2

Deterministic Slow–Fast Systems

A *slow–fast system* involves two kinds of dynamical variables, evolving on very different timescales. The ratio between the fast and slow timescale is measured by a small parameter ε. A slow–fast ordinary differential equation (ODE) is customarily written in the form[1]

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y) \;, \\ \dot{y} &= g(x,y) \;, \end{aligned} \tag{2.0.1}$$

where the components of $x \in \mathbb{R}^n$ are called *fast variables*, while those of $y \in \mathbb{R}^m$ are called *slow variables*. Rather than considering ε as a fixed parameter, it is of interest to study how the dynamics of the system (2.0.1) depends on ε, for all values of ε in an interval $(0, \varepsilon_0]$.

The particularity of a slow–fast ODE such as (2.0.1) is that, instead of remaining a system of coupled differential equations in the limit $\varepsilon \to 0$, it becomes an algebraic–differential system. Such equations are called *singularly perturbed*. Of course, it is always possible to convert (2.0.1) into a regular perturbation problem: The derivatives x' and y' of x and y with respect to the *fast time* $s = t/\varepsilon$ satisfy the system

$$\begin{aligned} x' &= f(x,y) \;, \\ y' &= \varepsilon g(x,y) \;, \end{aligned} \tag{2.0.2}$$

which can be considered as a small perturbation of the *associated system* (or *fast system*)

$$x' = f(x,\lambda) \;, \tag{2.0.3}$$

in which λ plays the rôle of a parameter. However, standard methods from perturbation theory only allow one to control deviations of the solutions of (2.0.2)

[1] In many applications, the right-hand side of (2.0.1) shows an explicit dependence on ε. We refrain from introducing this set-up here, but refer to Remark 2.1.4 concerning an equivalent reformulation.

from those of the associated system (2.0.3) for fast times s of order 1 at most, that is, for t of order ε. The dynamics on longer timescales has to be described by other methods, belonging to the field of *singular perturbation theory*.

The behaviour of the slow–fast system (2.0.1) is nonetheless strongly linked to the dynamics of the associated system (2.0.3). We recall in this chapter results from singular perturbation theory corresponding to the following situations.

- In Section 2.1, we consider the simplest situation, occurring when the associated system admits a hyperbolic equilibrium point $x^\star(\lambda)$ for all parameters λ in some domain. The set of all points $(x^\star(y), y)$ is called a *slow manifold* of the system. We state classical results by Tihonov and Fenichel describing the dynamics near such a slow manifold.
- In Section 2.2, we consider situations in which a slow manifold ceases to be hyperbolic, giving rise to a so-called *dynamic bifurcation*. These singularities can cause new phenomena such as bifurcation delay, relaxation oscillations, and hysteresis. We summarise results on the most generic dynamic bifurcations, including saddle–node, pitchfork and Hopf bifurcations.
- In Section 2.3, we turn to situations in which the associated system admits a stable periodic orbit, depending on the parameter λ. In this case, the dynamics of the slow variables is well approximated by averaging the system over the fast motion along the periodic orbit.

We will not consider more difficult situations, arising for more complicated asymptotic dynamics of the associated system (e.g., quasiperiodic or chaotic). Such situations have been studied in the deterministic case, but the analysis of their stochastic counterpart lies beyond the scope of this book.

2.1 Slow Manifolds

2.1.1 Definitions and Examples

We consider the slow–fast system (2.0.1), where we assume that f and g are twice continuously differentiable in a connected, open set $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$. The simplest situation occurs when the associated system admits one or several hyperbolic equilibrium points, i.e., points on which f vanishes, while the Jacobian matrix $\partial_x f$ of $x \mapsto f(x, \lambda)$ has no eigenvalue on the imaginary axis. Collections of such points define *slow manifolds* of the system. More precisely, we will distinguish between the following types of slow manifolds:

Definition 2.1.1 (Slow manifold). *Let $\mathcal{D}_0 \subset \mathbb{R}^m$ be a connected set of nonempty interior, and assume that there exists a continuous function $x^\star : \mathcal{D}_0 \to \mathbb{R}^n$ such that $(x^\star(y), y) \in \mathcal{D}$ and*

$$f(x^\star(y), y) = 0 \tag{2.1.1}$$

for all $y \in \mathcal{D}_0$*. Then the set* $\mathcal{M} = \{(x,y)\colon x = x^\star(y), y \in \mathcal{D}_0\}$ *is called a* slow manifold *of the system* (2.0.1)*.*

Let $A^\star(y) = \partial_x f(x^\star(y), y)$ *denote the* stability matrix *of the associated system at* $x^\star(y)$*. The slow manifold* $\mathcal{M}$ *is called*

- hyperbolic *if all eigenvalues of* $A^\star(y)$ *have nonzero real parts for all* $y \in \mathcal{D}_0$*;*
- uniformly hyperbolic *if all eigenvalues of* $A^\star(y)$ *have real parts which are uniformly bounded away from zero for* $y \in \mathcal{D}_0$*;*
- asymptotically stable *if all eigenvalues of* $A^\star(y)$ *have negative real parts for all* $y \in \mathcal{D}_0$*;*
- uniformly asymptotically stable *if all eigenvalues of* $A^\star(y)$ *have negative real parts which are uniformly bounded away from zero for* $y \in \mathcal{D}_0$*.*

Finally, a hyperbolic slow manifold $\mathcal{M}$ *is called* unstable *if at least one of the eigenvalues of the stability matrix* $A^\star(y)$ *has positive real part for some* $y \in \mathcal{D}_0$*.*

In the particular case of a one-dimensional slow variable, a slow manifold is also called *equilibrium branch* (the terminology comes from bifurcation theory). The graph of all equilibrium branches versus y is called *bifurcation diagram*.

In the adiabatic limit $\varepsilon \to 0$, the dynamics on the slow manifold is described by the so-called *reduced system* (or *slow system*)

$$\dot{y} = g(x^\star(y), y) \; . \tag{2.1.2}$$

However, it remains yet to be proved that the reduced system indeed gives a good approximation of the original system's dynamics. Before developing the general theory, let us consider a few examples.

Example 2.1.2 (Slowly varying parameters). Consider a dynamical system

$$x' = f_\mu(x) \; , \tag{2.1.3}$$

depending on a set of parameters $\mu \in \mathbb{R}^p$ (the prime always indicates derivation with respect to a fast time s). In an experimental set-up, it is often possible to modify one or several parameters at will, for instance an energy supply, an adjustable resistor, etc. Modifying parameters sufficiently slowly (in comparison to the relaxation time of the system) may allow measurements for different parameter values to be taken in the course of a single experiment. This procedure can be described mathematically by setting $\mu = h(\varepsilon s)$, for a given function $h : \mathbb{R} \to \mathbb{R}^p$. Introducing the slow time $t = \varepsilon s$, one arrives at the slow–fast system

$$\begin{aligned} \varepsilon\dot{x} &= f_{h(y)}(x) =: f(x,y) \; , \\ \dot{y} &= 1 \; . \end{aligned} \tag{2.1.4}$$

Equation (2.1.3) is of course the associated system of (2.1.4). If f_μ vanishes on some set of equilibrium points $x = x^\star_\mu$, then the slow manifold is given by the equation $x = x^\star_{h(y)} =: x^\star(y)$. Note that the reduced dynamics is trivial, since it is always given by the equation $\dot{y} = 1$.

Example 2.1.3 (Overdamped motion of a particle in a potential). The dynamics of a particle in $\mathbb{R}^d$, subject to a field of force deriving from a potential $U(x)$, and a viscous drag, is governed by the second-order equation

$$x'' + \gamma x' + \nabla U(x) = 0 \; . \tag{2.1.5}$$

One possible way to write this equation as a first-order system is

$$\begin{aligned} x' &= \gamma(y - x) \; , \\ y' &= -\frac{1}{\gamma}\nabla U(x) \; . \end{aligned} \tag{2.1.6}$$

Let us assume that the friction coefficient γ is large, and set $\varepsilon = 1/\gamma^2$. With respect to the slow time $t = \sqrt{\varepsilon} s$, the dynamics is governed by the slow–fast system

$$\begin{aligned} \varepsilon\dot{x} &= y - x \; , \\ \dot{y} &= -\nabla U(x) \; . \end{aligned} \tag{2.1.7}$$

The slow manifold is given by $x^\star(y) = y$. The stability matrix being simply $A^*(y) \equiv -\mathbb{1}$, the slow manifold is uniformly asymptotically stable, and the reduced dynamics is governed by the equation

$$\dot{y} = -\nabla U(y) \; , \tag{2.1.8}$$

or, equivalently, $\dot{x} = -\nabla U(x)$. This relation is sometimes called *Aristotle's law*, since it reflects the fact that at large friction, velocity is proportional to force, as if inertia were absent.

Remark 2.1.4. For simplicity, we assumed in (2.0.1) that the right-hand side does not explicitly depend on ε. This is no real constraint as we can always introduce a dummy variable for ε. We rewrite the slow–fast system

$$\begin{aligned} \varepsilon\dot{x} &= f(x, y, \varepsilon) \; , \\ \dot{y} &= g(x, y, \varepsilon) \; , \end{aligned} \tag{2.1.9}$$

as

$$\begin{aligned} \varepsilon\dot{x} &= f(x, y, z) \; , \\ \dot{y} &= g(x, y, z) \; , \\ \dot{z} &= 0 \, , \end{aligned} \tag{2.1.10}$$

and consider z as an additional slow variable. Thus slow manifolds for (2.1.9) are of the form $x^\star(y, \varepsilon)$.[2]

[2]Some authors do not allow for ε-dependent slow manifolds and consider $x^\star(y, 0)$, obtained by setting $\varepsilon = 0$, as the slow manifold.

The following example provides an application with ε-dependent right-hand side.

Example 2.1.5 (Stommel's box model). Simple climate models, whose dynamic variables are averaged values of physical quantities over some large volumes, or *boxes*, are called box models. Stommel's model gives a simple qualitative description of the North Atlantic thermohaline circulation. The fast variable is proportional to the temperature difference between a low- and a high-latitude box, and the slow variable is proportional to the salinity difference (see Section 6.2 for a more detailed description). Their dynamics is governed by the system

$$\begin{aligned} \varepsilon \dot{x} &= -(x-1) - \varepsilon x Q(x-y) , \\ \dot{y} &= \mu - y Q(x-y) , \end{aligned} \tag{2.1.11}$$

where the small parameter ε reflects the fact that the relaxation time for the temperature difference is much shorter than the one for the salinity difference. The parameter μ is proportional to the freshwater flux, while the function Q describes the Fickian mass exchange. Typical choices are $Q(z) = 1 + \eta|z|$ [Sto61][3] or $Q(z) = 1 + \eta^2 z^2$ [Ces94], where η, which depends on the volume of the boxes, is of order one.

The slow manifold is of the form $x^\star(y,\varepsilon) = 1 - \varepsilon Q(1-y) + \mathcal{O}(\varepsilon^2)$ for small ε and y from a bounded set, and is obviously uniformly asymptotically stable. The reduced dynamics is given by an equation of the form

$$\dot{y} = \mu - y Q(1-y) + \mathcal{O}(\varepsilon) . \tag{2.1.12}$$

For the above-mentioned choices of Q, depending on the values of μ and η, there can be up to three equilibrium points, one of which is unstable.

Example 2.1.6 (Van der Pol oscillator). The van der Pol oscillator is an electric circuit including a current-dependent resistor (see Section 6.1.2). Its (scalar) equation is

$$x'' + \gamma(x^2-1)x' + x = 0 . \tag{2.1.13}$$

For $\gamma = 0$, it reduces to an harmonic oscillator. For large γ, however, the dynamics becomes very far from harmonic. Proceeding as in Example 2.1.3, (2.1.13) can be transformed into the slow–fast system

$$\begin{aligned} \varepsilon \dot{x} &= y + x - \frac{x^3}{3} , \\ \dot{y} &= -x , \end{aligned} \tag{2.1.14}$$

where again $\varepsilon = 1/\gamma^2$ and $t = \sqrt{\varepsilon} s$. The associated system has up to three equilibria, and the slow manifold is a curve, given implicitly by the equation

[3] Note that Stommel's choice of Q does not satisfy our differentiability assumption. Since left and right derivatives exist, the system can nevertheless be studied by patching together solutions for $x < y$ and $x > y$.

