2

Concepts and Applications of Stochastic Ageing

2.1 Introduction

The concept of ageing is very important in reliability analysis. ‘No ageing’ means that the age of a component has no effect on the distribution of residual lifetime of the component. ‘Positive ageing’ (also known as ‘averse ageing’) describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a component. This situation is common in reliability engineering as components tend to become worse with time due to increased wear and tear. On the other hand, ‘negative ageing’ has an opposite effect on the residual lifetime. ‘Negative ageing’ is also known as ‘beneficial ageing’. Although this is less common, when a system undergoes regular testing and improvement, there are cases for which we have reliability growth phenomenon. Though we concentrate on positive ageing in this book, it is being understood that a parallel development of negative ageing can also be carried out.

Concepts of ageing describe how a component or system improves or deteriorates with age. Many classes of life distributions are categorized or defined in the literature according to their ageing properties. An important aspect of such classifications is that the exponential distribution is nearly always a member of each class. The notion of stochastic ageing plays an important role in any reliability analysis and many test statistics have been developed in the literature for testing exponentiality against different ageing alternatives. Our aim in this chapter is to provide an overview of these developments.

By ‘life distributions’ we mean those for which negative values do not occur, i.e., $F(x) = 0$ for $x < 0$. The nonnegative variate X is thought of as the time to failure (or death) of an electrical or mechanical component (or organism), but other interpretations may be possible – an inter-event time is normally necessarily positive.

In this chapter, we focus on classes of life distributions based on notions of ageing–IFR (increasing failure rate) is perhaps the best-known, but we shall meet several others also, and study their interrelationships whenever possible.

The chapter may serve as a continuation of the ageing concepts developed in the pioneering book Barlow and Proschan (1981), which was first printed by Holt, Reinhart and Winston in 1975.

The major parts of the current chapter are devoted to

- Introducing different ageing characteristics,
- Classifications of life classes based on various ageing characteristics and establishing their interrelationships,
- Failure rates of mixtures of distributions,
- Elementary properties of these life classes,
- Partial orderings of two life distributions based on comparison of their ageing properties.

From the definitions of the life distribution classes, results may be derived concerning such things as properties of systems (based upon properties of components), bounds for survival functions, moment inequalities, and algorithms for use in maintenance policies (Hollander and Proschan, 1984).

Most readers will know that statistical theory applied to distributions of lifetime lengths plays an important part in both the reliability engineering and the biometrics literature. We may also note a third applications area: Heckman and Singer (1986) review econometric work on duration variables (e.g., lengths of periods of unemployment, or time intervals between purchases of a certain good), much of which, they say, has borrowed freely and often uncritically from reliability theory and biostatistics.

Section 2.2 gives characterizations of lifetime distributions by their survival, failure rate or mean residual life functions. In Section 2.3, we list several commonly used life distributions together with their basic properties. In Section 2.4 we give formal definitions of ten basic ageing notions and their interrelationships together with a table of summary furnished with key references. Section 2.5 discusses the properties of some of these basic ageing classes and Section 2.6 is devoted to the non-monotonic failure rate classes such as the bathtub and upside-down bathtub life distributions, which are important in reliability applications. Section 2.7 briefly presents some additional but less known ageing classes. In Section 2.8, we consider failure rates of mixtures of life distributions. This has an important application in burn-in. Section 2.9 provides an introduction to partial ordering through which the strength of the ageing property of the two life distributions within the same class is compared. Section 2.10 considers briefly the matter of relative ageing of two life distributions. Relative ageing is really a form of partial ordering. We discuss in Section 2.11 how the relationship between the sth and the $(s+1)$th equilibrium distribution can be used to describe the relationship between the shape of the failure rate and the shape of mean residual life function of a distribution. Finally in Section 2.12, we tidy up the loose ends on stochastic ageing and the section ends with some remarks concerning future research directions that may bridge the theory and applications.

Abbreviations

The following table of acronyms and abbreviations will be a useful reference. Although this has largely been given in Chapter 1, the list here gives a more exhaustive coverage for ageing concepts.

Table 2.1. List of Ageing Class Abbreviations

Abbreviation	**Ageing Class**
BT (UBT)	Bathtub shape (Upside-down bathtub shape)
DMRL (IMRL)	Decreasing mean residual life (Increasing mean residual life)
HNBUE	Harmonically new better than used in expectation
(HNWUE)	(Harmonically new worse than used in expectation)
IFR (DFR)	Increasing failure rate (Decreasing failure rate)
IFRA (DFRA)	Increasing failure rate average (Decreasing failure rate average)
$\mathcal{L}$-class	Laplace class of distributions
NBU (NWU)	New better than used (New worse than used)
NBUE	New better than used in expectation
(NWUE)	(New worse than used in expectation)
NBUC	New better than used in convex ordering
(NWUC)	(New worse than used in convex ordering)
NBUFR	New better than used in failure rate
(NWUFR)	(New worse than used in failure rate)
NBUFRA	New better than used in failure rate average
(NWUFRA)	New worse than used in failure rate average
NBWUE	New better then worse than used in expectation
(NWBUE)	(New worse then better than used in expectation)

We note that NBUFRA is also known as NBAFR.

2.2 Characterizations of Lifetime Distributions

Rather than $F(t)$, we often think of $\bar{F}(t) = \Pr(X > t) = 1 - F(t)$, which is known as the survival function or reliability function. Here, X denotes the lifetime of a component, i.e., time to first failure. The expected value of X is denoted by μ. The function

$$\bar{F}(x \mid t) = \bar{F}(t+x)/\bar{F}(t), \quad x, t \geq 0, \tag{2.1}$$

represents the survival function of a unit of age t, i.e., the conditional probability that a unit of age t will survive for an additional x units of time. The expected value of the remaining (residual) life, at age t, is $\mu(t) = E(X - t \mid X > t)$ which may be shown to be $\int_0^\infty \bar{F}(x \mid t)\, dx$. It is obvious that $\mu(0) = \mu$.

When $F'(t) = f(t)$ exists, we can define the failure rate (hazard rate or force of mortality) of a component as

$$r(t) = f(t)/\bar{F}(t) \tag{2.2}$$

for t such that $\bar{F}(t) > 0$. This can also be written as

$$r(t) = \lim_{\Delta \to 0} \frac{\Pr(t \leq X < t + \Delta |\ t \leq X)}{\Delta}. \tag{2.3}$$

Thus for small Δ, $r(t)\Delta$ is approximately the probability of a failure occurring in $(t, t + \Delta]$ given no failure has occurred in $(0, t]$.

It follows that, if $r(t)$ exists, then

$$-\log \bar{F}(t) = \int_0^t r(x)\, dx \tag{2.4}$$

represents the cumulative failure (hazard) rate which may be designated by $H(t)$. Equivalently

$$\bar{F}(t) = \exp\left\{-\int_0^t r(x)\, dx\right\} = \exp\left\{-H(t)\right\}. \tag{2.5}$$

A lifetime distribution can also be characterized by its mean residual life (MRL) defined by

$$\mu(t) = E(X - t \,|\, X > t) \tag{2.6}$$

through

$$\bar{F}(t) = \frac{\mu}{\mu(t)} \exp\left\{-\int_0^t \mu(x)^{-1}\, dx\right\},\ t \geq 0. \tag{2.7}$$

We will discuss MRL more fully in Chapter 4.

In short, a lifetime distribution may be characterized by $\bar{F}(t)$, the conditional survival function $\bar{F}(x \,|\, t)$, $r(t)$ or $\mu(t)$. In addition, Galambos and Hagwood (1992) have shown that a life distribution may also be characterized by the second moment of the residual life $E[(X - t)^2 \,|\, X > t]$.

Remarks on terminology

Calling the function $r(t)$ the failure rate in (2.2) could cause some confusion if this terminology is not adequately explained. The confusion aries because another 'failure rate' is also used by some authors in the context of a point process of failures. We now follow the approach of Thompson (1981) to highlight this confusion and attempt to provide a distinction between the two concepts.

Let $N(t)$ denote the number of failures in the interval $(0, t]$. Set $M(t) = EN(t)$ and let $\xi(t) = M'(t)$ and so $\xi(t)$ is the instantaneous rate of change of the expected number of failure with respect to time; thus we may call $\xi(t)$ the failure rate of the process.

Another characteristic of interest in a failure process is

$$\lambda(t) = \lim_{\Delta \to 0} \frac{\Pr[N(t, t + \Delta) \geq 1]}{\Delta}. \tag{2.8}$$

If $\lambda(t)$ exists, then for small Δ, $\lambda(t)\Delta$ is approximately the probability of failure in the interval $(t, t + \Delta]$. Assuming the simultaneous failures do not occur (which is true for most applications), $\xi(t) = \lambda(t)$, if they exist.

Clearly, $r(t)$ is not the same as $\lambda(t)$ since the $r(t)\Delta$ as defined via (2.2) is (approximately) a conditional probability of a failure in $(t, t + \Delta]$ whereas $\lambda(t)\Delta$ is not conditional on the event prior to t.

Under the framework of a stochastic point process, Thompson (1981) discussed basic ways to characterize reliability. The distinction between the failure rate of a process, useful for repairable systems, and the failure rate of a distribution, useful for nonrepairable systems is drawn.

In the point process literature, the failure rate of the process $\xi(t)$ or $\lambda(t)$ is generally known as the intensity function. In reliability modeling, this is sometimes called the 'rate of occurrence of failure (ROCOF)' for repairable systems so that it is not to be confused with the traditional failure rate concept for the lifetime distribution. For further discussion, see Ascher and Feingold (1984).

Note that in the case of a homogeneous Poisson process, the failure rate of the process is λ which is also the failure rate of the the exponential distribution. We wish to emphasize here the 'failure rate' used in this book is the failure rate of a life distribution F defined in (2.2); it is *not* the failure rate of a point process of failures.

One of the reasons for our usage of the acronym 'failure rate' instead of 'hazard rate' in this book is that IHR (DHR) is rarely used in the literature on classification of life distributions. The 'near' universal use of the ageing notions such as IFR (DFR) is consistent with our choice in calling $r(t)$ the failure rate of a life distribution.

2.2.1 Shape of a Failure Rate Function

We assume that the failure rate function $r(t)$ is a real-valued differentiable function $r(t) : R^+ \to R^+$. As usual, by increasing we mean nondecreasing and by decreasing, we mean nonincreasing. $r(t)$ is said to be

(1) strictly increasing if $r'(t) > 0$ for all t and is denoted by I;

(2) strictly decreasing if $r'(t) < 0$ for all t and is denoted by D;

(3) bathtub shaped if $r'(t) < 0$ for $t \in (0, t_0)$, $r'(t_0) = 0$, $r'(t) > 0$ for $t > t_0$, and is denoted by BT;

(4) upside-down bathtub shaped if $r'(t) > 0$ for $t \in (0, t_0)$, $r'(t_0) = 0$, $r'(t) > 0$ for $t > t_0$, and is denoted by UBT;

(5) modified bathtub shaped if $r(t)$ is first increasing and then bathtub shaped, and is denoted by MBT;

(6) roller-coaster shaped if there exist n consecutive change points $0 < t_1 < t_2 < ... < t_n < \infty$ such that in each interval $[t_{j-1}, t_j]$, $1 \leq j \leq n+1$, where $t_0 = 0, t_{n+1} = \infty$, $r(t)$ is strictly monotone and it has opposite monotonicity in any two adjacent such intervals. For detailed description of physical basis for the roller-coaster failure shaped failure rate, see Wong (1988, 1989, 1991).

Remark 1: We wish to point out that the points at which the derivative of the failure rate function $r(t)$ or the mean residual life function $\mu(t)$ changes sign are called the 'change points' in this book. The term 'change point' is used in a different context in the statistical literature.

Remark 2: Some authors include $r(t) =$ constant in the middle interval for their definitions of BT and UBT. We will incorporate this more general and yet more realistic definition in Chapter 3.

Remark 3: A MBT shape may be considered as a curve increases at the beginning and then follows a bathtub shape, see for example, Gupta and Warren (2001). So a MBT curve can be considered as a roller-coaster curve.

Remark 4: The roller-coaster failure rate curve was first promoted by Wong (1989) who observed that the failure rates of many electronic systems have generally decreasing failure rates with failure humps on them. Thus, these failure rate curves manifested a roller-coaster shape. However, we have yet to find any published failure rate data of this shape that can be used for our study. Further, there is no well-known lifetime distribution that we know of which has a failure rate function that exhibits this shape.

It is convenient to extend the above shape definitions to an arbitrary function. To that end, we say that a function $g \in$ I, D, BT, UBT or MBT accordingly as its shape has the appropriate characteristics. For example, $g \in$ BT means that g is first decreasing and then increasing.

Many of the failure rate functions have complex expressions because of the integral in the denominator and thus the determination of the shape is not straightforward. Glaser (1980) presented a method to determine the shape of $r(t)$ with at most one turning point. His method uses the density function instead of the failure rate.

Note: A turning point of a function is a point at which the function has a local maximum or a local minimum.

Define

$$\eta(t) = -\frac{f'(t)}{f(t)}. \tag{2.9}$$

We will see later that this eta function plays an important role in our study of the failure rate function $r(t)$.

The relationships between $r(t)$ and $\eta(t)$ are given by

$$\frac{d}{dt}\log r(t) = r(t) - \eta(t) \tag{2.10}$$

and

$$\left[\frac{1}{r(t)}\right]' = \frac{\eta(t)}{r(t)} - 1. \tag{2.11}$$

Here we obviously assume that $f(t)$ is a twice differentiable positive density function on $(0, \infty)$.

The above equations also suggest that the turning point of $r(t)$ is a solution of the equation $\eta(t) = r(t)$. We can also verify that $\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \eta(t)$.

Theorem 2.1: (Glaser, 1980). Let $\eta(t)$ be defined as in (2.9).

(a) If $\eta(t) \in$ I, then $r(t)$ is of type I.

(b) If $\eta(t) \in$ D, then $r(t)$ is of type D.

(c) If $\eta(t) \in$ BT and (i) if there exists a y_0 such that $r'(y_0) = 0$, then $r(t)$ is of type BT and (ii) otherwise $r(t)$ is of type I.

(d) If $\eta(t) \in$ UBT and (i) if there exists a y_0 such that $r'(y_0) = 0$, then $r(t)$ is UBT and (ii) otherwise $r(t)$ is of type D.

Proof: Define the reciprocal of the failure rate by

$$g(t) = 1/r(t) = R(t)/f(t). \tag{2.12}$$

It follows that its derivative given in (2.11) may be written as

$$g'(t) = g(t)\eta(t) - 1 \tag{2.13}$$

where $\eta(t)$ is defined as above. Without going into detail, it can be shown that

$$g'(t) = \int_t^\infty [f(y)/f(t)][\eta(t) - \eta(y)]\, dy. \tag{2.14}$$

(It has been pointed out that the preceding equation implicitly requires that $f'(t)$ be integrable at infinity).

We can now proceed to prove the theorem.

(a) The assumption that $\eta'(t) > 0$ for all $t > 0$ implies, from (2.14), that $g'(t) < 0$ for all $t > 0$, which from (2.12), implies $r(t) \in$ I.

(b) $\eta'(t) < 0 \Rightarrow g'(t) > 0$ for all $t > 0$ so $r(t) \in$ D.

(c)(i) Let t_0 be the change point of η so that $\eta'(t_0)$=0. Claim $g''(y_0) < 0$. Since $g'(y_0) = 0$, it follows from (2.13) that $g''(y_0) = g(y_0)\eta'(y_0)$. Therefore, $g''(y_0) < 0 \Leftrightarrow \eta'(y_0) < 0 \Leftrightarrow y_0 < t_0$. Suppose $y_0 \geq t_0$. By (2.14) and the assumption, it is obvious that $g'(t) < 0$ for all $t > t_0$. Therefore,

$g'(y_0) < 0$, which is a contradiction. Thus $y_0 < t_0$ and $g''(y_0) < 0$. It is now obvious that there is a unique root in $(0, \infty)$ to $g(y) = 0$, i.e., $y = y_0$, and g attains a maximum at this point. This implies $r(t) \in$ BT with the turning point $t^* = y_0$.

(c)(ii) Here we have either $g'(t) > 0$ for all $t > 0$ or $g'(t) < 0$ for all $t > 0$. From (2.14) we have that $g'(t) < 0$ for all $t \geq t_0$. Therefore $g'(t) < 0$ for all $t > 0$ so $r(t) \in$ I.

(d) The proof is analogous to that of (c) and will be omitted here.

It is noted in Glaser (1980) that in the last two cases, determining the existence of y_0 leaves us with the original difficulty of evaluating the derivative of $r(t)$. However, we may simplify the problem in many situations with the following lemma.

Lemma 2.1: Let $\varepsilon = \lim_{t \to 0} f(t)$ and $\delta = \lim_{t \to 0} g(t)\eta(t)$, where $g(t) = 1/r(t)$.

- Suppose $\eta \in$ BT, then
  (a) if either $\varepsilon = 0$ or $\delta < 1$, then $r(t) \in$ I.
  (b) if either $\varepsilon = \infty$ or $\delta > 1$, then $r(t) \in$ BT.
- Suppose $\eta \in$ UBT, then
  (a) if either $\varepsilon = 0$ or $\delta < 1$, then $r(t) \in$ UBT.
  (b) if either $\varepsilon = \infty$ or $\delta > 1$, then $r(t) \in$ D.

Gupta (2001) used Glaser's theorem to determine the shapes of several lifetime distributions that include the lognormal, inverse Gaussian, mixture of inverse Gaussians, power quadratic exponential families, mixture of gammas, etc.

In the proof of Theorem 2.1 above, Glaser showed that the change point of $r(t)$ occurs before the change point of $\eta(t)$. This finding has an important impact on the relationship between the shape $\mu(t)$ and that of $r(t)$. We will follow up this matter in Section 4.5.

Extension of Glaser's Result

Gupta and Warren (2001) generalized the result of Glaser to the case where $r(t)$ has two or more turning points. To achieve this, they first gave the following theorem which relates the turning points of $r(t)$ with those of $\eta(t)$.

Theorem 2.2: Let $\eta(t)$ defined as on (2.9), i.e., $\eta(t) = -\frac{f'(t)}{f(t)}$ and $f(t)$ is a twice differentiable positive density on $(0, \infty)$. If $\eta'(t)$ has zeros at $z_1, z_2, ..., z_n$ (n finite) such that $z_1 < z_2 <, ..., < z_n$, then the the equation $r'(t) = 0$ has at most one solution on $[z_{k-1}, z_k]$ for $k = 1, ..., n$ with $z_0 = 0$. Thus $r(t)$ has at most n changes of monotonicity.

Proof: It follows from (2.14) that

$$g'(t)f(t) = \int_t^\infty f(y)[\eta(t) - \eta(y)]\, dy.$$

Since $f(t) > 0$ for all $t > 0$, the sign and zeros of g', and therefore of r', are completely determined by the integral of the right side of the equation. We next designate this integral by

$$s(t) = \int_t^{\infty} f(y)[\eta(t) - \eta(y)]\, dy. \tag{2.15}$$

We note that the zeros of s are precisely the critical (change) points of r. It can be verified that $s'(t) = \eta'(t)\bar{F}(t)$ so both the sign and zeros of s' and η' are the same. That is, both have identical monotonicity.

By the given assumption, η is monotonic on $[z_{k-1}, z_k]$. Since s and η have identical monotonicity, s is monotonic on each interval $[z_{k-1}, z_k]$, such that the expression $s(t) = 0$ has at most one solution on that interval. Using the fact that the zeros and sign of r' are determined by s, we conclude that $r'(t) = 0$ has at most one solution on $[z_{k-1}, z_k]$. Thus the proof is completed.

The following theorem of Gupta and Warren (2001) is a generalization of Glaser (1980) and is useful when the shape of η is known and the number of critical points of r is known.

