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Vector Algebra Survival Kit

Vector algebra is one of the most important topics in modern mathematics. The Czech mathematician Bernard Bolzano first developed the concept of a vector in 1804. This concept was further developed by the French mathematician Jean Argand, the German mathematician August Möbius, the Irish mathematician Sir William Hamilton (who is believed to have coined the term vector) and culminated in the development of the *theory of vectors* by the Polish mathematician Hermann Grassmann in 1844. The concept of the vector continued to develop through the late nineteenth century and early twentieth century until the American mathematician Josiah Gibbs introduced the *theory of vector analysis* in 1890 and the *theory of vector spaces* in 1930.

In our study of computer graphics we will frequently use vectors to solve a variety of problems in such diverse areas as geometric modelling, transformations, projections, visibility determination, lighting, shading and texturing, and the development of curves, surfaces and deformations. Thus it is important to gain a thorough understanding of vector algebra.

2.1 Some Basic Definitions and Notation

Let us start by defining the terms *scalar* and *vector*. A scalar is a quantity that is completely determined by a single numerical value, which consists of a possibly signed *magnitude* (i.e. a *real* number). For instance, quantities such as temperature, time, length, mass, and speed are all represented by scalars. A vector, on the other hand, is a quantity that is determined by a *direction* and a *magnitude*. It is a *directed* quantity represented by an arrow. The direction and the length of this arrow determine the vector's direction and magnitude, respectively.

In everyday language it is common to use quantities such as speed and velocity interchangeably. This use however is inaccurate as the speed of a vehicle represents the magnitude of its movement alone and gives no indication of its direction of movement, while the velocity of a vehicle represents both its magnitude and its direction of movement. A scalar is a single dimensional quantity, while a vector is a multidimensional quantity.

In this book we denote vectors in bold italic notation such as $\vec{\boldsymbol{v}}$, $\bar{\boldsymbol{v}}$ or $\boldsymbol{v}$, while scalars are denoted by Greek letters or non-bold italic characters such as λ or l.

A three-dimensional geometric vector can be seen as a translation in three-dimensional Euclidean space $\mathcal{E}^3$. Given a point $\boldsymbol{P}$ in $\mathcal{E}^3$, we may use a vector $\boldsymbol{v}$ to move this point to a new position $\boldsymbol{P}'$, as shown in Fig. 2.1. Sometimes we call the vector $\boldsymbol{v}$ a *displacement vector*, as it displaces point $\boldsymbol{P}$ to its new position $\boldsymbol{P}'$. The distance between points $\boldsymbol{P}$ and $\boldsymbol{P}'$ is called the *magnitude* of the vector $\boldsymbol{v}$ and it is denoted by $|\boldsymbol{v}|$ or the non-bold italic version of the vector name v. This distance is the same for all points $\boldsymbol{P}$ in $\mathcal{E}^3$. Thus, the original position of point $\boldsymbol{P}$ is immaterial. For all vectors $\boldsymbol{v}$ we say that $|\boldsymbol{v}| \geq 0$.

There exists a special vector whose magnitude is zero. This vector translates every point $\boldsymbol{P}$ onto itself, i.e. the position of the point is left unchanged and the magnitude of the vector is zero. We call this vector the *zero vector* or *null vector* and we denote it by $\vec{\mathbf{0}}$ or $\mathbf{0}$ when there is no notational ambiguity (i.e. when it can not be misinterpreted as a scalar). No direction is associated with the zero vector. Thus,

$$\left|\vec{\mathbf{0}}\right| = 0 \tag{2.1}$$

It follows that

$$|\boldsymbol{v}| > 0 \quad \text{for every } \boldsymbol{v} \neq \vec{\mathbf{0}} \tag{2.2}$$

A vector $\boldsymbol{v}$ whose magnitude is equal to one (i.e. $|\boldsymbol{v}|$=1) is called a *unit vector*.

From the above discussion it should be apparent that all non-zero vectors are characterised by their *direction* and their *magnitude* and that two vectors are equal if they have the same direction and the same magnitude.

Given two points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ in E^3, the straight line segment between $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, as well as the direction from $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$, is denoted by $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}$ or $\overline{\boldsymbol{P}_1\boldsymbol{P}_2}$ and is called a *directed segment*. See Fig. 2.2. The points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are called the *initial* and *terminal* points of the directed segment, respectively. The distance between the two points is called the *magnitude* of $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}$ and is denoted by $\left|\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}\right|$ or $|\boldsymbol{P}_1\boldsymbol{P}_2|$. We use the notation $(\boldsymbol{P}_1\boldsymbol{P}_2)$ to denote the signed (directed) distance between the two points. This of course means that when travelling along the line defined by the

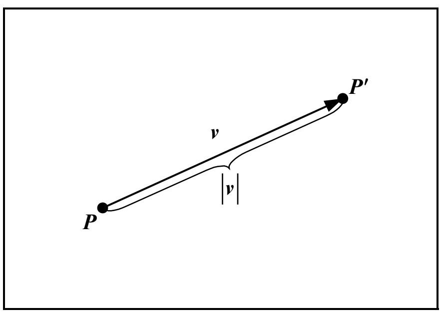

FIGURE 2.1. A displacement vector.

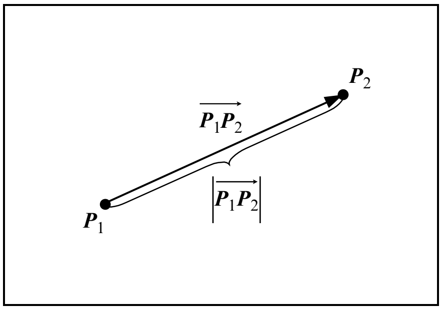

FIGURE 2.2. A directed segment.

points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, distances are taken to be positive in one direction and negative in the opposite direction. Thus,

$$(\boldsymbol{P}_2\boldsymbol{P}_1) = -(\boldsymbol{P}_1\boldsymbol{P}_2) \tag{2.3}$$

From the above discussion it should be apparent that all non-zero directed segments are defined by their direction and their magnitude. Two directed segments are said to be *equivalent* if they have the same direction and magnitude. We can extend the concept of the directed segment to include the case where the initial and terminal points are the same. Thus, $\overrightarrow{\boldsymbol{PP}}$ is a directed segment of zero magnitude and a non-unique direction.

