Preface

This book is based on a two-semester course in ordinary differential equations that I have taught to graduate students for two decades at the University of Missouri. The scope of the narrative evolved over time from an embryonic collection of supplementary notes, through many classroom tested revisions, to a treatment of the subject that is suitable for a year (or more) of graduate study.

If it is true that students of differential equations give away their point of view by the way they denote the derivative with respect to the independent variable, then the initiated reader can turn to Chapter 1, note that I write $\dot{x}$, not x', and thus correctly deduce that this book is written with an eye toward dynamical systems. Indeed, this book contains a thorough introduction to the basic properties of differential equations that are needed to approach the modern theory of (nonlinear) dynamical systems. But this is not the whole story. The book is also a product of my desire to demonstrate to my students that differential equations is the least insular of mathematical subjects, that it is strongly connected to almost all areas of mathematics, and it is an essential element of applied mathematics.

When I teach this course, I use the first part of the first semester to provide a rapid, student-friendly survey of the standard topics encountered in an introductory course of ordinary differential equations (ODE): existence theory, flows, invariant manifolds, linearization, omega limit sets, phase plane analysis, and stability. These topics, covered in Sections 1.1–1.8 of Chapter 1 of this book, are introduced, together with some of their important and *interesting* applications, so that the power and beauty of the subject is immediately apparent. This is followed by a discussion of linear systems theory and the proofs of the basic theorems on linearized stability in Chapter 2. Then, I conclude the first semester by presenting one or two realistic applications from Chapter 3. These applications provide a capstone for the course as well as an excellent opportunity to teach the mathematics graduate students some physics, while giving the engineering and physics students some exposure to applications from a mathematical perspective.

In the second semester, I introduce some advanced concepts related to existence theory, invariant manifolds, continuation of periodic orbits, forced oscillators, separatrix splitting, averaging, and bifurcation theory. Since there is not enough time in one semester to cover all of this material in depth, I usually choose just one or two of these topics for presentation in class. The material in the remaining chapters is assigned for private study according to the interests of my students.

My course is designed to be accessible to students who have only studied differential equations during one undergraduate semester. While I do assume some knowledge of linear algebra, advanced calculus, and analysis, only the most basic material from these subjects is required: eigenvalues and eigenvectors, compact sets, uniform convergence, the derivative of a function of several variables, and the definition of metric and Banach spaces. With regard to the last prerequisite, I find that some students are afraid to take the course because they are not comfortable with Banach space theory. These students are put at ease by mentioning that no deep properties of infinite dimensional spaces are used, only the basic definitions.

Exercises are an integral part of this book. As such, many of them are placed strategically within the text, rather than at the end of a section. These interruptions of the flow of the narrative are meant to provide an opportunity for the reader to absorb the preceding material and as a guide to further study. Some of the exercises are routine, while others are sections of the text written in "exercise form." For example, there are extended exercises on structural stability, Hamiltonian and gradient systems on manifolds, singular perturbations, and Lie groups. My students are strongly encouraged to work through the exercises. How is it possible to gain an understanding of a mathematical subject without doing some mathematics? Perhaps a mathematics book is like a musical score: by sight reading you can pick out the notes, but practice is required to hear the melody.

The placement of exercises is just one indication that this book is not written in axiomatic style. Many results are used before their proofs are provided, some ideas are discussed without formal proofs, and some advanced topics are introduced without being fully developed. The pure axiomatic approach forbids the use of such devices in favor of logical order. The other extreme would be a treatment that is intended to convey the ideas of the subject with no attempt to provide detailed proofs of basic results. While the narrative of an axiomatic approach can be as dry as dust, the excitement of an idea-oriented approach must be weighed against the fact that it might leave most beginning students unable to grasp the subtlety of the arguments required to justify the mathematics. I have tried to steer a middle course in which careful formulations and complete proofs are given for the basic theorems, while the ideas of the subject are discussed in depth and the path from the pure mathematics to the physical universe is clearly marked. I am reminded of an esteemed colleague who mentioned that a certain textbook "has lots of fruit, but no juice." Above all, I have tried to avoid this criticism.