Theorem 2.3:

1. Suppose $\eta \in$ UBT. Then
   (a) If r' has no zeros, then $r(t) \in$ I.
   (b) If r' has one zero, then $r(t)$ is strictly increasing except at one point or $r(t) \in$ B.
   (c) If r' has two zeros, then $r(t) \in$ BT.
2. Suppose η is bathtub then upside-down bathtub. Then
   (a) If r' has no zeros, then $r(t) \in$ I.
   (b) If r' has one zero, then $r(t)$ is strictly decreasing except at one point or $r(t) \in$ UBT.
   (c) If r' has two zeros, then $r(t)$ is bathtub then upside-down bathtub.

Proof: See Gupta and Warren (2001).

The Glaser's extension will be applicable when considering the gamma mixtures with common scale parameter in Section 2.8.

2.3 Ageing Distributions

There are many lifetime distributions that have been proposed. Below are a selected few that have appeared more frequently in the literature. We do not think a comprehensive study of these distributions is warranted in the current text as most of them can be found in Johnson et al. (1994, 1995). Thus, only basic properties that are related to reliability are briefly given below. Several of these distributions will be further studied within an appropriate context throughout Chapters 3–5.

2.3.1 Exponential

The exponential (or negative exponential) distribution is applied in a very wide variety of statistical procedures. Currently among the most prominent applications are in the field of life-testing. The density function is

$$f(t) = \lambda e^{-\lambda t},\ \lambda > 0, t \geq 0 \tag{2.16}$$

and

$$\bar{F}(x \mid t) = \bar{F}(x),\ \text{for all } x, t \geq 0$$

which means that its survival probability over an additional period of duration x is the same regardless of its present age. It describes a component that does not age with time. In case where such a simple structure is not adequate, a modification of the exponential distribution (often a Weibull distribution) is then used.

Also, it has a constant failure rate, i.e., $r(t) = \lambda,$ for all $t \geq 0$.

The exponential distribution is a special case of the gamma, Weibull, Gompertz, linear failure rate and the exponential-geometric distributions to be presented below. It is a common member of nearly every known ageing class. It plays an important role in tests of stochastic ageing which will be discussed in Chapter 7. Lastly, we note that

$$E(X) = \frac{1}{\lambda}, \quad \mathrm{var}(X) = \frac{\alpha}{\lambda^2}.$$

In general, the rth moment about zero is

$$\mu_r' = \frac{\Gamma(r+1)}{\lambda^r}.$$

2.3.2 Gamma

The density function of a standard two-parameter gamma distribution is

$$f(t) = \frac{\lambda^\alpha t^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda t},\ \alpha, \lambda > 0. \tag{2.17}$$

If $\alpha = 1$, (2.17) reduces to an exponential distribution discussed above. In fact, the gamma distribution can be constructed from the exponential by taking powers of the Laplace transform of the latter. If α is a positive integer, we have an Erlang distribution. Moreover, if $\alpha = \nu/2$, we obtain a chi-square distribution with ν degrees of freedom.

The gamma distribution appears naturally in the theory associated with normally distributed random variables as the distribution of the sum of squares of independent standard normal variables.

For general α, the distribution function does not have a closed form. However, when α is a positive integer, $F(t)$ may be written in a closed form as

$$F(t) = 1 - \sum_{i=0}^{\alpha-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}, \text{ for } t \geq 0. \tag{2.18}$$

The rth moment about zero of the gamma distribution is

$$\mu_r' = \frac{\Gamma(\alpha + r)}{\lambda^r \Gamma(\alpha)}, \; r = 1, 2, \ldots.$$

In particular,

$$E(X) = \frac{\alpha}{\lambda}, \quad \text{var}(X) = \frac{\alpha}{\lambda^2}.$$

It can be shown, by a change of variable, that

$$r(t)^{-1} = \int_0^\infty \left(1 + \frac{u}{t}\right)^{\alpha-1} e^{-\lambda u} du.$$

It follows that $r(t)^{-1}$ is increasing for $0 < \alpha \leq 1$, decreasing for $\alpha \geq 1$. Thus $r(t)$ is increasing for $\alpha \geq 1$ and decreasing for $0 < \alpha \leq 1$. The shape of $r(t)$ can be determined through Glaser's eta function easily since

$$\eta(t) = -\frac{f'(t)}{f(t)} = \lambda - \frac{\alpha - 1}{t}. \tag{2.19}$$

Here η is increasing for $\alpha > 1$, constant for $\alpha = 1$, and decreasing for $0 < \alpha < 1$ and thus the shape of $r(t)$ is confirmed as stated above.

We refer the reader to Johnson et al. (1994, Chapter 17) for other facets of this well known lifetime distribution.

Mixtures of gamma distributions will be considered in Section 2.8.2.

2.3.3 Truncated Normal

The density function of a (positively) truncated normal is given by

$$f(t) = \frac{1}{a\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}, \text{ for } 0 \leq t < \infty, \tag{2.20}$$

where $\sigma > 0$, $-\infty < \mu < \infty$, $a = \int_0^\infty (1/\sigma\sqrt{2\pi}) e^{-(t-\mu)^2/2\sigma^2} \, dt$.

The mean is

$$E(X) = \mu + \frac{\sigma\phi\left(\frac{-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{-\mu}{\sigma}\right)} = \mu + \frac{\sigma\phi\left(\frac{-\mu}{\sigma}\right)}{a}$$

where $\phi(\cdot), \Phi(\cdot)$ are, respectively, the density and distribution function of the standard normal random variable. Here μ is the mean of the normal distribution. Clearly, $E(X) > \mu$ and $\text{var}(X) < \sigma^2$. If $\mu - 3\sigma \gg 0$, then a is close to 1 and $E(X) \cong \mu$ and $\text{var}(X) \cong \sigma^2$.

Davis (1952), after examining failure data for a wide variety of items, has shown empirically that items manufactured and tested under close control may be fitted with truncated normal life distributions of the form (2.20).

We note that

$$\log f(t) = -\log(a\sigma\sqrt{2\pi}) - \frac{(t-\mu)^2}{2\sigma^2},\ t \geq 0 \tag{2.21}$$

is a concave function on $[0, \infty)$ and thus F is IFR (Barlow and Proschan, 1981, p. 77).

Though the expression for $r(t)$ is complicated, Navarro and Hernandez (2004) noted that the following:

1. $r'(t) = (r(t) - (t-\mu)/\sigma^2)r(t)$,
2. $r(t) > (t-\mu)/\sigma^2$,
3. $r(t)$ increases to ∞ as $t \to \infty$,
4. $\lim\limits_{t\to\infty} r'(t) = 1/\sigma^2$,

and other properties.

The distribution here is a singly truncated normal from below. We note that various other types of normal truncations have been investigated (see, e.g., Johnson et al. (1994, pp. 156-162).

2.3.4 Weibull

The Weibull distribution is named after the Swedish physicist Waloddi Weibull, who in 1939 used it to represent the distribution of the breaking strength of materials and in 1951 for a variety of other applications. It is perhaps the most frequently used lifetime model in the reliability literature. Hallinan (1993) gave a comprehensive review of its properties and applications. Chapter 21 of Johnson et al. (1994) is devoted to this distribution. A recent monograph by Murthy et al. (2003) gives nearly every facet regarding Weibull and its related distributions. The survival function of the two-parameter Weibull is

$$\bar{F}(t) = \exp\{-(\lambda t)^\alpha\},\ \alpha, \lambda > 0. \tag{2.22}$$

When $\alpha = 1$, the Weibull distribution reduces to an exponential distribution. In fact, if X has an exponential distribution with parameter 1, then $X^{1/\alpha}/\lambda$ has the survival function (2.22).

The rth moment about the zero of the Weibull distribution is

$$\mu'_r = \frac{\Gamma\left(\frac{r}{\alpha}+1\right)}{\lambda^r}.$$

In particular, the mean and variance are, respectively,

$$E(X) = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha}+1\right),$$

$$\mathrm{var}(X) = \frac{1}{\lambda^2}\left\{\Gamma\left(\frac{2}{\alpha}+1\right) - \Gamma\left(\frac{1}{\alpha}+1\right)^2\right\}.$$

An important characteristic of the Weibull distribution is that its failure rate $r(t)$ has a simple form:

$$r(t) = \alpha\lambda(\lambda t)^{\alpha-1}. \tag{2.23}$$

It follows that $r(t)$ is increasing in t for $\alpha \geq 1$ and decreasing for $\alpha \leq 1$.

Mixtures of the Weibull distribution are considered in Section 2.8 below. A more detailed study on the Weibull and its related distributions will be given in Chapter 5.

2.3.5 Lognormal

The lognormal distribution is sometimes called the antilognormal distribution. This alternative name has some logical basis in that it is not the distribution of the logarithm of a normal variable (this is not even always real) but of an exponential (that is, antilogarithm) function of such a variable. In other words, if $\log X$ has a normal distribution, then X is said to have a lognormal distribution. However, 'lognormal' is most commonly used and we will follow this practice.

The cdf of the lognormal distribution is given by

$$F(t) = \Phi\left\{\frac{\log t - \alpha}{\sigma}\right\},\ \sigma > 0, t \geq 0, \tag{2.24}$$

where $\Phi(\cdot)$ denotes the standardized normal distribution function. The density function is

$$f(t) = (t\sqrt{2\pi}\sigma)^{-1}\exp[-(\log t - \alpha)^2/2\sigma^2], \quad t \geq 0. \tag{2.25}$$

The rth moment of X about the origin is

$$\mu_r' = \exp\left(r\alpha + \frac{1}{2}r^2\sigma^2\right).$$

The failure rate function of the lognormal has been shown as

$$r(t) = \frac{(1/\sqrt{2\pi}t\sigma)\exp\left\{-(\log at)^2/2\sigma^2\right\}}{1 - \Phi\left\{\log(at)/\sigma\right\}}, \tag{2.26}$$

where $a = e^{-\alpha}$.

Although the expression of $r(t)$ is quite complicated, $\eta(t)$ is however quite simple, namely,

$$\eta(t) = -\frac{f'(t)}{f(t)} = \frac{1}{\sigma^2 t}(\sigma^2 + \log t - \alpha). \tag{2.27}$$

An application of Glaser's theorem shows that $r(t)$ is UBT. Also, $\lim_{t\to 0} r(t) = 0$ and $\lim_{t\to\infty} r(t) = 0$ (Sweet, 1990). For estimation of the change point, see Gupta et al. (1997).

Chapter 14 of Johnson et al. (1994) gives a full discussion on this distribution. Its failure rate and mean residual life will be discussed further in Chapter 3 and Chapter 4, respectively.

2.3.6 Birnbaum-Saunders

Birnbaum and Saunders (1969a,b) introduced a lifetime distribution

$$F(t) = \Phi\left\{\frac{1}{\alpha}\cdot\left[\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{t}{\beta}\right)^{-1/2}\right]\right\} = \Phi\left\{\frac{1}{\alpha}\xi\left(\frac{t}{\beta}\right)\right\},\ t > 0, \tag{2.28}$$

where $\xi(t) = t^{1/2} - t^{-1/2}$, $\alpha, \beta > 0$ and $\Phi(\cdot)$ denotes the cdf of the standard normal. The density function is given by

$$f(t) = (\alpha\beta)^{-1}(2\pi)^{-1/2}\xi'\left(\frac{t}{\beta}\right)\exp\left\{-\frac{1}{2\alpha^2}\xi^2\left(\frac{t}{\beta}\right)\right\},\ t > 0. \tag{2.29}$$

Desmond (1986) noted that in this distributional form, derived by Birnbaum-Saunders (1969a,b), had been previously obtained by Freudenthal and Shinozuka (1961) with a somewhat different parametrization.

The random variable X that corresponds to (2.28) is a simple transformation of the the standard normal variable

$$X = \beta\left[\tfrac{1}{2}U\alpha + \sqrt{(\tfrac{1}{2}U\alpha^2) + 1}\right]^2.$$

The above variable arises from a model representing the time to failure of material subject to a cyclically repeated stress pattern.

It can be shown that X has a Birnbaum-Saunders distribution if

$$\frac{1}{\alpha}\left(\sqrt{\frac{X}{\beta}} - \sqrt{\frac{\beta}{X}}\right) \tag{2.30}$$

has a standard normal. From this expression, Chang and Tang (1994a,b) proposed a simple random variate generating algorithm for this distribution.

Surprisingly, the mean and variance of X are quite simple. These are given, respectively, by

$$E(X) = \beta\left(\frac{1}{2}\alpha^2 + 1\right), \quad \mathrm{var}(X) = \beta^2\alpha^2\left(\frac{5}{4}\alpha^2 + 1\right).$$

The failure rate function $r(t)$ cannot be given explicitly. Since both the lognormal distribution and the Birnbaum-Saunders distribution can be derived from the normal distribution, we expect a similarity between the two in this respect. Indeed, a comparison between the failure rates of the Birnbaum and Saunders and the lognormal distribution was given in Nelson (1990). While the failure rate of Birnbaum and Saunders is zero at $t = 0$, then increases to a maximum for some t_0 and finally decreases to a finite positive value (i.e., $r(t) \in$ UBT) when $\beta = 1$ and $\alpha > 0.8$, the failure rate of the lognormal also has a UBT shape but decreases to zero. It was shown in Chang and Tang (1993) that $r(t) \in$ I when $\alpha \to 0$. Some recent work on this distribution can be found in Dupuis and Mills (1998), Rieck (1999) and Ng et al. (2003). The last discussed the maximum likelihood estimates and a modification of the moment estimates of the two parameters, and proposed a bias-correction method for these estimates. See Chapter 33 of Johnson et al. (1995) for a more detailed discussion on the properties of this distribution.

2.3.7 Inverse Gaussian

The name ‘inverse Gaussian’ was first applied to a certain class of distributions by Tweedie (1947), who noted the inverse relationship between the cumulant generating functions of these distributions and those of Gaussian (normal) distributions. The same class of distributions was derived by Wald (1947) as an asymptotic form of average sample number in sequential analysis and hence the distribution is also known as the Wald distribution. The inverse Gaussian distribution was popularized as a lifetime model by Chhikara and Folks (1977).

The density function of the inverse Gaussian is

$$f(t) = \sqrt{\frac{\lambda}{2\pi t^3}} \cdot \exp\left[-\frac{\lambda}{2\mu^2 t}(t-\mu)^2\right], \; \lambda > 0, t \geq 0. \tag{2.31}$$

The corresponding distribution function is

$$F(t) = \Phi\left\{\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} - 1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} + 1\right)\right\}. \tag{2.32}$$

The mean and variance of the distribution are, respectively,

$$E(X) = \mu, \quad \mathrm{var}(X) = \frac{\mu^3}{\lambda}.$$

Again, the expression for $r(t)$ is quite complicated. However, one can verify easily that

$$\eta(t) = \frac{3\mu^2 t + \lambda(t^2 - \mu^2)}{2\mu^2 t^2}. \tag{2.33}$$

It follows from Theorem 2.1 that that $r(t)$ is UBT. Further, $\lim_{t\to 0} r(t) = 0$ and $\lim_{t\to\infty} r(t) = c \neq 0$.

For further properties see Chhikara and Folks (1989) and Chapter 15 of Johnson et al. (1994).

2.3.8 Gompertz

Gompertz (1825) derived possibly the earliest probability model for human mortality. He postulated that "the average exhaustion of a man's power to avoid death to be such that at the end of equal infinitely small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of these intervals." From this hypothesis Gompertz deduced the force of mortality or the failure rate function as

$$r(t) = Bc^t,\ t \geq 0, B > 0, c \geq 0, \tag{2.34}$$

which, when solved as a differential equation, yields the survival function as

$$\bar{F}(t) = e^{-B(c^t-1)/\log c},\ t \geq 0. \tag{2.35}$$

The density function is easily obtained as

$$f(t) = Bc^t e^{-B(c^t-1)/\log c},\ t \geq 0, B > 0, c \geq 0. \tag{2.36}$$

It is clear that $r(t)$ increases (decreases) in t if $c > 1$ ($c < 1$). For $c = 1$, $r(t) = B$ showing that the Gompertz distribution includes the exponential as its special case.

In discussing reliability theory of ageing and longevity, Gavrilov and Gavrilova (2001) stated that while the Weibull distribution is more commonly applicable for failure times of technical devices, the Gompertz distribution is more common for biological systems.

2.3.9 Makeham

The survival function of the Makeham distribution is

$$\bar{F}(t) = \exp[-\alpha t + (\beta/\lambda)(e^{\lambda t} - 1)],\ t \geq 0, \alpha, \beta, \lambda > 0, \tag{2.37}$$

and its failure rate function is

$$r(t) = \alpha + \beta e^{\lambda t}. \tag{2.38}$$

It is clear that $r(t) \in$ I.

In the literature, the Makeham distribution is more often called the Gompertz-Makeham distribution. It is a generalization of the Gompertz distribution. Letting $c = e$ in (2.35), we clearly obtain a special case of (2.37). This distribution is widely used in life insurance, mortality studies and survival analysis in general. For a brief review, see Al-Hussaini et al. (2000).

2.3.10 Linear Failure Rate

The survival function of the linear failure rate function is given by

$$\bar{F}(t) = \exp\{-\lambda_1 t - \lambda_2 t^2/2\},\ \lambda_1, \lambda_2, t \geq 0 \tag{2.39}$$

with

$$r(t) = \lambda_1 + \lambda_2 t. \tag{2.40}$$

The linear failure rate distribution aries often in reliability literature probably because of its simple form.

This simple two-parameter model in the IFR class is a simple special case of the quadratic failure rate model (see Section 3.4.1) and a generalization of the exponential distribution in a direction distinct from the gamma and Weibull discussed in this section. While, in the IFR case, both gamma and Weibull require the failure rate to be zero at $t = 0$, the linear failure rate model has $r(0) = \lambda_1 > 0$, thus providing a gentler transition from the constant failure rate to the strict IFR property.

The linear failure rate distribution was motivated by its application to human survival data (Kodlin, 1967, Carbone et al., 1967). Its properties have been studied by several authors, notably Bain (1974) and Sen and Bhattacharyya (1995).

A basic structural property of the linear failure rate distribution of the minimum of two independent variables X_1 and X_2 having exponential (λ_1) and Rayleigh (λ_2) distributions whose survival functions are given above. This series structure provides a physical motivation in the framework of competing risks.

We will study the mixture of two linear failure rate distributions in Section 2.8.4.

2.3.11 Lomax Distribution

The Lomax distribution is also known as the Pareto of the second kind. The distribution may arise as a mixture distribution. Suppose X has an exponential distribution with parameter λ having density function

$$g(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta},\ x \geq 0,$$

then the resulting unconditional survival function of X is given by

$$\bar{F}(t) = (1 + \beta t)^{-\alpha}. \tag{2.41}$$

The λ here may be considered as the operating environment of a component which varies according to a gamma distribution.

We can easily verify that

$$E(X) = \frac{1}{\beta(\alpha - 1)}, \ \alpha > 1$$

and

$$\mathrm{var}(X) = \frac{\alpha}{(\alpha - 1)^2(\alpha - 2)}, \ \alpha > 2.$$

The failure rate function is given by

$$r(t) = \frac{(\alpha + 1)\beta}{1 + \beta t}. \tag{2.42}$$

It is easy to see that $r(t) \in \mathrm{D}$.

The special case $\beta = 1$ corresponds to the Burr XII distribution with $c = 1, k = \alpha$ (see below).

2.3.12 Log-logistic

The probability density function and the survival function are, respectively, given by

$$f(t) = \frac{k\rho(\rho t)^{k-1}}{[1 + (\rho t)^k]^2}, \ t > 0, \rho > 0, k > 0, \tag{2.43}$$

$$\bar{F}(t) = \frac{1}{1 + (\rho t)^k}. \tag{2.44}$$

The rth moment about zero of the log-logistic distribution is

$$\mu'_r = \frac{1}{k\rho} B\left(\frac{r}{k}, 1 - \frac{r}{k}\right).$$

It is easy to verify that the failure rate function is

$$r(t) = \frac{k\rho(\rho t)^{k-1}}{1 + (\rho t)^k}. \tag{2.45}$$

It can be shown easily that $r(t) \in \mathrm{D}$ when $k \le 1$; $r(t) \in \mathrm{UBT}$ when $k > 1$. The turning point of the failure rate function is given by

$$t^* = \frac{(k - 1)^{1/k}}{\rho}.$$

The log-logistic distribution has proved to be quite useful in analyzing survival data, see, e.g., Cox (1970), Cox and Oakes (1984), Bennett (1983), O'Quigley and Struthers (1982), and Gupta, Akman and Lvin (1999). Note that when the scale parameter $\rho = 1$, the log-logistic is also a special case of Burr XII below.