If a vector $\boldsymbol{v}$ moves a point $\boldsymbol{P}$ to $\boldsymbol{P}'$, then $\boldsymbol{v}$ has the same direction and magnitude as the directed segment $\overrightarrow{\boldsymbol{PP}'}$. The directed segment $\overrightarrow{\boldsymbol{PP}'}$ is said to be *representative* of the vector $\boldsymbol{v}$ with initial point $\boldsymbol{P}$. Similarly, given a point $\boldsymbol{P}'$ there is *one and only one* representative with terminating point $\boldsymbol{P}'$. Conversely, given a directed segment $\overrightarrow{\boldsymbol{PP}'}$ there is only one vector $\boldsymbol{v}$ represented by $\overrightarrow{\boldsymbol{PP}'}$.

Two directed segments are said to represent the *same* vector if and only if they are equivalent.

Confusion between a vector and its representative directed segment must be avoided as it leads to serious errors in vector algebra. Despite the fact that both the vector $\boldsymbol{v}$ and its representative directed segment $\overrightarrow{\boldsymbol{PP}'}$ have the same direction and magnitude, $\overrightarrow{\boldsymbol{PP}'}$ has an initial point associated with it, whereas $\boldsymbol{v}$ does not. For this reason a directed segment is sometimes called a *localised vector*.

If two non-zero vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ have the same direction they are said to be *equidirectional* or *parallel* and if they have an opposite direction they are said to be *opposite* or *antiparallel*. Two vectors that are either parallel or antiparallel are said to be *collinear*. Parallel vectors are denoted by $\boldsymbol{v}_1 \uparrow\uparrow \boldsymbol{v}_2$, antiparallel vectors are denoted by $\boldsymbol{v}_1 \uparrow\downarrow \boldsymbol{v}_2$ and collinear vectors are denoted by $\boldsymbol{v}_1 || \boldsymbol{v}_2$. If $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}'_1}$ and $\overrightarrow{\boldsymbol{P}_2\boldsymbol{P}'_2}$ are the representatives of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively, then the vectors are parallel if the lines $\boldsymbol{P}_1\boldsymbol{P}'_1$ and $\boldsymbol{P}_2\boldsymbol{P}'_2$ are coincident or parallel. It is convenient to think of the zero vector $\vec{\boldsymbol{0}}$ as being parallel to any vector.

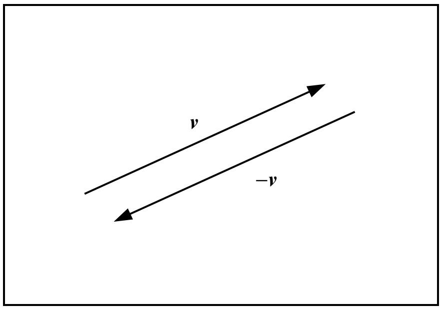

FIGURE 2.3. A vector and its inverse vector.

Given a non-zero vector $\boldsymbol{v}$, the vector with the same magnitude but an opposite direction is called the *negative* or *inverse* of $\boldsymbol{v}$ and is denoted by $-\boldsymbol{v}$. If $\overrightarrow{\boldsymbol{PP'}}$ is the representative of $\boldsymbol{v}$, then $\overrightarrow{\boldsymbol{P'P}}$ is representative of $-\boldsymbol{v}$. See Fig. 2.3. For any vector $\boldsymbol{v}$ we have:

$$-(-\boldsymbol{v}) = \boldsymbol{v} \tag{2.4}$$

2.2 Multiplication of a Vector by a Scalar

If $\boldsymbol{v}$ is a non-zero vector and α is a positive real number (scalar), then we define their product $\alpha\boldsymbol{v}$ to be the vector with the same direction as $\boldsymbol{v}$ and a magnitude $\alpha\,|\boldsymbol{v}|$ or αv. See Fig. 2.4. If α is a negative number, we define $\alpha\boldsymbol{v}$ to be $(-\alpha)(-\boldsymbol{v})$, so that $\alpha\boldsymbol{v}$ is the vector with a direction opposite to $\boldsymbol{v}$ (i.e. antiparallel to $\boldsymbol{v}$) and a magnitude $-\alpha\boldsymbol{v}$. We also define $0\boldsymbol{v}$ to be the zero vector for any vector $\boldsymbol{v}$, and $\alpha\vec{\boldsymbol{0}}$ to be the zero vector for any scalar value α. The operation of forming the product $\alpha\boldsymbol{v}$ is called *scalar multiplication* of $\boldsymbol{v}$ by α.

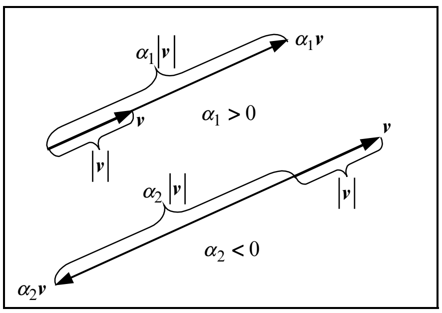

FIGURE 2.4. Scalar multiplication.

The following axioms and rules of vector algebra apply to the vector by scalar product for all vectors $\boldsymbol{v}$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and all scalars α and β:

Existence of the vector by scalar product:
$\alpha\boldsymbol{v}$ is a vector (A2.1)
Existence of the zero element:
$0\boldsymbol{v} = \boldsymbol{v}0 = \vec{\mathbf{0}}$ (A2.2)
Existence of the neutral element:
$1\boldsymbol{v} = \boldsymbol{v}1 = \boldsymbol{v}$ (A2.3)
Associative law:
$\alpha(\beta\boldsymbol{v}) = (\alpha\beta)\boldsymbol{v}$ (R2.1)
Distributive laws:
$(\alpha + \beta)\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\boldsymbol{v}$ (R2.2)
$\alpha(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \alpha\boldsymbol{v}_1 + \alpha\boldsymbol{v}_2$ (R2.3)

2.3 Vector Addition

If vector $\boldsymbol{v}_1$ moves point $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$ and vector $\boldsymbol{v}_2$ moves point $\boldsymbol{P}_2$ to $\boldsymbol{P}_3$, then the combined effect of $\boldsymbol{v}_1$ followed by $\boldsymbol{v}_2$ moves $\boldsymbol{P}_1$ to $\boldsymbol{P}_3$. See Fig. 2.5. The directed segment $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_3}$ represents a vector $\boldsymbol{v}_3$, which is unaffected by the choice of $\boldsymbol{P}_1$. So we obtain a unique vector $\boldsymbol{v}_3$ by combining $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. $\boldsymbol{v}_3$ is known as the *sum* of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$:

$$\boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2 \tag{2.5}$$

The sum of vectors is sometimes referred to as the *resultant vector*. The operation forming $\boldsymbol{v}_3$ from $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is called *vector addition*. Vector addition can also be expressed in terms of directed segments. Thus,

$$\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_3} = \overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2} + \overrightarrow{\boldsymbol{P}_2\boldsymbol{P}_3} \tag{2.6}$$

where $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_3}$ is the representative of $\boldsymbol{v}_1 + \boldsymbol{v}_2$.