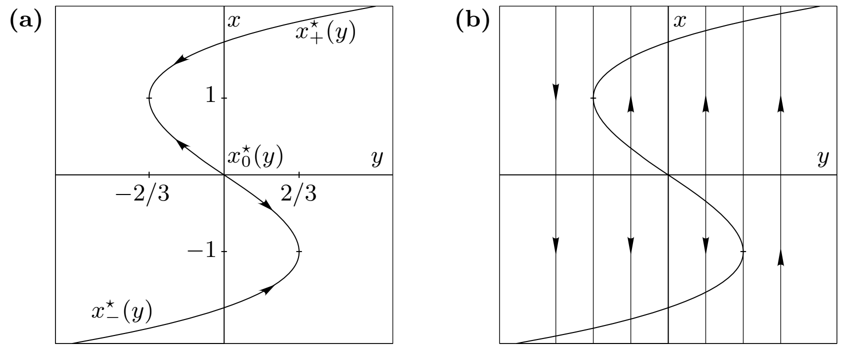

Fig. 2.1. Behaviour of the van der Pol equation in the singular limit $\varepsilon \to 0$: **(a)** the reduced dynamics (2.1.17) on the slow manifold, **(b)** the fast dynamics of the associated system $x' = y + x - \frac{1}{3}x^3$.

$x^3/3 - x = y$. The stability matrix, which is a scalar in this case, has value $1-x^2$, showing that points with $|x| > 1$ are stable while points with $|x| < 1$ are unstable. The slow manifold is thus divided into three equilibrium branches (Fig. 2.1),

$$\begin{aligned} x^\star_- &: (-\infty, \tfrac{2}{3}) \to (-\infty, -1) \;, \\ x^\star_0 &: (-\tfrac{2}{3}, \tfrac{2}{3}) \;\to (-1, 1) \;, \\ x^\star_+ &: (-\tfrac{2}{3}, \infty) \to (1, \infty) \;, \end{aligned} \tag{2.1.15}$$

meeting at two *bifurcation points* $\pm(1, -\frac{2}{3})$.

The reduced system is best expressed with respect to the variable x. Since on the slow manifold one has

$$-x = \dot{y} = (x^2 - 1)\dot{x} \;, \tag{2.1.16}$$

the reduced dynamics is governed by the equation

$$\dot{x} = -\frac{x}{x^2 - 1} \;, \tag{2.1.17}$$

which becomes singular in $x = \pm 1$. We will see below (Example 2.2.3 in Section 2.2.2) that the true dynamics for $\varepsilon > 0$ involves *relaxation oscillations*, in which slow and fast motions alternate (Fig. 2.5).

2.1.2 Convergence towards a Stable Slow Manifold

The first results showing that the reduced equation on a slow manifold may indeed give a good approximation to the full dynamics are due to Tihonov [Tih52] and Gradšteĭn [Gra53]. In particular, the following theorem on exponentially fast convergence of solutions to an ε-neighbourhood of a uniformly asymptotically stable slow manifold is contained in their results.

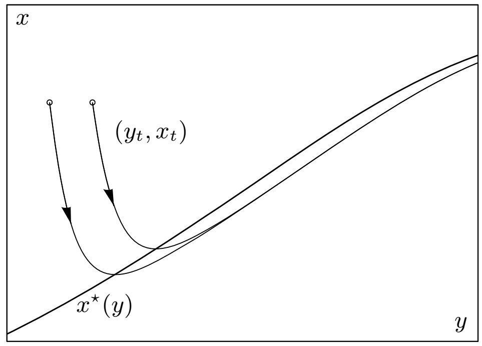

Fig. 2.2. Two orbits approaching a uniformly asymptotically stable slow manifold.

Theorem 2.1.7 (Convergence towards a slow manifold). *Let* $\mathcal{M} = \{(x,y)\colon x = x^\star(y), y \in \mathcal{D}_0\}$ *be a uniformly asymptotically stable slow manifold of the slow–fast system* (2.0.1)*. Let* f *and* g*, as well as all their derivatives up to order two, be uniformly bounded in norm in a neighbourhood* $\mathcal{N}$ *of* $\mathcal{M}$*. Then there exist positive constants* $\varepsilon_0, c_0, c_1, \kappa = \kappa(n), M$ *such that, for* $0 < \varepsilon \leqslant \varepsilon_0$ *and any initial condition* $(x_0, y_0) \in \mathcal{N}$ *satisfying* $\|x_0 - x^\star(y_0)\| \leqslant c_0$*, the bound*

$$\|x_t - x^\star(y_t)\| \leqslant M\|x_0 - x^\star(y_0)\|\mathrm{e}^{-\kappa t/\varepsilon} + c_1\varepsilon \tag{2.1.18}$$

holds as long as $y_t \in \mathcal{D}_0$*.*

This result shows that after a time of order $\varepsilon|\log\varepsilon|$, all orbits starting in a neighbourhood of order 1 of the slow manifold $\mathcal{M}$ will have reached a neighbourhood of order ε, where they stay as long as the slow dynamics permits (Fig. 2.2). This phenomenon is sometimes called "slaving principle" or "adiabatic reduction".

A simple proof of Theorem 2.1.7 uses Lyapunov functions. Since, for any $y \in \mathcal{D}_0$, $x^\star(y)$ is an asymptotically stable equilibrium point of the associated system, there exists a Lyapunov function $x \mapsto V(x,y)$, admitting a non-degenerate minimum in $x^\star(y)$, with $V(x^\star(y),y) = 0$, and satisfying

$$\langle \nabla_x V(x,y), f(x,y)\rangle \leqslant -2\kappa V(x,y) \tag{2.1.19}$$

in a neighbourhood of $x^\star(y)$, see, e.g., [Arn92, § 22]. One can choose $V(\cdot,y)$ to be a quadratic form in $x - x^\star(y)$, with $V(x,y)/\|x - x^\star(y)\|^2$ uniformly bounded above and below by positive constants. Taking into account the y-dependence of V and $x^\star$, one arrives at the relation

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}V(x_t,y_t) =: \varepsilon\dot{V}_t \leqslant -2\kappa V_t + const\,\varepsilon\sqrt{V_t}\;, \tag{2.1.20}$$

which implies (2.1.18).

2.1.3 Geometric Singular Perturbation Theory

While Tihonov's theorem is useful to describe the orbits' approach to a small neighbourhood of a uniformly asymptotically stable slow manifold, it does not provide a very precise picture of the dynamics in this neighbourhood. Fenichel has initiated a geometrical approach [Fen79], which allows for a description of the dynamics in terms of invariant manifolds. The simplest result of this kind concerns the existence of an invariant manifold near a hyperbolic (not necessarily stable) slow manifold.

Theorem 2.1.8 (Existence of an adiabatic manifold). *Let the slow manifold* $\mathcal{M} = \{(x,y)\colon x = x^\star(y), y \in \mathcal{D}_0\}$ *of the slow–fast system* (2.0.1) *be uniformly hyperbolic. Then there exists, for sufficiently small* ε*, a locally invariant manifold*

$$\mathcal{M}_\varepsilon = \{(x,y)\colon x = \bar{x}(y,\varepsilon), y \in \mathcal{D}_0\} , \tag{2.1.21}$$

where $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$*. In other words, if the initial condition is taken on* $\mathcal{M}_\varepsilon$*, that is,* $x_0 = \bar{x}(y_0,\varepsilon)$*, then* $x_t = \bar{x}(y_t,\varepsilon)$ *as long as* $y_t \in \mathcal{D}_0$*.*

We shall call $\mathcal{M}_\varepsilon$ an *adiabatic manifold*. The dynamics on $\mathcal{M}_\varepsilon$ is governed by the equation

$$\dot{y} = g(\bar{x}(y,\varepsilon), y) , \tag{2.1.22}$$

which reduces to (2.1.2) in the limit $\varepsilon \to 0$. By extension, we also call it *reduced system*. The deviations from the limiting system can now be treated by standard methods of regular perturbation theory.

Fenichel has in fact proved more general results, in particular on the existence of invariant manifolds associated with the stable and unstable manifolds of a family of hyperbolic equilibria of the fast system. These results, together with their many refinements, are known as *geometric singular perturbation theory*. See, for instance, [Jon95] for a review.

Theorem 2.1.8 can be proved by using the centre-manifold theorem (see, for instance, [Car81]). Indeed, by again viewing ε as a dummy dynamic variable, and using the fast time $s = t/\varepsilon$, the slow–fast system can be rewritten as

$$\begin{aligned} x' &= f(x,y) , \\ y' &= \varepsilon g(x,y) , \\ \varepsilon' &= 0 . \end{aligned} \tag{2.1.23}$$

Any point of the form $(x^\star(y), y, 0)$ is an equilibrium point of this system.[4] The linearisation of (2.1.23) around such a point has the structure

$$\begin{pmatrix} A^\star(y) & \partial_y f(x^\star(y),y) & 0 \\ 0 & 0 & g(x^\star(y),y) \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.1.24}$$

[4] There might also exist equilibrium points $(x^\star(y), y, \varepsilon)$ with $\varepsilon > 0$, namely if $g(x^\star(y),y) = 0$. At these points, slow and adiabatic manifold coïncide for *all* ε.

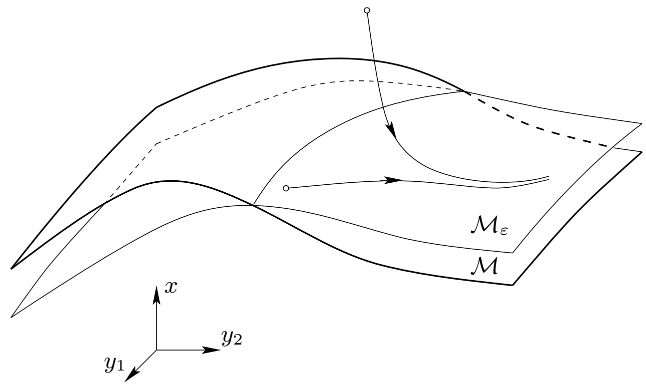

Fig. 2.3. An adiabatic manifold $\mathcal{M}_\varepsilon$ associated with a uniformly asymptotically stable slow manifold. Orbits starting in its vicinity converge exponentially fast to an orbit on the adiabatic manifold.

Hence it admits $m+1$ vanishing eigenvalues, while the eigenvalues of $A^\star(y)$ are bounded away from the imaginary axis by assumption. Thus the centre-manifold theorem (cf. [Car81, Theorem 1, p. 4]) implies the existence of an invariant manifold.

Two important properties of centre manifolds carry over to this special case. Firstly, if the slow manifold $\mathcal{M}$ is uniformly asymptotically stable, then the centre manifold $\mathcal{M}_\varepsilon$ is locally attractive, cf. [Car81, Theorem 2, p. 4]. This means that for each initial condition (x_0, y_0) sufficiently close to $\mathcal{M}_\varepsilon$, there exists a particular solution $\widehat{y}_t$ of the reduced equation (2.1.22) such that

$$\|(x_t, y_t) - (\widehat{x}_t, \widehat{y}_t)\| \leqslant M \|(x_0, y_0) - (\widehat{x}_0, \widehat{y}_0)\| \mathrm{e}^{-\kappa t/\varepsilon} \tag{2.1.25}$$

for some $M, \kappa > 0$, where $\widehat{x}_t = \bar{x}(\widehat{y}_t, \varepsilon)$. Thus, after a short time, the orbit (x_t, y_t) has approached an orbit on the invariant manifold.

The second property concerns approximations of $\bar{x}(y,\varepsilon)$. Using invariance, one easily sees from (2.0.1) that $\bar{x}(y,\varepsilon)$ must satisfy the partial differential equation

$$\varepsilon \partial_y \bar{x}(y,\varepsilon) g(\bar{x}(y,\varepsilon), y) = f(\bar{x}(y,\varepsilon), y) \;. \tag{2.1.26}$$

This equation is difficult to solve in general. However, the approximation theorem for centre manifolds (cf. [Car81, Theorem 3, p. 5]) states that, if f and g are sufficiently differentiable, then any power series in ε or y, satisfying (2.1.26) to order ε^k or y^k, is indeed an approximation of $\bar{x}(y,\varepsilon)$ to that order. It is thus possible to construct an approximate solution of (2.1.26) by inserting a power series in ε and equating like powers. The first orders of the expansion in ε are

$$\begin{aligned}\bar{x}(y,\varepsilon) &= x^\star(y) + \varepsilon A^\star(y)^{-1} \partial_y x^\star(y) g(x^\star(y), y) + \mathcal{O}(\varepsilon^2) \\ &= x^\star(y) - \varepsilon A^\star(y)^{-2} \partial_y f(x^\star(y), y) g(x^\star(y), y) + \mathcal{O}(\varepsilon^2) \;,\end{aligned} \tag{2.1.27}$$

where the last line follows from the implicit-function theorem, applied to the equation $f(x^\star(y), y) = 0$.