2.3.13 Burr XII

The Burr XII distribution was first introduced by Burr (1942). It includes the exponential, Weibull, and log-logistic distributions for particular limiting values of the parameters. Rodriguez (1977) and Tadikamalla (1980) explored in great detail the connection between the Burr XII distributions and other continuous distributions. Zimmer et al. (1998) and Ghitany and Al-Awadhi (2002) have discussed properties and reliability applications of Burr XII distribution having reliability function

$$\bar{F}(t) = \frac{1}{(1+t^c)^k}, \ k, c > 0, \ t > 0. \tag{2.46}$$

The corresponding density function is

$$f(t) = \frac{kct^{c-1}}{(1+t^c)^{k+1}}.$$

For $c = 1$, it becomes the Pareto distribution of the second kind (the Lomax).

The rth moment about zero of the Burr XII distribution is given by

$$\mu_r' = k\left[B\left(k - \frac{r}{c}, 1 + \frac{r}{c}\right)\right], \ k > \frac{r}{c},$$

where $B(p,q)$ is the beta function defined by $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$; so

$$\mu = k\left[B\left(k - \frac{1}{c}, 1 + \frac{1}{c}\right)\right]$$

and

$$\operatorname{var}(X) = k\left[B\left(k - \frac{2}{c}, 1 + \frac{2}{c}\right)\right] - \mu^2.$$

It is easy to verify that

$$r(t) = \frac{kct^{c-1}}{(1+t^c)}, \tag{2.47}$$

and

$$r'(t) = \frac{ct^{t-2}(c-1-t^c)}{(1+t^c)^2}. \tag{2.48}$$

For $c \leq 1$, the slope is always negative, for $c > 1$ the slope is positive for $t^c < c-1$ and negative for $t^c > c-1$. Thus $r(t)$ is D for $c \leq 1$ and UBT if $c > 2$. The maximum failure rate occurs at $t = (c-1)^{1/c}$.

Zimmer et al. (1998) have also shown that the Burr XII can approximate several useful reliability distributions (a fact that we have noted above). Watkins (1999) gave an algorithm for calculating the maximum likelihood estimates of the three-parameter Burr XII distribution. The algorithm exploits the link between this distribution and the two-parameter Weibull distribution, which merges as the limiting case of the former.

2.3.14 Exponential-geometric (EG) and Generalization

The exponential-geometric distribution is a special case of Marshall and Olkin's (1997) family of exponential distributions. The current name was apparently coined by Adamidis and Loukas (1998) who along with Marshall and Olkin (1997) studied its properties in detail.

The distribution may be obtained by compounding (mixing) an exponential distribution with a geometric distribution. The density function is

$$f(t) = \lambda(1-p)e^{-\lambda t}(1-pe^{-\lambda t})^{-2}, \ \lambda > 0, 1 < p < 1. \tag{2.49}$$

Here, λ is the scale parameter of an exponential distribution whereas p is the proportion parameter of the geometric distribution. The reliability function is given by

$$\bar{F}(t) = (1-p)e^{-\lambda t}(1-pe^{-\lambda t})^{-1}, \, t > 0 \tag{2.50}$$

and thus the failure rate function for the EG distribution is

$$r(t) = \lambda(1-pe^{-\lambda t})^{-1}. \tag{2.51}$$

It is easy to see that the above failure rate function is decreasing in t although the DFR property also follows from the results of Proschan (1963) on mixture. The initial failure rate $r(0) = \beta(1-p)^{-1}$ and the long-term failure rate $r(\infty) = \lambda$ which are both finite. In contrast, the failure rate of the Weibull distribution has $r(0) = \infty$ and $r(\infty) = 0$ when the shape parameter $\alpha < 1$. So the EG distribution could be an attractive alternative to the Weibull in the case when the long-term failure rate is finite.

The rth moment about zero is given by

$$\mu'_r = (1-p)r!(\lambda^r p)^{-1}L(p;r),$$

where $L(p;r) = \sum_{j=1}^{\infty} p^j j^{-r}$ is the polylogarithmic function which can be evaluated easily.

Adamidis and Loukas (1998) have considered the maximum likelihood estimates of the parameters p and λ and they gave an EM algorithm for the computation of these estimates.

As mentioned above, the EG distribution is a special case of the Marshall and Olkin (1997) family of distributions obtained by adding a parameter to the original survival function $\bar{G}(t)$ such that

$$\bar{F}(t) = \frac{\beta\bar{G}(t)}{1-(1-\beta)\bar{G}(t)}, \ -\infty < t < \infty, 0 < \beta < \infty. \tag{2.52}$$

For our purpose, we consider only lifetime random variables so $t > 0$.

The case $\bar{G}(t) = \exp(-\lambda t)$ was studied in detail, in particular, it was shown that

$$E(X) = -\frac{\beta \log \beta}{\lambda(1-\beta)},$$

and

$$\text{mode}(X) = \begin{cases} 0, & \beta \leq 2 : \\ \lambda^{-1}, & \beta \geq 2. \end{cases}$$

The failure rate function is

$$r(t) = \lambda(1 - (1-\beta)e^{-\lambda t})^{-1}$$

which is decreasing in t for $0 < \beta < 1$ and increasing in t for $\beta > 1$.

For $\beta = 1 - p < 1$, it reduces to the EG distribution discussed above. If $\beta = 1$, it becomes the exponential distribution. In view of its flexility, it is conceivable that this 2-parameter family of distributions may sometimes be a competitor to the Weibull and gamma families.

The case where $\bar{G}(t) = \exp\{-(\lambda t)^\alpha\}$ will be considered in Section 5.5 as an extended Weibull distribution.

2.4 Basic Concepts for Univariate Reliability Classes

2.4.1 Some Acronyms and Notions of Aging

The concepts of increasing and decreasing failure rates for univariate distributions have been found very useful in reliability theory. The classes of distributions having these ageing properties are designated as the IFR and DFR distributions, respectively, and have been extensively studied. Other classes such as 'increasing failure rate on average' (IFRA), 'new better than used' (NBU), 'new better than used in expectation' (NBUE), and 'decreasing mean residual life' (DMRL) have also been of much interest. For fuller accounts of these classes see, e.g., Bryson and Siddiqui (1969), Barlow and Proschan (1981), and Hollander and Proschan (1984).

A class that slides between NBU and NBUE, known as 'new better than used in convex ordering' (NBUC), has also attracted some interest recently.

The notion of 'harmonically new better than used in expectation' (HNBUE) was introduced by Rolski (1975) and studied by Klefsjö (1981, 1982). Further generalizations along this line were given by Basu and Ebrahimi (1984a). A class of distributions denoted by $\mathcal{L}$ has an ageing property that is based on the Laplace transform, and was put forward by Klefsjö (1983c). Deshpande et al. (1986) used stochastic dominance comparisons to describe positive ageing and suggested several new positive ageing criteria based on these ideas – see their paper and Section 2.7 for details. Two further classes, NBUFR ('new better than used in failure rate') and NBUFRA ('new better than used in failure rate average') require the absolute continuity of the distribution function, and have been discussed in Loh (1984a,b), Deshpande et al. (1986), Kochar and Wiens (1987), and Abouammoh and Ahmed (1988).

We are now ready to give formal definitions of ten basic reliability classes. Some of the ageing classes in this group are by no means the most important ones ones in terms of their applications. Some members are selected for historical reasons.

2.4.2 Definitions of Reliability Classes

Most of the reliability classes are defined in terms of the failure rate $r(t)$, conditional survival function $\bar{F}(x \mid t) = \frac{\bar{F}(x+t)}{\bar{F}(x)}$, or the mean residual life $\mu(t)$. All these three functions provide probabilistic information on the residual lifetime and hence ageing classes may be formed according to the behavior of the ageing effect on a component.

The ten reliability classes mentioned above are defined as follows.

Definition 2.1: F is said to be IFR if $\bar{F}(x \mid t)$ is decreasing in $0 \leq t < \infty$ for each $x \geq 0$. It is a decreasing failure rate (DFR) distribution if $\bar{F}(x \mid t)$ is increasing in t. F is IFR (DFR) iff $-\log \bar{F}(t)$ is convex (concave). When the density exists, IFR (DFR) is equivalent to $r(t) = f(t)/\bar{F}(t)$ being increasing (decreasing) in $t \geq 0$ (Barlow and Proschan, 1981, p. 54).

Definition 2.2: F is said to be IFRA if $-(1/t) \log \bar{F}(t)$ is increasing in $t \geq 0$. This is equivalent to $\bar{F}(\alpha t) \geq \bar{F}^{\alpha}(t)$, $0 < \alpha < 0$, $t \geq 0$ (Barlow and Proschan, 1981, p. 84). (The latter is equivalent to $-\log \bar{F}(t)$ being a star-shaped function; i.e., $-\log \bar{F}(\alpha t) \leq -\alpha \log \bar{F}(t)$. For more information about this notion, see Dykstra, 1985.) It is also equivalent to $\int_0^t r(t)\, dx/t$ increasing in $t \geq 0$, because of the fact that $-\log \bar{F}(t) = H(t) = \int_0^t r(x)\, dx$. It is a decreasing failure rate in average (DFRA) distribution if $-(1/t) \log \bar{F}(t)$ is decreasing in $t \geq 0$ or $\bar{F}(\alpha t) \leq \bar{F}^{\alpha}(t)$ for all $0 < \alpha < 1$.

Definition 2.3: F is said to be DMRL if the mean remaining life function $\mu(t) = \int_0^\infty \bar{F}(x \mid t)\, dx$ is decreasing in t, i.e., $\mu(s) \geq \mu(t)$ for $0 \leq s \leq t$. In other words, the older the device is, the smaller is its mean residual life (Bryson and Siddiqui, 1969). Similarly, F is said to be IMRL if $\mu(s) \leq \mu(t)$ for $0 \leq s \leq t$.

Definition 2.4: F is said to be new better than use (NBU) if $\bar{F}(x \mid t) \leq \bar{F}(x)$, i.e., $\bar{F}(x+t) \leq \bar{F}(x)\bar{F}(t)$ for $x, t \geq 0$. This means that a device of any particular age has a stochastically smaller remaining lifetime than does a new device (Barlow and Proschan, 1981). The definition here is also equivalent to $\log \bar{F}(x+t) \leq \log \bar{F}(x) + \log \bar{F}(t) \Leftrightarrow \int_0^t r(u)\, du \leq \int_x^{x+t} r(u)\, du$.

F is said to be new worse than used (NWU) if $\bar{F}(x+t) \geq \bar{F}(x)\bar{F}(t)$ for all $x, t \geq 0$.

Definition 2.5: F is said to be new better than used in expectation (NBUE) if $\int_0^\infty \bar{F}(x \mid t)\, dx \leq \mu$ for $t \geq 0$. This is equivalent to $\int_t^\infty \bar{F}(x)\, dx \leq \mu \bar{F}(t)$. This means that a device of any particular age has a smaller mean remaining lifetime than does a new device (Barlow and Proschan, 1981). F is said to be new worse than used in expectation (NWUE) if $\int_0^\infty \bar{F}(x\,|\,t)\,dx \geq \mu$ for all $t \geq 0$.

Definition 2.6: F is said to be harmonically new better than used (HNBUE) if $\int_t^\infty \bar{F}(x)\,dx \leq \mu \exp(-t/\mu)$ for $t \geq 0$. There is an alternative definition in terms of the mean residual life (Rolski, 1975). This is equivalent to $1/\{\frac{1}{t}\int_0^t \mu^{-1}(x)\,dx\} \leq \mu$. Similarly, F is said to be harmonically new worse than used (HNWUE) if $\int_t^\infty \bar{F}(x)\,dx \geq \mu \exp(-t/\mu)$ for $t \geq 0$.

Definition 2.7: F is said to be a Laplace class ($\mathcal{L}$)-distribution if for every $s \geq 0$, $\int_0^\infty e^{-st}\bar{F}(t)\,dt \geq \mu/(1+s)$. The expression $\mu/(1+s)$ can be written as for $\int_0^\infty \exp(-sx)\bar{G}(x)\,dx$, where $\bar{G}(x) = \exp(-x/\mu)$. This means that the inequality is one between the Laplace transforms of $\bar{F}$ and of an exponential survival function with the same mean as F (Klefsjö, 1983c).

Definition 2.8: F is said to be new better than used in failure rate (NBUFR) if $r(t) > r(0)$ for $t \geq 0$ (Deshpande et al., 1986). F is said to be new worse than used in failure rate (NWUFR) if $r(t) < r(0)$ for $t \geq 0$.

Definition 2.9: F is said to be new better than used in failure rate average (NBAFR or NBUFRA) if $r(0) \leq \frac{1}{t}\int_0^t r(x)\,dx$ for all $t \geq 0$ (Loh, 1984ab). Note that this is equivalent to $r(0) \leq \frac{-\log \bar{F}(t)}{t}, t \geq 0$. Similarly, F is said to be new worse than used in failure rate average (NWUFRA) if $r(0) \geq \frac{-\log \bar{F}(t)}{t}$ for $t \geq 0$.

Definition 2.10: F is said to be new better than used in convex ordering (NBUC) if $\int_y^\infty \bar{F}(t\,|\,x)\,dt \leq \int_y^\infty \bar{F}(t)\,dt$ for all $x, y \geq 0$ (Cao and Wang, 1991).

Using Laplace transforms, Block and Savits (1980a) established necessary and sufficient conditions for the IFR, IFRA, DMRL, NBU, and NBUE properties to hold.

There are many other ageing classes have been defined and some of these 'further' classes will be given in Section 2.7. The ten classes above are chosen because they are easily understood, intuitively appealing with known applications. Their relevance to reliability theory and survival analysis have been well documented in the literature, especially the first five classes.

2.4.3 Interrelationships

The following chain of implications exists among the ten ageing classes (adapted from Deshpande et al., 1986; Kochar and Wiens, 1987):

$$\begin{array}{ccccccccc}
\text{IFR} & \Rightarrow & \text{IFRA} & \Rightarrow & \text{NBU} & \Rightarrow & \text{NBUFR} & \Rightarrow & \text{NBUFRA} \\
\Downarrow & & & & \Downarrow & & & & \\
\Downarrow & & & & \text{NBUC} & & & & \\
\Downarrow & & & & \Downarrow & & & & \\
\text{DMRL} & & \Rightarrow & & \text{NBUE} & \Rightarrow & \text{HNBUE} & \Rightarrow & \mathcal{L}
\end{array}$$

We note that a partial chain IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU $\Rightarrow$ NBUE has long been established (Barlow and Proschan, 1981, p. 159). For completeness, a brief sketch of the proof is as follows.

(i) F is IFR if $-\log \bar{F}(x)$ is convex whereas F is IFRA if $\log \bar{F}(x)$ is a star-shaped function, i.e., if $-\log \bar{F}(\lambda x) \leq -\lambda \log \bar{F}(x)$ for $0 \leq \lambda \leq 1$ and $x \geq 0$. Since a convex function is stared-shaped so IFR $\Rightarrow$ IFRA.

$$
\begin{aligned}
F \text{ is IFRA} &\Rightarrow -(1/x)\log \bar{F}(x) \text{ is increasing } x \\
&\Leftrightarrow \bar{F}(x)^{\frac{1}{x}} \text{ is decreasins in } x \\
&\Leftrightarrow \bar{F}(x+y)^{\frac{1}{x+y}} \leq \bar{F}(x)^{\frac{1}{x}} \leq \bar{F}(y)^{\frac{1}{y}} \text{ assuming } x > y \\
&\Rightarrow \bar{F}(x+y)^{\frac{x+y}{x+y}} \leq \bar{F}(x)^{\frac{x+y}{x}} \\
&\Rightarrow \bar{F}(x+y) = \bar{F}(x)\bar{F}^{\frac{y}{x}}(x) \leq \bar{F}(y)\bar{F}^{\frac{y}{y}}(y) = \bar{F}(x)\bar{F}(y) \\
&\Leftrightarrow F \text{ is NBU.}
\end{aligned}
$$

(ii) Now F IFR implies that $\dfrac{\bar{F}(x+t)}{\bar{F}(t)}$ decreases in $t \geq 0$, in other words, $\dfrac{\bar{F}(x+t)}{\bar{F}(t)} \leq \dfrac{\bar{F}(x+s)}{\bar{F}(s)}$ for $t \geq s$. Integrating both sides with respect to x, we have $\mu(t) \leq \mu(s)$, i.e., F is DMRL.

(iii) Show NBU $\Rightarrow$ NBUC $\Rightarrow$ NBUE $\Rightarrow$ HNBUE $\Rightarrow \mathcal{L}$.

F is NBU $\Leftrightarrow \dfrac{\bar{F}(x+y}{\bar{F}(x)} \leq \bar{F}(y)$. Integrating both sides with respect to y so $\dfrac{\int_0^\infty \bar{F}(x+y)\,dy}{\bar{F}(x)} \leq \displaystyle\int_0^\infty \bar{F}(y)\,dy$, i.e., $\mu(x) \leq \mu$ and thus F is NBUE.

Next, F NBU implies $\bar{F}(x\,|\,t) \leq \bar{F}(x) \Rightarrow \displaystyle\int_y^\infty \bar{F}(x\,|\,t)\,dx \leq \int_y^\infty \bar{F}(x)\,dx$ for all $y \geq 0$ which implies that F is NBUC.

Letting $y = 0$, the preceding inequality reduces to the corresponding definition of NBUE showing that NBUC $\Rightarrow$ NBUE.

Next, if F is NBUE, we have $\mu(t) \leq \mu$ for all $t \geq 0$. Thus $\mu^{-1}(t) \geq \frac{1}{\mu}$. Therefore $\int_0^t \mu^{-1}(x)\,dx \geq t/\mu$ or equivalently $1/\{\frac{1}{t}\int_0^t \mu^{-1}(x)\,dx\} \leq \mu$ which implies F HNBUE.

Lastly, we want to show that if F HNBUE then $F \in \mathcal{L}$. It can be shown easily that

$$\int_0^\infty e^{-st}\bar{F}(t)\,dt = \mu - s\int_0^\infty e^{-st}\left(\int_t^\infty \bar{F}(x)\,dx\right)\,dt.$$

As $\int_t^\infty \bar{F}(x)\,dx \le \mu e^{-t/\mu}$ because of $F \in$ HNBUE, it follows from above equation that $\int_0^\infty e^{-st}\bar{F}(t)\,dt \ge \mu/(1+s\mu)$ so $F \in \mathcal{L}$ is proved.

(iv) We next want to show that NBU⇒ NBUFR ⇒ NBUFRA.

$$\begin{aligned}
F \text{ is NBU} &\Leftrightarrow \frac{\bar{F}(t+x)}{\bar{F}(t)} \le \bar{F}(x), x, t > 0 \\
&\Leftrightarrow \frac{\bar{F}(t+x) - \bar{F}(t)}{\bar{F}(t)} \le -F(x) \\
&\Rightarrow \lim_{x\to 0} \frac{F(x)}{x} \le \lim_{x\to 0} \frac{\bar{F}(t+x) - \bar{F}(t)}{x\bar{F}(t)} \\
&\Leftrightarrow f(0) \le \frac{f(t)}{\bar{F}(t)} \quad \text{or} \quad r(0) \le \frac{f(t)}{\bar{F}(t)} \\
&\Leftrightarrow F \text{ is NBUFR}
\end{aligned}$$

$$\begin{aligned}
F \text{ is NBUFR} &\Leftrightarrow r(t) < r(0) \\
&\Rightarrow \tfrac{1}{t}\textstyle\int_0^t r(x)\,dx < r(0) \\
&\Rightarrow F \text{ is NBUFRA}
\end{aligned}$$

so we have completed the proof of the chain.

