu^2 \alpha^2 + (\nu^1 Q^1 - a)^2. \end{aligned}$$

Set $\rho = \nu^1 Q^1 + \nu^2 Q^2 - a$. We check that

$$\nu^1 Q^1 - \nu^2 Q^2 = \frac{\nu^1 \alpha^1 - \nu^2 \alpha^2}{\rho - a},$$

and that ρ must be solution of the equation

$$\phi(\rho) = (\rho - a)^2(-3\rho^2 - 2a\rho + a^2 + 2M) + (\nu^1 \alpha^1 - \nu^2 \alpha^2)^2 = 0.$$

Assume that

$$a < \frac{s}{\sqrt{3}},$$

we can show that there exists only one solution such that $\rho > -a$. This solution is larger than $s/\sqrt{3}$ and $\sqrt{M/2}$. If

$$a > \frac{s}{\sqrt{3}},$$

then

$$\sqrt{\frac{M}{2}} < \frac{s}{\sqrt{3}},$$

we can show that for a sufficiently large, the equation may have 3 solutions larger than $\sqrt{M/2}$. It has only one larger than a .

As an example, consider the following model of "knowledge as a public good", see [Dockner et al. (2000), Section 9.5]. Two players contribute by investing in accumulating knowledge as a capital, whose evolution is governed by

$$\dot{x} = -\delta x + v^1 + v^2, \quad x(0) = x_0.$$

Each player faces an investment cost given by

$$\rho v^i + \frac{1}{2}(v^i)^2,$$

and benefits from the common knowledge according to an individual revenue given by

$$x(t)(a^i - x(t)).$$

Note that in this model the individual profit declines when the collective knowledge is sufficiently large, and can become negative. There is a saturation effect. The pay-off for each player (to be minimized) is given by

$$J^i(v^1, v^2) = \int_0^\infty e^{-rt} [\rho v^i + \frac{1}{2}(v^i)^2 - x(t)(a^i - x(t))] dt.$$

We begin with the closed-loop Nash equilibrium. We write the Dynamic Programming equations. We first consider the Hamiltonians

$$H^i(x, \mathbf{v}, q^i) = -x(a^i - x) + \rho v^i + \frac{1}{2}(v^i)^2 + q^i(-\delta x + v^1 + v^2)$$

and look for a Nash equilibrium in (v^1, v^2) . We obtain easily

$$u^i(\mathbf{q}) = -\rho - q^i$$

and it follows that

$$H^i(x, \mathbf{q}) = -x(a^i - x) - \frac{1}{2}\rho^2 - \frac{1}{2}(q^i)^2 - q^i(2\rho + \delta x) - q^1 q^2.$$

So the DP equations read

$$r\Psi^i = -x(a^i - x) - \frac{1}{2}\rho^2 - \frac{1}{2}(\Psi_x^i)^2 - \Psi_x^i(2\rho + \delta x) - \Psi_x^1 \Psi_x^2.$$

We look for quadratic solutions

$$\Psi^i = \frac{1}{2}Px^2 - \beta^i x - \gamma^i$$

and we obtain, by identification

$$\frac{3}{2}P^2 + P(\delta + \frac{r}{2}) - 1 = 0,$$

$$\beta^i(P + \delta + r) + P(\beta^1 + \beta^2) = a^i + 2P\rho,$$

$$r\gamma^i = \frac{1}{2}\rho^2 + \frac{1}{2}(\beta^i)^2 - 2\rho\beta^i + \beta^1\beta^2.$$