Example 2.1.9 (Overdamped motion of a particle in a potential – continued). Consider again the slow–fast system of Example 2.1.3. The adiabatic manifold satisfies the equation

$$\varepsilon\partial_y\bar{x}(y,\varepsilon)\nabla U(\bar{x}(y,\varepsilon)) = -y + \bar{x}(y,\varepsilon) \;, \tag{2.1.28}$$

and admits the expansion

$$\bar{x}(y,\varepsilon) = y + \varepsilon\nabla U(y) + 2\varepsilon^2\partial_{yy}U(y)\nabla U(y) + \mathcal{O}(\varepsilon^3) \;, \tag{2.1.29}$$

where $\partial_{yy}U(y)$ denotes the Hessian matrix of U. The dynamics of $x = \bar{x}(y,\varepsilon)$ on the adiabatic manifold is thus governed by the equation

$$\begin{aligned} \dot{x} &= \frac{y-x}{\varepsilon} \\ &= -\partial_y\bar{x}(y,\varepsilon)\nabla U(x) \\ &= -\big[\mathbb{1} + \varepsilon\partial_{xx}U(x) + \mathcal{O}(\varepsilon^2)\big]\nabla U(x) \;. \end{aligned} \tag{2.1.30}$$

This is indeed a small correction to the limiting equation $\dot{x} = -\nabla U(x)$.

One should note two facts concerning the power-series expansion of adiabatic manifolds, in the case where the fast and slow vector fields f and g are analytic. First, centre manifolds are in general not unique. However, all centre manifolds of a non-hyperbolic equilibrium share the same expansions, and thus differ typically only by exponentially small terms. Second, although these expansions are asymptotic expansions, they do not in general provide convergent series.

Example 2.1.10. Consider the slow–fast system

$$\begin{aligned} \varepsilon\dot{x} &= -x + h(y) \;, \\ \dot{y} &= 1 \;, \end{aligned} \tag{2.1.31}$$

where h is analytic. The slow manifold is given by $x^\star(y) = h(y)$ and is uniformly asymptotically stable. The adiabatic manifold satisfies the equation

$$\varepsilon\partial_y\bar{x}(y,\varepsilon) = -\bar{x}(y,\varepsilon) + h(y) \;, \tag{2.1.32}$$

and admits the asymptotic expansion

$$\bar{x}(y,\varepsilon) = h(y) - \varepsilon h'(y) + \varepsilon^2 h''(y) - \dots \;. \tag{2.1.33}$$

This series will not converge in general, since the norm of the kth derivative of h may grow like $k!$ by Cauchy's formula. In fact, (2.1.31) admits for each k the particular solution

$$x_t = \sum_{j=0}^{k-1} (-\varepsilon)^j h^{(j)}(y_t) - (-\varepsilon)^{k-1} \int_0^t \mathrm{e}^{-(t-s)/\varepsilon} h^{(k)}(y_s)\,\mathrm{d}s \;. \tag{2.1.34}$$

If $|h|$ is bounded by a constant M in a strip of width $2R$ around the real axis, Cauchy's formula shows that $|h^{(k)}| \leqslant MR^{-k}k!$, and the last term in (2.1.34) is bounded by $(\varepsilon/R)^k Mk!$. Using Stirling's formula, one finds that this remainder is smallest for $k \simeq R/\varepsilon$, and that for this value of k it is of order $\mathrm{e}^{-1/\varepsilon}$. This situation is quite common in analytic singularly perturbed systems.

2.2 Dynamic Bifurcations

An essential assumption in the previous section was the (uniform) hyperbolicity of the slow manifold, that is, the fact that no eigenvalue of the Jacobian matrix $A^\star(y) = \partial_x f(x^\star(y), y)$ approaches the imaginary axis. It can happen, however, that the slow dynamics of y on the adiabatic manifold takes the orbit to a region where this assumption is violated, that is, to a *bifurcation point* of the associated system. We have seen such a situation in the case of the van der Pol oscillator (Example 2.1.6), where the points $\pm(1, -\frac{2}{3})$ correspond to saddle–node bifurcations.

We will start, in Section 2.2.1, by showing how the dynamics near such a bifurcation point can be reduced to a suitable centre manifold, involving only bifurcating directions. Then we turn to a detailed description of the most generic bifurcations.

2.2.1 Centre-Manifold Reduction

We say that $(\widehat{x}, \widehat{y})$ is a *bifurcation point* of the slow–fast system

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y) \;, \\ \dot{y} &= g(x,y) \;, \end{aligned} \tag{2.2.1}$$

if $f(\widehat{x}, \widehat{y}) = 0$ and $\partial_x f(\widehat{x}, \widehat{y})$ has q eigenvalues on the imaginary axis, with $1 \leqslant q \leqslant n$. We consider here the situation where $q < n$, because otherwise no reduction is possible, and assume that the other $n-q$ eigenvalues of $\partial_x f(\widehat{x}, \widehat{y})$ have strictly negative real parts (in order to obtain a locally attracting invariant manifold).

We introduce coordinates $(x^-, z) \in \mathbb{R}^{n-q} \times \mathbb{R}^q$ in which the matrix $\partial_x f(\widehat{x}, \widehat{y})$ becomes block-diagonal, with a block $A^- \in \mathbb{R}^{(n-q)\times(n-q)}$ having eigenvalues in the left half-plane, and a block $A^0 \in \mathbb{R}^{q\times q}$ having eigenvalues on the imaginary axis. We consider again ε as a dummy dynamic variable, and write the system in slow time as

$$
\begin{aligned}
(x^-)' &= f^-(x^-, z, y) \;, \\
z' &= f^0(x^-, z, y) \;, \\
y' &= \varepsilon g(x^-, z, y) \;, \\
\varepsilon' &= 0 \;.
\end{aligned} \tag{2.2.2}
$$

This system admits $(\widehat{x}^-, \widehat{z}, \widehat{y}, 0)$ as an equilibrium point, with a linearisation having $q + m + 1$ eigenvalues on the imaginary axis (counting multiplicity), which correspond to the directions z, y and ε. In other words, z has become a slow variable near the bifurcation point. We can thus apply the centre-manifold theorem to obtain the existence, for sufficiently small ε and in a neighbourhood $\mathcal{N}$ of $(\widehat{z}, \widehat{y})$, of a locally attracting invariant manifold

$$
\widehat{\mathcal{M}}_\varepsilon = \{(x^-, z, y) \colon x^- = \bar{x}^-(z, y, \varepsilon), (z, y) \in \mathcal{N}\} \;. \tag{2.2.3}
$$

The function $\bar{x}^-(z, y, \varepsilon)$ satisfies $\bar{x}^-(\widehat{z}, \widehat{y}, 0) = \widehat{x}^-$, and is a solution of the partial differential equation

$$
\begin{aligned}
f^-(\bar{x}^-(z, y, \varepsilon), z, y) = {} & \partial_z \bar{x}^-(z, y, \varepsilon) f^0(\bar{x}^-(z, y, \varepsilon), z, y) \\
& + \varepsilon \partial_y \bar{x}^-(z, y, \varepsilon) g(\bar{x}^-(z, y, \varepsilon), z, y) \;.
\end{aligned} \tag{2.2.4}
$$

In particular, $\bar{x}^-(z, y, 0)$ corresponds to a centre manifold of the associated system.

By local attractivity, it is sufficient to study the $(q+m)$-dimensional *reduced system*

$$
\begin{aligned}
\varepsilon \dot{z} &= f^0(\bar{x}^-(z, y, \varepsilon), z, y) \;, \\
\dot{y} &= g(\bar{x}^-(z, y, \varepsilon), z, y) \;,
\end{aligned} \tag{2.2.5}
$$

which contains only bifurcating fast variables.

2.2.2 Saddle–Node Bifurcation

Assume that $\partial_x f(\widehat{x}, \widehat{y})$ has one vanishing eigenvalue, all its other eigenvalues having negative real parts, that is, $q = 1$. The most generic bifurcation then is the *saddle–node bifurcation*.

We will ease notations by choosing $\widehat{x} = 0$ and $\widehat{y} = 0$, writing x instead of z, and omitting the ε-dependence on the right-hand side of (2.2.5).[5] We thus arrive at the reduced system

$$
\begin{aligned}
\varepsilon \dot{x} &= \tilde{f}(x, y) \;, \\
\dot{y} &= \tilde{g}(x, y) \;,
\end{aligned} \tag{2.2.6}
$$

[5] This ε-dependence is harmless for the saddle–node bifurcation, which is structurally stable, but may play a nontrivial rôle for less generic bifurcations. We will examine examples of such situations, involving avoided bifurcations, in Chapter 4.

where now $x \in \mathbb{R}$. With a slight abuse of notation we also drop the tilde and write (2.2.6) as

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y) \;, \\ \dot{y} &= g(x,y) \;. \end{aligned} \tag{2.2.7}$$

The fast vector field f satisfies the bifurcation conditions

$$f(0,0) = 0 \qquad \text{and} \qquad \partial_x f(0,0) = 0 \;. \tag{2.2.8}$$

We will discuss here the case of a one-dimensional slow variable $y \in \mathbb{R}$, that is, $m = 1$, as for the van der Pol oscillator in Example 2.1.6. A saddle–node bifurcation occurs if

$$\partial_{xx} f(0,0) \neq 0 \qquad \text{and} \qquad \partial_y f(0,0) \neq 0 \;. \tag{2.2.9}$$

For the sake of definiteness, we will choose the variables in such a way that

$$\partial_{xx} f(0,0) < 0 \qquad \text{and} \qquad \partial_y f(0,0) < 0 \tag{2.2.10}$$

hold. This implies in particular that the slow manifold exists for $y < 0$. Finally, the additional assumption

$$g(0,0) > 0 \tag{2.2.11}$$

guarantees that trajectories starting near the stable slow manifold are driven towards the bifurcation point. The simplest example of this kind is the system

$$\begin{aligned} \varepsilon \dot{x} &= -x^2 - y \;, \\ \dot{y} &= 1 \;, \end{aligned} \tag{2.2.12}$$

where the slow manifold consists of a stable branch $\mathcal{M}_- = \{(x,y)\colon x = \sqrt{-y}, y < 0\}$ and an unstable branch $\mathcal{M}_+ = \{(x,y)\colon x = -\sqrt{-y}, y < 0\}$. The linearisation of f at these branches is given by $\mp 2\sqrt{-y}$.

For general systems satisfying (2.2.8), (2.2.10) and (2.2.11), a qualitatively similar behaviour holds in a neighbourhood of the bifurcation point $(0,0)$. By rescaling x and y, we may arrange for

$$\partial_{xx} f(0,0) = -2 \qquad \text{and} \qquad \partial_y f(0,0) = -1 \;. \tag{2.2.13}$$

Using the implicit-function theorem, one easily shows that there exists a neighbourhood $\mathcal{N}$ of $(0,0)$ such that

- there is an asymptotically stable slow manifold

$$\mathcal{M}_- = \{(x,y) \in \mathcal{N}\colon x = x^\star_-(y), y < 0\} \;, \tag{2.2.14}$$

  where $x^\star_-(y) = \sqrt{-y}\,[1 + \mathcal{O}_y(1)]$, and $\mathcal{O}_y(1)$ stands for a remainder $r(y)$ satisfying $\lim_{y\to 0} r(y) = 0$;

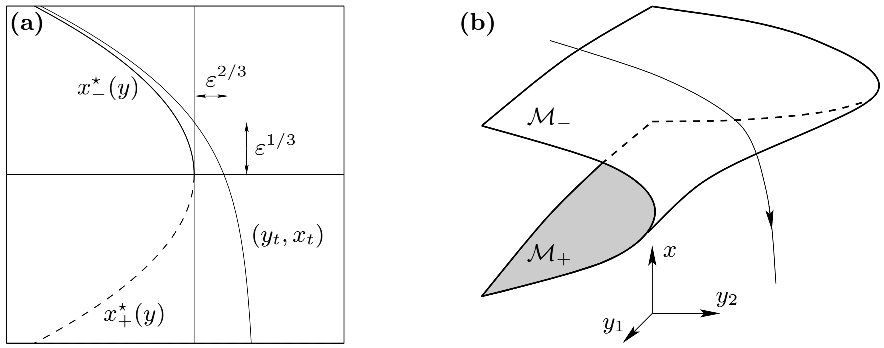

Fig. 2.4. Saddle–node bifurcation of a slow manifold. **(a)** Solutions track the stable branch $x^\star_-(y)$ at a distance growing up to order $\varepsilon^{1/3}$. They jump after a delay of order $\varepsilon^{2/3}$. **(b)** For multidimensional slow variables y, a saddle–node bifurcation corresponds to a fold in the slow manifold.

- the linearisation of f at $\mathcal{M}_-$ is given by

$$a^\star_-(y) := \partial_x f(x^\star_-(y), y) = -2\sqrt{-y}\,[1 + \mathcal{O}_y(1)] \; ; \tag{2.2.15}$$

- there is an unstable slow manifold

$$\mathcal{M}_+ = \{(x, y) \in \mathcal{N} \colon x = x^\star_+(y), y < 0\} \; , \tag{2.2.16}$$

  where $x^\star_+(y) = -\sqrt{-y}\,[1 + \mathcal{O}_y(1)]$;
- there are no other slow manifolds in $\mathcal{N}$;
- $g(x, y) > 0$ in $\mathcal{N}$.