In definitions 2.1–2.10 of Section 2.4.2, if we reverse the inequalities and interchange “increasing” and “decreasing”, we obtain the classes DFR, DFRA, NWU, IMRL, NWUE, HNWUE, $\bar{\mathcal{L}}$, NWUFR, NWUFRA, and NWUC. They satisfy the same chain of implications. These are sometimes referred to as the ‘dual’ classes and their roles are to define negative ageing effects to a device.

2.5 Properties of the Basic Ageing Classes

The properties of interest concerning ageing classes are mainly on

(1) Preservation or closure property of an given ageing class under the reliability operations of
   (a) Formation of coherent systems of independent components,
   (b) Addition of life lengths (convolution),
   (c) Mixtures of distributions,
(2) Reliability bounds,
(3) Whether any ageing class can arise from a shock model,
(4) Moment inequalities,

(5) Testing exponentiality against an ageing alternatives.

Item (4) will be dealt with in Section 2.5.5 whereas item (5) will be considered in Chapter 7 in detail.

2.5.1 Properties of IFR and DFR

The following properties of the IFR and DFR concepts can be found in Barlow and Proschan (1981), Patel (1983) and many others:

1. If X_1 and X_2 are both IFR, so is $X_1 + X_2$; but the DFR property is not so preserved.
2. A mixture of DFR distributions is also DFR; but this is not necessarily true for IFR distributions.
3. Parallel systems of identical IFR units are IFR.
4. Series systems of (not necessarily identical) IFR units are IFR.
5. Order statistics from an IFR distribution have IFR distributions, but this is not true for spacings from an IFR distribution; order statistics from a DFR distribution do not necessarily have a DFR distribution, but spacings from a DFR distribution are DFR.
6. The pdf of an IFR distribution need not be unimodal.
7. The pdf of a DFR distribution is a decreasing function.
8. If the rth moment (about zero) of a continuous life F distribution is known, the IFR lower bound on $\bar{F}(t)$ is
$$\bar{F}(t) \geq \begin{cases} e^{-\alpha t}, \text{ if } t < \mu_r^{1/r} \\ 0, \quad\ \ \text{if } t > \mu_r^{1/r}, \end{cases} \tag{2.53}$$
where $\alpha = [\Gamma(r+1)/\mu_r]^{1/r}$ (Barlow and Proschan, 1981, p.112). The bound is sharp.
The special case $r = 1$ is important in reliability applications as the first moment is usually easy to find or estimate.
9. Let F be DFR with mean μ, then
$$\bar{F}(t) \leq \begin{cases} e^{-t/\mu}, \text{ for } t \leq \mu; \\ \frac{\mu e^{-1}}{t}, \ \text{ for } t \geq \mu. \end{cases} \tag{2.54}$$

The IFR phenomenon is well understood and needs no further elaboration. To put in nontechnical terms, a device having IFR lifetime deteriorates with age, i.e., the age has an adverse effect on the device for if it has an IFR lifetime distribution.

Defying a common expectation, DFR phenomenon also occurs quite frequently. Generally speaking, a lifetime population is expected to exhibit decreasing failure rate (DFR) when its behavior over time is characterized by

- 'work hardening' (in engineering), and
- immunity in (biological organisms).

The term 'infant mortality phase' is sometimes used to described the DFR phenomenon over the early part of the life span. In a DFR population, 'age' is actually beneficial to a device or an organism. Improvement of reliability might have occurred by means of physical changes that caused self-improvement or simply it might have been due to population heterogeneity. Indeed, the DFR property is often inherent in mixtures of distributions.

2.5.2 Properties of IFRA

As $-\log \bar{F}(t)$ given in (2.4) represents the cumulative failure rate, the name given to this class is appropriate. Block and Savits (1976) showed that the IFRA is equivalent to $E^{\alpha}[h(X)] \leq E[h^{\alpha}(X/\alpha)]$ for all continuous nonnegative increasing functions h and all α such that $0 < \alpha < 1$.

This ageing notion is fully investigated in the book by Barlow and Proschan (1981). It is the smallest class containing the exponential distribution which is closed under the formation of coherent systems as well as under convolution. The IFRA closure theorem is pivotal to many of the results given in Barlow and Proschan (1981). The IFRA class is perhaps one of the more important ageing classes in reliability analysis. Curiously, interest about IFRA has seemed to wane in the recent time. It has been shown that a device subject to shocks governed by a Poisson process, which fails when the accumulated damage exceeds a fixed threshold, has an IFRA distribution (Esary et al., 1973).

One of the attractive properties that an IFRA (DFRA) distribution enjoys is that its reliability bound can be obtained in terms of its known quantile.

Theorem 2.4: Let F be IFRA (DFRA) with pth quantile ξ_p (i.e., $F(\xi_p) = p$). Then

$$\bar{F}(t) \begin{cases} \geq (\leq) e^{-\alpha t}, \text{ for } 0 \leq t < \xi_p \\ \leq (\geq) e^{-\alpha t}, \text{ for } t > \xi_p. \end{cases} \tag{2.55}$$

Proof: We note that the exponential survival probability $e^{-\alpha t}$ has the same pth quantile ξ_p as does F. Thus at least one crossing of $e^{-\alpha}$ by $\bar{F}(t)$ must occur at $t = \xi_p$. By the single crossing property of an IFRA distribution with $e^{-\lambda t}, \lambda > 0$ (Barlow and Proschan, 1981, p. 89), we conclude $\bar{F}$ crosses with the exponential survival function with the same quantile at most once from above (below).

Sengupta (1994) presented a unified derivation of the upper and lower bounds (in terms of finite moments) of IFR (DFR), IFRA (DFRA) or NBU (NWU) reliability functions. A table of bounds on $\bar{F}(t)$ based on the rth moments for various cases is also given. However, numerical methods are required to solve for the values of these bounds.

Recently, El-Bassiouny (2003) has shown that if F is IFRA, then for all integers $r \geq 0$, $k \geq 2$,

$$\nu_{(r+1)} \geq \frac{\mu'_{r+1}}{k^{r+1}} \tag{2.56}$$

where $\nu_{(r)} = E[\min(X_1, ..., X_k)]^r$, $\mu'_r = E(X_1^r)$ and $X_1, ...X_k$ are independent and identically distributed random variables.

Several moment inequalities for IFR (DFR) appeared much earlier and these will be given Section 2.5.5 below.

2.5.3 NBU and NBUE

Properties of NBU, NWU, NBUE, NWUE were also well documented in the book by Barlow and Proschan (1981). These classes of life distributions may arise from a consideration of shock models similar to those involving IFRA distributions. Barlow and Proschan (1981, Chapter 6) showed that these concepts are useful in the study of maintenance policies.

Closure property

Abouammoh and El-Neweihi (1986) show that the NBU class is closed under formation of parallel systems of i.i.d. components.

Probability bounds

The first three bounds below are found in Barlow and Proschan (1981, p. 188).

If F is NBU with $\bar{F}(t) = \alpha$ for a fixed value of t, then

$$\bar{F}(x) \begin{cases} \geq \alpha^{1/k} & \text{for} \quad \frac{t}{k+1} < x < \frac{t}{k},\ k = 0, 1, ..., \\ \leq \alpha^k & \text{for} \quad kt \leq x \leq (k+1)t, k = 0, 1, ... \end{cases}$$

The bounds are sharp.

If F is now NWU, then

$$\bar{F}(x) \begin{cases} \leq \alpha^{1/(k+1)} & \text{for} \quad \frac{t}{k+1} < x < \frac{t}{k},\ k = 0, 1, ..., \\ \geq \alpha^{k+1} & \text{for} \quad kt \leq x \leq (k+1)t, k = 0, 1, ... \end{cases}$$

The bounds are sharp.

If F is NBUE with mean μ, then

$$\bar{F}(t) \geq \begin{cases} 1 - \frac{t}{\mu}, & t \leq \mu; \\ 0, & t > \mu. \end{cases}$$

The bounds are sharp.

If F is NWUE with mean μ, then

$$\bar{F}(t) \leq \mu/(\mu + t),\ t \geq 0$$

(Haines and Singpurwalla, 1974).

The next two bounds were given by Launer (1984).

(i) If F is NBUE with mean μ and variance σ^2, then

$$\bar{F}(t) \geq \begin{cases} (\sigma^2 + \mu^2 - t^2)/(\sigma^2 + (\mu + t)^2 - t^2), & t \leq \sqrt{\mu_2'}; \\ 0, & t > \sqrt{\mu_2'}. \end{cases}$$

(ii) If F is now NWUE, then

$$\bar{F}(t) \leq \begin{cases} (\sigma^2 + \mu^2)/(\sigma^2 + (\mu + t)^2), & 0 < t \leq 2\sigma^2/\mu; \\ \sigma^2/(\sigma^2 + t^2), & 2\sigma^2/\mu \leq t. \end{cases}$$

Launer (1984) obtained other bounds that are based on $E(X^r|X > t)$ and $E(X^r|X \leq t)$. In general, the usefulness of a bound depends on (i) how easy a bound it can be estimated from data and (ii) how sharp the bound is.

Replacement models

Consider a system operating over an indefinite period of time. Upon failure, repair (or replacement) is performed, requiring negligible time. The successive intervals between failures are independent, identically distributed random variables $X_1, X_2, ...$ of a renewal process. Let $N(t)$ denote the number of renewals (replacements) in $(0, t]$ and $M(t)$ the expected number of renewals (replacements) in $(0, t]$, i.e., $E(N(t)) = M(t)$.

Theorem 2.5: Let $E(X) = \mu < \infty$ be the mean lifetime of a component.

(a) If F is NBUE , then

$$\frac{t}{\mu} - 1 \leq M(t) \leq \frac{t}{\mu}. \tag{2.57}$$

(b) If F is NWUE, then

$$M(t) \geq \frac{t}{\mu}. \tag{2.58}$$

Proof: The following is essentially an abridged version of the proof given in Barlow and Proschan (1981, pp. 169-171).

(a) Let $S_{N(t)}$ denote the time to the kth renewal if $N(t) = k, k \geq 0$ so the $(k+1)$th renewal must occur after time t. Thus $S_{N(t)+1} - t \geq 0$. It follows from the classical renewal theory that $E(S_{N(t)+1} - t) = \mu[M(t) + 1] - t \geq 0$ which gives $M(t) \geq \frac{t}{\mu} - 1$. (This is true irrespective of the ageing class it belongs to.)

On the other hand, F NBUE implies the stationary distribution $\hat{F}(t) = (1/\mu) \int_0^t \bar{F}(x)\, dx \geq F(t)$. This means the expected time to the first renewal under the the stationary renewal process is smaller than under the ordinary renewal process. Hence, $\hat{M}(t) \geq M(t)$ where $\hat{M}(t)$ denotes the expected number of renewals in a stationary renewal process which is given by $\frac{t}{\mu}$. It now follows immediately that $M(t) \leq \frac{t}{\mu}$ so the proof of (a) is completed.

(b) Suppose now F is NWUE so that $\hat{F}(t) = (1/\mu) \int_0^t \bar{F}(x)\, dx \leq F(t)$. Using the second part of the proof for (a) but with the inequality reversed, we conclude that $M(t) \geq \frac{t}{\mu}$. See Barlow and Proschan (1981, p. 171).

Chen (1994) showed that the distributions of these classes may be characterized through certain properties of the corresponding renewal functions.

Cheng and He (1989) studied the reliability bounds on NBUE and NWUE classes and Cheng and Lam (2002) obtained reliability bounds on NBUE from first two known moments.

NBU-t_0 and NWU-t_0 Classes

Without getting side-tracked from discussing the ten basic classes, we now introduce a new life class which is obtained by relaxing the conditions for NBU (NWU) class somewhat. Let $t_0 \geq 0$.

Definition 2.11: We say that a life distribution F is new better than used at t_0 (NBU-t_0) if

$$\bar{F}(x + t_0) \leq \bar{F}(x)\bar{F}(t_0), \text{ for all } x \geq 0. \tag{2.59}$$

The class was first introduced in Hollander et al. (1985). The dual notion of new worse than used at t_0 is defined analogously by reversing the first inequality in the preceding equation. Several non-parametric tests dealing with this class have been proposed in the literature. It is interesting to note that apart from the exponential, there are some other distributions that belong to the boundary members of NBU-t_0 and NWU-t_0 classes. Though not listed along with the ten classes in Section 2.4.2, the NBU-t_0 (NWU-t_0) class has been frequently discussed and referred to so it could be considered as an important ageing class in reliability. Park (2003) gave a detailed review of its properties and applications.

HNBUE and HNWUE

Klefsjö (1982) obtained the properties of HNBUE and HNWUE classes of life distributions. Basu and Ebrahimi (1986) gave a survey on these classes. Cheng and Lam (2001) gave reliability bounds on HNBUE life distributions with the first two known moments.

Though mathematically elegant, we have yet to find any meaningful or significant applications for HNBUE or HNWUE classes.

NBUC and NWUC

NBUC and NWUC classes were first defined by Cao and Wang (1991). These authors showed that neither NBUC nor NWUC is closed under mixture or formation of series systems. We have seen earlier that NBUC is sandwiched between NBU and NBUE, i.e.,

$$\text{NBU} \Rightarrow \text{NBUC} \Rightarrow \text{NBUE} \Rightarrow \text{HNBUE}.$$

Hendi et al. (1993) have shown that the new better than used in convex ordering is closed under the formation of parallel systems with independent and identically distributed components. Both Li et al. (2000) and Pellerey and Patakos (2002) showed that this closure property holds for nonidentical parallel components as well. Besides, Li et al. (2000) also presented a lower bound of the reliability function for this class based upon the mean and the variance. Hu and Xie (2002) gave a new proof of the closure property of NBUC under convolution (they corrected the errors of Cao and Wang, 1991 and Li and Kochar, 2001 regarding this closure property).

NBUFR (NWUFR)

Abouammoh and Ahmed (1988) showed that every k-out-of-n system, $1 \leq k \leq n$, has the NBUFR property.

Gohout and Kuhnert (1997) showed that NBUFR class is closed under the formation of coherent systems of independent components. El-Bassiouny et al. (2004) derived a moment inequality for the class of NBUFR (NWUFR):

$$\mu'_r \geq (\leq) \frac{f(0)\mu'_{r+1}}{r+1},\ f(0) > 0, r \geq 0,$$

where $\mu'_s = E(X^s), s \geq 0$ and $f(t)$ is the density function of the distribution.

Laplace classes $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\bar{\mathcal{L}}}$

Suppose $F \in \mathcal{L}$ with mean μ, Sengupta (1995) showed that

(a) $\alpha_t \leq \bar{F}(t) \leq 1$ if $t \leq \mu$,
(b) $0 \leq \bar{F}(t) \leq 1 - \alpha_t$, if $t \geq \mu$,

where

$$\alpha_t = \inf\left\{\alpha : \inf_{s>0}[e^{st/\mu} - (1+s)(1-\alpha+e^{-s(1-t/\mu)/\alpha})] \geq 0\right\}.$$

The bounds are sharp. It appears the the actual computations of bounds are nontrivial.

On the other hand, let $F \in \bar{\mathcal{L}}$ with mean μ, Sengupta (1995) showed that

$$0 \leq \bar{F}(t) \leq \begin{cases} \inf\limits_{s>0} \dfrac{s}{(s+t/\mu)(1-e^{-s})} & \text{if } t \leq 2\mu \\ \mu/t & \text{if } t \geq 2\mu \end{cases}$$

Again, the bounds are sharp.

We are of the opinion that this ageing concept has limited application.

2.5.4 DMRL and IMRL

The properties DMRL and IMRL classes will be studied in Chapter 4 in details. For time being, it suffices to state that the DMRL class is closed under formation of parallel systems of i.i.d. components (Abouammoh and El-Neweihi, 1986).

2.5.5 Summary of Preservation Properties of Classes of Distributions

One may wish to know under what reliability operations a given class of life distributions is preserved.

Here we consider the reliability operations of

(a) Formation of coherent systems,
(b) Addition of life lengths (convolution of distributions),
(c) Mixture of life distributions,
(d) Mixture of non-crossing life distributions,

applied to several basic classes of life distributions. The following table adapted from Park (2003) provides a summary of these preserving (or non-preserving) properties under reliability operations. Park's table was itself an update of the original table given by Barlow and Proschan (1981, p.104, 187).

Table 2.2. Preservation Under Reliability Operations

Class of life distribution	Formation of coherent structure	Convolution of life distributions	Mixture of life distributions	Mixture of non-crossing life distributions
IFR	Not closed	Closed	Not closed	Not closed
IFRA	Closed	Closed	Not closed	Not closed
NBU	Closed	Closed	Not closed	Not Closed
NBUE	Not closed	Closed	Not closed	Not closed
DMRL	Not closed	Not closed	Not closed	Not closed
HNBUE	Not closed	Closed	Not closed	Not closed
NBU-t_0	Closed	Not closed	Not closed	Not closed
DFR	Not closed	Not closed	Closed	Closed
DFRA	Not closed	Not closed	Closed	Closed
NWU	Not closed	Not closed	Not closed	Closed
NWUE	Not closed	Not closed	Not closed	Closed
IMRL	Not closed	Not closed	Closed	Closed
HNWUE	Not closed	Not closed	Closed	Not closed
NWU-t_0	Not closed	Not cloded	Not closed	Closed

2.5.6 Moments Inequalities

Moment inequalities for elementary ageing classes have been in the literature for many years. The following results are found in Barlow and Proschan (1981, p. 116, 187).

Theorem 2.6: Let F be a continuous distribution with known mean μ and μ'_r be the rth moment about zero. Let $\lambda_r = \mu'_r/\Gamma(r+1)$.

(i) Let F be IFRA (DFRA). Then

$$\begin{aligned} \mu'_r &\leq (\geq)\Gamma(r+1)\mu^r \quad \text{for } 0 < r \leq 1, \\ \mu'_r &\geq (\leq)\Gamma(r+1)\mu^r \quad \text{for } 1 < r \leq \infty. \end{aligned} \tag{2.60}$$

The bounds are sharp.

(ii) Let F be NBU (NWU) and $\lambda_r = \mu'_r/\Gamma(r+1)$, the normalized rth moment. Then

$$\lambda_{r+s} \leq (\geq)\lambda_r\lambda_s$$

for $r \geq 0, s \geq 0$.

(iii) Let F be NBUE (NWUE), then

$$\lambda_{r+1} \leq (\geq)\lambda_r\lambda_1, \quad \text{for } r \geq 0.$$

Proof: We largely follow the approach of Barlow and Proschan (1981) in the proof below.

(i) First, F IFRA is equivalent to F is star-shaped with respect to G where $G(t)$ is an exponential distribution (see Barlow and Proschan 1981, p.107 and also Section 10.3 for a definition for 'F is star-shaped with respect to G'.) So $\bar{F}$ crosses $\bar{G}$ at most once. Now if $\bar{G}(t) = \exp\left(-t/\lambda_s^{1/s}\right)$, then

$$\int_0^\infty t^s dG(t) = \int_0^\infty (t^s/\lambda_s^{1/s}) \exp\left(t/\lambda_r^{1/s}\right) dt = \mu'_s = \int_0^\infty t^s dF(t),$$

so $\bar{F}$ crosses G exactly once. Now, it can be shown that

$$\int_0^\infty \psi(x)x^{s-1}\bar{F}(x)\,dx \leq \int_0^\infty \psi(x)x^{s-1}\bar{G}(x)\,dx;\; s > 0 \tag{2.61}$$

if ψ is increasing. The inequality in (2.61) reverses if F is DFRA.

