Robert Gilmer's work on semigroup rings

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1 Introduction

Group rings, and more generally semigroup rings, have played an important role in modern algebra and topology. In this article, we are interested in Robert Gilmer's pioneering work on semigroup rings. This includes his two papers with T. Parker [30, 31] on divisibility properties in semigroup rings, submitted in March and May of 1973, respectively; his semigroup ring example of a two-dimensional non-Noetherian UFD [24], submitted in May of 1973; his work with J. T. Arnold on the (Krull) dimension of semigroup rings [12], submitted in September of 1975; and his book *Commutative Semigroup Rings* [25], finished in the summer of 1983 and published in 1984. Arnold and Parker (see [45]) were both PhD students of Gilmer.

In the introduction, we give a leisurely motivation for semigroup rings and establish notation. In the second section, we cover Gilmer's work with T. Parker on divisibility properties in semigroup rings. In the third section, we discuss Gilmer's construction of a two-dimensional non-Noetherian UFD, his work with J. T. Arnold on the dimension of a semigroup ring, and his book on semigroup rings. In the final section, we consider generalizations to Krull semigroup rings, graded rings, and divisibility properties in semigroups. We also discuss the (t-)class group and Picard group of monoid domains.

The polynomial ring $\mathbb{Q}[X]$ over the field $\mathbb{Q}$ of rational numbers is a PID. Varying the coefficients produces different ring-theoretic properties. For example, the polynomial ring $D[X]$ over an integral domain D is a UFD (resp., GCD-domain, Krull domain, PVMD) if and only if D is a UFD (resp., GCD-domain, Krull domain, PVMD). One often tries to prove that $D[X]$ satisfies a certain property $\mathcal{P}$ if and only if D satisfies property $\mathcal{P}$. Sometimes this holds; other times it does not. For example, because of dimension constraints, $D[X]$ is a PID or a Dedekind domain only in the trivial case when D is a field.

In $\mathbb{Q}[X]$, the exponents are nonnegative integers. Instead of just varying the coefficients, why not also vary the set S of exponents? For example, if we let $S = \mathbb{Z}$, then we get the Laurent polynomial ring $\mathbb{Q}[X, X^{-1}]$. So how should we vary the exponents? We will follow the notation of Northcott [44] which emphasizes that semigroup rings are generalized polynomial rings. Let R be a commutative ring with $1 \neq 0$. Then $R[X;S]$ will be the ring of all formal polynomials $\sum r_\alpha X^\alpha$ with each $\alpha \in S$ and $r_\alpha \in R$, almost all $r_\alpha = 0$, addition defined by $\sum r_\alpha X^\alpha + \sum s_\alpha X^\alpha = \sum (r_\alpha + s_\alpha) X^\alpha$, and with multiplication defined using the distributive law and $(r_\alpha X^\alpha)(r_\beta X^\beta) = r_\alpha r_\beta X^{\alpha+\beta}$. Let $\alpha, \beta, \gamma \in S$. Since $X^\alpha X^\beta = X^{\alpha+\beta}$, the set S must be closed under addition. The commutative and associative laws in $R[X;S]$, $X^\alpha X^\beta = X^\beta X^\alpha$ and $X^\alpha(X^\beta X^\gamma) = (X^\alpha X^\beta)X^\gamma$, yield $\alpha + \beta = \beta + \alpha$ and $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ in S, respectively. Also, we want 1 to be X^0; so S should be an additive commutative monoid. We call $R[X;S]$ a *semigroup* (or *monoid*) *ring.*

We usually want $R[X;S]$ to be an integral domain; so R would have to be an integral domain. If $\alpha + \beta = \alpha + \gamma$ in S, then $X^\alpha X^\beta = X^\alpha X^\gamma$ in $R[X;S]$ would yield $X^\beta = X^\gamma$, and hence $\beta = \gamma$. Thus S must be a cancellative monoid. Also, S must be *torsionfree*, in the sense that $n\alpha = n\beta$ for n a positive integer and $\alpha, \beta \in S$ implies that $\alpha = \beta$ (since $X^\alpha - X^\beta$ divides $X^{n\alpha} - X^{n\beta}$ in $R[X;S]$). Conversely, let R be an integral domain and S an additive commutative torsionfree cancellative monoid. Then S may be totally ordered, and hence it is easily seen that the product of two nonzero elements in $R[X;S]$ is nonzero. Thus we have shown the following theorem (also see [23] or [25, Theorem 8.1]).

Theorem 1.1. *The semigroup ring $R[X;S]$ is an integral domain if and only if R is an integral domain and S is a commutative torsionfree cancellative monoid.*

Given an additive commutative cancellative monoid S, let $\langle S \rangle = \{s - t \mid s, t \in S\}$ be its quotient group. The fact that S is torsionfree is equivalent to $\langle S \rangle$ being torsionfree, i.e., S is a submonoid of a torsionfree abelian group. Let $U(S)$ be the set of invertible elements of S; then $U(S)$ is the maximal subgroup of S and $U(S) = S \cap -S$.

In the integral domain case, it is easy to determine the units of $R[X;S]$. They are precisely the monomials rX^α, where $r \in U(R)$ and $\alpha \in U(S)$. If $R[X;S]$ is not an integral domain, then it is of interest to investigate special types of elements of $R[X;S]$ such as units, zero-divisors, nilpotent elements, and idempotent elements. Gilmer has investigated these in joint papers with R. Heitmann [28], T. Parker [46], and M. Teply [33, 34] (also see [25, Chapter 2]).