$\mathcal{M}_-$ being uniformly asymptotically stable for negative y, bounded away from 0, there is an adiabatic manifold associated with it which, in this $(1+1)$–dimensional setting, is simply a particular solution of the system. When approaching the bifurcation point, two effects make it increasingly difficult for this solution to track the slow manifold:

- The stable manifold $\mathcal{M}_-$ becomes less and less attracting,
- and $\mathcal{M}_-$ has a vertical tangent at the bifurcation point.

The interesting consequence is that trajectories no longer track the stable manifold at a distance of order ε, but of order $\varepsilon^{1/3}$. After a delay of order $\varepsilon^{2/3}$, they leave the neighbourhood $\mathcal{N}$ in the direction of negative x-values (Fig. 2.4). The following notation will be useful to formulate such scaling laws.

Notation 2.2.1 (Scaling behaviour). *For functions $x_1(t, \varepsilon)$ and $x_2(t, \varepsilon)$, defined for t in a specified interval I and $0 < \varepsilon \leqslant \varepsilon_0$, we indicate by*

$$x_1(t, \varepsilon) \asymp x_2(t, \varepsilon) \tag{2.2.17}$$

the existence of two constants $c_\pm > 0$ such that

$$c_- x_2(t,\varepsilon) \leqslant x_1(t,\varepsilon) \leqslant c_+ x_2(t,\varepsilon) \tag{2.2.18}$$

for all $t \in I$ *and* $0 < \varepsilon \leqslant \varepsilon_0$.

For instance, we have $x^\star_-(y) \asymp \sqrt{-y}$ and $a^\star_-(y) \asymp -\sqrt{-y}$. The announced behaviour of the orbits is made precise in the following theorem.

Theorem 2.2.2 (Dynamic saddle–node bifurcation). *Choose an initial condition* $(x_0, y_0) \in \mathcal{N}$ *with constant* $y_0 < 0$ *and an* x_0 *satisfying* $x_0 - x^\star_-(y_0) \asymp \varepsilon$. *Then there exist constants* $c_1, c_2 > 0$ *such that*

$$x_t - x^\star_-(y_t) \asymp \frac{\varepsilon}{|y_t|} \qquad \textit{for } y_0 \leqslant y_t \leqslant -c_1\varepsilon^{2/3} , \tag{2.2.19}$$

$$x_t \asymp \varepsilon^{1/3} \qquad \textit{for } -c_1\varepsilon^{2/3} \leqslant y_t \leqslant c_2\varepsilon^{2/3} . \tag{2.2.20}$$

Moreover, for any sufficiently small constant $L > 0$, x_t *reaches* $-L$ *at a time* $t(L)$ *such that* $y_{t(L)} \asymp \varepsilon^{2/3}$.

Proof. Since $\dot y > 0$ in $\mathcal{N}$ by assumption, we may use y instead of t as time variable, and study the equation

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(x,y)}{g(x,y)} =: \bar f(x,y) . \tag{2.2.21}$$

It is straightforward to check that $\bar f$ satisfies the same properties (2.2.8) and (2.2.10) as f (before rescaling), and therefore, it is sufficient to study the slow–fast system (2.2.7) for $g \equiv 1$. Again we may assume by rescaling x and y that (2.2.13) holds.

- For $y \leqslant -c_1\varepsilon^{2/3}$, where c_1 remains to be chosen, we use the change of variables $x = x^\star_-(y) + z$ and a Taylor expansion, yielding the equation

$$\varepsilon \frac{\mathrm{d}z}{\mathrm{d}y} = a^\star_-(y) z + b(z,y) - \varepsilon \frac{\mathrm{d}x^\star_-(y)}{\mathrm{d}y} , \tag{2.2.22}$$

  where $|b(z,y)| \leqslant Mz^2$ in $\mathcal{N}$ for some constant M. The properties of $x^\star_-(y)$ and $a^\star_-(y)$, cf. (2.2.14) and (2.2.15), imply the existence of constants $c_\pm > 0$ such that

$$\varepsilon \frac{\mathrm{d}z}{\mathrm{d}y} \leqslant -\sqrt{-y}\biggl[c_- - \frac{M}{\sqrt{-y}} z\biggr] z + \varepsilon \frac{c_+}{\sqrt{-y}} . \tag{2.2.23}$$

  We now introduce the time

$$\tau = \inf\biggl\{ t \geqslant 0 \colon (x_t, y_t) \notin \mathcal{N} \text{ or } |z_t| > \frac{c_-}{2M}\sqrt{-y_t} \biggr\} \in (0, \infty] . \tag{2.2.24}$$

  Note that $\tau > 0$ for sufficiently small ε, since $-y_0$ is of order 1 and z_0 is of order ε by assumption. For $t \leqslant \tau$, we have

$$\varepsilon \frac{\mathrm{d}z}{\mathrm{d}y} \leqslant -\frac{c_-}{2}\sqrt{-y}\, z + \varepsilon \frac{c_+}{\sqrt{-y}} \;, \tag{2.2.25}$$

and thus, solving a linear equation,

$$\begin{aligned} z_t \leqslant{}& z_0 \mathrm{e}^{-c_-[(-y_0)^{3/2}-(-y_t)^{3/2}]/3\varepsilon} \\ &+ c_+ \int_{y_0}^{y_t} \frac{1}{\sqrt{-u}} \mathrm{e}^{-c_-[(-u)^{3/2}-(-y_t)^{3/2}]/3\varepsilon} \,\mathrm{d}u \;, \end{aligned} \tag{2.2.26}$$

as long as $t \leqslant \tau$ and $y_t < 0$. The integral can be evaluated by a variant of the Laplace method, see Lemma 2.2.8 below. We obtain the existence of a constant $K = K(c_-, c_+)$ such that

$$z_t \leqslant K \frac{\varepsilon}{|y_t|} \tag{2.2.27}$$

for $t \leqslant \tau$ and $y_t < 0$. We remark in passing that the corresponding argument for the lower bound in particular shows that $z_t > 0$.

Now choosing $c_1 = (3KM/c_-)^{2/3}$, we find that for all $t \leqslant \tau$ satisfying $y_t \leqslant -c_1\varepsilon^{2/3}$, we also have $z_t \leqslant c_-\sqrt{-y_t}/3M$, so that actually $t < \tau$ whenever $y_t \leqslant -c_1\varepsilon^{2/3}$. Thus (2.2.27) holds for all t satisfying $y_t \leqslant -c_1\varepsilon^{2/3}$, which proves the upper bound in (2.2.19). The lower bound is obtained in a similar way.

- For $-c_1\varepsilon^{2/3} \leqslant y_t \leqslant c_2\varepsilon^{2/3}$, where c_2 will be chosen below, we use the fact that we rescaled x and y in such a way that

$$f(x,y) = -x^2 - y + \mathcal{O}(y) + \mathcal{O}(x^2) \tag{2.2.28}$$

holds. The subsequent scaling $x = \varepsilon^{1/3} z$, $y = \varepsilon^{2/3} s$ thus yields the equation

$$\frac{\mathrm{d}z}{\mathrm{d}s} = -z^2 - s + \mathcal{O}_\varepsilon(1) \;, \tag{2.2.29}$$

which is a small perturbation of a solvable Riccati equation. In fact, setting $z(s) = \varphi'(s)/\varphi(s)$ in (2.2.29) without the error term yields the linear second order equation $\varphi''(s) = -s\varphi(s)$, whose solution can be expressed in terms of Airy functions.

In particular, there exist constants $c_3 > c_2 > 0$ such that $z(s)$ remains positive, of order 1, for $s \leqslant c_2$, and reaches negative values of order 1 for $s = c_3$. Returning to the variables (x,y), we conclude that (2.2.20) holds, and that $x_t \asymp -\varepsilon^{1/3}$ as soon as $y_t = c_3\varepsilon^{2/3}$.

- For $y_t \geqslant c_3\varepsilon^{2/3}$, the trivial estimate $f(x,y) \leqslant -(1-\kappa)x^2$ can be employed to show that x_t reaches any negative value of order 1 after yet another time span of order $\varepsilon^{2/3}$, which concludes the proof. □

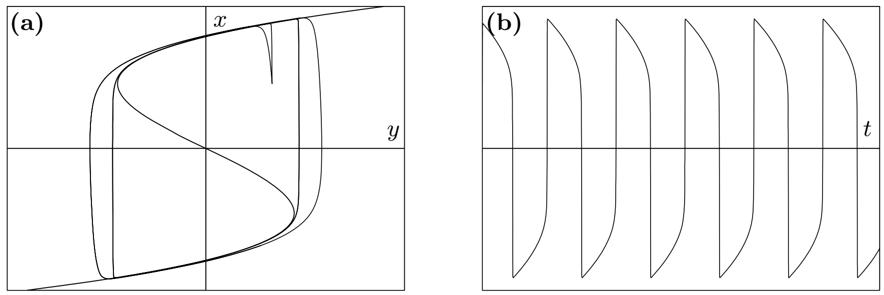

Fig. 2.5. **(a)** Two solutions of the van der Pol equations (2.1.13) (*light curves*) for the same initial condition $(1, \frac{1}{2})$, for $\gamma = 5$ and $\gamma = 20$. The heavy curve is the slow manifold $y = \frac{1}{3}x^3 - x$. **(b)** The graph of x_t (again for $\gamma = 20$) displays relaxation oscillations.

Example 2.2.3 (Van der Pol oscillator – continued). As already mentioned, the system

$$\begin{aligned} \varepsilon \dot{x} &= y + x - \frac{x^3}{3} \,, \\ \dot{y} &= -x \,, \end{aligned} \tag{2.2.30}$$

admits two saddle–node bifurcation points $\pm(1, -\frac{2}{3})$. Consider an orbit starting near the stable slow manifold $x = x^\star_+(y)$. This orbit will approach an ε-neighbourhood of the slow manifold in a time of order $\varepsilon|\log \varepsilon|$. Since $\dot{y} < 0$ for positive x, the bifurcation point $(1, -\frac{2}{3})$ is reached in a time of order 1.

Theorem 2.2.2, applied to the variables $(x - 1, -(y + \frac{2}{3}))$, shows that x_t "jumps" as soon as y_t has reached a value $-\frac{2}{3} - \mathcal{O}(\varepsilon^{2/3})$. Once the orbit has left a small neighbourhood of the bifurcation point, f is negative and bounded away from zero. Solutions of the associated system $x' = y + x - \frac{x^3}{3}$ enter a neighbourhood of order 1 of the slow manifold $x = x^\star_-(y)$ in a fast time of order 1, while y changes only by $\mathcal{O}(\varepsilon)$ during the same (fast) time, so that by an argument of regular perturbation theory, x_t also approaches $x^\star_-(y_t)$ in a time of order ε.

We can then apply the same analysis as before, showing that the orbit tracks the branch $x = x^\star_-(y)$ until reaching the bifurcation point $(-1, \frac{2}{3})$ and jumping back to the first branch. In fact, using a Poincaré section and the exponential contraction near adiabatic manifolds, one easily shows that orbits approach a periodic orbit. (This is known to hold for any value of the damping coefficient.) The resulting trajectory displays alternating slow and fast motions, called *relaxation oscillations* (Fig. 2.5).

Similar phenomena have been analysed for more general systems, including multidimensional slow variables. The difference is that, the Poincaré map no longer being one-dimensional, orbits do not necessarily approach a periodic one. A detailed analysis of such situations is given in [MR80] and [MKKR94].

2.2.3 Symmetric Pitchfork Bifurcation and Bifurcation Delay

We consider the reduced slow–fast system

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y) , \\ \dot{y} &= g(x,y) , \end{aligned} \tag{2.2.31}$$

for $x, y \in \mathbb{R}$, again under the assumptions

$$f(0,0) = \partial_x f(0,0) = 0 \qquad \text{and} \qquad g(0,0) > 0 . \tag{2.2.32}$$

In this section we assume that f and g are of class $\mathcal{C}^3$. If, unlike in the case of the saddle–node bifurcation, $\partial_{xx} f(0,0)$ vanishes, several new kinds of bifurcations can occur. These bifurcations are not generic, unless f and g are restricted to belong to some smaller function space, for instance functions satisfying certain symmetry conditions. We consider here the case where f is odd in x and g is even in x:

$$f(-x,y) = -f(x,y) \qquad \text{and} \qquad g(-x,y) = g(x,y) \tag{2.2.33}$$

for all (x,y) in a neighbourhood $\mathcal{N}$ of $(0,0)$. We say that a *supercritical symmetric pitchfork bifurcation* occurs at $(0,0)$ if

$$\partial_{xy} f(0,0) > 0 \qquad \text{and} \qquad \partial_{xxx} f(0,0) < 0 . \tag{2.2.34}$$

The simplest example of such a system is

$$\begin{aligned} \varepsilon\dot{x} &= yx - x^3 , \\ \dot{y} &= 1 . \end{aligned} \tag{2.2.35}$$

The line $x = 0$ is a slow manifold, which changes from stable to unstable as y changes from negative to positive. For positive y, there are two new branches of the slow manifold, of equation $x = \pm\sqrt{y}$, which are stable (Fig. 2.6).