We take the positive solution

$$P = -\frac{r + 2\delta}{6} + \sqrt{\left(\frac{r + 2\delta}{6}\right)^2 + \frac{2}{3}}$$

and the closed-loop controls are given by

$$u^i(x, t) = -Px + \beta^i - \rho.$$

Applying these controls we get the trajectory

$$\dot{y} = -(\delta + 2P)y + \beta^1 + \beta^2 - 2\rho,$$

which has a stable solution since $P > 0$. The stable solution is

$$\bar{y} = \frac{\beta^1 + \beta^2 - 2\rho}{2P + \delta}.$$

We next consider open-loop controls. We assume to simplify that $a^1 = a^2 = a$. We write the Maximum Principle necessary conditions. Thanks to our simplification, the two adjoint variables coincide. We get the system

$$\dot{y} = -\delta y - 2(\rho + p), \quad y(0) = x_0,$$

$$-\dot{p} + (r + \delta)p = -a + 2y,$$

$$u^i = -(\rho + p).$$

The solution of this system is easily obtained as follows

$$p = Qy - q,$$

where Q is a solution of

$$2Q^2 + (r + 2\delta)Q - 2 = 0.$$

Note that $Q > P$. Next q is the solution of

$$-\dot{q} + q(r + \delta + 2Q) = a + 2Q\rho.$$

The corresponding trajectory is defined by

$$\dot{y} + y(\delta + 2Q) = -2\rho + 2q, \quad y(0) = x_0.$$

It has also a stable solution, given by

$$\hat{y} = \frac{2q - 2\rho}{\delta + 2Q},$$

where q is given by

$$q = \frac{a + 2Q\rho}{r + \delta + 2Q}.$$

We can show that

$$\hat{y} = \frac{2(a - \rho(r + \delta))}{\delta(r + \delta) + 4},$$

$$\bar{y} = \frac{2(a - \rho(r + \delta) - \rho P)}{\delta(r + \delta) + 4 + P\delta},$$

and conclude that

$$\bar{y} < \hat{y}.$$

The economic interpretation of this inequality is the following. Each player has interest to benefit from the other player's investment and contribute the less possible. In the closed-loop case, one can make more use of the common education than in the open-loop case, resulting in a lower steady state.

From the economic considerations one can conjecture that the steady state should improve in the case of a cooperative game. We shall verify this property. In the cooperative game formulation we take as common objective function the sum of each player objective function

$$J(v^1, v^2) = \int_0^\infty e^{-rt} [\rho(v^1 + v^2) + \frac{1}{2}((v^1)^2 + (v^2)^2) - 2x(t)(a - x(t))] dt,$$

with the trajectory

$$\dot{x} = -\delta x + v^1 + v^2, \quad x(0) = x_0.$$

We can write the Maximum Principle for the cooperative game, and obtain

$$\dot{y} = -\delta y - 2(\rho + p), \quad y(0) = x_0,$$

$$-\dot{p} + (r + \delta)p = -2a + 4y,$$

$$u^i = -(\rho + p).$$

Let us check simply the steady state, given by the relations

$$\delta y = 2(\rho + p),$$

$$(r + \delta)p = -2a + 4y.$$

This leads to a steady state defined by

$$y^* = 2 \frac{2a - (r + \delta)}{8 + \delta(r + \delta)}$$

and it is easy to check that

$$y^* > \hat{y}.$$

Existence Result

We go back to equations (2.2). To simplify, we assume that the coefficients do not depend on time. Recalling the notation ν^i , the equations are

$$(2.5) \quad \begin{aligned} -\dot{Q}^1 - 2aQ^1 + \nu^1(Q^1)^2 + 2\nu^2Q^1Q^2 &= \alpha^1, \\ -\dot{Q}^2 - 2aQ^2 + \nu^2(Q^2)^2 + 2\nu^1Q^1Q^2 &= \alpha^2 \\ Q^1(T) &= \gamma^1, \quad Q^2(T) = \gamma^2. \end{aligned}$$

It is of interest to mimic the stationary case and introduce

$$\begin{aligned} \rho &= \nu^1Q^1 + \nu^2Q^2 - a, \\ \sigma &= \nu^1Q^1 - \nu^2Q^2. \end{aligned}$$

We obtain the equations

$$\begin{aligned} -2\dot{\rho} + 3\rho^2 + 2a\rho &= 2M + a^2 + \sigma^2, \quad \rho(T) = N - a, \\ -\dot{\sigma} &= \nu^1\alpha^1 - \nu^2\alpha^2 - \sigma(\rho - a), \quad \sigma(T) = \nu^1\gamma^1 - \nu^2\gamma^2, \end{aligned}$$

which reduce to the algebraic equation for ρ in the stationary case. We want to prove the following result

THEOREM 4.1 *There exist a positive solution of equations (2.2)*

We have seen in the stationary case that there may be several solutions. So we do not claim uniqueness.