The above result is sometimes referred to as the *triangle rule*. This rule states that if two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are represented in direction and magnitude by two

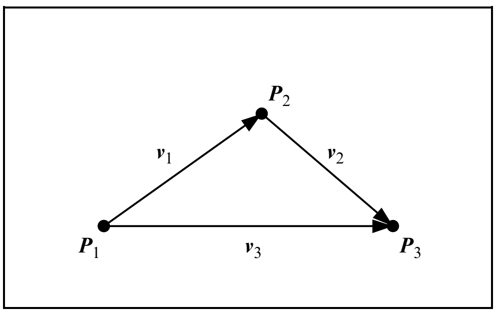

FIGURE 2.5. Vector addition.

consecutive sides of a triangle, then their sum $\boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2$ is represented in direction and magnitude by the third side (i.e. the closing side) of this triangle. The direction of the resultant vector is as shown in Fig. 2.5, i.e. from the initial point of $\boldsymbol{v}_1$ to the terminal point of $\boldsymbol{v}_2$.

The triangle rule can be generalised to the *polygon rule*, which deals with the summation of n vectors. In this case we arrange the vectors in such a fashion that the initial point of each vector (to be added to the sum) is placed at the terminal point of the vector preceding it in the summation, thus forming a *vector polygon*. Then the *resultant vector* is the vector with the same initial point as the first vector in the summation and the same terminal point as the last vector in the summation. The direction of the resultant vector $\boldsymbol{r}$ is as shown in Fig. 2.6a. If the vector polygon is closed, then the resultant vector is the zero vector. See Fig. 2.6b

The *difference* $\boldsymbol{v}_1 - \boldsymbol{v}_2$ of two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is defined as the vector:

$$\boldsymbol{v}_3 = \boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{v}_1 + (-\boldsymbol{v}_2) \tag{2.7}$$

The operation that forms $\boldsymbol{v}_3$ from $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is called *vector subtraction*. See Fig. 2.7.

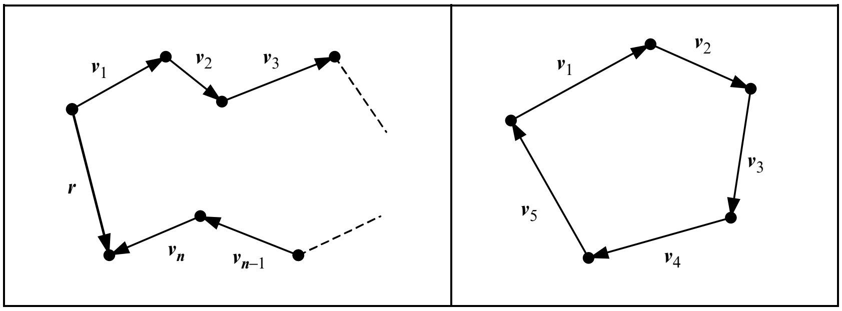

FIGURE 2.6. (a) The polygon rule. (b) A closed vector polygon.

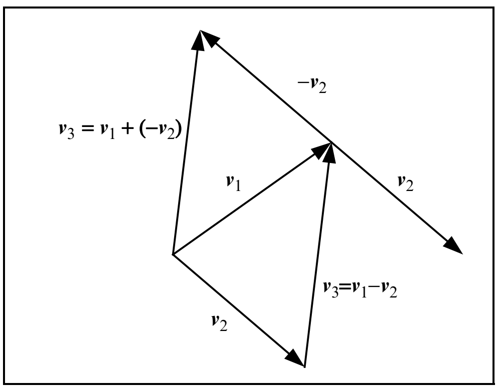

FIGURE 2.7. Vector subtraction.

The following axioms and rules of vector algebra apply to vector addition hold for all vectors $\boldsymbol{v}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$:

Existence of the vector sum:
$\boldsymbol{v}_1 + \boldsymbol{v}_2$ is a vector (A2.4)

Existence of the neutral element:
$$\vec{\boldsymbol{0}} + \boldsymbol{v} = \boldsymbol{v} + \vec{\boldsymbol{0}} = \boldsymbol{v} \tag{A2.5}$$

Existence of the inverse element:
$$\boldsymbol{v} + (-\boldsymbol{v}) = (-\boldsymbol{v}) + \boldsymbol{v} = \vec{\boldsymbol{0}} \tag{A2.6}$$

Commutative law:
$$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_2 + \boldsymbol{v}_1 \tag{R2.4}$$

Associative law:
$$(\boldsymbol{v}_1 + \boldsymbol{v}_2) + \boldsymbol{v}_3 = \boldsymbol{v}_1 + (\boldsymbol{v}_2 + \boldsymbol{v}_3) \tag{R2.5}$$

Change of detection rule:
$$-(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \boldsymbol{v}_2 - \boldsymbol{v}_1 \tag{R2.6}$$

2.4 Position Vectors and Free Vectors

The *position vector* of a point $\boldsymbol{P}$ in $\mathcal{E}^3$ relative to an origin $\boldsymbol{O}$ is defined to be the vector representing the directed segment $\overrightarrow{\boldsymbol{OP}}$. The position vector of a point $\boldsymbol{P}$ is sometimes referred to as a *bound vector*, as its initial point is fixed at the origin $\boldsymbol{O}$ and its terminal point is fixed at the point $\boldsymbol{P}$. The position vector $\boldsymbol{p}$ of a point $\boldsymbol{P}$ relative to the origin $\boldsymbol{O}$ is sometimes denoted by:

$$\vec{\boldsymbol{p}}(\boldsymbol{P}) = \overrightarrow{\boldsymbol{OP}} \tag{2.8}$$

In contrast a *free vector* or an *orientation vector* is any *unbound* vector that is free to be translated in an arbitrary fashion.