Similar properties hold for general systems satisfying (2.2.32), (2.2.33) and (2.2.34). By rescaling x and y, one can always arrange for $\partial_{xy} f(0,0) = 1$ and $\partial_{xxx} f(0,0) = -6$. It is then straightforward to show that in a sufficiently small neighbourhood $\mathcal{N}$ of $(0,0)$,

- all points on the line $x = 0$ belong to a slow manifold, with stability "matrix"
$$\partial_x f(0,y) =: a(y) = y\,[1 + \mathcal{O}(y)] , \tag{2.2.36}$$
and thus the manifold is stable for $y < 0$ and unstable for $y > 0$, provided $\mathcal{N}$ is small enough;
- there are two other stable slow manifolds of the form $\{x = \pm x^\star(y), y > 0\}$, where $x^\star(y) = \sqrt{y}[1 + \mathcal{O}_y(1)]$; the linearisation of f at these manifolds satisfies
$$\partial_x f(\pm x^\star(y), y) =: a^\star(y) = -2y[1 + \mathcal{O}_y(1)] ; \tag{2.2.37}$$

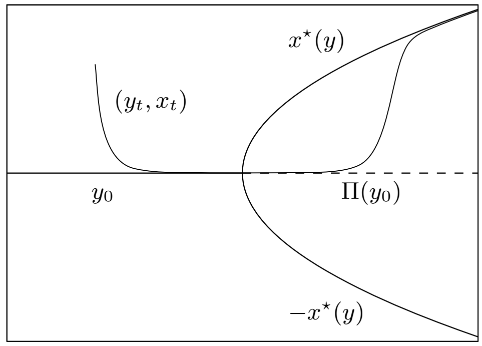

Fig. 2.6. Dynamic pitchfork bifurcation. Solutions starting in (x_0, y_0) with $x_0 > 0$ and $y_0 < 0$ track the unstable manifold $\{x = 0,\ y > 0\}$ until $y_t = \Pi(y_0)$, before jumping to the stable branch $x^\star(y)$.

- there are no other slow manifolds in $\mathcal{N}$;
- $g(x, y) > 0$ in $\mathcal{N}$.

Consider now an orbit starting in $(x_0, y_0) \in \mathcal{N}$, with $x_0 > 0$ and $y_0 < 0$. It reaches an ε-neighbourhood of the slow manifold in $x = 0$ in a time of order $\varepsilon|\log \varepsilon|$. The new feature in this situation is that $x_t \equiv 0$ is itself a particular solution of (2.2.31), and thus the slow manifold is actually an adiabatic manifold. For this reason, x_t approaches 0 exponentially closely, and when the slow manifold becomes unstable at $y = 0$, a time of order 1 is needed for x_t to move away from 0 and approach $x^\star(y_t)$ (Fig. 2.6). This phenomenon is known as *bifurcation delay*.

In order to quantify this delay, one can define, for negative y_0 such that $(0, y_0) \in \mathcal{N}$, a *bifurcation delay time*

$$\Pi(y_0) = \inf\left\{ y_1 > 0\colon \int_{y_0}^{y_1} \frac{a(y)}{g(0,y)} \, \mathrm{d}y > 0 \right\} . \qquad (2.2.38)$$

In the particular case of the system (2.2.35), $\Pi(y_0) = -y_0$. In the general case, one easily sees that $\Pi(y_0) = -y_0 + \mathcal{O}(y_0^2)$. The following theorem states that trajectories starting in $(x_0, y_0) \in \mathcal{N}$ track the line $x = 0$ approximately until $y_t = \Pi(y_0)$.

Theorem 2.2.4 (Dynamic pitchfork bifurcation). *Let ε and $\mathcal{N}$ be sufficiently small, and choose an initial condition $(x_0, y_0) \in \mathcal{N}$ such that $x_0 > 0$ and $y_0 < 0$. Then there exist $c_1 > 0$ and*

$$\begin{aligned} y_1 &= y_0 + \mathcal{O}(\varepsilon|\log\varepsilon|) , \\ y_2 &= \Pi(y_0) - \mathcal{O}(\varepsilon|\log\varepsilon|) , \\ y_3 &= y_2 + \mathcal{O}(\varepsilon|\log\varepsilon|) , \end{aligned} \qquad (2.2.39)$$

such that

$$0 \leqslant x_t \leqslant \varepsilon \qquad \text{for } y_1 \leqslant y_t \leqslant y_2\;, \tag{2.2.40}$$

$$|x_t - x^\star(y_t)| \leqslant c_1\varepsilon \qquad \text{for } y_t \geqslant y_3\;, \tag{2.2.41}$$

for all times t for which $(x_t, y_t) \in \mathcal{N}$.

Proof.

- Since $f(x, y_0) < 0$ for positive x, solutions of the associated system $x' = f(x, y_0)$ reach any neighbourhood of order 1 of $x = 0$ in a fast time of order ε. Thus, by regular perturbation theory, the orbit of the slow–fast system starting in (x_0, y_0) enters any neighbourhood of order 1 of the slow manifold $x = 0$ in a slow time of order ε. From there on, Tihonov's theorem shows that an ε-neighbourhood is reached in a time of order $\varepsilon|\log\varepsilon|$, and thus $x_{t_1} = \varepsilon$ at a time t_1 at which $y_{t_1} = y_1$.
- First note that we can write

$$f(x, y) = x\bigl[a(y) + b(x, y)x^2\bigr] \tag{2.2.42}$$

for some bounded function b. Let $\tau = \inf\{t > t_1 \colon x_t > \varepsilon\}$. For $t_1 \leqslant t \leqslant \tau$, we have

$$\varepsilon\frac{\mathrm{d}x}{\mathrm{d}y} = \biggl[\frac{a(y)}{g(0, y)} + \mathcal{O}(\varepsilon^2)\biggr]x\;, \tag{2.2.43}$$

and thus there is a constant $M > 0$ such that

$$\varepsilon\mathrm{e}^{[\alpha(y_t, y_1) - M\varepsilon^2]/\varepsilon} \leqslant x_t \leqslant \varepsilon\mathrm{e}^{[\alpha(y_t, y_1) + M\varepsilon^2]/\varepsilon}\;, \tag{2.2.44}$$

where

$$\alpha(y, y_1) = \int_{y_1}^{y} \frac{a(u)}{g(0, u)}\,\mathrm{d}u\;. \tag{2.2.45}$$

This shows that x_t remains smaller than ε until a time t_2 at which $\alpha(y_{t_2}, y_1) = \mathcal{O}(\varepsilon^2)$, so that $y_{t_2} = y_2 = \Pi(y_0) + \mathcal{O}(\varepsilon|\log\varepsilon|)$.
- By continuity of f, for any sufficiently small constant c_0, $f(x, y) > \frac{1}{2}xa(y_2)$ for $|x| \leqslant c_0$ and $y_2 \leqslant y \leqslant y_2 + c_0$. Thus x_t is larger than $\varepsilon\mathrm{e}^{a(y_2)(t-t_2)/\varepsilon}$, and reaches c_0 after a time of order $\varepsilon|\log\varepsilon|$. Using the associated system and the fact that f is positive, bounded away from 0 for x in any compact interval contained in $(0, x^\star(y))$, one concludes that x_t reaches any neighbourhood of order 1 of the equilibrium branch $x^\star(y)$ after a time of order ε. From there on, Tihonov's theorem allows to conclude the proof. □

Remark 2.2.5. The proof shows that a similar bifurcation delay exists for any system for which $f(0, y)$ is identically zero. Only the behaviour after leaving the unstable equilibrium branch may differ, as it depends on the global behaviour of f.

2.2.4 How to Obtain Scaling Laws

The analysis of the dynamic saddle–node bifurcation in Section 2.2.2 revealed a nontrivial ε-dependence of solutions near the bifurcation point, involving fractional exponents as in $\varepsilon^{1/3}$ and $\varepsilon^{2/3}$. These exponents were related to the behaviour in $\sqrt{-y}$ of slow manifolds near the bifurcation point.

It is in fact possible to give a rather complete classification of the scaling laws that may occur for one-dimensional bifurcations. Such a classification involves two steps: Firstly, analyse the power-law behaviour of slow manifolds near a bifurcation point; secondly, study the equation governing the distance between solutions of the slow–fast system and the slow manifold.

The first step is best carried out with the help of the bifurcation point's *Newton polygon*. Consider again the scalar equation

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}y} = f(x,y) \tag{2.2.46}$$

where f is of class $\mathcal{C}^r$ for some $r \geqslant 2$, with all derivatives uniformly bounded in a neighbourhood $\mathcal{N}$ of the origin. We also require the bifurcation conditions

$$f(0,0) = \partial_x f(0,0) = 0 \tag{2.2.47}$$

to hold. The function f admits a Taylor expansion in $\mathcal{N}$ of the form

$$f(x,y) = \sum_{\substack{j,k\geqslant 0\\ j+k\leqslant r}} f_{jk} x^j y^k + R(x,y) , \qquad f_{jk} = \frac{1}{j!k!} \partial_{x^j y^k} f(0,0) , \tag{2.2.48}$$

with a remainder $R(x,y) = \mathcal{O}(\|(x,y)\|^r)$, and $f_{00} = f_{10} = 0$. Assume for simplicity[6] that $f_{r0} \neq 0$ and $f_{0r} \neq 0$. Then the Newton polygon $\mathcal{P}$ is obtained by taking the convex envelope of the set of points

$$\left\{(j,k) \in \mathbb{N}_0^2 \colon f_{jk} \neq 0 \text{ or } j+k \geqslant r\right\} . \tag{2.2.49}$$

Assume that f vanishes on a curve of equation $x = x^\star(y)$, defined for y in an interval of the form $I = (-\delta, 0]$, $[0,\delta)$ or $(-\delta,\delta)$, such that $|x^\star(y)| \asymp |y|^q$ for some $q > 0$. We will call $\{(x,y) \colon x = x^\star(y), y \in I\}$ an *equilibrium branch* with exponent q. It is well known that the Newton polygon then admits a segment of slope $-q$. For instance, in the case of the saddle–node bifurcation studied in Section 2.2.2, there are two equilibrium branches with exponent $1/2$, and $\mathcal{P}$ has two vertices $(2,0)$ and $(0,1)$, connected by a segment of slope $-1/2$.

The second step is to examine the behaviour of the distance $z = x - x^\star(y)$ between solutions of (2.2.46) and the equilibrium branch. The dynamics of z_t is governed by an equation of the form

[6] This assumption is made to avoid situations such as $f(x,y) = x^{5/2} - y$, in which $r = 2$ but $f_{20}(x,y) \equiv 0$, where the relation between Newton's polygon and slow manifold does not work.

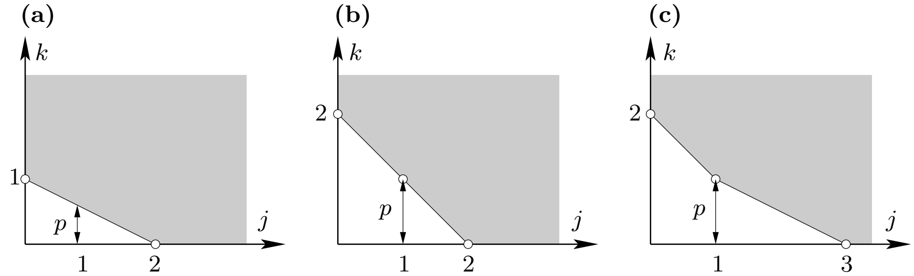

Fig. 2.7. Newton's polygon for **(a)** the saddle–node, **(b)** the transcritical and **(c)** the asymmetric pitchfork bifurcation. The saddle–node bifurcation has branches of exponent $q = 1/2$, for which $p = 1/2$. The transcritical bifurcation has branches of exponent $q = 1$, for which $p = 1$. The asymmetric pitchfork bifurcation has two types of branches, with exponents $q = 1$ and $q = 1/2$, both with $p = 1$.

$$\varepsilon \frac{\mathrm{d}z}{\mathrm{d}y} = a(y)z + b(y,z) - \varepsilon \frac{\mathrm{d}x^\star(y)}{\mathrm{d}y} , \qquad (2.2.50)$$

where $a(y) = \partial_x f(x^\star(y), y)$ is the linearisation of f at the equilibrium branch, and b is of order z^2. The behaviour of $a(y)$ near $y = 0$ can also be deduced from the Newton polygon $\mathcal{P}$. Indeed, Taylor's formula implies that

$$\partial_x f(x^\star(y), y) = \sum_{\substack{j\geqslant 1, k\geqslant 0\\ j+k\leqslant r}} j f_{jk} x^\star(y)^{j-1} y^k + \mathcal{O}(|y|^r + |y|^{rq}) . \qquad (2.2.51)$$

Generically (that is, unless some unexpected cancellation occurs) we will thus have $|a(y)| \asymp |y|^p$, where

$$p = \inf\bigl\{ q(j-1) + k \colon j \geqslant 1,\, k \geqslant 0,\, j+k \leqslant r \text{ and } f_{jk} \neq 0 \bigr\} . \qquad (2.2.52)$$

Geometrically speaking, p is the ordinate at 1 of the tangent to Newton's polygon with slope $-q$ (Fig. 2.7).