Note that the polynomial ring $R[\{X_\alpha\}]$ is the semigroup ring $R[X; \oplus_\alpha \mathbb{Z}_+]$ and the Laurent polynomial ring $R[\{X_\alpha, X_\alpha^{-1}\}]$ is the group ring $R[X; \oplus_\alpha \mathbb{Z}]$. More generally, let A be a subring of $R[\{X_\alpha\}]$ generated by monomials over R. Then $A = R[X;S]$, where $S = \{(n_\alpha) \in \oplus_\alpha \mathbb{Z}_+ \mid \sum X_\alpha^{n_\alpha} \in A\}$. In particular, a subring A of $R[X]$ generated by monomials over R is $R[X;S]$ for $S = \{n \in \mathbb{Z}_+ \mid X^n \in A\}$. We view semigroup rings as a generalization of polynomial rings. The reason that things work so nicely in the polynomial ring case is that $S = \mathbb{Z}_+$ is the nicest possible semigroup. Although polynomial rings are semigroup rings, we are primarily interested in the general case when S is not $\oplus_\alpha \mathbb{Z}_+$.

Let D be an integral domain with quotient field K and S a commutative torsionfree cancellative monoid. Then $K[X;S] = D[X;S]_T$ and $D[X;\langle S\rangle] = D[X;S]_{T'}$, where $T = D \setminus \{0\}$ and $T' = \{X^\alpha \mid \alpha \in S\}$ are multiplicative subsets of $D[X;S]$. Also, note that $D[X;S] = K[X;S] \cap D[X;\langle S\rangle]$. We can thus sometimes reduce questions about monoid domains to group rings or to monoid domains over a field, often using a "Nagata-type" theorem (i.e., if R_T satisfies a certain property $\mathcal{P}$ for a "nice" multiplicative set T, then R also satisfies property $\mathcal{P}$). Since S is totally ordered, $D[X;S]$ becomes a graded integral domain with $\deg(dX^\alpha) = \alpha$ for $0 \neq d \in D$ and $\alpha \in S$. Thus graded ring techniques often play an important role in studying semigroup rings (cf. Section 4).

In $R[X;S]$, we can vary both the coefficients and the exponents. Thus semigroup rings provide a very handy way to construct examples since ring-theoretic properties of $R[X;S]$ are determined by properties of both R and S, and hence we have much more freedom than in polynomial rings. This is very pretty mathematics which illustrates the interplay between ring-theoretic and semigroup-theoretic techniques. It has certainly played a major role in my research activity.

For notation, R will be a commutative ring with nonzero identity and $U(R)$ its group of units, D will be an integral domain with quotient field $qf(D)$, and K will be a field. The dimension of R, $\dim(R)$, will always mean Krull dimension, and $\mathrm{char}(R)$ will be the characteristic of R. We let S denote a commutative cancellative monoid, written additively, with group of invertible elements $U(S)$ and quotient group $\langle S\rangle$. Let G denote an abelian group (usually torsionfree) and $\mathrm{rank}(G) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$. For a set A, let $A^* = A \setminus \{0\}$; and for a partially ordered monoid S, let S_+ be its set of nonnegative elements. As usual, $\mathbb{Z}$ and $\mathbb{Q}$ will denote the integers and rational numbers, respectively. For more on semigroups, see [25, 37]; and for abelian groups, see [20, 21]. For any undefined notions or notation, see Gilmer's "other" book [27]; see [19] for Krull domains. In most cases, we will cite both the original reference and the corresponding result in [25].

2 Divisibility in Semigroup Rings

In this section, we discuss Gilmer's two papers with T. Parker [30, 31] on divisibility properties in semigroup rings. The main goal of [30] is to determine necessary and sufficient conditions for $D[X;S]$ to be a UFD. But first we consider GCD-domains. For most of our results, we will first consider the group ring case, and then the general monoid ring result.

Theorem 2.1. *Let D be an integral domain and G a torsionfree abelian group. Then $D[X;G]$ is a GCD-domain if and only if D is a GCD-domain.*

Proof. We may reduce to the case where G is finitely generated, and hence free. In this case, the result follows easily from the well-known polynomial ring case. For more details, see either [30, Proposition 5.1 and Theorem 6.1] or [25, Theorems 14.1 and 14.2].

In particular, $K[X;G]$ is a GCD-domain for any field K and any torsionfree abelian group G. We next consider the case for monoid domains. Let $R = K[X^2, X^3] = K[X;S]$, where K is a field and $S = \{0, 2, 3, 4, \ldots\} \subset \mathbb{Z}_+$. Then R is not a GCD-domain since X^5 and X^6 have no GCD in R. In analogy for integral domains, define a torsionfree cancellative monoid S to be a *GCD-monoid* if each pair of elements of S has a GCD(equivalently, an LCM). Then any free abelian monoid or torsionfree abelian group is a GCD-monoid. However, $S = \{0, 2, 3, 4, \ldots\}$ is not a GCD-monoid since 5 and 6 have no GCD in S. Our next result is somewhat typical in that $D[X;S]$ satisfies a certain ring-theoretic property $\mathcal{P}$ if and only if D satisfies property $\mathcal{P}$ and S satisfies the additive monoid analog of property $\mathcal{P}$ (also see Section 4).

Theorem 2.2. *Let D be an integral domain and S a torsionfree cancellative monoid. Then $D[X;S]$ is a GCD-domain if and only if D is a GCD-domain and S is a GCD-monoid.*

Proof. If $D[X;S]$ is a GCD-domain, then D must be a GCD-domain and S a GCD-monoid. The converse follows from Theorem 2.1 using a "Nagata-type" theorem that $D[X;S]_T = D[X;\langle S\rangle]$, where $T = \{X^\alpha \mid \alpha \in S\}$, is a GCD-domain implies that $D[X;S]$ is a GCD-domain. See either [30, Theorems 6.1 and 6.4] or [25, Theorems 14.1 and 14.5] for more details.