2.5 The Vector Equation of a Line

Lines in $\mathcal{E}^3$ can be represented by equations involving position vectors. Let $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ be two points on a line, having position vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, respectively,

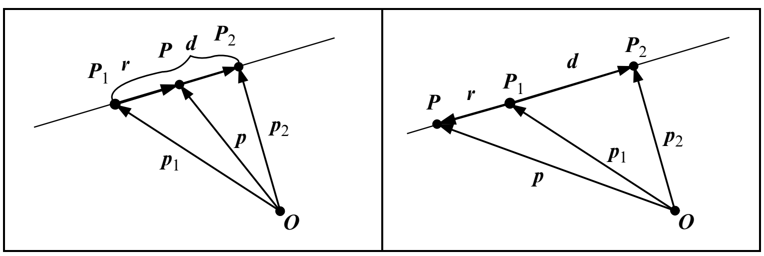

FIGURE 2.8. A line passing through two points.

relative to some origin $\boldsymbol{O}$. From Fig. 2.8 we see that:

$$\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2} = \overrightarrow{\boldsymbol{P}_1\boldsymbol{O}} + \overrightarrow{\boldsymbol{O}\boldsymbol{P}_2} \tag{2.9}$$

which represents the *direction vector*:

$$\begin{aligned} \boldsymbol{d} &= (-\boldsymbol{p}_1) + \boldsymbol{p}_2 \\ &= \boldsymbol{p}_2 - \boldsymbol{p}_1 \end{aligned} \tag{2.10}$$

Now, let $\boldsymbol{P}$ be a point whose position vector relative to the origin $\boldsymbol{O}$ is $\boldsymbol{p}$. This point will lie on the line $\boldsymbol{P}_1\boldsymbol{P}_2$ *if and only if* $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}}$ and $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}$ represent collinear vectors. Since $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}}$ represents vector $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{p}_1$, it follows that point $\boldsymbol{P}$ lies on the line $\boldsymbol{P}_1\boldsymbol{P}_2$ if and only if $\boldsymbol{p} - \boldsymbol{p}_1 = t(\boldsymbol{p}_2 - \boldsymbol{p}_1)$ for some scalar t, i.e. if and only if

$$\boldsymbol{p} = (1-t)\boldsymbol{p}_1 + t\boldsymbol{p}_2 \tag{2.11}$$

This is the *parametric vector equation of the line* $\boldsymbol{P}_1\boldsymbol{P}_2$ relative to some origin $\boldsymbol{O}$. Here t is a parameter that can assume any real value. It is worth noting that at point $\boldsymbol{P}_1$ $t = 0$, at point $\boldsymbol{P}_2$ $t = 1$, at points preceding the directed segment $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}$ $t < 0$ and at points following it $t > 1$.

2.6 Linear Dependence/Independence of Vectors

If $\boldsymbol{a}$ is a non-zero vector, then any vector $\boldsymbol{v}$ which is collinear to the vector $\boldsymbol{a}$ is of the form $\alpha\boldsymbol{a}$ for some scalar α. When vectors $\boldsymbol{a}$ and $\boldsymbol{v}$ are parallel then $\alpha = |\boldsymbol{v}| / |\boldsymbol{a}|$, when vectors $\boldsymbol{a}$ and $\boldsymbol{v}$ are antiparallel then $\alpha = -|\boldsymbol{v}| / |\boldsymbol{a}|$ and finally when $\boldsymbol{v} = \vec{\boldsymbol{0}}$ then $\alpha = 0$.

If vector $\boldsymbol{v}$ is of the form $\alpha\boldsymbol{a}$ (i.e. $\boldsymbol{v} = \alpha\boldsymbol{a}$), then we say that vector $\boldsymbol{v}$ is *linearly dependent* on vector $\boldsymbol{a}$. This definition remains valid even if $\boldsymbol{a} = \vec{\boldsymbol{0}}$. When $\boldsymbol{a} \neq \vec{\boldsymbol{0}}$, vector $\boldsymbol{v}$ is linearly dependent on vector $\boldsymbol{a}$ if and only if vectors $\boldsymbol{v}$ and $\boldsymbol{a}$ are collinear.

Next, assume that we have two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that are non-collinear and are represented by the directed segments $\overrightarrow{\boldsymbol{OA}}$ and $\overrightarrow{\boldsymbol{OB}}$, as shown in Fig. 2.9.

Given a vector $\boldsymbol{v}$ represented by the directed segment $\overrightarrow{\boldsymbol{OV}}$, where point $\boldsymbol{V}$ lies on the plane $\boldsymbol{AOB}$, then we say that the vector $\boldsymbol{v}$ is *coplanar* with vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ if and only if there exist scalars α and β such that:

$$\boldsymbol{v} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} \tag{2.12}$$

In such a case, we say that the vector $\boldsymbol{v}$ is *linearly dependent* on vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and that vector $\boldsymbol{v}$ is formed as a *linear combination* of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Further we say that vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{v}$ are *coplanar vectors*. Thus, any vector lying on a

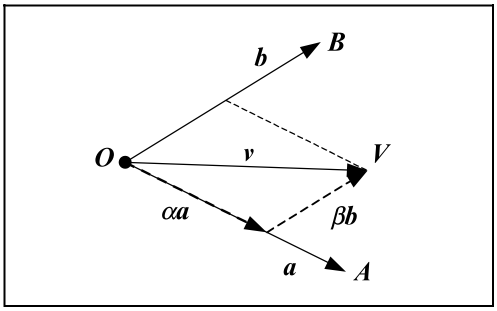

FIGURE 2.9. Linear dependence on two vectors.

plane defined by two other vectors is linearly dependent on these vectors, i.e. it can be expressed as the sum of multiples of these vectors.

Finally, assume that we have three non-coplanar vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ represented by the directed segments $\overrightarrow{\boldsymbol{OA}}$, $\overrightarrow{\boldsymbol{OB}}$, $\overrightarrow{\boldsymbol{OC}}$, respectively, as shown in Fig. 2.10. The relationship of these vectors implies that points $\boldsymbol{O}$, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are not coplanar.

A vector $\boldsymbol{v}$ is said to be linearly dependent on vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ if and only if there exist scalars α, β, γ such that:

$$\boldsymbol{v} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \tag{2.13}$$

Thus any vector of $\mathcal{E}^3$ is linearly dependent on three non-coplanar vectors.

In general, given a set of n non-zero vectors $\{\boldsymbol{v}_i\}_{i=1}^n$ and a set of scalars $\{\alpha_i\}_{i=1}^n$, there exists a linear combination of these vectors that vanishes, i.e. equal to the null vector:

$$\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \cdots + \alpha_n\boldsymbol{v}_n = \vec{\boldsymbol{0}} \tag{2.14}$$

If this linear combination of vectors vanishes with at least one of the scalars $\alpha_i \neq 0$, then we say that these vectors are *linearly dependent*. If however this linear combination of vectors only vanishes when all scalars $\{\alpha_i = 0\}_{i=1}^n$, then we say that these vectors are *linearly independent*.

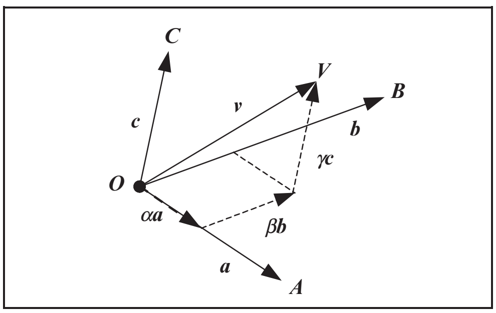

FIGURE 2.10. Linear dependence on three vectors.