Definition 2.2.6 (Tame equilibrium branch). *Let I be an interval of the form $I = (-\delta, 0]$, $[0, \delta)$ or $(-\delta, \delta)$, and assume that f vanishes on an equilibrium branch $\{(x,y) \colon x = x^\star(y), y \in I\}$ with exponent $q > 0$. The equilibrium branch is called* tame *if $|\partial_x f(x^\star(y), y)| \asymp |y|^p$ with p given by* (2.2.52).

Example 2.2.7.

- If $f(x,y) = -x^2 - y$, then there are two tame equilibrium branches with exponent $q = 1/2$, and such that $p = 1/2$.
- If $f(x,y) = (x-y)^2$, then there is an equilibrium branch of equation $x = y$, whose exponent is $q = 1$. This branch in not tame since $a(y) = \partial_x f(y,y)$ is identically zero, and thus $p = \infty$.

The rational numbers q and p are usually sufficient to determine the scaling behaviour of solutions near a bifurcation point. An essential rôle is played by the following variant of the Laplace method, which allows to analyse solutions of (2.2.50) when the nonlinear term $b(y,z)$ is absent.

Lemma 2.2.8. *Fix constants $y_0 < 0$, and $\zeta_0, c, p, q > 0$. Then the function*

$$\zeta(y,\varepsilon) = \mathrm{e}^{c|y|^{p+1}/\varepsilon}\left[\varepsilon\zeta_0\mathrm{e}^{-c|y_0|^{p+1}/\varepsilon} + \int_{y_0}^{y}|u|^{q-1}\mathrm{e}^{-c|u|^{p+1}/\varepsilon}\,\mathrm{d}u\right] \tag{2.2.53}$$

satisfies

$$\zeta(y,\varepsilon) \asymp \begin{cases} \varepsilon|y|^{q-p-1} & \text{for } y_0 \leqslant y \leqslant -\varepsilon^{1/(p+1)}\;, \\ \varepsilon^{q/(p+1)} & \text{for } -\varepsilon^{1/(p+1)} \leqslant y \leqslant 0\;. \end{cases} \tag{2.2.54}$$

Using similar ideas as in the proof Theorem 2.2.2, we obtain the following general result on the behaviour of solutions tracking a stable, tame equilibrium branch, approaching a bifurcation point. If $x^\star(y)$ is not identically zero, we may assume that it is decreasing near the bifurcation point, otherwise we change x to $-x$.

Theorem 2.2.9 (Scaling behaviour near a bifurcation point). *Assume that $x^\star : (-\delta, 0] \to \mathbb{R}_+$ describes a tame equilibrium branch of f, with exponent $q > 0$, which is stable, that is, the linearisation $a(y) = \partial_x f(x^\star(y), y)$ satisfies $a(y) \asymp -|y|^p$, where p is given by (2.2.52). Fix an initial condition $(x_0, y_0) \in \mathcal{N}$, where $y_0 \in (-\delta, 0)$ does not depend on ε and $x_0 - x^\star(y_0) \asymp \varepsilon$. Then there is a constant $c_0 > 0$ such that the solution of (2.2.46) with initial condition (x_0, y_0) satisfies*

$$x_t - x^\star(y_t) \asymp \varepsilon|y_t|^{q-p-1} \qquad \text{for } y_0 \leqslant y_t \leqslant -c_0\varepsilon^{1/(p+1)}\;. \tag{2.2.55}$$

If, moreover, there is a constant $\delta_1 > 0$ such that $f(x,y) < 0$ for $-\delta \leqslant y \leqslant 0$ and $0 < x - x^\star(y) \leqslant \delta_1$, then there exists $c_1 > 0$ such that

$$x_t \asymp \varepsilon^{q/(p+1)} \qquad \text{for } -c_0\varepsilon^{1/(p+1)} \leqslant y_t \leqslant c_1\varepsilon^{1/(p+1)}\;. \tag{2.2.56}$$

Proof.

- For $y_0 \leqslant y_t \leqslant -c_0\varepsilon^{1/(p+1)}$, where c_0 will be determined below, we have

$$\varepsilon\frac{\mathrm{d}z}{\mathrm{d}y} \leqslant -c_-|y|^p z + b(y,z) + \varepsilon c_+|y|^{q-1} \tag{2.2.57}$$

  for some constants $c_\pm > 0$. We know that $|b(y,z)| \leqslant Mz^2$ for some $M > 0$, but we will need a better bound in the case $p > q$. In fact, Taylor's formula implies that there is a $\theta \in [0,1]$ such that

$$\begin{aligned} z^{-2}b(y,z) &= \frac{1}{2}\partial_{xx}f(x^\star(y) + \theta z, y)\;, \\ &= \frac{1}{2}\sum_{\substack{j\geqslant 2, k\geqslant 0\\ j+k\leqslant r}} j(j-1)f_{jk}[x^\star(y) + \theta z]^{j-2}y^k + R(y)\;, \end{aligned} \tag{2.2.58}$$

where $R(y) = \mathcal{O}(x^\star(y)^{r-2} + z^{r-2} + y^r)$. Using the definition (2.2.52) of p, we obtain that whenever $|z| \leqslant L|y|^q$ for some $L > 0$, there is a constant $M(L)$ such that

$$|b(y,z)| \leqslant M(L)|y|^{p-q}z^2 \; . \tag{2.2.59}$$

Since $M(L)$ is a nondecreasing function of L, we can choose L such that $2LM(L) \leqslant c_-$. In order to prove the upper bound in (2.2.55), we introduce

$$\tau = \inf\bigl\{t \geqslant 0 \colon |z_t| > L|y_t|^q\bigr\} \; . \tag{2.2.60}$$

For $t \leqslant \tau$, it easily follows from (2.2.57) and the definition of $M(L)$ that

$$\varepsilon \frac{\mathrm{d}z}{\mathrm{d}y} \leqslant -\frac{c_-}{2}|y|^p z + \varepsilon c_+ |y|^{q-1} \; , \tag{2.2.61}$$

so that, solving a linear equation and using Lemma 2.2.8,

$$z_t \leqslant K\varepsilon |y_t|^{q-p-1} \tag{2.2.62}$$

for some $K > 0$, as long as $y_t \leqslant -\varepsilon^{1/(p+1)}$. The conclusion is then obtained as in the proof of Theorem 2.2.2, taking $c_0^{p+1} = 1 \wedge (4MK/c_-)$. The lower bound is obtained in a similar way.

- For $-c_0\varepsilon^{1/(p+1)} \leqslant y_t \leqslant c_1\varepsilon^{1/(p+1)}$, where c_1 is determined below, we use the scaling $x = \varepsilon^{q/(p+1)}\tilde{x}$, $y = \varepsilon^{1/(p+1)}\tilde{y}$, yielding the equation

$$\begin{aligned}
\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{y}} &= \varepsilon^{-(p+q)/(p+1)} f\bigl(\varepsilon^{q/(p+1)}\tilde{x}, \varepsilon^{1/(p+1)}\tilde{y}\bigr) \; , \\
&= \sum_{\substack{j,k\geqslant 0 \\ j+k\leqslant r}} f_{jk}\varepsilon^{[(j-1)q+k-p]/(p+1)}\tilde{x}^j\tilde{y}^k + \tilde{r}(x,y) \; ,
\end{aligned} \tag{2.2.63}$$

for some remainder $\tilde{r}(x,y)$. By (2.2.52), we have $(j-1)q + k \geqslant p$ whenever $f_{jk} \neq 0$ or $j + k \geqslant r - 1$, and thus the right-hand side of (2.2.63) is of order 1 at most. In fact, we have

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{y}} = \sum f_{jk}\tilde{x}^j\tilde{y}^k + \mathcal{O}_\varepsilon(1), \tag{2.2.64}$$

where the sum is taken over all vertices of the Newton polygon belonging to a segment of slope $-q$.

If $f(x,y) < 0$ above $x^\star(y)$, x_t must decrease as long as $y_t < 0$, and it cannot cross $x^\star(y_t)$. Hence $\tilde{x} \asymp 1$, whenever $\tilde{y} = 0$. Since (2.2.64) does not depend on ε to leading order, $\tilde{x}$ remains positive and of order 1 for $\tilde{y}$ in a sufficiently small interval $[0, c_1]$. Going back to original variables, (2.2.56) is proved. □

The behaviour after y_t has reached the bifurcation value 0 depends on the number of equilibrium branches originating from the bifurcation point. In

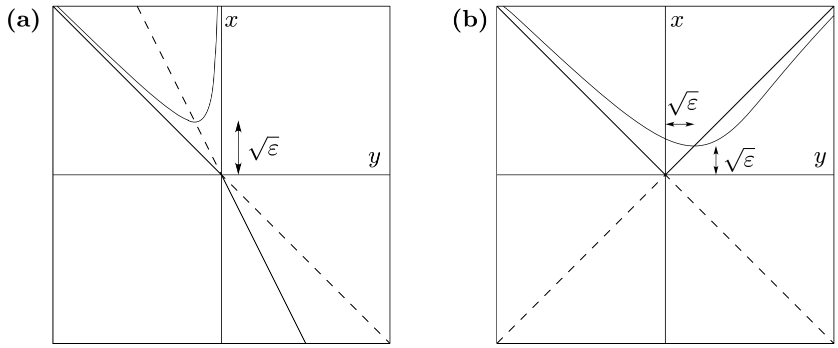

Fig. 2.8. Dynamic transcritical bifurcation. **(a)** If the unstable branch lies above the decreasing stable branch, solutions cross the unstable branch for $y \asymp -\sqrt{\varepsilon}$ and explode soon thereafter. **(b)** If the unstable branch lies below the decreasing stable branch, solutions cross the new stable branch for $y \asymp \sqrt{\varepsilon}$, before approaching it again like ε/y.

the case of the indirect saddle–node bifurcation, no equilibrium branch exists near the origin for positive y, and thus the orbits jump to some other region of phase space (or explode). If one or several stable equilibrium branches exist for $y > 0$, the behaviour of solutions can be analysed by methods similar to those of Theorem 2.2.9. Rather then attempting to give a general classification of possible behaviours, we prefer to discuss two of the most generic cases.

Example 2.2.10 (Transcritical bifurcation). Assume that $f \in \mathcal{C}^2$ satisfies, in addition to the general bifurcation condition (2.2.47), the relation $f_{01} = 0$, but with $f_{20}, f_{11}, f_{02} \neq 0$. The Newton polygon has vertices $(2,0)$, $(1,1)$ and $(0,2)$, connected by two segments of slope 1 (Fig. 2.7b). We thus take $q = 1$ and look for equilibrium branches of the form

$$x = Cy(1 + \rho(y)) , \qquad \rho(0) = 0 . \tag{2.2.65}$$

Inserting this in the equation $f(x,y) = 0$ and taking $y \to 0$, we obtain the condition

$$f_{20}C^2 + f_{11}C + f_{02} = 0 \tag{2.2.66}$$

for C. Thus there are three cases to be considered, depending on the value of the discriminant $\Delta = f_{11}^2 - 4f_{20}f_{02}$.

1. If $\Delta > 0$, then (2.2.66) admits two solutions $C_+ > C_-$. For each of these solutions, the implicit-function theorem, applied to the pair (y, ρ), shows that there is indeed a unique equilibrium branch of the form (2.2.65). A simple computation of $\partial_x f(x,y)$ on each of these branches shows that they have opposite stability, and exchange stability as y passes through 0.
2. If $\Delta = 0$, then (2.2.66) admits one solution, but the implicit-function theorem cannot be applied. The behaviour will depend on higher order terms. The second part of Example 2.2.7 belongs into this category.

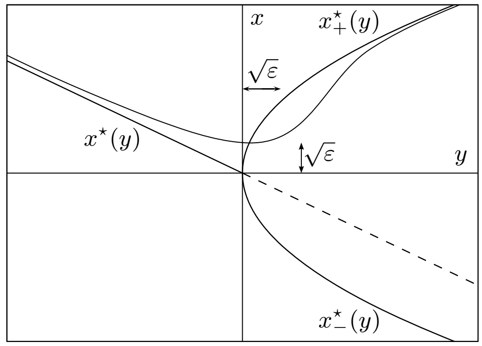

Fig. 2.9. Asymmetric dynamic pitchfork bifurcation. Solutions tracking the decreasing stable branch switch to the new branch $x^\star_+(y)$ after a time of order $\sqrt{\varepsilon}$.