An integral domain D is a UFD if and only if D is a GCD-domain and D satisfies the ascending chain condition on principal ideals (ACCP). Note that $R = K[X;\mathbb{Q}]$ is a GCD-domain for any field K by Theorem 2.1, but R is not a UFD since ACCP fails. For example, we have the strictly ascending chain of principal ideals $(1-X) \subset (1-X^{1/2}) \subset (1-X^{1/4}) \subset \cdots$ in R, which corresponds to the strictly ascending chain of cyclic subgroups $\langle 1\rangle \subset \langle 1/2\rangle \subset \langle 1/4\rangle \subset \cdots$ in $\mathbb{Q}$. Similarly, $K[X;\mathbb{Q}_+]$ is a GCD-domain, but not a UFD.

Let G be a torsionfree abelian group. Recall that every nonzero element of G has *type* $(0,0,0,\ldots)$ means that for each $0 \neq g \in G$, there is a largest positive integer n_g such that the equation $n_g x = g$ is solvable in G (see [21, Section 85]). This is equivalent to each rank-one subgroup of G is cyclic (free), or more suggestively, G satisfies ACC on cyclic subgroups, or ACC on cyclic submonoids [25, Theorem 14.10]. This property plays an important role in properties related to chain conditions since (as above) a strictly ascending chain of cyclic subgroups $\langle g_1\rangle \subset \langle g_2\rangle \subset \cdots$ in G gives rise to a strictly ascending chain of principal ideals $(1-X^{g_1}) \subset (1-X^{g_2}) \subset \cdots$ in $D[X;G]$.

Note that any subgroup of a torsionfree abelian group which satisfies ACC on cyclic subgroups also satisfies ACC on cyclic subgroups. In particular, if S is a torsionfree cancellative monoid such that $\langle S\rangle$ satisfies ACC on cyclic subgroups, then so does $U(S)$. However, the converse is false since $S = \mathbb{Q}_+$ has $U(S) = 0$ which certainly satisfies ACC on cyclic subgroups, but $\langle S\rangle = \mathbb{Q}$ does not. Although the results in [30, 31] are stated using the type $(0, 0, 0, \ldots)$ terminology, we use the more suggestive ACC on cyclic subgroups or cyclic submonoids terminology as in [25].

Theorem 2.3. *Let D be an integral domain and G a torsionfree abelian group. Then $D[X; G]$ is a UFD if and only if D is a UFD and G satisfies ACC on cyclic subgroups.*

Proof. If $D[X; G]$ is a UFD, then D must be a UFD, and G satisfies ACC on cyclic subgroups by the above remarks. The converse is much more difficult and finally reduces to the case where D is an algebraically closed field. For more details, see either [30, Theorem 7.13] or [25, Theorem 14.16].

As in the GCD-domain case, we may define the analog of unique factorization for a torsionfree cancellative monoid. We call such a monoid a *factorial monoid*. Note that a factorial monoid has the form $G \oplus F_+$, where G is any torsionfree abelian group and $F = \oplus_\alpha \mathbb{Z}$ is a free abelian group with the usual product order. We are now ready for the main result of both [30] and this section. The Krull domain analog of Theorem 2.4 will be discussed in Section 4.

Theorem 2.4. *Let D be an integral domain and S a torsionfree cancellative monoid. Then $D[X; S]$ is a UFD if and only if D is a UFD, S is a factorial monoid, and $U(S)$ satisfies ACC on cyclic subgroups.*

Proof. Again, the "$\Rightarrow$" implication is fairly clear. The converse follows from Theorem 2.3 via a "Nagata-type" theorem as in the proof of Theorem 2.2. See either [30, Theorem 7.17] or [25, Theorem 14.16] for more details.

Theorem 2.4 just says that a factorial monoid domain looks like the ring $D[X; G][\{Y_\alpha\}]$, where D is a UFD, G is a torsionfree abelian group which satisfies ACC on cyclic subgroups, and $\{Y_\alpha\}$ is a family of indeterminates. Note that if S is a factorial monoid, then $U(S)$ satisfies ACC on cyclic subgroups if and only if $\langle S\rangle$ satisfies ACC on cyclic subgroups, if and only if S satisfies ACC on cyclic submonoids.

As a corollary of Theorem 2.3, the group ring $D[X; G]$ satisfies ACCP if and only if D satisfies ACCP and G satisfies ACC on cyclic subgroups ([30, Corollary 7.14] or [25, Theorem 14.17]). What about $D[X; S]$? Several partial results are given in [30, pages 77 and 82]. For example, $D[X; S]$ satisfies ACCP if D satisfies ACCP, S satisfies ACC on cyclic submonoids, and $\langle S\rangle$ satisfies ACC on cyclic subgroups. However, the converse fails. Let $S = \{q \in \mathbb{Q} \mid q \geq 1\} \cup \{0\}$, and let K be a field. Then one easily checks that $K[X; S]$ satisfies ACCP, S satisfies ACC on cyclic submonoids, but $\langle S\rangle = \mathbb{Q}$ does not satisfy ACC on cyclic subgroups. This gives an easy example to show that ACCP is not preserved by localization since $K[X; \mathbb{Q}] = K[X; S]_T$, where

$T = \{ X^\alpha \mid \alpha \in S\}$. Other chain conditions in semigroup rings are investigated in [25, 26].

Article [30] concludes with a characterization of when $D[X;S]$ is a PID (or Dedekind domain or Euclidean domain). This happens only in the trivial case when D is a field and S is isomorphic to either $\mathbb{Z}_+$ or $\mathbb{Z}$, i.e., $D[X;S]$ is either $K[X]$ or $K[X,X^{-1}]$ for some field K.

Theorem 2.5. *Let D be an integral domain and S a nonzero torsionfree cancellative monoid. Then the following statements are equivalent.*

(1) *$D[X;S]$ is a Euclidean domain.*
(2) *$D[X;S]$ is a PID.*
(3) *$D[X;S]$ is a Dedekind domain.*
(4) *D is a field and S is isomorphic to either $\mathbb{Z}_+$ or $\mathbb{Z}$.*

Proof. This follows since $D[X;S]$ must be integrally closed and one-dimensional. For more details, see either [30, Theorem 8.4] or [25, Theorem 13.8].