Now, let $0 < r < s$, it follows from the rth moment about zero of an exponential distribution that

$$\mu'_r = r\int_0^\infty x^{r-1}\bar{F}(x)\,dx = r\int_0^\infty x^{r-1}\exp\left(-x/\lambda_r^{1/r}\right)dx, \tag{2.62}$$

i.e.,

$$\int_0^\infty x^{r-1}\bar{F}(x)\,dx = \int_0^\infty x^{r-1}\exp\left(-x/\lambda_r^{1/r}\right)dx.$$

Let $\psi(x) = x^{s-r}$ and applying (2.61), we obtain

$$\lambda_s = \frac{\mu_s'}{\Gamma(s+1)} = \frac{1}{\Gamma(s+1)}\int_0^\infty sx^{s-1}\bar{F}(x)\,dx$$

$$\leq (\geq)\int_0^\infty \frac{sx^{s-1}}{\Gamma(s+1)}\exp\left(-x/\lambda_r^{1/r}\right)dx = \lambda_r^{s/r},$$

that is

$$\lambda_r^{1/r} \geq (\leq)\lambda_s^{1/s}, \text{ for } 0 < r < s.$$

Letting $s = 1$, we prove the $\mu_r' = \Gamma(r+1) \geq (\leq)\mu^r$ for $0 < r < 1$. The inequalities are reversed for $1 \leq r < \infty$.

(ii) If F is NBU, then $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$ so for $r, s \geq 0$,

$$\int_0^\infty\int_0^\infty x^r y^s \bar{F}(x+y)\,dx\,dy \leq \int_0^\infty x^r\bar{F}(x)\,dx\int_0^\infty y^s\bar{F}(y)\,dy.$$

Applying (2.62), the right-hand side is $\mu_{r+1}\mu_{s+1}/(r+1)(s+1)$.

Letting $x + y = u$ and $y = v$, the left-hand side of the above equation becomes

$$\int_0^\infty\int_0^u \bar{F}(u)(u-v)^r v^s\,dv\,du = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+3)}\mu_{r+s+2}'.$$

It now follows immediately $\lambda_{r+s+2} \leq \lambda_{r+1}\lambda_{s+1}$.

If F is NWU, then $\lambda_{r+s+2} \geq \lambda_{r+1}\lambda_{s+1}$.

(iii) Since F is NBUE, $\int_x^\infty \bar{F}(u)\,du \leq \mu\bar{F}(x)$. Thus, for all $r \geq 0$,

$$\int_0^\infty x^r\int_x^\infty \bar{F}(u)\,du\,dx \leq \mu\int_0^\infty x^r\bar{F}(x)\,dx,$$

i.e.,

$$\int_0^\infty \bar{F}(u)\int_0^u x^r\,dx\,du \leq \mu\int_0^\infty x^r\bar{F}(x)\,dx.$$

The left-hand side can be evaluated by integration by parts giving $\mu_{r+2}'/(r+2)$ whereas the right-hand side is $\mu_{r+1}'\mu$ because of (2.62). Thus $\lambda_{r+2} \leq \lambda_{r+1}\lambda_1$ is now proved.

The case for NWUE can be proved similarly. We note that an important feature of these inequalities is their simplicity.

Interest in moment inequalities for several classes has been rekindled in recent years. The main objective of these inequalities is to formulate test statistics for testing a a distribution is from a particular ageing class. Unfortunately, some of these have complicated expressions and those listed below are some special cases which we think would be of interest to the readers.

Ahmad (2001) presented moment inequalities for IFR, NBU, NBUE and HNBUE. Recently, Ahmad and Mugdadi (2004) also provided similar inequalities for IFRA, NBUC and DMRL classes. In the following, we assume that $\{X_i\}$ are i.i.d. with distribution function F; r and s are integers. We also let $\nu_{(r)} = E[\min(X_1, X_2)]^r$.

(i) F IFR (Ahmad, 2001)

$$2^{(r+2)(r-1)/2}\nu_{(r)} \geq r!\mu^r,\ r \geq 2$$

and

$$\nu_{(2r+2)} \geq \binom{2r+2}{r+1}\left(\frac{1}{2}\right)^{(2r+2)} \left\{\mu'_{r+1}\right\}^2.$$

(ii) F NBU (Ahmad, 2001)

For integer $k \geq$ and $r_i \geq 0$, $i = 1, 2, ..., k$

$$\left(\sum_{i=1}^{k} r_i + k\right)! \prod_{i=1}^{k} \mu_{r_i+1} \geq \prod_{i=1}^{k} (r_i+1)!\mu_{r_1+\ldots r_k+k}. \tag{2.63}$$

For $r_1 = r$, $r_2 = s$, $k = 2$, the above inequality reduces to

$$(r+s+2)!\mu'_{r+1}\mu'_{s+1} \geq (r+1)!(s+1)!\mu'_{r+s+2} \tag{2.64}$$

which is equivalent to (ii) of Theorem 2.6 above. Thus (2.63) is a generalization of the result (ii) of the theorem mentioned. In view of its complexity, one doubts if the above moment inequality would generate wide applications.

(iii) F NBUE (Ahmad, 2001)

$$\mu'_{r+1}\mu \geq \mu'_{r+2}/(r+2),\ \ r \geq 0.$$

(This is the same as (iii) of Theorem 2.6 above)

(iv) F HNBUE (Ahmad, 2001)

$$\mu^{r+2} \geq \mu'_{r+2}/(r+2)!,\ \ r \geq 0. \tag{2.65}$$

The proof of (2.65) is straightforward by noting that F HNBUE implies $\int_x^\infty \bar{F}(u)\,du \leq \mu e^{-x/\mu}$ so

$$\int_0^\infty x^r \int_x^\infty \bar{F}(u)\,du\,dx \leq \mu \int_0^\infty x^r e^{-x/\mu}\,dx.$$

By exchanging the order of integration and applying (2.62), the left-hand side is $\mu'_{r+2}/[(r+2)(r+1)]$. The integral on the right is related the $(r+1)$th moment (about zero) of the exponential distribution so the result follows immediately.

(v) F IFRA (Ahmad and Mugdadi, 2004)

$$\mu'_{r+1} \geq E\left\{\min\left(\frac{X_1}{\alpha}, \frac{X_2}{1-\alpha}\right)^{r+1}\right\}, r \geq 0, 0 < \alpha < 1, .$$

(vi) F NBUC (Ahmad and Mugdadi, 2004)

$$(r+2)!(s+1)!\mu'_{r+s+3} \leq (r+s+3)!\mu'_{r+2}\mu'_{s+1}, \; r, s \geq 0.$$

We note that this inequality is equivalent to (2.64) which holds for NBU distributions.

(vii) F DMRL (Ahmad and Mugdadi, 2004)

$$(r+1)E[X_1\{\min(X_1, X_2)\}^r] \geq (r+2)\nu_{(2)}^{(r+1)}, \; r \geq 0.$$

Abu-Youssef (2002) obtained a simple bound $\nu_{(2)} \geq (\leq)\frac{\mu^2}{2}$ if F is DMRL (IMRL).

2.5.7 Scaled TTT Transform and Characterizations of Ageing Classes

The concept of the total time on test (TTT) processes was first defined by Barlow and Campo (1975). The TTT transform has been found useful to study the ageing properties of the underlying distribution and at the same time can be applied to solve geometrically some stochastic maintenance problems.

Let F be an lifetime distribution and define

$$F^{-1}(p) = \inf\{x : F(x) \geq p\}, \, p \in [0, 1). \tag{2.66}$$

Let us define

$$H_F^{-1}(p) = \int_0^{F^{-1}(p)} \bar{F}(x)\, dx, p \in [0, 1]. \tag{2.67}$$

If the mean lifetime μ is finite, then

$$H_F^{-1}(1) = \int_0^{F^{-1}(1)} \bar{F}(x)\, dx = \mu. \tag{2.68}$$

The scaled total time on test transform (scaled TTT transform) is defined by

$$\phi(p) = H_F^{-1}(p)/H_F^{-1}(1) = H_F^{-1}(p)/\mu. \tag{2.69}$$

It follows from the above definition that $\phi(p), p \in [0,1]$ is equivalent to the equilibrium distribution of the probability distribution function F, if F is non-arithmetic. The curve $\phi(p)$ versus $p \in [0,1]$, is called the scaled TTT curve.

Classifications of ageing distributions

It follows from (2.67) that

$$\frac{d}{dp} H_F^{-1}(p)|_{p=F(t)} = \frac{1}{r(t)},\ t > 0, p \in [0,1]. \tag{2.70}$$

Theorem 2.7 Let F be a continuous lifetime distribution.

(i) F is IFR (DFR) if and only if $\phi(p)$ is concave (convex) in $p \in [0,1]$.

(ii) F is IFRA (DFRA) if and only if $\phi(p)/p$ is decreasing (increasing) in $p \in [0,1]$.

(iii) F is NBUE (NWUE) if and only if $\phi(p) \geq p$ ($\phi(p) \leq p$) for $p \in [0,1]$.

(iv) F is DMRL (IMRL) if and only if $(1-\phi(p))/(1-p)$ is decreasing (increasing) in $p \in [0,1]$.

(v) F is HNBUE (HNWUE) if and only if $\phi(p) \leq 1 - \exp\{-F^{-1}(p)/\mu\}$ ($\phi(p) \geq 1 - \exp\{-F^{-1}(p)/\mu\}$) for $p \in [0,1]$.

(vi) $F \in$ BT (UBT) if ϕ has only one reflection point u_0 such that $0 < u_0 < 1$ and it is convex (concave) on $[0, u_0]$ and concave (convex) on $[u_0, 1]$.

Proof: Without loss of generality, we assume $F(t)$ is absolutely continuous.

(i) The result was independently proved by Barlow and Campo (1975) and Lee and Thompson (1976). Assuming $F(t)$ is absolutely continuous so that $p = F(t)$ implies $t = F^{-1}(p)$. Using the chain rule, we can easily verify that $\phi(p)$ is concave (convex) in p implies $r(t)$ is increasing (decreasing) in t so that the result (i) is proved.

(ii) The result was also due to Barlow and Campo (1975). The proof was nontrivial.

(iii) The NBUE (NWUE) characterization was made by Bergman (1977). This follows from

$$\phi(p) \geq (\leq)\, p \Leftrightarrow \int_0^t \bar{F}(x)\,dx \geq (\leq)\, \mu F(t) \Leftrightarrow \int_t^\infty \bar{F}(x)\,dx \leq (\geq)\, \mu \bar{F}(t).$$

Langberg et al. (1980a) gave more detailed results on IFRA (DFRA) and NBU (NWU).

(iv) The result on DMRL (IMRL) was from Klefsjö (1982). We note $(1 - \phi(p))/(1-p)$ is equivalent to $\mu(t)/\mu$ so the result follows immediately.

(v) The result on HNBUE (HNWUE) was also obtained by Klefsjö (1982). This follows easily from $1 - \phi(p) = \int_t^\infty \bar{F}(x)\,dx/\mu$.

(vi) The result on bathtub distributions was proved by Barlow and Campo (1975). The proof follows from (i) that F is DFR iff ϕ is convex and F is IFR iff ϕ is concave so ϕ lies below the 45°-line in its leftmost part and above the line in its rightmost part.

Result (vi) was used by Aarset (1987) and Xie (1989) to derive test statistics for testing exponetiality against BT distributions.

Note that several of test statistics that are based on the scaled TTT transform will be presented in Chapter 7.

2.6 Non-monotonic Failure Rates and Non-monotonic Mean Residual Lives

Survival and failure times are frequently modelled by increasing or decreasing failure rate distributions. While this is appropriate for many cases, it may be inappropriate if the course of a disease is such that the mortality reaches a peak after some finite period and then declines slowly. Gupta and Warren (2001) gave two such examples:

- In a study of curability of breast cancer, Langlands et al. (1979) found that the peak of mortality occurred after about three years.
- Bennett (1983) analyzed the data from Veterans Administration lung cancer presented by Prentice (1973) and showed that the empirical failure rates for both low PS and high PS groups are non-monotonic. (PS = Potassium sulfide)

Thus, we need to analyze such data sets with appropriate lifetime models that have non-monotonic failure rates $r(t)$.

We postpone a full discussion on these ageing classes to the next chapter. Here we present a brief preview only.

2.6.1 Non-monotonic Failure Rates

A failure rate function falls into one of the four categories: (a) monotonic failure rates if $r(t)$ is either increasing or decreasing; (b) bathtub type failure rate if $r(t)$ has a bathtub (BT) or an upside-down bathtub (UBT) shape; and (c) modified bathtub failure rate if $r(t)$ is first increasing and then bathtub; and (d) generalized bathtub failure rate if $r(t)$ is a polynomial, or has a roller-coaster shape or some generalization.

Lai et al. (2001) give an overview on the class of bathtub shaped failure rate (BT) distributions. We will devote a a fuller discussion on this class of life distributions in Chapter 3.

2.6.2 Non-monotonic Mean Residual Lives

Recall from (2.6), the mean residual lifetime is defined as $\mu(t) = E(X - t \mid X > t)$ which is equivalent to $\int_0^\infty \bar{F}(x \mid t)\, dx = \int_t^\infty \bar{F}(x)\, dx / \bar{F}(t)$.

Recall also from Definition 2.3 that F is said to be DMRL if the mean remaining life function $\int_0^\infty \bar{F}(x \mid t)\, dt$ is decreasing in x. That is, the older the device is, the smaller is its mean residual life and hence $\mu(t)$ is monotonic. However, in many real life situations, the mean residual lifetime is non-monotonic and thus there arise several ageing notions defined in terms of the non-monotonic behavior of $\mu(t)$.

Guess et al. (1986) also defined a life class known as the increasing then decreasing mean residual life (IDMRL). To put in simply, F is IDMRL if $\mu(t) \in$ UBT. The dual class of 'decreasing initially, then increasing mean residual life' (DIMRL) has its $\mu(t) \in$ BT. We will discuss in Chapter 4 various facets of mean residual life, in particular, how the shapes of $r(t)$ and $\mu(t)$ are interrelated.

2.7 Some Further Classes of Ageing

Ageing concepts are proliferating in the literature. In addition to those lifetime classes defined above, there are a number of other ageing classes that have been investigated over the years. Without giving details, we just present their acronyms, definitions and references below.

- IFR(2) (Increasing failure rate of second order) iff
$$\int_0^x \frac{\bar{F}(u+s)\, du}{\bar{F}(s)} \geq \int_0^x \frac{\bar{F}(u+t)}{\bar{F}(t)}\, du, \text{for all } x \geq 0, t \geq s.$$
See Deshpande et al. (1986), Franco et al. (2001).
Clearly IFR $\Rightarrow$ IFR(2) $\Rightarrow$ DMRL.
- NBU(2) (New better than used of second order) iff
$$\int_0^x \bar{F}(u)\, du \geq \int_0^x \frac{\bar{F}(t+u)\, du}{\bar{F}(t)} \text{ for all } x, t \geq 0.$$
See Deshpande et al. (1986), Franco et al. (2001), Li and Kochar (2001), Hu and Xie (2002), Li (2004).
Clearly NBU $\Rightarrow$ NBU(2) $\Rightarrow$ NBUE.
- HNBUE(3) (Harmonic new better than used of third order) iff
$$\int_x^\infty \int_t^\infty \bar{F}(u)\, du\, dt \leq \mu^2 e^{-x/\mu} \text{ for all } x, t \geq 0.$$
See Deshpande et al. (1986).

 Clearly HNBUE $\Rightarrow$ HNBUE(2).

- DMRLHA (Decreasing mean residual life in harmonic average) iff

$$\left[(1/t)\int_0^t (1/\mu(x)\,dx\right]^{-1} \text{ is decreasing in } t.$$

See Deshpande et al. (1986).
It can be shown that DMRL $\Rightarrow$ DMRLHA $\Rightarrow$ NBUE.

- SIFR (Stochastically increasing failure rate)
Let Y be a random variable with cdf F and mean μ; $X_i's$ are i.i.d. exponential random variables with the same mean μ and the $X_i's$ are independent such that $X_0 \equiv 0$. Then F is said to be stochastically increasing failure rate if

$$\Pr\left(Y \geq \sum_{i=0}^{k+1} X_i / Y \geq \sum_{i=0}^{k} X_i\right) \leq \Pr\left(Y \geq \sum_{i=0}^{k} X_i / Y \geq \sum_{i=0}^{k-1} X_i\right)$$

for all $k = 1, 2, ...$ See Singh and Deshpande (1985).

- SNBU (Stochastically new better than used). With the preceding assumptions, F is said to be stochastically new better than used if

$$\Pr\left(Y \geq \sum_{i=0}^{k+1} X_i / Y \geq \sum_{i=0}^{k} X_i\right) \leq \Pr(Y \geq X_{k+1})$$

for all $k = 1, 2, ...$ See Singh and Deshpande (1985).
It has been shown that (i) IFR $\Rightarrow$ SIFR, (ii) NBU $\Rightarrow$ SNBU, and (iii) SIFR $\Rightarrow$ SNBU.

- BMRL-t_0 (Better mean residual life at t_0 class). The mean life declines during the time 0 to t_0, and thereafter is no longer greater than what it was at t_0. See Kulasekera and Park (1987).

- DVRL (Decreasing variance of residual life) iff $\sigma^2(t) \leq \sigma^2(s)$ for all $s \leq t$ where $\sigma^2(t) = \mathrm{var}(X - t|X \geq t)$ is the variance of the residual life. See Launer (1984).

- DPRL-α (Decreasing 100α percentile residual life) iff the α-percentile residual life $q_{\alpha,F}(t)$ defined by

$$q_\alpha(t) = \inf\{x : F_t(x) \geq \alpha\}, 0 < \alpha < 1, F(0) = 0$$

is decreasing in $t \in [0, T)$. Here $F_t = 1 - \bar{F}(t+x)/\bar{F}(t), x \geq 0$. See Joe and Proschan (1984).
F is IFR $\Leftrightarrow$ F is DPRL-α for all $0 < \alpha < 1$.

- NBUP-α (New better than used with respect to the 100α percentile) iff $F(0) = 0$ and $q_\alpha(t) \leq q_\alpha(0)$ for all $t \in [0, T)$. See Joe and Proschan (1984).
It is easy to see that

* DPRL-$\alpha \Rightarrow$ NBUP-α for any $0 < \alpha < 1$.
* NBU $\Leftrightarrow$ NBUP-α for all $0 < \alpha < 1 \Rightarrow$ NBUE.

Joe and Proschan (1984) provided other chains of relationships but we will not elaborate on them here.

- IFR $* t_0$ (IFR after t_0) iff $\bar{F}(bx) \geq \bar{F}^b(x)$ for all $x \geq t_0 > 0$ and all $t_0/x \leq b \leq 1$. See Li and Li (1998). It is easy to follow that IFR $\Rightarrow$ IFR $* t_0$.
- NBU $* t_0$ (NBU after t_0) iff $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$ for all $x \geq 0$ and $y \geq t_0 > 0$. See Li and Li (1998).
It is easy to follow that NBU $\Rightarrow$ NBU $* t_0$.
- UBA (used better than aged) if $\bar{F}(x+t) \geq \bar{F}(x)e^{-t/\mu(\infty)}$ for all $x, t \geq 0$ and UBAE (used better than aged in expectation) if $\mu(t) \geq \mu(\infty)$ for all $t \geq 0$ assuming $0 < \mu(\infty) < \infty$. See Alzaid (1994), Willmot and Cai (2000) and Ahmad (2004).
It has been shown that DMRL $\Rightarrow$ UBA $\Rightarrow$ UBAE $\Leftarrow$ DVRL.

Nearly every one of these additional ageing classes is sandwiched in between two well-known classes discussed in Section 2.4.2. There are rarely any known distributions that belong to these further classes which are not already in the established classes. Apart from the DVRL class, we do not find these ageing concepts intuitive or easily interpretable. It is conceivable that more meaningful applications may emerge in future.

2.8 Failure Rates of Mixtures of Distributions

Interest on the ageing behaviour of mixtures has a long history (Barlow and Proschan, pp. 161-164). Mixtures arise from heterogeneous populations. A typical case is where a population consists of two subpopulations (which may be referred to as components of the mixtures). Mixtures also arise when we pool data from several distributions to enlarge the sample, for example. Mixtures are important in burn-in (see Block and Savits, 1997). Although mixtures of DFR distributions are always DFR, some mixtures of IFR may also be DFR. A well-known 'border line' example by Proschan (1963) exhibits the strict DFR property of a mixture of exponential distributions that have constant failure rates. Barlow (1985) and Mi (1998a), respectively, gave a Bayesian and non-Bayesian explanation of this unexpected phenomenon. Gurland and Sethuraman (1995) considered various types of finite and continuous mixtures of IFR distributions and developed conditions for such mixtures to be 'ultimately' DFR.