3. If $\Delta < 0$, then $(0,0)$ is an isolated equilibrium point of f.

Let us consider the case $\Delta > 0$ in more detail. Assume that the stable equilibrium branch approaching the origin from the left is decreasing. The situation depends on the location of the unstable equilibrium branch, whether it lies above or below the stable one (Fig. 2.8):

- If the unstable branch lies above the stable one, which happens for $f_{20} > 0$, then Theorem 2.2.9, applied for $q = p = 1$, shows that x_t tracks the equilibrium branch at a distance of order $\varepsilon/|y_t|$ until $y = -c_0\sqrt{\varepsilon}$. Thereafter, the trajectory escapes from a neighbourhood of the bifurcation point.
- If the unstable branch lies below the stable one, which happens for $f_{20} < 0$, then the theorem can still be applied and shows, in addition, that x_t stays of order $\sqrt{\varepsilon}$ until $y = c_1\sqrt{\varepsilon}$. For later times, an analysis of the equation governing the distance between solutions and the new stable equilibrium branch shows that x_t approaches that branch like ε/y_t.

Example 2.2.11 (Asymmetric pitchfork bifurcation). Assume that $f \in \mathcal{C}^3$ satisfies $f_{01} = f_{20} = 0$, but $f_{30}, f_{11}, f_{02} \neq 0$. Then the Newton polygon has three vertices $(3,0)$, $(1,1)$ and $(0,2)$, connected by two segments of slope $1/2$ and 1 (Fig. 2.7c). A similar analysis as in the previous example shows the existence of an equilibrium branch $x^\star(y) = -(f_{02}/f_{11})y[1 + \mathcal{O}_y(1)]$, existing for both positive and negative y, and of two equilibrium branches $x^\star_\pm(y) \asymp \sqrt{|y|}$, existing either for $y < 0$ or for $y > 0$, depending on the sign of f_{11}/f_{30}.

Consider the case where the branch $x^\star(y)$ is decreasing, stable for $y < 0$ and unstable for $y > 0$, and the branches $x^\star_\pm(y)$ exist for $y > 0$. (Then the latter two branches are necessarily stable, see Fig. 2.9.) The simplest example of this kind is $f(x,y) = (x+y)(y-x^2)$.

Theorem 2.2.9, applied for $q = p = 1$, shows that x_t tracks the equilibrium branch $x^\star(y)$ at a distance of order $\varepsilon/|y_t|$ until $y = -c_0\sqrt{\varepsilon}$, and remains of order $\sqrt{\varepsilon}$ until $y = c_1\sqrt{\varepsilon}$. At this time, x_t is at a distance of order $\varepsilon^{1/4}$

from the equilibrium branch $x^\star_+(y)$. Since $f > 0$ for $x^\star(y) < x < x^\star_+(y)$, x_t increases. By the same method as in the previous example, one shows that x_t approaches $x^\star_+(y_t)$ like $\varepsilon/y_t^{3/2}$. Note in particular that, unlike in the case of the symmetric pitchfork bifurcation, there is no macroscopic delay.

2.2.5 Hopf Bifurcation and Bifurcation Delay

A *Hopf bifurcation* occurs if the Jacobian matrix $\partial_x f(\widehat{x}, \widehat{y})$ at the bifurcation point $(\widehat{x}, \widehat{y})$ has a pair of conjugate eigenvalues $\pm\mathrm{i}\omega_0$ on the imaginary axis (with $\omega_0 \neq 0$). We assume here that all other eigenvalues of $\partial_x f(\widehat{x}, \widehat{y})$ have negative real parts. We can thus consider the reduced system

$$\begin{aligned} \varepsilon\dot{x} &= f(x, y) \;, \\ \dot{y} &= g(x, y) \;, \end{aligned} \tag{2.2.67}$$

for $x \in \mathbb{R}^2$, where $f(\widehat{x}, \widehat{y}) = 0$ and $\partial_x f(\widehat{x}, \widehat{y})$ has the eigenvalues $\pm\mathrm{i}\omega_0$.

The implicit-function theorem implies the existence, in a neighbourhood $\mathcal{D}_0 \subset \mathbb{R}^m$ of $\widehat{y}$, of a slow manifold $\{(x, y) \colon x = x^\star(y), y \in \mathcal{D}_0\}$, with $x^\star(\widehat{y}) = \widehat{x}$. Let us denote by $a(y) \pm \mathrm{i}\omega(y)$ the eigenvalues of $\partial_x f(x^\star(y), y)$, where $a(0) = 0$ and $\omega(0) = \omega_0$. The associated system $x' = f(x, \lambda)$ admits $x^\star(\lambda)$ as an equilibrium (of focus type), which is stable if $a(\lambda) < 0$ and unstable if $a(\lambda) > 0$. Depending on second- and third-order terms of the Taylor expansion of f, there is (generically)

- either a stable periodic orbit near $x^\star(\lambda)$ for $a(\lambda) > 0$ and no invariant set near $x^\star(\lambda)$ for $a(\lambda) < 0$ (*supercritical* Hopf bifurcation),
- or an unstable periodic orbit near $x^\star(\lambda)$ for $a(\lambda) < 0$ and no invariant set near $x^\star(\lambda)$ for $a(\lambda) > 0$ (*subcritical* Hopf bifurcation).

We refer to [GH90, p. 152] for the precise relation between criticality and coefficients of the Taylor expansion.

The reduced system on the slow manifold is given by

$$\dot{y} = g(x^\star(y), y) \;. \tag{2.2.68}$$

We are interested in the following situation. Let Y_t be the solution of (2.2.68) such that $Y_{t^\star} = \widehat{y}$ for some $t^\star \in \mathbb{R}$, and assume that $a(Y_t)$ changes sign from negative to positive as t crosses $t^\star$, with positive velocity, that is, $\langle \nabla a(\widehat{y}), g(\widehat{x}, \widehat{y}) \rangle > 0$. One can then construct an affine change of variables $x = x^\star(y) + S(y)(z, \overline{z})^{\mathrm{T}}$, such that the complex variable z satisfies an equation of the form

$$\varepsilon\dot{z} = \big[a(Y_t) + \mathrm{i}\omega(Y_t)\big] z + b(z, \overline{z}, t, \varepsilon) - \varepsilon W(Y_t) \frac{\mathrm{d}x^\star(Y_t)}{\mathrm{d}t} \;, \tag{2.2.69}$$

where $b(z, \overline{z}, t, \varepsilon) = \mathcal{O}(|z|^2 + \varepsilon|z|)$, and $W(Y_t)$ is the first line of $S(Y_t)^{-1}$.

If $x^\star(y) \equiv \widehat{x}$ identically in y, then $z_t \equiv 0$ is a particular solution of (2.2.69), and the situation is very similar to the one encountered when studying the symmetric pitchfork bifurcation in Section 2.2.3. Solutions starting near the slow manifold at a time t_0 for which $a(Y_{t_0}) < 0$ approach zero exponentially closely, and need a macroscopic *bifurcation delay* time after $t = t^\star$ before leaving a neighbourhood of order one of the origin. This delay time is given by an expression similar to (2.2.38).

The new feature of the Hopf bifurcation is that if f and g are *analytic*,[7] then a delay persists even when $x^\star(y)$ depends on y. This can be understood as a stabilising effect of fast oscillations around the slow manifold. The system (2.2.69) and its solution can be analytically continued to a complex neighbourhood of $t = t^\star$, so that we can define the analytic function

$$\Psi(t) = \int_{t^\star}^{t} \bigl[a(Y_s) + \mathrm{i}\omega(Y_s)\bigr] \,\mathrm{d}s \,. \tag{2.2.70}$$

For real $t_0 < t^\star$, we define

$$\Pi(t_0) = \inf\bigl\{t > t^\star \colon\ \operatorname{Re}\Psi(t_0) = \operatorname{Re}\Psi(t)\bigr\} \,. \tag{2.2.71}$$

Neishtadt has proved the following result [Neĭ87, Neĭ88].

Theorem 2.2.12 (Dynamic Hopf bifurcation). *There exist* buffer times *$t_- < t^\star$ and $t_+ = \Pi(t_-) > t^\star$ and constants $c_1, c_2 > 0$ with the following property. Any solution of (2.2.67) starting sufficiently close to $(x^\star(Y_{t_0}), Y_{t_0})$ satisfies*

$$\|(x_t, y_t) - (x^\star(Y_t), Y_t)\| \leqslant c_1\varepsilon \tag{2.2.72}$$

for $t_0 + c_2\varepsilon|\log\varepsilon| \leqslant t \leqslant t_1 - c_2\varepsilon|\log\varepsilon|$, where $t_1 = \Pi(t_0) \wedge t_+$. If, moreover, $\|x_{t_0} - x^\star(Y_{t_0})\|$ is of order 1, then x_t leaves a neighbourhood of order 1 of the slow manifold at a time $t_1 + \mathcal{O}(\varepsilon|\log\varepsilon|)$.

This result means that for $t_- \leqslant t_0 \leqslant t^\star - \mathcal{O}(1)$, solutions experience a bifurcation delay until time $\Pi(t_0)$, which is similar to the delay occurring for the symmetric pitchfork bifurcation. For $t_0 \leqslant t_-$, however, this delay saturates at t_+. The computation of the buffer times $t_\pm$ is discussed in [DD91] and in [Nei95]. Roughly speaking, (t_-, t_+) are the points furthest apart on the real axis that can be connected by a path of constant $\operatorname{Re}\Psi$, satisfying certain regularity assumptions.

Example 2.2.13. Assume that $a(Y_t) = t - t^\star$ and $\omega(Y_t) = \omega_0 > 0$. Then

$$\operatorname{Re}\Psi(t) = \frac{1}{2}\bigl[(\operatorname{Re}t - t^\star)^2 - (\operatorname{Im}t + \omega_0)^2 + \omega_0^2\bigr] \tag{2.2.73}$$

is constant on hyperbolas centred in $(t^\star, -\mathrm{i}\omega_0)$. We have $\Pi(t) = 2t^\star - t$, and the buffer times are given by $t_\pm = t^\star \pm \omega_0$.

[7] Strictly speaking, the reduced system (2.2.67) is usually not analytic, even when the original system is, because of lack of regularity of centre manifolds. This is why the analysis in [Neĭ87, Neĭ88] does not use centre manifolds, but keeps track of all original variables. The results are, however, the same.

Note finally that analyticity is crucial for the existence of a delay if $x^\star(y)$ depends on y. Neishtadt has shown that the delay is usually destroyed even by $\mathcal{C}^\infty$-perturbations.

2.3 Periodic Orbits and Averaging

The case where the slow–fast system admits an asymptotically stable slow manifold corresponds to the simplest possible asymptotic dynamics of the associated (or fast) system: All trajectories are attracted by an equilibrium point. The reduced (slow) dynamics is then simply obtained by projecting the equations of motion on the slow manifold.

If the asymptotic dynamics of the fast system is more complicated than stationary, how should one proceed to determine the effective dynamics of the slow variables? For quite a general class of systems, one can show that the effective dynamics can be obtained by averaging the vector field $g(x, y)$ with respect to the invariant measure describing the asymptotic fast dynamics. We focus here on the case where the asymptotic fast motion is periodic.

2.3.1 Convergence towards a Stable Periodic Orbit

Assume that the associated system

$$\frac{\mathrm{d}x}{\mathrm{d}s} \equiv x' = f(x, y_0) \tag{2.3.1}$$

has, for each fixed value of y_0 in an open set $\mathcal{D}_0$, a periodic solution $\gamma^\star(s, y_0)$, with period $T(y_0)$. We further assume that this orbit is *asymptotically stable*. Recall that the stability of a periodic orbit is related to the linear system

$$\xi' = \partial_x f(\gamma^\star(s, y_0), y_0)\xi \; . \tag{2.3.2}$$

Let $U(s, s_0)$ denote the principal solution of this system, that is, such that $\xi_s = U(s, s_0)\xi_{s_0}$. The eigenvalues of the monodromy matrix $U(T(y_0), 0)$ are called *characteristic multipliers*, and their logarithms are called *characteristic exponents* or *Lyapunov exponents*. One of the multipliers is equal to 1, and the corresponding eigenvector is the tangent vector to the orbit at $s = 0$. We will require that the $n - 1$ other characteristic multipliers have a modulus strictly less than unity, which implies the orbit's asymptotic stability.

We further assume that $\gamma^\star(s, y_0)$ is continuous in both variables (note that the origin of time can be chosen arbitrarily on the orbit), and that there are positive constants T_1, T_2 such that $T_1 \leqslant T(y) \leqslant T_2$ uniformly for $y \in \mathcal{D}_0$.