2.7 Vector Bases

Given an ordered set of three non-coplanar vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, we call this a *basis* of E^3. The order of the vectors is significant. If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is a basis of E^3, then $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{a}$ is a different basis of E^3. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ forming the basis are called *base vectors*. Every other vector in E^3 can be expressed as a linear combination of a given basis.

In general, a basis of an n-dimensional *vector space* E^n is defined by a set of n linearly independent vectors $\{\boldsymbol{v}_i\}_{i=1}^n$ in E^n. Every other vector $\boldsymbol{v}$ in this vector space can be expressed as a linear combination of the base vectors:

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n \tag{2.15}$$

where $\{\alpha_i\}_{i=1}^n$ are scalars.

2.8 The Components of a Vector

Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be a basis. Then vector $\boldsymbol{v}$ is linearly dependent on the base vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and there exist a unique triple of scalars α, β, γ such that $\boldsymbol{v} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$. The scalars α, β, γ are called the *components of vector* $\boldsymbol{v}$ *relative to the basis* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.

Let $\boldsymbol{v}$ be the set of all the vectors $\boldsymbol{v}$ and $\boldsymbol{C}$ be the set of all the triples of components $[\alpha, \beta, \gamma]$ where α, β, γ are scalars. There is a *one to one correspondence* between sets $\boldsymbol{v}$ and $\boldsymbol{C}$ associating each vector $\boldsymbol{v}$ with a triple of components $[\alpha, \beta, \gamma]$. Thus a triple of components is uniquely determined by $\boldsymbol{v}$ and given the triple $[\alpha, \beta, \gamma]$ there is only one vector v that satisfies the equation $\boldsymbol{v} = \alpha\,\boldsymbol{a} + \beta\,\boldsymbol{b} + \gamma\,\boldsymbol{c}$. Using this correspondence between vectors and the ordered triples of scalars we can prove properties of vectors by working in terms of their components rather than the vectors themselves, remembering that the basis relative to which a vector is expressed uniquely determines its components. Thus, changing the basis relative to which a vector is expressed changes its components.

Let us now see how multiplication of a vector by a scalar and vector addition can be defined in terms of the components of vectors.

2.8.1 Multiplication of a Vector by a Scalar

Given a vector $\boldsymbol{v}$ with components $[\alpha, \beta, \gamma]$ relative to some basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and a scalar s, then the vector by a scalar product sv in terms of its components is given by:

$$s[\alpha, \beta, \gamma] = [s\alpha, s\beta, s\gamma] \tag{2.16}$$

2.8.2 Vector Addition

Given two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ with components $[\alpha_1, \beta_1, \gamma_1]$ and $[\alpha_2, \beta_2, \gamma_2]$ relative to some basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, then the vector sum $\boldsymbol{v}_1 \pm \boldsymbol{v}_2$ in terms of its components is given by:

$$[\alpha_1, \beta_1, \gamma_1] \pm [\alpha_2, \beta_2, \gamma_2] = [\alpha_1 \pm \alpha_2, \beta_1 \pm \beta_2, \gamma_1 \pm \gamma_2] \tag{2.17}$$

2.8.3 Vector Equality

Given two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ with components $[\alpha_1, \beta_1, \gamma_1]$ and $[\alpha_2, \beta_2, \gamma_2]$ relative to some basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, then the vectors are said to be equal if their corresponding components are equal, i.e.

$$[\alpha_1, \beta_1, \gamma_1] = [\alpha_2, \beta_2, \gamma_2] \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 \wedge \beta_1 = \beta_2 \wedge \gamma_1 = \gamma_2 \tag{2.18}$$

The components $[\alpha, \beta, \gamma]$ of a vector $\boldsymbol{v}$ can either be written in row or in column form, i.e.

$$[\alpha, \beta, \gamma] \text{ or } \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

The row form is said to be the *transpose* of the column form and vice versa, i.e.

$$[\alpha, \beta, \gamma]^T = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \text{ and } \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}^T = [\alpha, \beta, \gamma]$$

These two representations are equivalent and the choice of one over the other is a matter of taste.

2.9 Orthogonal, Orthonormal and Right-Handed Vector Bases

A basis is said to be *orthogonal* if its base vectors are mutually perpendicular and it is said to be *orthonormal* if its base vectors are mutually perpendicular unit vectors.

Given an orthogonal basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and any vector $\boldsymbol{v}$, the directed segments $\overrightarrow{\boldsymbol{OA}}$, $\overrightarrow{\boldsymbol{OB}}$, $\overrightarrow{\boldsymbol{OC}}$, $\overrightarrow{\boldsymbol{OV}}$ represent the $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{v}$ vectors, respectively, as shown in Fig. 2.11.

Let $\boldsymbol{P}$ be the foot of the perpendicular from point $\boldsymbol{V}$ to the plane $\boldsymbol{AOB}$ (i.e. the projection of point $\boldsymbol{V}$ on the plane $\boldsymbol{AOB}$) and let $\boldsymbol{Q}$ be the foot of the perpendicular from point $\boldsymbol{P}$ to the line $\boldsymbol{OA}$. From this figure we see that:

$$\overrightarrow{\boldsymbol{OV}} = \overrightarrow{\boldsymbol{OQ}} + \overrightarrow{\boldsymbol{QP}} + \overrightarrow{\boldsymbol{PV}} \tag{2.19}$$

If vector $\boldsymbol{v}$ has components $[\alpha, \beta, \gamma]$ relative to the orthogonal basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, then $\overrightarrow{\boldsymbol{OV}}$ represents the vector sum $\alpha\,\boldsymbol{a} + \beta\,\boldsymbol{b} + \gamma\,\boldsymbol{c}$. But the directed segments $\overrightarrow{\boldsymbol{OQ}}$, $\overrightarrow{\boldsymbol{QP}}$, $\overrightarrow{\boldsymbol{PV}}$ have the same directions as $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, respectively. Thus $\overrightarrow{\boldsymbol{OQ}}$, $\overrightarrow{\boldsymbol{QP}}$,

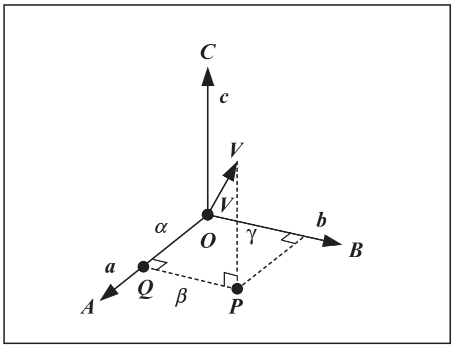

FIGURE 2.11. An orthogonal vector basis.