There were two immediate sequels to [30]. First, in [31], Gilmer and Parker considered several additional divisibility properties and also allowed the coefficient rings to have zero-divisors. Secondly, in [24], Gilmer used results from [30] to construct a two-dimensional non-Noetherian UFD; this example will be discussed in the next section.

We first state the main results from [31] in the integral domain setting. These results are similar to that for PIDs in Theorem 2.5, but there is a little more freedom on the monoid S since these rings need not be Noetherian. Recall that we have observed that $K[X;\mathbb{Q}]$ and $K[X;\mathbb{Q}_+]$ are both GCD-domains for any field K, but are not UFDs. They are also Bezout domains since they are ascending unions of PIDs (for example, $K[X;\mathbb{Q}] = \bigcup_{n=1}^{\infty} K[X;(1/n!)\mathbb{Z}]$). It will be convenient to call a monoid S a *Prüfer submonoid of* $\mathbb{Q}$ if $S = G \cap \mathbb{Q}_+$, where G is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. This just means that S is the union of an ascending sequence of cyclic submonoids [25, Theorem 13.5].

Theorem 2.6. *Let D be an integral domain and S a nonzero torsionfree cancellative monoid. Then the following statements are equivalent.*

(1) *$D[X;S]$ is a Bezout domain.*
(2) *$D[X;S]$ is a Prüfer domain.*
(3) *D is a field and S is isomorphic to either a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ or a Prüfer submonoid of $\mathbb{Q}$.*

Proof. See either [31, Theorem] or [25, Theorem 13.6] for details.

We next allow R to have zero-divisors, but S will still be a torsionfree cancellative monoid. In this case, the only change is that "field" gets replaced by "von Neumann regular ring". We say that a commutative ring R is a *Prüfer ring* if each finitely generated regular ideal of R is invertible and that R is a *Bezout ring* if each finitely generated ideal of R is principal.

Theorem 2.7. *Let R be a commutative ring and S a nonzero torsionfree cancellative monoid. Then the following statements are equivalent.*

(1) *$R[X;S]$ is a Bezout ring.*
(2) *$R[X;S]$ is a Prüfer ring.*
(3) *R is von Neumann regular and S is isomorphic to either a subroup of $\mathbb{Q}$ containing $\mathbb{Z}$ or a Prüfer submonoid of $\mathbb{Q}$.*

Proof. For details, see either [31, Corollary 3.1] or [25, Theorem 18.9].

In [31], Gilmer and Parker also determined when a monoid ring is either an almost Dedekind domain or a general ZPI-ring. Recall that an integral domain D is an *almost Dedekind domain* if D_M is a Noetherian valuation domain for each maximal ideal M of D, and that a commutative ring R is a *general ZPI-ring* if each ideal of R is a finite product of prime ideals, equivalently, if R is a finite direct sum of Dedekind domains and special principal ideal rings.

Theorem 2.8. *Let D be an integral domain and S a nonzero torsionfree cancellative monoid. Then $D[X;S]$ is an almost Dedekind domain if and only if D is a field and S is isomorphic to either $\mathbb{Z}_+$ or a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ such that if char$(D) = q$ is nonzero, then $1/q^k \notin S$ for some positive integer k.*

Proof. See either [31, Theorem], or [25, Corollary 20.15] for the group ring case.

For example, for any field K, the monoid domain $K[X;\mathbb{Q}_+]$ is an almost Dedekind domain which is not a Dedekind domain. They also gave more technical conditions for almost Dedekind semigroup rings; the interested reader may consult [31, Theorem 4.2].

Theorem 2.9. *Let R be a commutative ring and S a nonzero torsionfree cancellative monoid. Then $R[X;S]$ is a general ZPI-ring if and only if R is a finite direct sum of fields and S is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_+$. In particular, $R[X;S]$ is a general ZPI-ring if and only if $R[X;S]$ is a principal ideal ring.*

Proof. If $R[X;S]$ is a general ZPI-ring, then it is a Noetherian Prüfer ring. The result then follows from Theorems 2.5 and 2.7 since R is a finite direct sum of fields. The converse is clear. See either [31, Theorem 5.1 and Corollary 5.1] or [25, Theorem 18.10] for more details.

Several related conditions are also investigated in [25, Sections 18 and 19]. For example, in Theorem 2.7, we may add the equivalence that $R[X;S]$ is arithmetical (recall that a ring T is *arithmetical* if $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all ideals A, B, C of T) [25, Theorem 18.9]. In [25, Section 19], arithmetical monoid rings are studied in the case where the cancellative monoid S is not torsionfree. The treatment of these topics is considerably reorganized in [25] from that in [31] (see the comments in [25, page 251]).

3 Non-Noetherian UFDs, Dimension Theory, and the Book

In this section, we first discuss Gilmer's example of a two-dimensional non-Noetherian UFD. It is a direct application of results in [30] and is one of my favorite examples in ring theory. Later, we will discuss Gilmer's work with J. T. Arnold on the dimension theory of semigroup rings and his book on semigroup rings.

The standard example of a non-Noetherian UFD is $R = K[\{X_n\}_{n=1}^{\infty}]$, the polynomial ring over a field K in infinitely many indeterminates. Unfortunately, R has infinite Krull dimension. So what's an example of a finite-dimensional non-Noetherian UFD? Examples of 3-dimensional non-Noetherian quasilocal UFDs in characteristic 0 and 2 were given by J. David [17, 18] in 1972–1973. Note that a one-dimensional UFD is a PID, and hence Noetherian. So what about the two-dimensional case?