We turn now to the full slow–fast system

$$\begin{aligned} \varepsilon\dot{x} &= f(x, y) \; , \\ \dot{y} &= g(x, y) \; . \end{aligned} \tag{2.3.3}$$

It seems natural that this system should admit solutions (x_t, y_t) for which x_t is close to the rapidly oscillating function $\gamma^\star(t/\varepsilon, y_t)$. The dynamics of y_t would then be governed by an equation of the form

$$\dot{y}_t \simeq g(\gamma^\star(t/\varepsilon, y_t), y_t) \;, \tag{2.3.4}$$

which has a rapidly oscillating right-hand side. The general philosophy of averaging is to bring into consideration the *averaged system*

$$\dot{\bar{y}} = \bar{g}(\bar{y}) := \frac{1}{T(\bar{y})} \int_0^{T(\bar{y})} g(\gamma^\star(s, \bar{y}), \bar{y}) \, \mathrm{d}s \;. \tag{2.3.5}$$

Introducing $\Gamma^\star(\theta, y) = \gamma^\star(T(y)\theta, y)$ allows to rewrite the averaged slow vector field as

$$\bar{g}(\bar{y}) = \int_0^1 g(\Gamma^\star(\theta, \bar{y}), \bar{y}) \, \mathrm{d}\theta \;. \tag{2.3.6}$$

We assume that the solution $\bar{y}_t$ of (2.3.5) with initial condition $\bar{y}_0 = y_0$ stays in $\mathcal{D}_0$ for $0 \leqslant t \leqslant t_1$. Under these assumptions, Pontryagin and Rodygin have proved the following result [PR60].

Theorem 2.3.1 (Dynamics near a slowly varying periodic orbit). *Let x_0 be sufficiently close to $\Gamma^\star(\theta_0, y_0)$ for some θ_0. Then there exists a function Θ_t, satisfying the relation*

$$\varepsilon \dot{\Theta}_t = \frac{1}{T(\bar{y}_t)} + \mathcal{O}(\varepsilon) \;, \tag{2.3.7}$$

such that the estimates

$$\begin{aligned} x_t &= \Gamma^\star(\Theta_t, \bar{y}_t) + \mathcal{O}(\varepsilon) \;, \\ y_t &= \bar{y}_t + \mathcal{O}(\varepsilon) \end{aligned} \tag{2.3.8}$$

hold for $\mathcal{O}(\varepsilon|\log\varepsilon|) \leqslant t \leqslant t_1$. The error terms $\mathcal{O}(\varepsilon)$ in (2.3.7) *and* (2.3.8) *are uniform in t on this time interval.*

We shall sketch the proof in the case of a two-dimensional fast variable $x \in \mathbb{R}^2$. Let $\mathrm{n}(\theta, y)$ be the outward unit normal vector to the periodic orbit at the point $\Gamma^\star(\theta, y)$. In a neighbourhood of the periodic orbit, the dynamics can be described by coordinates (θ, r) such that

$$x = \Gamma^\star(\theta, y) + r \, \mathrm{n}(\theta, y) \;. \tag{2.3.9}$$

On the one hand, we have

$$\varepsilon \dot{x} = f(\Gamma^\star(\theta, y), y) + A(\theta, y) \, \mathrm{n}(\theta, y) r + \mathcal{O}(r^2) \;, \tag{2.3.10}$$

where $A(\theta, y) = \partial_x f(\Gamma^\star(\theta, y), y)$. On the other hand, we can express $\varepsilon \dot{x}$ as a function of $\dot{\theta}$ and $\dot{r}$ by differentiating (2.3.9) and using the equation for $\dot{y}$.

Projecting on $\mathrm{n}(\theta, y)$ and on the unit tangent vector to the orbit yields a system of the following form, equivalent to (2.3.3):

$$
\begin{aligned}
\varepsilon\dot{\theta} &= \frac{1}{T(y)} + b_\theta(\theta, r, y, \varepsilon) \;, \\
\varepsilon\dot{r} &= f_r(\theta, r, y, \varepsilon) \;, \\
\dot{y} &= g(\Gamma(\theta, y) + r\,\mathrm{n}(\theta, y), y) \;.
\end{aligned}
\tag{2.3.11}
$$

The functions b_θ and f_r can be computed explicitly in terms of A, $\Gamma^\star$, n, and their derivatives with respect to y. They both vanish for $r = \varepsilon = 0$, and in particular the linearisation $\partial_r f_r(\theta, 0, y, 0)$ depends only on $A(\theta, y)$. In a neighbourhood of $r = 0$, $\dot{\theta}$ is thus positive and we can consider the equations

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= T(y)\frac{f_r(\theta, r, y, \varepsilon)}{1 + T(y)b_\theta(\theta, r, y, \varepsilon)} \;, \\
\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \varepsilon T(y)\frac{g(\Gamma(\theta, y) + r\,\mathrm{n}(\theta, y), y)}{1 + T(y)b_\theta(\theta, r, y, \varepsilon)} \;,
\end{aligned}
\tag{2.3.12}
$$

instead of (2.3.11). Averaging the right-hand side over θ yields a system of the form

$$
\begin{aligned}
\frac{\mathrm{d}\bar{r}}{\mathrm{d}\theta} &= T(\bar{y})\bigl[\bar{a}(\bar{y})\bar{r} + \mathcal{O}(\varepsilon)\bigr] \;, \\
\frac{\mathrm{d}\bar{y}}{\mathrm{d}\theta} &= \varepsilon\bigl[\bar{g}(\bar{y}) + \mathcal{O}(\bar{r}) + \mathcal{O}(\varepsilon)\bigr] \;,
\end{aligned}
\tag{2.3.13}
$$

where $\bar{a}(\bar{y}) < 0$. (In fact, $\bar{a}(y)$ is the Lyapunov exponent of the periodic orbit $\gamma^\star(\cdot, y)$.) This is again a slow–fast system, in which θ plays the rôle of fast time. It follows thus from Tihonov's theorem that $\bar{r}$ approaches a slow manifold $\bar{r}^\star(y) = \mathcal{O}(\varepsilon)$. For such $\bar{r}$, we are in a situation to which the standard averaging theorem can be applied to show that $r_t - \bar{r}_t$ and $y_t - \bar{y}_t$ both remain of order ε up to "times" θ_t of order $1/\varepsilon$.

2.3.2 Invariant Manifolds

Theorem 2.3.1 is the equivalent of Tihonov's theorem for slow manifolds. In order to give a more precise description of the dynamics in a neighbourhood of the family of slowly varying periodic orbits, it is useful to have an analogue of Fenichel's theorem as well, on the existence of an invariant manifold tracking the family of periodic orbits.

To construct this invariant manifold, one can proceed as follows. Let Π be the "time"($\theta = 1$)-Poincaré map associated with the equation (2.3.12). That is, if the trajectory with initial condition (r, y) at $\theta = 0$ passes through the point $(\widehat{r}, \widehat{y})$ at $\theta = 1$, then by definition $(\widehat{r}, \widehat{y}) = \Pi(r, y)$. If we add ε as a dummy variable, the Poincaré map can be written in the form

$$\begin{aligned}\widehat{r} &= R(r, y, \varepsilon) \;, \\ \widehat{y} &= y + \varepsilon Y(r, y, \varepsilon) \;, \\ \widehat{\varepsilon} &= \varepsilon \;.\end{aligned} \tag{2.3.14}$$

Since for $\varepsilon = 0$, one recovers the dynamics of the associated system, one necessarily has $R(0, y, 0) = 0$, and $\partial_r R(0, y, 0)$ has the same eigenvalues as the monodromy matrix $U(T(y), 0)$ of the periodic orbit, except for the eigenvalue 1 associated with the variable θ.

Thus every point of the form $(0, y, 0)$ is a fixed point of the Poincaré map, and the linearisation of Π around each of these points admits 1 as an eigenvalue of multiplicity $m+1$, while the other $n-1$ eigenvalues are strictly smaller than 1. The centre-manifold theorem thus yields the existence of an invariant manifold of equation $r = \varepsilon \bar{r}(y, \varepsilon)$, that is,

$$\varepsilon \bar{r}\Big(y + \varepsilon Y\big(\bar{r}(y, \varepsilon), y, \varepsilon\big), \varepsilon\Big) = R\big(\varepsilon \bar{r}(y, \varepsilon), y, \varepsilon\big) \;. \tag{2.3.15}$$

A perturbative calculation shows that

$$\bar{r}(y, \varepsilon) = \big[\mathbb{1} - \partial_r R(0, y, 0)\big]^{-1} \partial_\varepsilon R(0, y, 0) + \mathcal{O}(\varepsilon) \;. \tag{2.3.16}$$

We can now return to the original equation (2.3.12). The set of images under the flow of the invariant manifold $r = \varepsilon \bar{r}(y, \varepsilon)$ defines a cylinder-shaped invariant object, whose parametrisation we can denote as $r = \varepsilon \bar{r}(\theta, y, \varepsilon)$. In the original x-variables, the cylinder is given by the equation

$$\bar{\Gamma}(\theta, y, \varepsilon) = \Gamma^\star(\theta, y) + \varepsilon \bar{r}(\theta, y, \varepsilon)\, \mathrm{n}(\theta, y) \;. \tag{2.3.17}$$

The dynamics on the invariant cylinder is then governed by the reduced equations

$$\begin{aligned}\varepsilon \dot{\theta} &= \frac{1}{T(y)} + b_\theta(\theta, \varepsilon \bar{r}(\theta, y, \varepsilon), y, \varepsilon) \;, \\ \dot{y} &= g\big(\bar{\Gamma}(\theta, y, \varepsilon), y\big) \;.\end{aligned} \tag{2.3.18}$$

Note that the term $b_\theta(\theta, \varepsilon \bar{r}(\theta, y, \varepsilon), y, \varepsilon)$ is at most of order ε, so that we recover the fact that $\varepsilon \dot{\theta}$ is ε-close to $1/T(y)$.

Bibliographic Comments

The fact that solutions of a slowly forced system (or, more generally, fast modes) tend to track stable equilibrium states of the corresponding frozen system is an old idea in physics, where it appears with names such as adiabatic elimination, slaving principle, or adiabatic theorem.

First mathematical results on slow–fast systems of the form considered here go back to Tihonov [Tih52] and Gradšteĭn [Gra53] for equilibria, and

Pontryagin and Rodygin [PR60] for periodic orbits. For related results based on asymptotic series for singularly perturbed systems, see also Nayfeh [Nay73], O'Malley [O'M74, O'M91], and Wasow [Was87]. Similar ideas have been used in quantum mechanics [Bog56], in particular for slowly time-dependent Hamiltonians [Nen80, Ber90, JP91].

The geometric approach to singular perturbation theory was introduced by Fenichel [Fen79]. See in particular [Jon95, JKK96] for generalisations giving conditions on $\mathcal{C}^1$-closeness between original and reduced equations.

The dynamic saddle–node bifurcation was first studied by Pontryagin [Pon57]. Later, Haberman [Hab79] addressed this problem and related ones, using matched asymptotic expansions. The theory of relaxation oscillations in several dimensions has been developed in [MR80, MKKR94]. The power-law behaviour of solutions near saddle–node bifurcation points was later rediscovered by physicists when studying bistable optical systems [JGRM90, GBS97]. See also [Arn94] for an overview.

Dynamic pitchfork and transcritical bifurcations were first studied in the generic case by Lebovitz and Schaar [LS77, LS75], as well as Haberman [Hab79]. The phenomenon of bifurcation delay for symmetric pitchfork bifurcations was observed in a laser in [ME84]. However, the same phenomenon for Hopf bifurcations was first pointed out by Shishkova [Shi73], and later analysed in detail by Neishtadt [Neĭ87, Neĭ88, Nei95]. Similar phenomena for maps undergoing period doubling bifurcations have been studied in [Bae95], while the case of a Hopf bifurcation of a periodic orbit has been considered in [NST96].

Since these early results, dynamic bifurcations have been analysed by a great variety of methods. These methods include nonstandard analysis (see, e.g., [Ben91] for a review, in particular [DD91] for bifurcation delay, and [FS03] for recent results); blow-up techniques [DR96, KS01] and a boundary function method [VBK95]. The approach based on the Newton polygon, presented in Section 2.2.4, was introduced in [Ber98, BK99].

The method of averaging was already used implicitly in celestial mechanics, where it allows to determine the secular motion of the planets' orbits, by eliminating the fast motion along the Kepler ellipses. The method was developed in particular by van der Pol, Krylov, Bogoliubov and Mitropol'skiĭ.

We did not mention another interesting characteristic of systems with slowly varying periodic orbits, the so-called geometric phase shifts. These occur, e.g., when the system's parameters are slowly modified, performing a loop in parameter space before returning to their initial value. Then the position on the orbit changes by a phase, whose first-order term in the speed of parameter variation depends only on the geometry of the loop [Ber85, KKE91]. The same phenomenon occurs in quantum mechanics (see for instance [Ber90, JKP91]).

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