$\overrightarrow{\boldsymbol{PV}}$ represent the vectors $\alpha\boldsymbol{a}$, $\beta\boldsymbol{b}$, $\gamma\boldsymbol{c}$, respectively. From this we can deduce that the magnitudes of the components α, β, γ are equal to the magnitudes of the orthogonal projections of the directed segment $\overrightarrow{\boldsymbol{OV}}$ onto the lines $\boldsymbol{OA}$, $\boldsymbol{OB}$, $\boldsymbol{OC}$, respectively. The sign of α is positive if points $\boldsymbol{V}$ and $\boldsymbol{A}$ are on the same side of the plane $\boldsymbol{BOC}$ and negative if they are on opposite sides of this plane. Similarly for the signs of β and γ.

For a non-orthogonal basis we can not obtain the components of a vector, with respect to this basis, by orthogonal projections as we have done above. In a later section we resolve this problem using triple scalar products.

Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be three non-zero non-coplanar vectors, represented by the directed segments $\overrightarrow{\boldsymbol{OA}}$, $\overrightarrow{\boldsymbol{OB}}$, $\overrightarrow{\boldsymbol{OC}}$, respectively, form a basis, as shown in Fig. 2.12.

Suppose that there is a right-handed screw aligned with the directed segment $\overrightarrow{\boldsymbol{OC}}$ and pointing in the same direction as $\overrightarrow{\boldsymbol{OC}}$. A clockwise rotation of the screw will cause it to advance along $\overrightarrow{\boldsymbol{OC}}$ and will also cause $\overrightarrow{\boldsymbol{OA}}$ and $\overrightarrow{\boldsymbol{OB}}$ to rotate so that $\overrightarrow{\boldsymbol{OA}}$ rotates towards the original position of $\overrightarrow{\boldsymbol{OB}}$. Alternatively, suppose that the observer's eye is situated at point $\boldsymbol{C}$ and is looking along $\overrightarrow{\boldsymbol{OC}}$ towards the point $\boldsymbol{O}$.

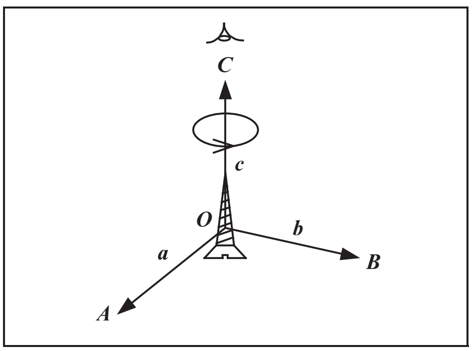

FIGURE 2.12. A right-handed basis.

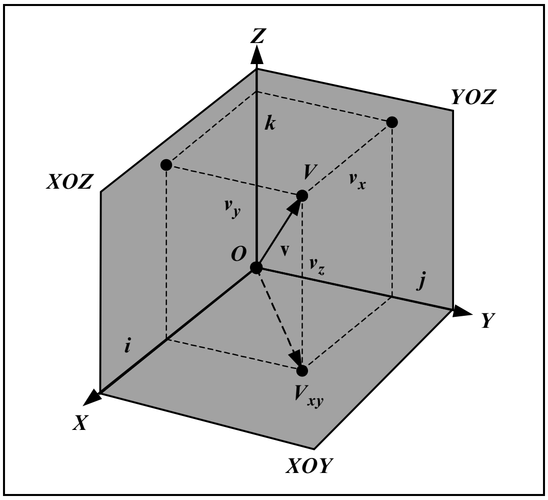

FIGURE 2.13. A rectangular Cartesian coordinate system.

A counter-clockwise rotation about $\overrightarrow{\boldsymbol{OC}}$ will cause $\overrightarrow{\boldsymbol{OA}}$ and $\overrightarrow{\boldsymbol{OB}}$ to rotate so that $\overrightarrow{\boldsymbol{OA}}$ rotates towards the original position of $\overrightarrow{\boldsymbol{OB}}$. Any ordered set of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ that are arranged thus is called a *right-handed system* of vectors or a *right-handed basis*.

2.10 Cartesian Bases and Cartesian Coordinates

A basis that is both orthonormal and right-handed is called a *Cartesian basis*.

Let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be a Cartesian basis, let $\boldsymbol{O}$ be any fixed point and let the directed segments $\overrightarrow{\boldsymbol{OX}}$, $\overrightarrow{\boldsymbol{OY}}$, $\overrightarrow{\boldsymbol{OZ}}$ represent the base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, respectively, as shown in Fig. 2.13.

If $\boldsymbol{v}$ is the point with position vector $\boldsymbol{v}$ relative to the origin $\boldsymbol{O}$, the components of vector $\boldsymbol{v}$ relative to the basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the perpendicular distances of point $\boldsymbol{V}$ from the planes $\boldsymbol{YOZ}$, $\boldsymbol{XOZ}$, $\boldsymbol{XOY}$, respectively, with the sign conventions described in Section 2.9.

In this section we abandon the conventions we have adopted earlier of denoting the components of vectors by Greek letters. We write v_x, v_y, v_z for the components of the vector $\boldsymbol{v}$ relative to some Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. Thus,

$$\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k} \tag{2.20}$$

By expressing the position vectors of points in this way, we establish a *one to one* correspondence between vectors and ordered triples of real numbers. Here the triple $[v_x, v_y, v_z]$ corresponds to point $\boldsymbol{V}$ with position vector $\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}$. Such a correspondence is called a *rectangular Cartesian coordinate system* and the components of the position vector are called the *coordinates* of the point to which they correspond.

The coordinates of the origin $\boldsymbol{O}$ are [0, 0, 0], the coordinates of a point on the line $\boldsymbol{OX}$ are given by $[x, 0, 0]$, the coordinates of a point on the plane $\boldsymbol{XOY}$ are given by $[x, y, 0]$ and finally the coordinates of a point in space are given by $[x, y, z]$. Similar properties hold for the other *coordinate axes* and *major planes*.

It is sometimes convenient to use vector notation rather than coordinate notation. When doing so, we often write $\boldsymbol{v} = \left[v_x, v_y, v_z\right]$ to mean the same thing as $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$. We frequently abbreviate the expression *the point with coordinates* $\left[v_x, v_y, v_z\right]$ by the expression *the point* $\left[v_x, v_y, v_z\right]$ and the expression *the point with position vector* $\boldsymbol{v}$ by the expression *the point* $\boldsymbol{v}$. Such abbreviations are valid so long as only one coordinate system is involved.