The idea is to construct a group ring $D[X;G]$ which is a two-dimensional UFD, but not Noetherian. Recall that $D[X;G]$ is Noetherian if and only if D is Noetherian and G is finitely generated (i.e., free of finite rank) [25, Theorem 7.7]. So first we need to know a little about the Krull dimension of a group ring. Let G be a torsionfree abelian group with $\text{rank}(G) = \gamma$. Then there is free abelian subgroup F of G with $\text{rank}(F) = \gamma$ and G/F is a torsion group. Hence $R[X;F] \subseteq R[X;G]$ is an integral extension, and thus $\dim(R[X;G]) = \dim(R[X;F])$. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of indeterminates with $|A| = \gamma$. Then $\dim(R[\{X_\alpha, X_\alpha^{-1}\}_{\alpha\in A}]) = \dim(R[\{X_\alpha\}_{\alpha\in A}])$, and hence $\dim(R[X;G]) = \dim(R[X;F]) = \dim(R[\{X_\alpha\}_{\alpha\in A}])$. As a special case, if D is either a Prüfer domain or a Noetherian integral domain and G is a torsionfree abelian group of finite rank n, then $\dim(D[X;G]) = \dim(D)+n$. In particular, if K is a field, then $\dim(K[X;G]) = \text{rank}(G)$.

By the above paragraph, we need to find a torsionfree abelian group G of rank two which is not finitely generated, but satisfies ACC on cyclic subgroups. Which, if any, abelian groups G satisfy these conditions? Fortunately, there is such a rank-two abelian group G. So in this case, $R = K[X;G]$ is a two-dimensional non-Noetherian UFD for any field K. Let L be a rank-two torsionfree abelian group which is not free (and hence not finitely generated), but every rank-one subgroup of L is free (cyclic), and hence L satisfies ACC on cycic subgroups. Such an abelian group exists (see [47] or [21, Section 88]). Let $L_n = L \oplus \mathbb{Z}^n$ for each integer $n \geq 0$. Then L_n is not finitely generated, $\text{rank}(L_n) = n+2$, and L_n satisfies ACC on cyclic subgroups.

Theorem 3.1. *Let K be a field. Then $K[X;L_n]$ is a non-Noetherian UFD of Krull dimension $n+2$.*

Proof. By Theorem 2.3, $R = K[X;L_n]$ is a UFD. By our earlier remarks, R is not Noetherian since L_n is not finitely generated and $\dim(R) = \text{rank}(L_n) = n+2$. See [31, Theorem 4] for more details.

If $\operatorname{char}(K) = p > 0$, then the group ring $R = K[X; L_n]$ in Theorem 3.1 may be localized at a suitable maximal ideal M so that the quasilocal domain R_M is an n-dimensional UFD, but not Noetherian.

Theorem 3.2. *Let p be prime and $n \geq 2$ an integer. Then there is a non-Noetherian quasilocal UFD of Krull dimension n and characteristic p.*

Proof. See [31, Theorem 4] for more details.

Theorem 3.2 leaves open the characteristic 0 case for quasilocal domains. In [14, Theorem D and Example], J. W. Brewer, D. L. Costa, and E. L. Lady showed that for each integer $n \geq 2$, there is a non-Noetherian quasilocal UFD with characteristic 0 and Krull dimension n. (Brewer was also a PhD student of Gilmer.) Their example is based on a localization of the group ring $\mathbb{Z}[G]$, where $G = L$ when $n = 2$ and $G = \mathbb{Z}[1/p] \oplus \mathbb{Z}[1/p] \oplus \mathbb{Z}[1/p]$ for p a prime when $n \geq 3$. In fact, they showed that the technique used in Theorem 3.2 of localizing a group ring over a field will not work in characteristic 0 [14, Theorem A]. Several other examples of 3-dimensional non-Noetherian quasilocal UFDs have been given in the literature (see [11]).

We next briefly discuss Gilmer's work with J. T. Arnold [12] on computing the Krull dimension of $R[X; S]$. This generalizes earlier work on Krull dimension mentioned in this section and does not assume that the cancellative monoid S is torsionfree. Theorem 3.3 reduces the calculation of the Krull dimension of a semigroup ring to that of a group ring, which in turn reduces it to the calculation of the Krull dimension of a polynomial ring since $\dim(R[X; G]) = \dim(R[\{X_\alpha\}_{\alpha \in A}])$, where $|A| = \operatorname{rank}(G)$. In [12], they also extended several results about chains of prime ideals in polynomial rings to semigroup rings $R[X; S]$, where S is a finitely generated torsionfree cancellative monoid.

Theorem 3.3. *Let R be a commutative ring and S a cancellative monoid with quotient group G. Then* $\dim(R[X; S]) = \dim(R[X; G])$.

Proof. This is proved in several reductions; first to the case where R is a finite-dimensional integral domain and S is finitely generated and torsionfree, and then to showing that $\dim(R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]) = \dim(R[X_1, \ldots, X_n, h_1, \ldots, h_j])$, where $h_1, \ldots, h_j$ are pure monomials in $X_1, \ldots, X_n$. For more details, see either [12] or [25, Theorem 21.4].

We conclude this section with a short discussion of Gilmer's book *Commutative Semigroup Rings* [25]. It was written when most of the topics were available only in their original research articles, and it is still the only other reference for many of these topics. Like *Multiplicative Ideal Theory* [27], it is still *the* reference in the field. It gives a unified, self-contained treatment of semigroups and semigroup rings. Many proofs are modified or simplified, sometimes to correct gaps of previous proofs in the literature.

The book consists of 25 sections grouped in 5 chapters. The chapters are (I) Commutative semigroups, (II) Semigroup rings and their distinguished elements, (III) Ring-theoretic properties of monoid domains, (IV) Ring-theoretic properties of monoid rings, and (V) Dimension theory and the isomorphism problems. Except for the second part of Chapter (V), the chapter titles are fairly self-explanatory and much of their content is discussed in this article. The "isomorphism problems" concerns the question of when does $R_1[X;S] \cong R_2[X;S]$ (resp., $R[X;S] \cong R[X;T]$) imply that $R_1 \cong R_2$ (resp., $S \cong T$) (also see [32]).