The density function of a mixture from two subpopulations with density functions f_1 and f_2 is simply given by

$$f(t) = pf_1(t) + (1-p)f_2(t), \;\; t \geq 0, 0 \leq p \leq 1; \tag{2.71}$$

and thus the survival function of a mixture is also a mixture of the two survival functions, i.e.,

$$\bar{F}(t) = p\bar{F}_1(t) + (1-p)\bar{F}_2(t). \tag{2.72}$$

The mixture failure rate $r(t)$ obtained from failure rates $r_1(t)$ and $r_2(t)$ associated with f_1 and f_2, respectively, can be expressed as

$$r(t) = \frac{pf_1(t) + (1-p)f_2(t)}{p\bar{F}_1(t) + (1-p)\bar{F}_2(t)} \tag{2.73}$$

where $f_i(t), \bar{F}_i(t)$ are the probability density and survival function of the distribution having failure rate $r_i(t), i = 1, 2$.

Let $r(t)$ in (2.73) be expressed as

$$r(t) = h(t)r_1(t) + (1-h(t))r_2(t) \tag{2.74}$$

where $h(t) = 1/(1+g(t)), \quad g(t) = (1-p)\bar{F}_2(t)/[p\bar{F}_1(t)]$. Clearly $0 \le h(t) \le 1$.

We can easily generalize the above equation to accommodate mixtures of k subpopulations giving

$$r(t) = \frac{\sum_{i=1}^{k} p_i f_i(t)}{\sum_{i=1}^{k} p_i \bar{F}_i(t)} \tag{2.75}$$

where $i = 1, 2, ...k$, $0 < p_i < 1$, $\sum_{i=1}^{k} p_i = 1, k \ge 2$.

2.8.1 Mixture of Two DFR Distributions

A mixture of two DFR is again DFR; this result has been proved by Barlow et al. (1963) and other authors but we think Gupta and Warren's (2001) approach is simpler.

On differentiation of (2.74), we can be verify that

$$r'(t) = h(t)r_1'(t) + (1-h(t))r_2'(t) - h(t)(1-h(t))(r_1(t) - r_2(t))^2. \tag{2.76}$$

(Navarro and Hernnandez (2004) noted that the original expression (3.2) in Gupta and Warren (2001) was incorrect.) Since $0 \le h(t) \le 1$, it is obvious from (2.76) that the mixtures of DFR is DFR. Also Theorem 1 of Gurland and Sethuraman (1995) can now be easily obtained.

2.8.2 Possible Shapes of $r(t)$ When Two Subpopulations Are IFR

The behavior of mixtures of IFR distributions is unintuitive. It is easy to find examples of subpopulations with increasing failure rate whose mixture can have either increasing, decreasing or other shapes.

The following are taken from Block, Li and Savits (2003a) to illustrate that a variety of shapes can occur even for a simple mixture of two IFR subpopulations.

Example 2.1 We consider two IFR Weibull distributions $f_1(t) = 2t \exp\left\{-t^2\right\}$ and $f_2(t) = 3t^2 \exp\left\{-t^3\right\}$ with $p = 0.5$. In this case $r(t)$ is IFR.

The next example show that a mixture of two IFR distribution gives rise to a DFR distribution

Example 2.2 Let $r_1(t) = 1 - \exp\{-5t\}$, $r_2(t) = 6 - \exp\{-5t\}$ with $p = 0.5$. We note that $r_1(t)$ strictly increases to 1 and $r_2(t)$ strictly increases to 6. However, $r(t) \in \text{DFR}$ strictly decreases to 1.

Example 2.3 Take $f_1(t) = \exp\{-t\}$ to be exponential, $f_2(t) = 16t \exp\{-4t\}$ to be an IFR gamma and let $p = 0.5$. In this case, $r(t) \in \text{UBT}$, i.e., $r_m(t)$ has an upside-down bathtub shape.

Example 2.4 Let $f_1(t) = 4 \exp\{-4t\}$ be exponential, $f_2(t) = t \exp\{-t\}$ be an IFR gamma and $p = 0.5$. Then $r(t) \in \text{BT}$.

Example 2.5 Consider two Weibull distributions, $f_1(t) = 2t \exp\left\{-t^2\right\}$ and $f_2(t) = 4t^3 \exp\left\{-t^4\right\}$ with $p = 0.5$. Both $r_1(t)$ and $r_2(t)$ increase to ∞. The mixture failure rate $r(t) \in \text{MBT}$. This phenomenon is also noted in Jiang and Murthy (1998).

The above examples show how the shape of the mixture failure rate varies as the subpopulation failure rates and the mixing proportion $p = 0.5$ remains fixed. The next example keeps the failure rates of the subpopulation fixed while varying the mixing proportion p.

Example 2.6 Consider two increasing linear failure rates $r_1(t) = t + 1$ and $r_2(t) = 4t + 5$. Block, Li and Savits (2003a) showed that

- $r(t) \in \text{MBT}$ for $p = 0.1$,
- $r(t) \in \text{BT}$ for $p = 0.65$,
- $r(t) \in \text{IFR}$ for $p = 0.95$.

Gurland and Sethuraman (1995) gave a necessary and sufficient condition for a mixture of two IFR distributions to be DFR. However, the condition does not appear to be easily verified.

We next consider mixtures of two life distributions from the same family of distributions possibly with different parameters.

2.8.3 Mixture of Two Gamma Densities with a Common Scale Parameter

Consider the mixture of two gamma densities:

$$f(t) = pf_1(t) + (1-p)f_2(t), \tag{2.77}$$

where

$$f_i(t) = \frac{\lambda^{\alpha_i} t^{\alpha_i - 1}}{\Gamma(\alpha_i)} e^{-\lambda t}, t > 0,\ \alpha_i, \lambda > 0, i = 1, 2. \tag{2.78}$$

Assuming $\alpha_1 < \alpha_2$, Glaser (1980) was able to determine the shape of the failure rate of the distribution specified by (2.77) in all cases except for one: $\alpha_1 > 1, \alpha_2 - \alpha_1 > 0$ with $\alpha_1 - 1 < (\alpha_2 - \alpha_1 - 1)^2/4$. For this case, he conjectured that the mixture density is IFR.

Gupta and Warren (2001) generalized the result of Glaser (1980) and also disproved the above conjecture. We now summarize the results of Glaser (1980) and that of Gupta and Warren (2001) as follows.

Theorem 2.8: Assuming $\alpha_1 < \alpha_2$, the gamma mixture has the following failure rate shapes:

(i) $0 < \alpha_1 < \alpha_2 \leq 1$ implies that F is DFR,
(ii) $0 < \alpha_1 < 1 < \alpha_2$ implies that F is BT,
(iii) $\alpha_1 = 1 < \alpha_2 \leq 2$ implies that F is IFR,
(iv) $\alpha_1 = 1, 2 < \alpha_2$ implies that F is BT,
 and for $1 < \alpha_1 < \alpha_2$,
(v) $\sqrt{\alpha_2 - 1} - \sqrt{\alpha_1 - 1} \leq 1$ implies that F is IFR, while
(vi) $\sqrt{\alpha_2 - 1} - \sqrt{\alpha_1 - 1} > 1$ implies that F is IFR or MBT.

Proof: The first five cases were proved by Glaser (1980) by considering the behavior of $\eta'(t)$ with $\eta(t) = f'(t)/f(t)$ as defined in (2.9). Case (vi) was incorrectly conjectured by Glaser but proved by Gupta and Warren (2001) via Theorem 2.3 above.

Remark: We have not listed the case when $\alpha_1 = \alpha_2 = 1$. This corresponds to the mixture of two exponential distributions. If we assume the scale parameters are the same as in the theorem, then the mixture has an exponential which is both IFR and DFR. However, if the two shape parameters are different, then the mixture has a DFR distribution as observed by Proschan (1963).

2.8.4 Mixture of Two Weibull Distributions

Jiang and Murthy (1998) categorized the possible shapes of failure rate function for a mixture of any two Weibull distributions in terms of five parameters. The failure rate shape can be one of eight different types including IFR, DFR, MBT, UBT and 'roller-coaster' shaped. They showed that, among other authors, this mixture distribution F cannot have a BT failure rate. They also stated that the mixture failure rates from two strictly IFR Weibull distributions with the same shape parameter can be either MBT or IFR. However, they did not explicitly classify the two possibilities. Wondmagegnehu (2004) developed the work of Jiang and Murthy (1998) further but assumed the two Weibull distributions to be strictly IFR.

Let $\bar{F}_1(t) = \exp\{-\theta_1 t^\alpha\}$ and $\bar{F}_2(t) = \exp\{-\theta_2 t^\alpha\}$ be the survival functions of two Weibull distributions so that $r_1(t) = \theta_1 \alpha t^{\alpha-1}$ and $r_2(t) = \theta_2 \alpha t^{\alpha-1}$ where $\alpha > 1$ and $\theta_2 > \theta_1 > 0$. Set $\beta = \theta_2/\theta_1$ and define

$$\omega_1 = \alpha(\beta-1) + \sqrt{\alpha^2(\beta-1)^2 + 4(\alpha-1)^2\beta}$$

and

$$\omega_2 = 2\alpha(\beta-1)\exp\left\{\frac{(\alpha-1)(\beta+1) + \sqrt{\alpha^2(\beta-1)^2 - 4(\alpha-1)^2\beta}}{\alpha(\beta-1)}\right\}.$$

Theorem 2.9: Let ω_1 and ω_2 be defined as above and p be the mixing proportion. Further, we define

$$\xi = \frac{\omega_1}{\omega_1 + \omega_2}.$$

Then the mixture failure rate $r(t)$ has

(a) a modified bathtub (MBT) shaped failure rate when $0 < p < \xi$ and

(b) an increasing failure rate (IFR) when $\xi \leq p < 1$.

Proof: From (2.73), it is easy to verify that

$$r(t) = \theta_1 \alpha t^{\alpha-1} + \frac{(1-p)(\beta-1)\theta_1 \alpha t^{\alpha-1}}{p e^{\theta_1(\beta-1)t^\alpha} + (1-p)}. \tag{2.79}$$

Letting $z = e^{\theta_1(\beta-1)t^\alpha}$, we have $t = [[1/\{\theta_1(\beta-1)\}]\log z]^{1/\alpha}$. Substituting this expression of t and $b = p(1-p)$ into (2.79) gives

$$r^*(z) = \left[\frac{bz+\beta}{bz+1}\right]\left(\frac{1}{\theta_1(\beta-1)}\log z\right)^{(\alpha-1)/\alpha}$$

where $r^*(t) = r(t)/(\theta_1\alpha)$.

Both $r(t)$ and $r^*(z)$ have the same monotonicity in the corresponding domains $\{t : t \geq 0\}$ and $\{z : z \geq 0\}$, respectively. It is now easier to study the shape of $r(t)$ via $r^*(z)$. Taking logarithm of $r^*(z)$ and then differentiating it with respect to z, we find

$$\frac{d}{dz}\log r^*(z) = \frac{K(z)}{\alpha z(\log z)(bz+\beta)(bz+1)},$$

where

$$K(z) = (\alpha-1)(bz+\beta)(bz+1) - b\alpha(\beta-1)z\log z.$$

We then examine the derivative $K'(z) = 2b(\alpha - 1)z[b - h(z)]$. The theorem can then be established, after several tedious steps, by considering the behaviour of $h(t)$. See Wondmagegnehu (2004) for a complete proof.

Wondmagegnehu (2004) also used several examples to illustrate possible shapes that the mixture failure rate can encounter when the two Weibull distributions have different shape and scale parameters.

2.8.5 Mixtures of Two Positively Truncated Normal Distributions

Navarro and Hernandez (2004) have considered the shape of the failure rate of the mixture of two positively truncated normal distributions given in Section 2.3.3. The method is based on the s-order equilibrium distribution of a renewal process defined by Fagiuoli and Pellerey (1993) and will be given in Section 2.9.1 below.

For a truncated normal distribution with parameter μ and σ, it is noted by Navarro and Hernandez (2004) that its eta function defined by the negative value of the derivative of the density over the density is given by $(t-\mu)/\sigma^2$. Let f_i be the two truncated normal densities with parameters μ_i and $\sigma_i, i = 1, 2$, and f be the density of the mixture and $\eta = -f'/f$. Also, let

$$w(t) = \frac{1}{1 + \alpha(t)},$$

where

$$\alpha(t) = \frac{1-p}{p}\frac{f_2(t)}{f_1(t)}.$$

Assuming that $\sigma_1 = \sigma_2$ and letting $\delta = \sigma_1^2/(\mu_2 - \mu_1)^2$, the authors showed that

1. If $\delta > 1/4$, then $r(t) \in$ I.
2. If $\delta \leq 1/4, w(0) \geq 0$, and $w(0)(1 - w(0) < \delta$, then $r(t) \in$ I.
3. If $\delta \leq 1/4, w(0) \geq 0$, and $w(0)(1 - w(0) \geq \delta$, $r(t) \in$ I or BT.
4. If $\delta \leq 1/4, w(0) < 1/2$, and $w(0)(1 - w(0) > \delta$, then $r(t) \in$ I or BT.
5. If $\delta \leq 1/4w(0) < 1/2$, and $w(0)(1 - w(0) \leq \delta$, then $r(t) \in$ I, BT or MBT.

Moreover, the change points of η are determined by $w(t)(1 - w(t)) = 0$.

The key ingredient of the proof for the above results hinges on the fact that if $\sigma_1 = \sigma_2$,

$$\sigma_1^2\eta'(t) = 1 - (1 - w(t))w(t)\frac{(\mu_2 - \mu_1)^2}{\sigma_1^2}$$

and $\eta'(t) \geq 0$ iff $(1 - w(t))w(t) \leq \delta$.

Navarro and Hernandez (2004) also obtained a general result on the shape of $r(t)$ when the variances are not equal and $\eta_2(t) \geq \eta_1(t)$. The proof now requires the theory of s-order equilibrium distribution of the renewal process mentioned which will be further developed in Section 2.9 and Section 2.11.

2.8.6 Mixtures of Two Increasing Linear Failure Rate Distributions

Block, Savits and Wondmagegnehu (2003) gave explicit conditions which delineate the possible shapes of the failure rate function for the mixture of two IFR linear failure rate distributions.

Let the two increasing linear failure rates be given by, respectively,

$$r_1(t) = c_1 t + d_1, \quad r_2(t) = c_2 t + d_2,$$

where, without loss of generality, we assume $c_2 \geq c_1 > 0$ and $d_2, d_1 \geq 0$. Thus the expressions for the two component survival functions are, respectively,

$$\bar{F}_1(t) = \exp\left\{-\frac{c_1 t^2}{2} - d_1 t\right\},$$

and

$$\bar{F}_2(t) = \exp\left\{-\frac{c_2 t^2}{2} - d_2 t\right\}.$$

Substituting the expressions for $\bar{F}_1(t), \bar{F}_2(t)$, $r_1(t), r_2(t)$ into (2.74), we have

$$r(t) = (c_1 t + d_1) + \frac{(1-p)[c_1(\gamma-1)t + a]}{p\exp((c_1/2)(\gamma-1)^2 + at) + (1-p)}, \tag{2.80}$$

where $\gamma = c_2/c_1 \geq 1, c_2 \geq c_1 > 0$, $a = d_2 - d_1$ and $0 < p < 1$. It turns out all the possible shapes are determined by these parameters: $\gamma = c_2/c_1$, $\delta = a/\sqrt{c_1}$ and p. Define the following parameters:

$$\alpha_1 = \frac{(\delta^2 + \gamma - 1) - \sqrt{(\delta^2 - \gamma - 1)^2 - 4\gamma}}{2\delta^2},$$

$$\alpha_1 = \frac{(\delta^2 + \gamma - 1) + \sqrt{(\delta^2 - \gamma - 1)^2 - 4\gamma}}{2\delta^2},$$

$$\alpha_3 = \frac{(\gamma-1) + \beta}{(\gamma-1) + \beta + \exp\{((\gamma+1) - \delta^2 + 2\beta)/2(\gamma-1)\}}$$

where $\beta = \sqrt{(\gamma-1)^2 + \gamma}$.

Non-crossing linear failure rates

We now consider the mixture failure rate for two non-crossing linear failure rates. Its possible shapes can be summarized as follows:

Theorem 2.10: (Block, Savits and Wondmagegnehu, 2003). Consider the mixture failure rate $r(t)$ given in (2.50) for two non-crossing linear failure rates $r_1(t) = c_1 t + d_1$ and $r_2(t) = c_2 t + d_2$ such that $c_2 > c_1 > 0$ and $d_2 > d_1 \geq 0$. Recall the expressions for α_1, α_2 and α_3 are given as above.

(i) If $\delta \leq \sqrt{\gamma} + 1$, then
  (a) $r(t) \in$ BT if $0 < p < \alpha_3$, or
  (b) $r(t) \in$ IFR if $\alpha_3 < p < 1$.

(ii) If $\sqrt{\gamma} + 1 < \delta < \sqrt{(\gamma + 1) + 2\beta}$, then
  (a) $r(t) \in$ MBT if $0 < p < \alpha_1$,
  (b) $r(t) \in$ BT if $\alpha_1 < p \leq \alpha_2$,
  (c) $r(t) \in$ MBT if $\alpha_2 < p < \alpha_3$, or
  (d) $r(t) \in$ IFR if $\alpha_3 \leq p < 1$.

(iii) If $\delta \geq \sqrt{(\gamma + 1) + 2\beta}$, then
  (a) $r(t) \in$ MBT if $0 < p < \alpha_1$,
  (b) $r(t) \in$ BT if $\alpha_1 < p \leq \alpha_2$, or
  (c) $r(t) \in$ IFR if $\alpha_2 \leq p < 1$.

Proof: The proof is rather lengthy. See Block, Savits and Wondmagegnehu (2003).

Linear failure rates with same slope

We now assume the two failure rate functions have the same slope, that is, $c_1 = c_2 = c$. Define

$$\delta = \frac{a}{\sqrt{c}} > 0, \quad \zeta_1 = \frac{1 - \sqrt{1 - 4/\delta^2}}{2}, \quad \zeta_2 = \frac{1 + \sqrt{1 - 4/\delta^2}}{2}.$$

Block, Savits and Wondmagegnehu (2003) showed that

(i) If $0 < \delta \leq 2$, then $r(t) \in$ IFR for all $p \in (0, 1)$.

(ii) If $\delta > 2$, then
  (a) $r(t) \in$ MBT if $p < \zeta_1$;
  (b) $r(t) \in$ BT if $\zeta_1 < p < \zeta_2$;
  (c) $r(t) \in$ IFR if $\zeta_2 < p < 1$.

Two failure rates with the same y-intercepts

We now consider the case $d_1 = d_2 = d$. Define

$$\xi = \frac{(\gamma - 1) + \beta}{(\gamma - 1) + \beta + \exp\{((\gamma + 1) - \delta^2 + 2\beta)/2(\gamma - 1)\}}$$

where $\beta = \sqrt{(\gamma - 1)^2 + \gamma}$. Block, Savits and Wondmagegnehu (2003) showed that

(i) $r(t) \in$ MBT if $0 < p < \xi$;

(ii) $r(t) \in$ IFR if $\xi < p < 1$.

Mixtures of crossing failure rates

Two linear failure rates may cross at the point $t_0 = -a/c_1(\gamma - 1)$ so that $r_1(t) > r_2(t)$ for all $t \in [0, t_0)$ and $r_1(t) < r_2(t)$ for all $t > t_0$. Block, Savits and Wondmagegnehu (2003) delineated all possible shapes that the mixture failure rate $r(t)$ can take assuming the two linear failure rates intersects at t_0.

2.8.7 Mixtures of an IFR Distribution with an Exponential Distribution

Gurland and Sethuraman (1995) gave the following definition:

Definition 2.12: An IFR distribution is said to be MRE if its mixture with an exponential is 'ultimately' DFR for some mixing proportion p. This means that for sufficiently large t, says $t \geq t_0$, the mixture failure rate is decreasing.