2.11 The Length of a Vector

Let $\left[v_x, v_y, v_z\right]$ be the components of a vector $\boldsymbol{v}$ related to some Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, as shown in Fig. 2.13. Let point $\boldsymbol{V}_{xy}$ be the foot of the perpendicular of point $\boldsymbol{V}$ onto the $\boldsymbol{XOY}$ plane. From this figure we see that:

$$\begin{aligned} |\boldsymbol{v}|^2 &= \left|\overrightarrow{\boldsymbol{OV}}\right|^2 \\ &= \left|\overrightarrow{\boldsymbol{OV}_{xy}}\right|^2 + v_z^2 \end{aligned}$$

But:

$$\begin{aligned} \left|\overrightarrow{\boldsymbol{OV}_{xy}}\right|^2 &= v_x^2 + v_y^2 \\ \therefore\ |\boldsymbol{v}|^2 &= v_x^2 + v_y^2 + v_z^2 \\ \therefore\ |\boldsymbol{v}| &= \sqrt{v_x^2 + v_y^2 + v_z^2} \end{aligned} \tag{2.21}$$

2.12 The Scalar Product of Vectors

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors represented by the directed segments $\overrightarrow{\boldsymbol{OA}}$ and $\overrightarrow{\boldsymbol{OB}}$, respectively. The angle θ between the two vectors is the angle between their representatives, as seen in Fig. 2.14. We assume that $0 \leq \theta \leq \pi$. If $\theta = 0$, then the two vectors have the same direction, if $\theta = \pi$, then the two vectors point in opposite directions and if $\theta = \frac{\pi}{2}$, then the two vectors are perpendicular.

The *scalar product* or *dot product* or *inner product* $\boldsymbol{a} \odot \boldsymbol{b}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as:

$$\boldsymbol{a} \odot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta \tag{2.22}$$

If $a = \vec{\boldsymbol{0}}$ or $b = \vec{\boldsymbol{0}}$, then the scalar product $\boldsymbol{a} \odot \boldsymbol{b}$ is defined to be zero.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors, then their scalar product simplifies to $\boldsymbol{a} \odot \boldsymbol{b} = \cos\theta$.

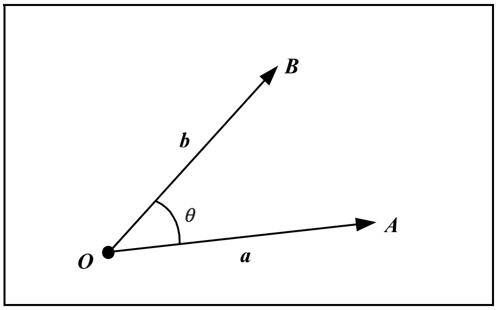

FIGURE 2.14. The angle between two vectors.

If $a \perp b$ (i.e. $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular), then their scalar product is $\boldsymbol{a} \odot \boldsymbol{b} = 0$, as $cos\left(\frac{\pi}{2}\right) = 0$.

From the above it should be apparent that the scalar product of two vectors is a scalar. This implies that the scalar product is only defined for two vectors. This further implies that the only power of vectors that is defined is the square power, i.e. $\boldsymbol{a}^2 = \boldsymbol{a} \odot \boldsymbol{a} = |\boldsymbol{a}|^2$.

Applying Eq. (2.22) to the base vectors of an orthonormal basis $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ we obtain:

$$\begin{array}{c} \boldsymbol{i} \odot \boldsymbol{i} = \boldsymbol{j} \odot \boldsymbol{j} = \boldsymbol{k} \odot \boldsymbol{k} = 1 \\ \text{and} \\ \boldsymbol{i} \odot \boldsymbol{j} = \boldsymbol{j} \odot \boldsymbol{k} = \boldsymbol{i} \odot \boldsymbol{k} = 0 \end{array} \tag{2.23}$$

since $cos\,(0) = 1$ and $cos\left(\frac{\pi}{2}\right) = 0$.

The above result can be written in a more condensed form using *Kronecker's* symbol that summarises orthonormality:

$$\delta_{l,m} = \boldsymbol{v}_l \odot \boldsymbol{v}_m = \begin{cases} 1, & l = m \\ 0, & otherwise \end{cases}$$
$$\text{with} \quad l, m \in \{1, 2, 3\} \quad \text{and} \quad \text{where} \quad \boldsymbol{v}_1 = \boldsymbol{i},\ \boldsymbol{v}_2 = \boldsymbol{j},\ \boldsymbol{v}_3 = \boldsymbol{k} \tag{2.24}$$

The following axioms and vector algebra rules apply to the scalar product of vectors for all vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}$, and all scalars α:

Existence of the scalar product:

$\boldsymbol{v}_1 \odot \boldsymbol{v}_2$ is a scalar (A2.7)

Powers of a vector:

$\boldsymbol{v}^2 = \boldsymbol{v} \odot \boldsymbol{v} = |\boldsymbol{v}|^2$ (A2.8)

Commutative law:

$\boldsymbol{v}_1 \odot \boldsymbol{v}_2 = \boldsymbol{v}_2 \odot \boldsymbol{v}_1$ (R2.7)

Distributive laws:

$(\alpha \boldsymbol{v}_1) \odot \boldsymbol{v}_2 = \boldsymbol{v}_1 \odot (\alpha \boldsymbol{v}_2) = \alpha\,(\boldsymbol{v}_1 \odot \boldsymbol{v}_2)$ (R2.8)

$\boldsymbol{v}_1 \odot (\boldsymbol{v}_2 + \boldsymbol{v}_3) = \boldsymbol{v}_1 \odot \boldsymbol{v}_2 + \boldsymbol{v}_1 \odot \boldsymbol{v}_3$ (R2.9)

2.13 The Scalar Product Expressed in Terms of its Components

If $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ is a basis and if $\boldsymbol{a}$, $\boldsymbol{b}$ are vectors with components $\left[a_x, a_y, a_z\right]$ and $\left[b_x, b_y, b_z\right]$, respectively, relative to this basis, then the scalar product $\boldsymbol{a} \odot \boldsymbol{b}$ (in terms of its components) is defined as:

$$\begin{aligned}
\boldsymbol{a} \odot \boldsymbol{b} &= \left(a_x\boldsymbol{p} + a_y\boldsymbol{q} + a_z\boldsymbol{r}\right) \odot \left(b_x\boldsymbol{p} + b_y\boldsymbol{q} + b_z\boldsymbol{r}\right) \\
&= a_x b_x \boldsymbol{p} \odot \boldsymbol{p} + a_y b_y \boldsymbol{q} \odot \boldsymbol{q} + a_z b_z \boldsymbol{r} \odot \boldsymbol{r} + \left(a_y b_z + a_z b_y\right) \boldsymbol{q} \odot \boldsymbol{r} \\
&\quad + \left(a_z b_x + a_x b_z\right) \boldsymbol{r} \odot \boldsymbol{p} + \left(a_x b_y + a_y b_x\right) \boldsymbol{p} \odot \boldsymbol{q}
\end{aligned} \tag{2.25}$$

When dealing with an orthonormal basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, using equation (2.23) we can simplify the above definition to:

$$\boldsymbol{a} \odot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z \tag{2.26}$$

2.14 Properties and Applications of the Scalar Product

In this section, unless otherwise stated or implied, we will assume that the components of all vectors used are defined with respect to a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

2.14.1 The Magnitude of a Vector Using its Components

Using Axiom (A2.8) we can now define the *magnitude*, *length* or *norm* of a vector in terms of the scalar product as:

$$|\boldsymbol{v}| = \sqrt{\boldsymbol{v} \odot \boldsymbol{v}} = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{2.27}$$

Which accords with the definition we have given in Eq. (2.21).