Proof. It is better to work with $z = \rho + a$. Thus we got the system

$$\begin{aligned} -\dot{z} + \frac{3}{2}z^2 - 2az &= M + \sigma^2, \quad z(T) = N \\ -\dot{\sigma} &= \nu^1\alpha^1 - \nu^2\alpha^2 - \sigma(z - 2a), \quad \sigma(T) = \nu^1\gamma^1 - \nu^2\gamma^2. \end{aligned}$$

Note that we recover Q^1, Q^2 from z, σ , by the formulas

$$Q^1 = \frac{z + \sigma}{2}, \quad Q^2 = \frac{z - \sigma}{2}.$$

We prove a priori estimates. We first interpret z as the Riccati equation of a control problem. Indeed, consider the control problem

$$\dot{x} = ax + \frac{3}{2}v, \quad x(t) = x$$

$$K_{x,t}(v(\cdot)) = \frac{1}{2} \left[\int_t^T \left(\left(M + \frac{\sigma^2(s)}{2} \right) x^2(s) + \frac{3}{2} v^2(s) \right) ds + N x^2(T) \right]$$

then it is easy to check that

$$\frac{1}{2}z^2(t)x^2 = \min_{v(\cdot)} K_{x,t}(v(\cdot)).$$

It follows immediately that $z(t) > 0$. In addition we can write

$$\frac{1}{2}z^2(t)x^2 \leq K_{x,t}(0).$$

Calling

$$\sigma_\infty = \sup_{\{0 \leq t \leq T\}} |\sigma(t)|,$$

we get the following inequality

$$0 \leq z(t) \leq \exp 2aT \left(\frac{M}{2a} + \frac{\sigma^2}{4a} + N \right).$$

Now $\sigma(t)$ is the solution of a linear equation. So we have the explicit formula

$$\begin{aligned} \sigma(t) = (\gamma^1 \nu^1 - \gamma^2 \nu^2) \exp - \int_t^T (z - 2a)(s) ds + (\alpha^1 \nu^1 - \alpha^2 \nu^2) \\ \times \int_t^T \left(\exp - \int_t^s ((z - 2a)(\tau) d\tau \right) ds. \end{aligned}$$

Since $z > 0$, it follows easily that

$$\begin{aligned} \sigma(t) = (\gamma^1 \nu^1 - \gamma^2 \nu^2) \exp - \int_t^T (z - 2a)(s) ds + (\alpha^1 \nu^1 - \alpha^2 \nu^2) \\ \times \int_t^T \left(\exp - \int_t^s ((z - 2a)(\tau) d\tau \right) ds. \end{aligned}$$

We obtain

$$\sigma_\infty \leq \exp 2aT \left(|\gamma^1 \nu^1 - \gamma^2 \nu^2| + \frac{|\alpha^1 \nu^1 - \alpha^2 \nu^2|}{2a} \right).$$

So we have obtained a priori bounds on $\sigma(t), z(t)$. They have been obtained provided $z(t)$ can be interpreted as the infimum of a control problem. Now if we consider a local solution $\sigma(t), z(t)$, near T , i.e. defined in $t \in (T - \epsilon, T]$ for ϵ sufficiently small, we obtain a positive solution for z , since $z(T) > 0$. Moreover the Control interpretation can be easily obtained, and thus the bounds are valid on this small interval. Since the bounds do not depend on ϵ , we can expand the solution beyond $T - \epsilon$, and in fact in $[0, T]$. We deduce Q^1, Q^2 . They are positive near T , and by extension, they are positive. This concludes the proof of Theorem 4.1.

References

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