4 Generalizations

In this final section, we discuss three types of extensions or generalizations of Gilmer's work on semigroup rings. The first is to other divisibility properties for monoid domains, with emphasis on when a monoid domain is a Krull domain. The second is to graded integral domains and their divisibility properties, and the third is to divisibility in monoids.

After determining when a monoid domain is a UFD or a Dedekind domain, the next natural question is: when is $D[X;S]$ a Krull domain, and if so, how do we calculate its divisor class group $Cl(D[X;S])$? Again, we first state the group ring case, which is due to R. Matsuda [40].

Theorem 4.1. *Let D be an integral domain and G a torsionfree abelian group. Then $D[X;G]$ is a Krull domain if and only if D is a Krull domain and G satisfies ACC on cyclic subgroups. Moreover, if $D[X;G]$ is a Krull domain, then $Cl(D[X;G]) = Cl(D)$.*

Proof. Suppose that $R = D[X;G]$ is a Krull domain with $qf(D) = K$. Then D is a Krull domain since $D = R \cap K$, and G satisfies ACC on cyclic subgroups since R satisfies ACCP. Conversely, if G satisfies ACC on cyclic subgroups, then $K[X;G]$ is a UFD by Theorem 2.3. The f-adic discrete valuations on $qf(R)$ induced by the irreducible elements f of the UFD $K[X;G]$ together with the discrete valuations on $qf(R)$ induced by the height-one prime ideals of the Krull domain D show that R is a Krull domain. The divisor class group result follows from Nagata's Theorem [19, Corollary 7.2]. For more details, see either [40, Propositions 3.3 and 5.3] or [25, Theorems 15.1, 15.4, and 16.2].

Theorem 4.1, together with ideas from the previous section, can be used to construct a 3-dimensional non-Noetherian Krull domain with any given divisor class group. Let G be an abelian group and D a Dedekind domain with class group G (such a D exists by Claborn's Theorem [19, Theorem 14.10]). Let $L = L_0$ be as in Theorem 3.1. Then $R = D[X;L]$ is a Krull domain with $Cl(R) = Cl(D) = G$ by Theorem 4.1, and R is non-Noetherian with $\dim(R) = 3$ for reasons discussed in the previous section.

An integral domain is a Krull domain if and only if it is completely integrally closed and satisfies ACC on integral v-ideals. We thus define a torsionfree cancellative monoid to be a *Krull monoid* if it is completely integrally closed and satisfies ACC on integral v-ideals (for other equivalent conditions, see [15, 25, 37]). Our next theorem is due to L. G. Chouinard [15].

Theorem 4.2. *Let D be an integral domain with quotient field K and S a torsionfree cancellative monoid. Then $D[X;S]$ is a Krull domain if and only if D is a Krull domain, S is a Krull monoid, and $U(S)$ satisfies ACC on cyclic subgroups. Moreover, $Cl(D[X;S]) = Cl(D) \oplus Cl(K[X;S])$ and $Cl(K[X;S])$ is independent of the field K.*

Proof. See either [15, Theorem 1] or [25, Theorem 15.6] for details. The "moreover" statement is from [3, Proposition 7.3].

Krull monoids have the form $G \oplus T$, where G is any torsionfree abelian group and T is a submonoid of a free abelian group $F = \oplus_\alpha \mathbb{Z}$ with the usual product order such that $T = \langle T \rangle \cap F_+$. Thus a Krull monoid domain is just a subring of a polynomial ring over a Krull group ring generated by monomials. For a Krull monoid S, it is easy to see that $U(S)$ satisfies ACC on cyclic subgroups if and only if $\langle S \rangle$ satisfies ACC on cyclic subgroups.

Let K be a field and $S \subseteq F_+$ a Krull monoid with $S = \langle S \rangle \cap F_+$ (here $F = \oplus_\alpha \mathbb{Z}$ is a free abelian group with the usual product order and each pr_α is the natural projection map) such that the $pr_\alpha|_{\langle S \rangle}$'s are distinct essential valuations of S. Then $Cl(K[X;S]) \cong F/\langle S \rangle$ (see [15, Theorem 2], [25, Section 16], and [4]). This fact may be used to show that any abelian group G is the divisor class group of a quasilocal Krull domain. Let K be a field and G an abelian group. Then one can construct a Krull domain $R = K[X;S]$ with $Cl(R) = G$ [15, Corollary 2]. In fact, one may then localize R to obtain a quasilocal Krull domain A with $Cl(A) = G$. See [4] for some specific calculations.

An integral domain D is a *Prüfer v-multiplication domain* (PVMD) if the monoid of finite-type v-ideals of D forms a group under v-multiplication. Thus a Krull domain or a Prüfer domain is a PVMD. Analogously, we define a torsionfree cancellative monoid S to be a *PVMD monoid* if the monoid of finite type v-ideals of S forms a group under v-multiplication. Theorem 4.2 then generalizes to PVMDs.

Theorem 4.3. *Let D be an integral domain, S a torsionfree cancellative monoid, and G a torsionfree abelian group. Then $D[X;S]$ is a PVMD if and only if D is a PVMD and S is a PVMD monoid. In particular, $D[X;G]$ is a PVMD if and only if D is a PVMD.*

Proof. The first "$\Rightarrow$" implication, the "in particular" statement, and several other results about PVMD semigroup rings were proved by S. Malik (see [38, Chapter 14] and [39]). The converse of the first implication was also conjectured by Malik, and it was proved in [2, Proposition 6.5].

By Theorems 2.6, 4.1, and 4.3, the group rings $\mathbb{Z}[X;\mathbb{Q}]$, $\mathbb{Q}[X;\mathbb{Z}\oplus\mathbb{Q}] = \mathbb{Q}[X;\mathbb{Q}][Y]$, and $\mathbb{Q}[X;\mathbb{Q}\oplus\mathbb{Q}]$ are all two-dimensional PVMDs which are neither Krull domains nor Prüfer domains.