Application of the implicit function theorem is a recurring theme in the book. For example, the implicit function theorem is used to prove the rectification theorem and the fundamental existence and uniqueness theorems for solutions of differential equations in Banach spaces. Also, the basic results of perturbation and bifurcation theory, including the continuation of subharmonics, the existence of periodic solutions via the averaging method, as well as the saddle node and Hopf bifurcations, are presented as applications of the implicit function theorem. Because of its central role, the implicit function theorem and the terrain surrounding this important result are discussed in detail. In particular, I present a review of calculus in a Banach space setting and use this theory to prove the contraction mapping theorem, the uniform contraction mapping theorem, and the implicit function theorem.

This book contains some material that is not encountered in most treatments of the subject. In particular, there are several sections with the title "Origins of ODE," where I give my answer to the question "What is this good for?" by providing an explanation for the appearance of differential equations in mathematics and the physical sciences. For example, I show how ordinary differential equations arise in classical physics from the fundamental laws of motion and force. This discussion includes a derivation of the Euler–Lagrange equation, some exercises in electrodynamics, and an extended treatment of the perturbed Kepler problem. Also, I have included some discussion of the origins of ordinary differential equations in the theory of partial differential equations. For instance, I explain the idea that a parabolic partial differential equation can be viewed as an ordinary differential equation in an infinite dimensional space. In addition, traveling wave solutions and the Galërkin approximation technique are discussed. In a later "origins" section, the basic models for fluid dynamics are introduced. I show how ordinary differential equations arise in boundary layer theory. Also, the ABC flows are defined as an idealized fluid model, and I demonstrate that this model has chaotic regimes. There is also a section on coupled oscillators, a section on the Fermi–Ulam–Pasta experiments, and one on the stability of the inverted pendulum where a proof of linearized stability under rapid oscillation is obtained using Floquet's method and some ideas from bifurcation theory. Finally, in conjunction with a treatment of the multiple Hopf bifurcation for planar systems, I present a short introduction to an algorithm for the computation of the Lyapunov quantities as an illustration of computer algebra methods in bifurcation theory.

Another special feature of the book is an introduction to the fiber contraction principle as a powerful tool for proving the smoothness of functions that are obtained as fixed points of contractions. This basic method is used first in a proof of the smoothness of the flow of a differential equation where its application is transparent. Later, the fiber contraction principle appears in the nontrivial proof of the smoothness of invariant manifolds at a rest point. In this regard, the proof for the existence and smoothness of stable and center manifolds at a rest point is obtained as a corollary of a more general existence theorem for invariant manifolds in the presence of a "spectral gap." These proofs can be extended to infinite dimensions. In particular, the applications of the fiber contraction principle and the Lyapunov–Perron method in this book provide an introduction to some of the basic tools of invariant manifold theory.

The theory of averaging is treated from a fresh perspective that is intended to introduce the modern approach to this classical subject. A complete proof of the averaging theorem is presented, but the main theme of the chapter is partial averaging at a resonance. In particular, the "pendulum with torque" is shown to be a universal model for the motion of a nonlinear oscillator near a resonance. This approach to the subject leads naturally to the phenomenon of "capture into resonance," and it also provides the necessary background for students who wish to read the literature on multifrequency averaging, Hamiltonian chaos, and Arnold diffusion.

I prove the basic results of one-parameter bifurcation theory—the saddle node and Hopf bifurcations—using the Lyapunov–Schmidt reduction. The fact that degeneracies in a family of differential equations might be unavoidable is explained together with a brief introduction to transversality theory and jet spaces. Also, the multiple Hopf bifurcation for planar vector fields is discussed. In particular, and the Lyapunov quantities for polynomial vector fields at a weak focus are defined and this subject matter is used to provide a link to some of the algebraic techniques that appear in normal form theory.