Gurland and Sethuraman (1995) also provided several examples that are MRE, some of these are now listed below.

Examples

Exponential distribution

It is a well known that the mixture of two exponential distributions is DFR (Proschan, 1963).

Gamma distribution

Let the density function of the IFR gamma distribution be

$$f_1(t) = \frac{\lambda_1^\alpha}{\Gamma(\alpha)} e^{-\lambda_1 t} t^{\alpha - 1}, \ \lambda_1 > 0, \alpha > 1, t > 0$$

and $f_2(t) = \lambda_2 e^{-\lambda_2 t}$. Then the gamma mixture with exponential is MRE for large t. Further, if $\lambda_1 > \lambda_2$, the mixture is DFR.

Weibull distribution

Let $\bar{F}_1(t) = e^{-\theta t^\alpha}$, $\theta, \alpha > 0, t > 0$ be the survival function of the Weibull distribution with failure rate given by $r_1(t) = \theta\alpha t^{\alpha-1}$. Gurland and Sethuraman (1995) showed that for $\alpha > 1$ the Weibull distribution is MRE. Let $r(t,p)$ be the failure rate of the mixture $p\bar{F}_1(t)+(1-p)\bar{F}_2(t)$ of a Weibull with an exponential having density $f_2(t) = \lambda e^{-\lambda t}$. Thus $r(t,p) = pr_1(t) + (1-p)r_2(t) = p\theta\alpha t^{\alpha-1} + (1-p)\lambda$. Also, there is a turning point $t_0 = t_0(p)$ such that $r(t,p)$ is decreasing for $t \geq t_0$. Using $\alpha = 3, \theta = 2.5$ and $\lambda = .25$, they gave plots of $r(t,p)$ for $p = .05, .1, .5, .7, .9, .95$. They observed that the turning point $t_0(p)$ of $r(t,p)$ decreases as p increases, because the exponential plays an increasingly important role in the mixture and, accordingly, fewer items have failed. We also observe that $r(t,p)$ tends to λ as $t \to \infty$.

The authors also gave an intuitive explanation of the MRE phenomenon having $r(t,p)$ decreasing for large t. This is because the early failure times of the mixture come from the distribution with larger failure rates, so that the larger failure times (in the tail of mixture) come from the distribution that has smaller failure rate in the tail. They also sounded a warning that the practice of pooling several IFR distributions may reverse the IFR property of the individual samples to a DFR property.

2.8.8 Failure Rate of Finite Mixture of Several Components Belonging to the Same Family

Al-Hussaini and Sultan (2001) gave a comprehensive review on reliability and failure rates of mixture models. Seven finite mixture models in which the components belonging to the same family of distributions are investigated. These are

1. Mixtures of normal components.
2. Mixtures of lognormal components.
3. Mixtures of inverse Gaussian components.
4. Mixtures of exponential components.
5. Mixtures of Rayleigh components.
6. Mixtures of Weibull components.
7. Mixtures of Gompertz components.

Three of this list have already been considered above. Plots of failure rates (with selected parameter values) of the mixture of two components are also given in Al-Hussaini and Sultan (2001). We observe that, nearly every one of these figures has an upside-down bathtub shape. This is reminiscent of the findings of Gurland and Sethuraman (1995) who stated in their Introduction section that ‘ many standard families of IFR distributions exhibit the property that the mixtures of two distributions from the same family are ultimately DFR’. We need, however, to put this statement in perspective, as we have seen in Section 2.8.1 that several other shapes are also possible for a mixture of two lifetime components.

Note: Al-Hussaini and Sultan (2001) also considered mixture models with components belonging to different families.

2.8.9 Initial and Final Behavior of Failure Rates of Mixtures

Block et at. (2001) reviewed the behavior of the failure rate of mixtures $r_m(t)$ of several components. This review mainly draws on the results of Block and Joe (1997), Block et al. (1993) and Block, Li and Savits (2003a). We now summarize them below.

Initial behavior of $r(t)$

Let the failure rate of the finite mixtures be given by

$$r(t) = \frac{\sum_{i=1}^{n} p_i f_i(t)}{\sum_{i=1}^{n} p_i \bar{F}_i(t)}. \tag{2.81}$$

Block, Li and Savits (2003a) showed that

$$r(0+) = \sum_{i=1}^{n} p_i r_i(0+) \tag{2.82}$$

assuming $r(0+) = f_i(0+)$ exists; and

$$r'(t) = \sum_{i=1}^{n} p_i f_i'(0+) + \left[\sum_{i=1}^{n} f_i(0+)\right]^2, \tag{2.83}$$

assuming both $r(0+)$ and $f_i'(0+)$ exist. Note that we have used $r_i(0+)$ to denote $\lim_{t\downarrow 0} r_i(t)$. The other limits are also defined similarly.

The authors also showed that the initial behavior of the failure rate r_ϕ of a system of several components is similar to the failure of the mixture.

Asymptotic behavior of $r_m(t)$

The asymptotic behavior of mixtures of exponentials has been studied by Clarotti and Spizzichino (1990) and more generally by Block et al. (1993).

(a) The first general result on mixtures of distributions is that, under mild conditions, the asymptotic limit of the failure rate of a mixture is the same as the limit of strongest component.

(b) The second result, due to Block and Joe (1997), is that for failure rates, which asymptotically behave like the ratios of polynomials, the eventual monotonicity of the mixture is the same as the monotonicity of the strongest component. Furthermore, if the failure rate of the strongest component is increasing, so is the failure rate of the mixture.

A similar result holds for systems except the role of the strongest subpopulation is replaced by strongest minimal path set.

Block, Li and Savits (2003a) gave a definite result concerning the limit of the failure rate of a mixture of two components. The result is now stated as below.

Theorem 2.11: Consider a mixture of two subpopulations with failure rates $r_1(t) \to \lambda \in [0, \infty)$ and $r_2(t) \to \infty$ such that

$$r(t) = \frac{pf_1(t) + (1-p)f_2(t)}{p\bar{F}_1(t) + (1-p)\bar{F}_2(t)}$$

(as given in (2.73)) converges to ξ as $t \to \infty$. Then ξ must be finite and equal to λ.

Proof: Set $\bar{F}(t) = p\bar{F}_1(t) + (1-p)\bar{F}_2(t)$. From the assumption, we deduce that

$$-\frac{\log \bar{F}(t)}{t} = \frac{1}{t}\int_0^t r(u)\,du \to \xi$$

and

$$\frac{\log \bar{F}_1(t)}{t} = \frac{1}{t}\int_0^t r_1(u)\,du \to \lambda$$

as $t \to \infty$. Since $\log \bar{F}(t) \geq \log\big(p\bar{F}_1(t)\big) = \log p + \log \bar{F}_1(t)$, we find that

$$\xi = -\lim_{t\to\infty} \frac{\log \bar{F}(t)}{t} \leq -\liminf_{t\to\infty} \left\{\frac{\log p}{t} + \frac{\log \bar{F}_1(t)}{t}\right\} = \lambda.$$

Thus, $\xi \leq \lambda < \infty$.

On the other hand, since $r_2(t) - r_1(t) \to \infty$,

$$\frac{\bar{F}_2(t)}{\bar{F}_1(t)} = \exp\left\{-\int_0^t [r_2(t) - r_1(t)]\,du\right\} = \mathrm{O}(e^{-Kt})$$

for all $K > 0$. In particular, $\bar{F}_2(t)/\bar{F}_1(t) \to 0$ as $t \to \infty$. It now follows that

$$\frac{f(t)}{\bar{F}_1(t)} = r(t)\left\{\frac{p\bar{F}_1(t) + (1-p)\bar{F}_2(t)}{\bar{F}_1(t)}\right\} \to p\xi$$

where $f(t) = pf_1(t) + (1-p)f_2(t)$. As $f(t) \geq pf_1(t)$, we have

$$p\xi = \lim_{t\to\infty}\left[\frac{f(t)}{\bar{F}_1(t)}\right] \geq \limsup_{t\to\infty}\left[\frac{pf_1(t)}{\bar{F}_1(t)}\right] = p\lambda,$$

i.e., $\xi \geq \lambda$ so $\xi = \lambda$.

Corollary 2.1: Let $r(t)$ be the failure rate of a finite mixture of subpopulations with failure rates $r_i(t)$ satisfying $r_i(t) \to a_i \in [0,\infty], 1 \leq n$, as $t \to \infty$. Then either $r(t) \to \alpha = \min_{1\leq i\leq n} a_i$, or $r(t)$ does not converge.

Proof: See Corollary 2.1 of Block et al. (2003a).

2.8.10 Continuous Mixtures of Distributions

Shaked and Spizzichino (2001) gave a review on the failure rate function of a continuous mixture.

Continuous mixtures of DFR distributions

It has been shown, see for example, Barlow and Proschan (1981, p. 103), that continuous mixtures of DFR distributions is DFR. More specifically, let F_α be the distribution function with parameter α. Suppose α itself is a random variable with distribution $M(\alpha)$, then the resultant distribution (generally known as the mixture distribution) is expressed as

$$F(t) = \int_{-\infty}^{\infty} F_\alpha(t)\, dM(\alpha).$$

M is called the mixing distribution. If each F_α is DFR, then mixture distribution F is DFR irrespective of the mixing distribution. The proof utilizes the fact that the hazard transform of a mixture is concave.

For example, let $\bar{F}_\alpha(t)$ be an exponential survival function with α being a gamma random variable having density $m(\alpha) = [\beta^\gamma \alpha^{\gamma-1}/\Gamma(\gamma)]e^{-\beta\alpha}$, $0 \leq \alpha < \infty$. Then the resulting distribution has a Pareto survival function $\bar{F}(t) = (1+t/\beta)^{-\gamma}$ which is DFR. A partial converse to this result is stated by Gleser (1989) showing that a gamma distribution with scale parameter λ and shape parameter $\beta \leq 1$ (i.e., having decreasing failure rate) can be expressed as a scale mixture of exponential distributions

$$f(x) = \int_0^\infty g_\lambda(\gamma)\gamma e^{-\gamma x}\, d\gamma,\ \lambda > 0,$$

where

$$g_\lambda(\gamma) = \frac{(\gamma-\lambda)^{-\beta}\lambda^\beta}{\gamma\Gamma(1-\beta)\Gamma(\beta)},\ \gamma \geq \lambda.$$

Continuous mixtures of IFR distributions

We have seen earlier that a mixture of two IFR distributions may be IFR, DFR or other ageing classes. Gurland and Sethuraman (1994) have given some examples of finite mixtures of rapidly increasing failure rate distributions but the resultant mixture distributions ultimately having decreasing failure rate.

Lynch (1999) gave some general conditions under which the IFR property is preserved by continuous mixtures. His result can be restated as follows. Let $\{\bar{F}(t\,|\,\theta) : \theta \geq 0\}$ be a family of survival functions with univariate parameter $\theta \geq 0$. Let M be the mixing distribution on $[0, \infty)$. The resultant mixture survival function is as below:

$$\bar{F}_M(t) = \int \bar{F}(t\,|\,\theta)\, dM(\theta).$$

The main result of Lynch (1999) is that if the mixing distribution M has an IFR distribution and if $\bar{F}(t\,|\,\theta)$ is log concave in the variables (t, θ), then $F_M(t)$ is IFR.

Block, Li and Savits (2003b) showed that Lynch's result is a special case of Savits (1985) with the correct interpretation. They also showed that similar closure theorems are possible for other ageing classes such as IFRA, NBU and DMRL.

Finkelstein and Esaulova (2001) considered several types of continuous (infinite) mixtures of IFR distribution. In particular, the corresponding limiting behavior of the mixture failure rate function is analyzed for the specific case of mixing which can be interpreted in terms of the proportional hazard model. It is found that under certain assumptions the mixture failure rate decreases to zero as $t \to \infty$.

2.9 Partial Orderings and Generalized Partial Orderings

Partial orderings of two life distributions have been studied quite extensively. Essentially, we are comparing two lifetime variables X and Y in terms of their failure rates $r_F(t)$ and $r_G(t)$, density functions $f(t)$ and $g(t)$, survival functions $\bar{F}(t)$ and $\bar{G}(t)$, mean residual lives $\mu_F(t)$ and $\mu_G(t)$, or other ageing characteristics. Ageing classes can often be characterized by some partial orderings. For example, in Barlow and Proschan (1981, pp.105-107), IFR and IFRA classes are characterized by 'convex ordering' and 'star-shaped ordering', respectively. However, these partial orderings do not fit in with the main body of of our approach so we will not discuss them till Section 10.3. Several authors have studied partial orderings and stochastic dominance, for example, Desphande et al. (1986), Kochar and Wiens (1987), Singh (1989), Fagiuoli and Pellerey (1993), Shaked and Shanthikumar (1994) and several others.

We now give definitions for several of these basic partial orderings. These are selected on the basis that they are easily understood and there is a chain of implications for these orderings to indicate their relative stringency.

Definition 2.13: X is said to be greater than Y in likelihood ratio ordering ($X \geq_{\text{LR}} Y$) if $f(t)/g(t)$ is increasing in $t \geq 0$.

Definition 2.14: X is said to be greater than Y in weak likelihood ratio ordering ($X \geq_{\text{WLR}} Y$) if $f(t)/g(t) \geq f(0)/g(0)$ for all $t \geq 0$.

Definition 2.15: X is said to be greater than Y in failure rate ordering ($X \geq_{\text{FR}} Y$) if $r_F(t) \leq r_G(t)$ for all $t \geq 0$ or $\bar{F}(t)/\bar{G}(t)$ is increasing in $t \geq 0$.

Definition 2.16: X is said to be greater than Y in stochastic ordering ($X \geq_{\text{ST}} Y$) if $\bar{F}(t) \geq \bar{G}(t)$, for all $t \geq 0$.

Definition 2.17: X is said to be greater than Y in mean residual ordering ($X \geq_{\text{MR}} Y$) if $\mu_F(t) \geq \mu_G(t)$, for all $t \geq 0$.

It is found that $X \geq_{\text{MR}} Y$ if and only if $\int_t^\infty \bar{F}(x)\,dx / \int_t^\infty \bar{G}(x)\,dx$ is increasing in $t \geq 0$.

Definition 2.18: X is said to be greater than Y in harmonic average mean residual ordering ($X \geq_{\text{HAMR}} Y$) if $\int_t^\infty \bar{F}(x)\,dx/\mu_F \geq \int_t^\infty \bar{G}(x)\,dx/\mu_G$, for all $t \geq 0$.

Definition 2.19: X is said to be greater than Y in variance residual life ordering ($X \geq_{\text{VR}} Y$) if $\int_t^\infty \int_x^\infty \bar{F}(u)\,du\,dx / \int_t^\infty \int_x^\infty \bar{G}(u)\,du\,dx$ is increasing for all $t \geq 0$.

Definition 2.20: X is said to be greater than Y in convex ordering ($X \geq_{\text{CX}} Y$) if $\int_t^\infty \bar{F}(x)\,dx \geq \int_t^\infty \bar{G}(x)\,dx$, for all $t \geq 0$. It is sometimes known as variable ordering (Ross, 1983). The convex ordering defined here does not appear to be equivalent to the 'convex ordering' given in Barlow and Proschan (1981, p.106).

Definition 2.21: X is said to be greater than Y in concave ordering ($X \geq_{\text{CV}} Y$) if $\int_0^t \bar{F}(x)\,dx \geq \int_0^t \bar{G}(x)\,dx$, for all $t \geq 0$.

Singh (1989) gave a chain of implications between the first eight partial orderings .

It appears that the stochastic ordering and failure rate ordering are two most important ones. For applications for redundancy applications in series, parallel and k-out-of-n systems, see Boland et al. (1992, 1998), Shaked and Shanthikumar (1995) and Boland (1998). Also, a chain of implications will be presented in Table 2.3 connecting these concepts and some of these generalized partial orderings to be defined in the next subsection. We note that there are other partial orderings introduced from different view points by various authors. A summary of these partial orderings is given in Chapter 33 of Johnson et al. (1995). Navarro et al. (1997) also studied some stochastic partial orderings between two doubly truncated variables.

2.9.1 Generlized Partial Orderings

There is a proliferation of generalized partial orderings in the literature over the recent years. Several of these definitions gave rise to new ageing classes some of which are presented in Section 2.7 (e.g., IFR(2), NBU(2) and HNBUE(3)). Here, our discussion are selective rather than inclusive. Fagiuoli and Pellerey (1993) defined several additional partial orderings based on the survival function of an equilibrium distribution of a random variable.

Let X be an absolutely continuous non-negative random variable with distribution function $F(t)$ and differentiable density function $f(t)$. Let $\mu_F(t)$ denote the mean of the lifetime variable X with cdf F. The equilibrium distribution corresponding to the lifetime variable X is defined as

$$E_F(t) = \int_0^t \bar{F}(x)\,dx/\mu_F, \tag{2.84}$$

and we will denote the survival equilibrium function by the function

$$\bar{E}_F(t) = 1 - E_F(t) = \int_t^\infty \bar{F}(x)\,dx/\mu_F. \tag{2.85}$$

We now define the survival functions of the equilibrium distributions recursively:

$$\bar{T}_0(X,x) = f(x),\ \bar{T}_{-1}(X,x) = -f'(x) \tag{2.86}$$

and

$$\bar{T}_s(X,x) = \frac{\int_x^\infty \bar{T}_{s-1}(X,u)\,du}{\mu_{s-1}(X)},\ \text{for integer } s \geq 1, \tag{2.87}$$

where

$$\mu_s(X) = \int_0^\infty \bar{T}_s(X,x)\,dx, s \geq 0. \tag{2.88}$$

The functions $\bar{T}_s(X,x)$ were first introduced by Fagiuoli and Pellerey (1993). Clearly, $\mu_0(X) = 1$. It follows from (2.87) and (2.88) that $\bar{T}_1(X,x) = \bar{F}(x)$, so $\mu_1(X) = E(X)$. Note also $\bar{T}_s(X,0) = 1, s \geq 1$. Further, $\bar{T}_2(X,x)$ is the survival function of the equilibrium distribution of X from which we deduce that $\bar{T}_s(X,x)$ is the survival function of the equilibrium distribution of a distribution with survival function $\bar{T}_{s-1}(X,x)$.

Nanda et al. (1996) have established some interesting properties regarding moments for s-order equilibrium distributions:

$$\mu_0(X)\mu_1(X)...\mu_s(X) = \frac{E(X^s)}{s!},\ s = 0,1,... \tag{2.89}$$

from which we obtain

$$\mu_s = \frac{E(X^s)}{sE(X^{s-1})},\ s = 1,2,...$$

In addition, Fagiuoli and Pellery (1993) also defined what we call the s-order failure rate function

$$r_s(X,x) = \frac{\bar{T}_{s-1}(x)}{\int_x^\infty \bar{T}_{s-1}(u\,du)} = -\frac{\frac{d}{dx}\bar{T}_s(x)}{\bar{T}_s(x)},\ s \geq 0 \tag{2.90}$$

and $r_0(X,x) = \frac{-f'(x)}{f(x)}$ when $f'(x)$ exists. For $s = 1$, $r_1(X,x) = \frac{f(x)}{\bar{F}(x)} = r_F(x)$, the failure rate function that corresponds to X. For $s = 2$, $r_2(X,x) = (\mu_F(x))^{-1}$, the reciprocal of the mean residual life function of X.

Suppose Y is also another random variable with distribution function $G(y)$ and density $g(y)$ having similar properties as X. We can now define four partial orderings in X and Y by comparing their respective equilibrium distributions.

The following four definitions are generalizations of Definitions 2.15, 2.16, 2.20 and 2.21.

Definition 2.22: X is said to be greater than Y in s-FR ordering ($X \geq_{s-\mathrm{FR}} Y$) if $\bar{T}_s(X,t)/\bar{T}_s(Y,t)$ is increasing in $t \geq 0$. This was shown to be equivalent to $r_s(X,x) \leq r_s(Y,t)$, for all $t \geq 0$.