For any integral domain D, let $T(D)$ be the group of t-invertible fractional t-ideals of D under t-multiplication, and let $Prin(D)$ be its subgroup of nonzero principal fractional ideals. Then the *(t-)class group* of D is the abelian group $Cl(D) = T(D)/Prin(D)$. Let $Inv(D) \subseteq T(D)$ be the subgroup of invertible ideals of D. Then $Pic(D) = Inv(D)/Prin(D)$, the *Picard group* or *ideal class group* of D, is a subgroup of $Cl(D)$. If D is either a Prüfer domain or a one-dimensional integral domain, then $Cl(D) = Pic(D)$, and $Cl(D)$ is the usual divisor class group if D is a Krull domain. The class group is important because ring-theoretic properties of D are often reflected in group-theoretic properties of $Cl(D)$. For example, if D is a PVMD, then $Cl(D) = 0$ (resp., is torsion) if and only if D is a GCD-domain (resp., AGCD-domain). Recall that D is an *almost GCD-domain* (AGCD-domain) if for any $0 \neq a, b \in D$, there is a positive integer $n = n(a,b)$ such that $a^nD \cap b^nD$ is principal. For more on the class group, see the survey article [7].

We next discuss the class group of a monoid domain. As a first step, S. Gabelli [22] showed that $Cl(D) = Cl(D[X])$ if and only if D is integrally closed. In analogy with Theorem 4.2 for Krull domains, our next result, due to S. El Baghdadi, L. Izelgue, and S. Kabbaj [13], gives a very satisfactory answer for the class group of an integrally closed monoid domain.

Theorem 4.4. *Let D be an integral domain with quotient field K and S a torsionfree cancellative monoid. If $D[X;S]$ is integrally closed, then $Cl(D[X;S]) = Cl(D) \oplus Cl(K[X;S])$ and $Cl(K[X;S])$ is independent of the field K.*

Proof. For details, see [13, Corollaries 2.8 and 2.10].

For the non-integrally closed monoid domain case, we include a result from [9]. Recall that an additive submonoid S of $\mathbb{Z}_+$ is called a *numerical semigroup* if $\mathbb{Z}_+ \setminus S$ is finite.

Theorem 4.5. *Let D be an integral domain with quotient field K and S a numerical semigroup. Then $Cl(D[X;S]) = Cl(D[X]) \oplus Pic(K[X;S])$. In particular, if D is integrally closed, then $Cl(D[X;S]) = Cl(D) \oplus Pic(K[X;S])$.*

Proof. Let $N = \{\, X^\alpha \mid \alpha \in S \,\}$ and $T = D^*$. Then the natural homomorphism $Cl(D[X;S]) \longrightarrow Cl(D[X;S]_N) \oplus Cl(D[X;S]_T) = Cl(D[X,X^{-1}]) \oplus Cl(K[X;S])$, given by $[I] \mapsto ([I_N],[I_T])$, is an isomorphism. Also, $Cl(D[X]) = Cl(D[X,X^{-1}])$ for any integral domain D, and $Cl(K[X;S]) = Pic(K[X;S])$ since $K[X;S]$ is one-dimensional. For more details, see [9, Theorem 5]. The "in particular" statement follows from the result of Gabelli [22] mentioned above.

Note that $Pic(K[X;S])$ in Theorem 4.5 may be computed (for example, by using the Mayer-Vietoris exact sequence for (U, Pic)). As a special case, we have that $Cl(D[X^2,X^3]) = Cl(D[X]) \oplus K$.

We next consider the Picard group of a monoid domain. In [29, Theorem 1.6], Gilmer and R. Heitmann showed that $Pic(D) = Pic(D[X])$ if and only if D is seminormal (recall that an integral domain D is *seminormal* if $x^2, x^3 \in D$ for $x \in qf(D)$ implies $x \in D$). Analogously, define a torsionfree cancellative monoid S to be *seminormal* if $2x, 3x \in S$ for $x \in \langle S \rangle$ implies $x \in S$. Then $D[X;S]$ is seminormal if and only if D and S are seminormal [2, Corollary 6.2].

Theorem 4.6. *Let D be an integral domain and S a nonzero torsionfree cancellative monoid. Then $Pic(D) = Pic(D[X;S])$ if and only if $D[X;S]$ is seminormal and $Pic(D) = Pic(D[X;U(S)])$. Moreover, if $U(S) \neq 0$, then $Pic(D) = Pic(D[X;S])$ if and only if $Pic(D) = Pic(D[X;\mathbb{Z}])$.*

Proof. This result is from [6, Corollary]. Also see [5].

Chouinard also determined the projective modules over certain Krull monoid domains [16]. This was later generalized by J. Gubeladze [36] to monoid domains of the form $D[X;S]$, where D is a PID and S is seminormal. R. G. Swan [48] has given a detailed exposition of Gubeladze's work; our next result is [48, Theorem 1.1].

Theorem 4.7. *Let D be a Dedekind domain and S a torsionfree cancellative monoid. Then all finitely generated projective $D[X;S]$-modules are extended from D if and only if S is seminormal.*

Closedness properties usually behave fairly well in that $D[X;S]$ often satisfies a property $\mathcal{P}$ if and only D satisfies $\mathcal{P}$ and S satisfies the additive semigroup analog of $\mathcal{P}$. For example, this holds for integrally closed, completely integrally closed, root closed, and seminormal. This is because these properties are "homogeneous" in the sense that a graded integral domain R satisfies them if and only if R satisfies them for homogeneous elements (see below). However, as we have seen, things do not behave as well for chain conditions. Many other ring-theoretic properties have been studied for semigroup rings (see [8, 25]). Much of this work has been done by R. Matsuda; we cite only [40, 41]. The interested reader should check Math Reviews (or MathSciNet) for more of his work.