Since almost all of the topics in this book are covered elsewhere, there is no claim of originality on my part. I have merely organized the material in a manner that I believe to be most beneficial to my students. By reading this book, I hope that you will appreciate and be well prepared to use the wonderful subject of differential equations.

Columbia, Missouri — Carmen Chicone
June 1999

Preface to the Second Edition

This edition contains new material, new exercises, rewritten sections, and corrections.

There are at least three nontrivial mathematical errors in the first edition: The proof of the Trotter product formula (Theorem 2.24) is valid only in case $e^{A+B} = e^A e^B$; the Floquet theorem (Theorem 2.47) on the existence of logarithms for matrices is valid only if the square of the real matrix in question has all positive eigenvalues; and the proof of the smoothness of invariant manifolds (Theorem 4.1) has a gap because the continuity of a certain fiber contraction with respect to its base space is assumed. The first two errors were pointed out by Mark Ashbaugh, the third by Mohamed ElBialy. These and many other less serious errors are corrected.

While much of the narrative has been revised, the most substantial additions and revisions not already mentioned are the following: the introductory Section 1.9.3 on contraction is rewritten to include a discussion of the continuity of fiber contractions and a more informative first application of the fiber contraction theorem, which is the proof of the smoothness of the solution of the functional equation $F \circ \phi - \phi = G$ (Theorem 1.234); Section 3.1 on the Euler-Lagrange equation is rewritten and expanded to include a more detailed discussion of Hamilton's theory, a presentation of Noether's Theorem, and several new exercises on the calculus of variations; Section 3.2 on classical mechanics has been revised by including more details; the application (in Section 3.5) of Floquet theory to the stability of the inverted pendulum is rewritten to incorporate a more elegant dimensionless model; a new Section 4.3.3 introduces the Lie derivative and applies it to prove the Hartman-Grobman theorem for flows; multidimensional continuation theory for periodic orbits in the presence of first integrals is discussed in the new Section 5.3.8, the basic result on the continuation of manifolds of periodic orbits in the presence of first integrals in involution is proved, and the Lie derivative is used again to characterize commuting flows; and the subject of dynamic bifurcation theory is introduced in a new Section 8.4 where the fundamental idea of delayed bifurcation is presented with applications to the pitchfork bifurcation and bursting.

Over 160 new exercises are included, most with multiple parts. While a few routine exercises are provided where I expect them to be helpful, most of the exercises are meant to challenge students on their understanding of the theory, stimulate interest, extend topics introduced in the narrative, and point the way to applications. Also, most exercises now have lettered parts for easy identification of portions of exercises for homework assignments.

As described in the Preface, the core first graduate course in ODE is contained in selections from the first three chapters. The instructor should budget class time so that all of the language and basic concepts of the subject (existence theory, flows, invariant manifolds, linearization, omega limit sets, phase plane analysis, and stability) are introduced and some applications are discussed in detail.

In my experience, sensitivity to the preparation of students is essential for a successful first graduate course in differential equations. Although the prerequisites are minimal, there are certainly some students who are unprepared for the challenges of a course based on this book if their exposure to differential equations is limited to no more than one undergraduate course where they studied only solution methods for linear second order equations. I have included some review (see Exercise 1.6) to serve as a bridge from their first course to this book. In addition, I often use some class time to review a few fundamental concepts (especially, the derivative as a linear transformation, compactness, connectedness, uniform convergence, linear spaces, eigenvalues, and Jordan canonical form) before they are encountered in context.

The second edition contains plenty of material for second semester courses, master's projects, and reading courses. Professionals might also find something of value.

I remain an enthusiastic teacher of the rich and important subject of differential equations. I hope that instructors will find this book a useful addition to their class design and preparation, and students will have a clear and faithful guide during their quest to learn the subject.

Columbia, Missouri — Carmen Chicone
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