Definition 2.23: X is said to be greater than Y in s-ST ordering ($X \geq_{s-\mathrm{ST}} Y$) if

$$\frac{\bar{T}_s(X,t)}{\bar{T}_s(Y,t)} \geq \frac{\bar{T}_s(X,0)}{\bar{T}_s(Y,0)},\ \text{for all } t \geq 0.$$

This is equivalent to $\bar{T}_s(X,t) \geq \bar{T}_s(Y,t),\ s \geq 1$.

Definition 2.24: X is said to be greater than Y in s-CV ordering ($X \geq_{s-\mathrm{CV}} Y$) if

$$\int_0^t \frac{\bar{T}_s(X,x)\,dx}{\bar{T}_s(X,0)} \geq \int_0^t \frac{\bar{T}_s(Y,x)\,dx}{\bar{T}_s(Y,0)},\ \text{for all } t \geq 0.$$

This is equivalent to

$$\int_0^t \bar{T}_s(X,u)\,du \geq \int_0^t \bar{T}_s(Y,u)\,du,\ s \geq 1.$$

Definition 2.25: X is said to be greater than Y in s-CX ordering ($X \geq_{s-\mathrm{CX}} Y$) if

$$\int_t^\infty \frac{\bar{T}_s(X,x)\,dx}{\bar{T}_s(X,0)} \geq \int_t^\infty \frac{\bar{T}_s(Y,x)\,dx}{\bar{T}_s(Y,0)},\ \text{for all } t \geq 0.$$

This is equivalent to

$$\int_t^\infty \bar{T}_s(X,u)\,du \geq \int_t^\infty \bar{T}_s(Y,u)\,du,\ s \geq 1.$$

We observe that the following equivalence relationships between the classical and generalized partial orderings :

0-FR$\Leftrightarrow$LR, 1-FR$\Leftrightarrow$FR, 2-FR$\Leftrightarrow$MR, 3-FR$\Leftrightarrow$VR, 1-CX$\Leftrightarrow$CX,

0-ST$\Leftrightarrow$WLR, 1-ST$\Leftrightarrow$ST, 2-ST$\Leftrightarrow$HAMR, 1-CV$\Leftrightarrow$CV.

These equivalences show that the generalized orderings defined in Definitions 22–25 are indeed generalizations of those orderings given from Definition 2.13 through Definition 2.21. Although the approach of defining generalized orderings through the survival functions of the s-order equilibrium distribution of X is mathematically novel, we feel that it does not have the same intuitive appeal that is prevalent in the more basic stochastic ordering concepts. On the other hand, we will see in Section 2.11 that the concept of sth order equilibrium distribution can play an important role in fostering a link between the shapes of the two important reliability measures, namely $r(t)$ and $\mu(t)$.

2.9.2 Connections Among the Partial Orderings

We now proceed to give the relationships among some classical orderings as well as with the generalized partial orderings we have just defined. Fagiuoli and Pellerey (1993) have shown that

s-FR $\Rightarrow$ $(s+1)$-FR; s-FR$\Rightarrow$ s-ST$\Rightarrow$ s-CV; s-ST$\Rightarrow$ s-CX.

The proofs of these implications follow directly from the definitions. Fagiuoli and Pellerey (1993) further showed that $(s+1)$-ST $\Rightarrow$ s-CV.

Combining all these relationships, we may summarize them in the following table (bearing in mind 1-FR$\Leftrightarrow$ FR, 1-ST$\Leftrightarrow$ST,1-CX$\Leftrightarrow$CX, and 1-CV$\Leftrightarrow$CV):

Table 2.3. Chains of relationships between partial orderings

LR	$\Rightarrow$	FR	$\Rightarrow$	MR	$\Rightarrow$	VR	$\Rightarrow \cdots$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$	
WLR	$\Rightarrow$ 0-CX $\Leftarrow$	ST	$\Rightarrow$ CX $\Leftarrow$	HAMR	$\Rightarrow$ 2-CX $\Leftarrow$	3-ST	$\Rightarrow$3-CX
$\downarrow$		$\Downarrow$		$\Downarrow$		$\downarrow$	
0-CV		CV		2-CV		3-CV	

2.9.3 Generalized Ageing Properties Classification

In this subsection, we discuss ageing classifications of equilibrium distributions and their relationships to the generalized partial orderings. Recall, the failure rate function that corresponds to the survival function $\bar{T}_s(X,t)$ as defined in (2.90) is:

$$r_s(X,t) = \frac{\bar{T}_s(X,t)}{\int_t^\infty \bar{T}_{s-1}(X,x)\,dx} = \frac{\frac{d}{dt}\bar{T}_s(X,t)}{T_s(X,t)}.$$

Averous and Meste (1989) proposed generalized ageing properties classification (a)–(c) below based on s-tailweight. Fagiuoli and Pellerey (1993) presented same definitions as given by Averous and Meste (1989) but based on $r_s(X,t)$ and $\bar{T}_s(X,t)$ instead. This latter approach seems to be more in line with the traditional way of defining ageing concepts through the failure rate function $r(t)$ and the survival function $\bar{F}(t)$. Thus the concepts to be presented below are in parallel to those basic concepts discussed in Section 2.4.

Definition 2.26: Let s be a non-negative integer and X be a lifetime random variable.

(a) X is said to be s-IFR (s-DFR) if $r_s(X,t)$ is increasing (decreasing) in $t \geq 0$. This is equivalent to $\dfrac{\bar{T}_s(X,x+t)}{\bar{T}_s(X,t)}$ is decreasing (increasing) in t for each $x \geq 0$.

(b) X is said to be s-IFRA (s-DFRA) if $\int_0^t r_s(X,x)\,dx/t$ is increasing (decreasing) in $t \geq 0$. or equivalently, $\left(\bar{T}_s(t)/\bar{T}_s(0)\right)^{1/t}$ is increasing in t. Note that $\bar{T}_s(0) = 1$ for $s \geq 1$.

(c) X is said to be s-NBU (s-NWU) if

$$\bar{T}_s(X,x+t)\bar{T}_s(X,0) \leq (\geq)\, \bar{T}_s(X,x)\bar{T}_s(X,t) \text{ for all } x,t \geq 0.$$

For $s \geq 1$, this becomes

$$\bar{T}_s(X,x+t) \leq (\geq)\, \bar{T}_s(X,x)\bar{T}_s(X,t).$$

(d) X is said to be s-NBUFR (s-NWUFR) if $r_s(X,t) \geq (\leq)\, r_s(X,0)$, for all $t \geq 0$.

(e) X is said to be s-NBAFR (s-NWAFR) if $\int_0^t r_s(X,x)\,dx/t \geq (\leq)\, r_s(X,0)$ for all $t \geq 0$.

(f) X is said to be s-NBUCV (s-NWUCV) if

$$\bar{T}_s(X,0)\int_0^t \bar{T}_s(X,x+y)\,dy \leq (\geq)\, \bar{T}_s(X,x)\int_0^t \bar{T}_s(X,y)\,dy \text{ for all } x,t \geq 0.$$

For $s \geq 1$, this becomes

$$\int_0^t \bar{T}_s(X,x+y)\,dy \leq (\geq)\, \bar{T}_s(X,x)\int_0^t \bar{T}_s(X,y)\,dy.$$

(g) X is said to be s-NBUCX (s-NWUCX) if

$$\bar{T}_s(X,0)\int_t^\infty \bar{T}_s(X,x+y)\,dy \leq (\geq)\, \bar{T}_s(X,x)\int_t^\infty \bar{T}_s(X,y)\,dy, \text{ for all } x,t \geq 0.$$

For $s \geq 1$, this becomes

$$\int_t^\infty \bar{T}_s(X, x+y)\,dy \le (\ge)\, \bar{T}_s(X, x) \int_t^\infty \bar{T}_s(X, y)\,dy.$$

The last two ageing classifications were due to Fagiuoli and Pellerey (1993).

We note the equivalence between the generalized ageing concepts and the classical ageing classes as follows:

0-IFR	$\Leftrightarrow$	ILR	1-IFR	$\Leftrightarrow$	IFR
2-IFR	$\Leftrightarrow$	DMRL	3-FR	$\Leftrightarrow$	DVRL
1-IFRA	$\Leftrightarrow$	IFRA	1-NBUFR	$\Leftrightarrow$	NBUFR
2-NBUFR	$\Leftrightarrow$	NBUE	2-NBAFR	$\Leftrightarrow$	HNBUE

All the abbreviations in the table above have been introduced in Sections 2.4–2.7 and this section except ILR. This is defined as X having an increasing likelihood ratio. It is also known as X having a PF_2 (Pólya frequency of order 2) density. We note that ILR $\Rightarrow$ IFR. For a definition of a PF_2 density class, see for example, Barlow and Proschan (1981, p. 76).

Fagiuoli and Pellerey (1993) also established the equivalence of the generalized partial orderings between X and its residual lifetime X_t and the generalized ageing classes.

Hu et al. (2001) gave connections among some generalized orderings, and characterized s-FR and s-ST orderings in terms of residual lives as well as in terms of equilibrium distributions, respectively. The s-CX and s-CV orderings were also both characterized by the equilibrium distributions and by the Laplace transforms.

Despite of their mathematical nicety, it is unclear how these generalized orderings can be applied effectively in reliability given their apparent lack of a meaningful interpretation.

2.9.4 Applications of Partial Orderings

First and higher order stochastic dominances which are essentially partial orderings have important applications in econometrics (Whitmore, 1970). Examples of applications may be found in Barlow and Proschan (1981), Stoyan (1983) and in Ross (1983), where partial orderings are used, respectively, in reliability context, in queues, and in other stochastic processes. Singh and Jain (1989) and Fagiuoli and Pellerey (1993) have proposed an application to stochastic comparison between two devices that are subjected to Poisson shock models.

Design engineers are well aware that a system where active spare allocation is made at the component level has a lifetime stochastically larger than the corresponding system where active spare allocation is made at the system level. Boland and El-Neweihi (1995) investigated this principle in hazard rate (failure rate) ordering and demonstrated that it does not hold in general. However, they discovered that for a 2-out-of-n system with independent and identical components and spares, active spare allocation at the component level is superior to active spare allocation at the system level. They conjectured that such a principle holds in general for a k-out-of n system. Singh and Singh (1997) have proved that for a k-out-of-n system where components and spares have independent and identical life distributions, active spare allocation at the component is superior to active spare allocation at the system level in likelihood ratio ordering. This is stronger than failure (hazard) rate ordering and thus establishing the conjecture of Boland and El-Neweihi (1995).

Boland and El-Neweihi (1995) have also established that the active spare allocation at the component level is better than the active spare allocation at the system level in hazard rate (failure rate) ordering for a series system when the components are matching although the components may not be identically distributed. Boland (1998) gave an example to show that the failure rate (hazard rate) comparison is what people really mean when they compare the performance of two products. For more on the failure rate (hazard rate) and other stochastic orders and their applications, the readers should consult Shaked and Shanthikumar (1994).

Apart from the basic partial orderings such as the likelihood ratio ordering, stochastic ordering, failure rate ordering and the mean residual life ordering, we have not found too many applications for the others.

2.10 Relative Ageing

Sengupta and Deshpande (1994) have studied three types of relative ageing of two life distributions. The first of these relative ageing concepts is the partial ordering originally proposed by Kalashnikov and Rachev (1986) which is defined as follows:

Definition 2.27: Let F and G be the distribution functions of the random variables X and Y, respectively. X is said to be *ageing* faster than Y (written as $X \prec_c Y$) if the random variable $\Lambda_G(X) = -\log \bar{G}(X)$ has an increasing failure rate.

If the failure rates $r_F(t)$ and $r_G(t)$ both exist with $r_F(t) = \frac{f(t)}{1-F(t)}$ and $r_G(t) = \frac{g(t)}{1-G(t)}$, then the above definition is equivalent to $\frac{r_F(t)}{r_G(t)}$ being an increasing function of t.

Another relative ageing defined in Sengupta and Deshpande (1994) is as follows:

Definition 2.28: $X \prec_* Y$ (X is ageing faster than Y in average) if $Z = \Lambda_G(X)$ is IFRA .

We observe that there are three types of 'X ages faster than Y' depending on whether $r_F(t)$ dominates $r_G(t)$ or being dominated by $r_G(t)$ or $r_F(t)$ crosses $r_G(t)$ from below. In the case of $r_F(t) \leq r_G(t)$, then X is said to be greater than Y in failure rate ordering according to Definition 2.15.

Remarks

If X ages faster than Y, it is equivalent to Y ages slower than X.

- Suppose now X ages slower than Y, i.e., $r_F(t)/r_G(t)$ is decreasing in t, Corollary 6 of Hu et al. (2001) shows that if $X \leq_{\text{FR}} Y$, then $X \leq_{\text{LR}} Y$ (see Definition 2.13 for LR ordering). Note that $X \leq_{\text{LR}} Y \Rightarrow X \leq_{\text{FR}} Y$ without a condition.
- Suppose $r_s(X,t)/r_s(Y,t)$ is decreasing in t, Theorem 5 of Hu et al. (2001) shows that if $X \leq_{s-\text{FR}} Y$, then $X \leq_{(s-1)-\text{FR}} Y$ for $s \geq 1$. (See Definition 2.22 for s-FR ordering.) This is a converse of the result in Section 2.9.2. which says if $X \leq_{(s-1)\text{FR}} Y \Rightarrow X \leq_{(s)-\text{FR}} Y$.

In Lai and Xie (2003) some results on relative ageing of two parallel structures were established. It is observed that the relative ageing property may be used to allocate resources and for failure identification when two components (systems) having the same mean. In particular, if 'X ages faster than Y' and that they have the same mean, then $\text{var}(X) \leq \text{var}(Y)$. Several examples are given therein. In particular, it is shown that when two Weibull distributions have the same mean, the one that ages faster has a smaller variance.

2.11 Shapes of η Function for s-order Equilibrium Distributions

Recall in Section 2.9, the equilibrium distribution function of a lifetime variable X with cdf F is defined in (2.84), i.e.,

$$E_F(t) = \int_0^t F(x)\,dx/\mu$$

so the density function is given $\bar{F}(t)/\mu$. Section 2.9 also defines the survival functions of the equivalent distributions recursively via

$$\bar{T}_0(X,t) = f(t),\ \bar{T}_{-1}(X,t) = f'(t)$$

and

$$\bar{T}_s(X,t) = \frac{\int_t^\infty \bar{T}_{s-1}(X,x)\,dx}{\mu_{s-1}(X)},\ \text{for integer } s \geq 1,$$

where

$$\mu_s(X) = \int_0^\infty \bar{T}_s(X,x)\,dx, s \geq 0.$$

It follows that (2.90) and the preceding two equations that

$$r_s(X,t) = \frac{\bar{T}_{s-1}(X,t)}{\int_t^\infty \bar{T}_{s-1}(X,x)\,dx}\ s \geq 1.$$

Now let us define an s-order η function by

$$\eta_s(X,t) = \frac{\bar{T}_{s-2}(X,t)}{\int_t^\infty \bar{T}_{s-2}(X,x)\,dx} \tag{2.91}$$

We assume that $\lim_{x\to\infty} f(x) = 0$.

It is now clear that

$$\eta_s(X,t) = r_{s-1}(X,t), s \geq 1. \tag{2.92}$$

For $s = 1$, $\eta_1(X,t) = r_0(X,t) = -\dfrac{f'(t)}{f(t)}$. As we are now dealing with one variable X only so we may suppress the argument X giving

$$\eta_s(t) = r_{s-1}(t), s \geq 1. \tag{2.93}$$

Thus,

$\eta_1(t) = \eta(t) = -\dfrac{f'(t)}{f(t)}$, $r_1(t) = r(t) = \frac{f(t)}{\bar{F}(t)}$,

$\eta_2(t) = r(t)$, $r_2(t) = 1/\mu(t)$.

Now the relationships between the shapes of $\eta(t)$ and the shapes of $r(t)$ have already been established by Glaser (1980) (our Theorem 2.1) and by Gupta and Warren (2001) (our Theorem 2.2). The same relationships between the shapes of $\eta_2(t) = r(t)$ of the equilibrium distribution of X and of the shapes of $r_2(t) = 1/\mu(t)$ also hold. These results can be generalized to establish the relationship between $\eta_s(t)$ and $r_s(t) = \eta_{s+1}(t)$.

Based on the above observations, Navarro and Hernandez (2004) gave the following theorems:

Theorem 2.12: If $E(X^{s+1}) < \infty$ for $s = 0, 1, 2, ...$, then

(a) $\eta_s \in$ I (D) $\Rightarrow \eta_{s+1} \in$ I (D);
(b) $\eta_s \in$ BT (UBT) $\Rightarrow \eta_{s+1} \in$ BT or I (UBT or D).

Proof: It follows directly from Theorem 2.1 and equation (2.92).

Theorem 2.13: If $E(X^{s+1}) < \infty$ for $s = 0, 1, 2, ...$, then $\eta'_{s+1}(t) = 0$ has at most one solution on the closed interval $[z_{k-1}, z_k]$, where $z_0 = 0 < z_1 < ... < z_n$ are the zeros of $\eta'_s(t)$ and $\eta'_{s+1}(t) = 0$ does not have any solution in (z_n, ∞).

Proof: It follows directly from Theorem 2.2 and equation (2.93).

The relationship between $\eta_2(t)$ and $\eta_3(t)$ will be used to establish between the relationships between the shapes $r(t)$ and $\mu(t)$ in Theorem 4.2 of Chapter 4.

Mi (2004) also studied (i) the shape of $\eta_2(t) = r(t)$ when $\eta_1(t) = -f'(t)/f(t)$ has a roller-coaster shape with a finite number of change points (ii) the shape of $\eta_3(t) = 1/\mu(t)$ when η_2 has a roller-coaster shape. However, Mi's (2004) did not use the s-order equilibrium distribution approach.

Instead, he defined $\eta_1(t) = N_1(t)/D_1(t) = N_1(t)/\int_t^\infty N_1(x)\,dx$ and $\eta_2(t) = N_2(t)/D_2(t) = D_1(t)/D_2(t) = D_1(t)/\int_t^\infty D_1(x)\,dx$ and established their relationships assuming $N_1(t)$ is integrable on $[0, \infty)$.

We will revisit these results when we consider the shape of $\mu(t)$ when the shape of $r(t)$ is known in Chapter 4.

2.12 Concluding Remarks on Ageing

The study of length of life of human beings, organisms, structures, materials, etc., is of great importance in the actuarial, biological, engineering and medical sciences. It is clear that research on ageing properties (univariate, bivariate, and multivariate) is currently being vigorously pursued. Many of the univariate definitions do have physical interpretations such as arising from shock models. The simple IFR, IFRA, NBU, NBUE, DMRL etc have been shown to be very useful in reliability related decision making, such as replacement and maintenance studies.

While positive ageing concepts are well understood, negative ageing concepts (life improved by age) are less intuitive. Nevertheless, negative ageing phenomenon does occur quite frequently. There have been cases reported by several authors where the failure rate functions decrease with time. Sample examples are the business mortality (Lomax, 1954), failures in the air-conditioning equipment of a fleet of Boeing 720 aircrafts or in semiconductors from various lots combined (Proschan, 1963), and the life of integrated circuit modules (Sanuders and Myhre, 1983). Gerchak (1984) reported that "Studies conducted in various social disciplines discovered that, the longer individuals remain in a state, the lower the chances of their leaving the state in subsequent periods." In general, a population is expected to exhibit decreasing failure rate (DFR) when its behaviors over time is characterized by 'work hardening' (in engineering terms), or 'immunity' (in biological terms). Modern phenomenon of DFR includes reliability growth (in software reliability).

Non-monotonic ageing concepts have been found useful in many reliability and survival analysis such as burn-in time decision. Applications of mean residual life concepts will be given in Chapter 4 whereas applications of bathtub (upside-down) shaped ageing will be presented in Chapter 3. We also refer our readers to Barlow and Proschan (1981), Bergman (1985) and Newby (1986) for other applications.

We envisage that these concepts are of interest not only to reliability modelers but also to the mainstream reliability practitioners.

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