We have already given several instances in this article where semigroup rings have been used to construct examples (also see the examples in [8, Section 6]). Another of my favorite examples is (the localization of) a monoid domain used by A. Grams [35] to construct an atomic integral domain which does not satisfy ACCP (recall that an integral domain is *atomic* if each nonzero nonunit is a product of irreducible elements). (Grams was also a PhD student of Gilmer.)

Much of the work on monoid domains generalizes to the context of graded integral domains. By a *(S-)graded integral domain*, we mean an integral domain R graded by a torsionfree cancellative monoid S. That is, $R = \oplus_{\alpha \in S} R_\alpha$, where each R_α is an additive abelian subgroup of R and $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for all $\alpha, \beta \in S$. Each nonzero $x \in R_\alpha$ is homogeneous of $\deg(x) = \alpha$. Let H be the set of nonzero homogeneous elements of R. Then H is a submonoid of R^* under multiplication. We call R_H the *homogeneous quotient field* of R. It is $\langle S \rangle$-graded in the natural way by $\deg(r/s) = \deg(r) - \deg(s)$ for $r, s \in H$, and every nonzero homogeneous element of R_H is a unit. Also, R_H is a completely integrally closed GCD-domain [3, Propositions 3.2 and 3.3] and $R_H \cong (R_H)_0^\gamma[X; \langle S \rangle]$, a twisted group ring over the field $(R_H)_0$. Moreover, R_H is a Laurent polynomial ring over the field $(R_H)_0$ (and hence is a PID) when R is $\mathbb{Z}_+$- or $\mathbb{Z}$-graded.

The monoid domain $R = D[X; S]$ is S-graded with $\deg(dX^\alpha) = \alpha$ for each $d \in D^*$ and $\alpha \in S$. In this case, $H = \{ dX^\alpha \mid d \in D^* \text{ and } \alpha \in S\}$, and thus R has homogeneous quotient field $R_H = K[X; G]$, where $K = qf(D)$ and $G = \langle S \rangle$. However, $R = D[X; S]$ is a very special graded ring in that it is an inert extension of $R_0 = D$ (an extension $A \subseteq B$ of integral domains is *inert* if whenever $xy \in A$ for $x, y \in B$, then $x = ru$ and $y = su^{-1}$ for some $r, s \in A$ and $u \in U(B)$) and $R_\alpha = DX^\alpha \cong D$ for each $\alpha \in S$. Other graded domain constructions that have received considerable attention include the $A + XB[X]$ construction (see [8]).

Given a divisibility property, we can define the corresponding homogeneous divisibility property in the obvious manner. For example, we say that R is a *graded GCD-domain* if any two nonzero homogeneous elements of R have a (necessarily homogeneous) GCD, R is a *graded UFD* if every nonzero nonunit homogeneous element of R is a product of (necessarily homogeneous) prime elements, R is a *graded Krull domain* if R is completely integrally closed with respect to homogeneous elements of R_H and R satisfies ACC on homogeneous integral v-ideals, and R is a *graded PVMD* if the monoid of homogeneous finite-type v-ideals of R forms a group under v-multiplication. We can ask if R satisfies a given divisibilty property if and only if either R^* or H satisfies the corresponding homogeneous divisibility property. Our next result gives some examples, for others, see [1, 2, 8].

Theorem 4.8. *Let R be a graded integral domain and H its monoid of nonzero homogeneous elements. Then*

(a) *R is a GCD-domain if and only if R is a graded GCD-domain (i.e., H is a GCD-monoid).*
(b) *R is a UFD if and only if R is a graded UFD (i.e., H is a factorial monoid) and R_H is a UFD.*
(c) *R is a Krull domain if and only if R is a graded Krull domain and R_H is a Krull domain.*
(d) *R is a PVMD if and only if R is a graded PVMD.*

Proof. See [1, Theorems 3.4, 4.4, and 5.8] for parts (a), (b), and (c), respectively. Part (d) is proved in [2, Theorem 6.4].

Since R_H is always a GCD-domain, parts (a) and (d) have the same form as (b) and (c) in Theorem 4.8. Also, if R is $\mathbb{Z}_+$- or $\mathbb{Z}$-graded, then R is a UFD (resp., Krull domain) if and only if R is graded UFD (resp., Krull domain) since R_H is a PID. However, $K[X;\mathbb{Q}]$ is a graded UFD for any field K, but not a UFD. Theorems 2.2, 2.4, 4.2, and 4.3 follow easily from Theorem 4.8, see [1, Propositions 3.5, 4.7, and 5.11] and [2, Proposition 6.5] for details.

One can also consider conditions on the nonzero homogeneous ideals of a graded integral domain R. This leads to the study of the homogeneous class group $HCl(R)$ and the homogeneous Picard group $HPic(R)$ (see [2, 3, 7, 8, 10, 13]).

Recently, there has been considerable activity on generalizing ring-theoretic properties to the context of semigroups or monoids. This comes about for (at least) two reasons. First, we have seen that $R[X;S]$ satisfies a certain ring-theoretic property often implies that S satisfies the corresponding additive monoid property. For example, we have seen that if $D[X;S]$ is a GCD-domain (resp., UFD, Krull domain, PVMD, seminormal domain), then S is a GCD (resp., factorial, Krull, PVMD, seminormal) monoid. This holds for many more properties. In fact, R. Matsuda [42, 43] has recast much of *Multiplicative Ideal Theory* [27] in the context of torsionfree cancellative monoids. Again, the interested reader should consult Math Reviews (or MathSciNet) for other work of Matsuda.

Secondly, divisibility properties of an integral domain D are often equivalent to the corresponding divisibility properties in the multiplicative monoid D^*. For example, D is a UFD (resp., GCD-domain, Krull domain) if and only if D^* is a factorial (resp., GCD, Krull) monoid. Thus, nowadays much of the research on non-unique factorization in integral domains is done in the more general setting of commutative cancellative monoids. For recent such work and additional references, see [8] and F. Halter-Koch's book *Ideal Systems* [37].

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