2.14.2 Normalising a Vector

Given a non-zero vector $\boldsymbol{v}$ we can *normalise* this vector (i.e. cause it to become a unit-vector) by dividing the vector by its magnitude. This *normalised vector* is given by:

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \left[\frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}}, \frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}}, \frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}}\right] \tag{2.28}$$

The magnitude of a normalised vector is $\left|\hat{\boldsymbol{v}}\right| = 1$.

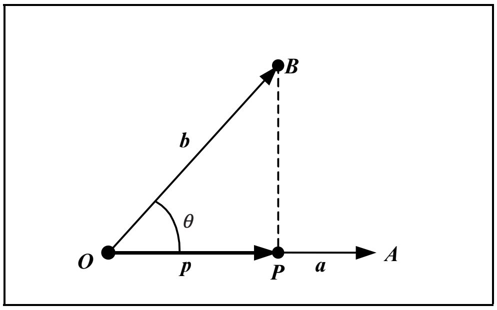

FIGURE 2.15. Projection of a vector onto another.

2.14.3 The Projection of a Vector onto Another

Suppose that $\boldsymbol{a}$, $\boldsymbol{b}$ are non-zero and non-collinear vectors represented by the directed segments $\overrightarrow{\boldsymbol{OA}}$, $\overrightarrow{\boldsymbol{OB}}$, respectively, as shown in Fig. 2.15. Let $\boldsymbol{P}$ be the foot of the perpendicular from point $\boldsymbol{B}$ to the line $\boldsymbol{OA}$. Then $\overrightarrow{\boldsymbol{OP}}$ is the projection of $\overrightarrow{\boldsymbol{OB}}$ onto $\overrightarrow{\boldsymbol{OA}}$ and vector $\boldsymbol{p}$ represents the projection of vector $\boldsymbol{b}$ onto vector $\boldsymbol{a}$. Vector $\boldsymbol{p}$ is known as the *projection vector*. From the diagram it should be apparent that the magnitude of $\boldsymbol{p}$ is not affected by the magnitude of $\boldsymbol{a}$.

Initially, let us suppose that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ are unit vectors. Using simple trigonometry we determine that in this case the magnitude of the projection vector $\boldsymbol{p}$ is $|\boldsymbol{p}| = \cos\theta$ and the vector itself is given by:

$$\boldsymbol{p} = \boldsymbol{a} \cdot \cos\theta \tag{2.29}$$

Next, let us assume that vector $\boldsymbol{a}$ is a unit vector and $\boldsymbol{b}$ is any non-zero vector. In this case the magnitude of the projection vector $\boldsymbol{p}$ is $|\boldsymbol{p}| = |\boldsymbol{b}|\cos\theta$ and the vector itself is given by:

$$\boldsymbol{p} = \boldsymbol{a} \cdot |\boldsymbol{b}|\cos\theta \tag{2.30}$$

Finally, let us relax all restrictions and assume that $\boldsymbol{a}$, $\boldsymbol{b}$ are any non-zero vectors. In this case the magnitude of the projection vector $\boldsymbol{p}$ is $|\boldsymbol{p}| = |\boldsymbol{b}|\cos\theta$ (as before) but now the vector itself is given by:

$$\boldsymbol{p} = \hat{\boldsymbol{a}} \cdot |\boldsymbol{b}|\cos\theta = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}|\boldsymbol{b}|\cos\theta$$

$$\therefore \ \boldsymbol{p} = \boldsymbol{a} \cdot \frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}\cos\theta \tag{2.31}$$

Observe that above we have normalised vector $\boldsymbol{a}$ before we scaled it by the magnitude of the projection vector $|\boldsymbol{p}|$.

If vectors $\boldsymbol{a}$, $\boldsymbol{b}$ are parallel, then we define the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ to be $\boldsymbol{b}$ itself.

2.14.4 The Cosine of the Angle Between two Vectors

Given two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ their scalar product is given by $\boldsymbol{a} \odot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$. Solving this equation for *cos* θ we obtain:

$$\cos\theta = \frac{\boldsymbol{a} \odot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}} \tag{2.32}$$

2.14.5 The Scalar Product of Collinear Vectors

Assume that we have two non-zero collinear vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

If the vectors are parallel, then their scalar product is given by:

$$\boldsymbol{a} \odot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \quad \Leftrightarrow \quad \boldsymbol{a} \uparrow\uparrow \boldsymbol{b} \tag{2.33}$$

since $cos\,(0) = 1$.

If the vectors are antiparallel, then their scalar product is given by:

$$\boldsymbol{a} \odot \boldsymbol{b} = -\,|\boldsymbol{a}|\,|\boldsymbol{b}| \quad \Leftrightarrow \quad \boldsymbol{a} \uparrow\downarrow \boldsymbol{b} \tag{2.34}$$

since $cos\,(\pi) = -1$.

2.14.6 The Scalar Product of Orthogonal Vectors

Given two non-zero orthogonal vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the cosine of the angle between them will be zero and thus their scalar product *vanishes* (i.e. it is zero). This is both *a necessary and sufficient condition*. This means that the converse is also true, i.e. if the scalar product of two vectors vanishes, then the vectors are orthogonal. We say that:

$$\boldsymbol{a} \odot \boldsymbol{b} = 0 \quad \Leftrightarrow \quad \boldsymbol{a} \perp \boldsymbol{b} \tag{2.35}$$

since $cos\left(\frac{\pi}{2}\right) = 0$.

2.15 The Direction Ratios and Direction Cosines of a Vector

Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be a Cartesian basis. Suppose that the vector $\boldsymbol{v}$ is a non-zero vector, with components $\left[v_x, v_y, v_z\right]$ relative to the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ basis, that is represented by the directed segment $\overrightarrow{\boldsymbol{OV}}$, as seen in Fig. 2.16. Let point $\boldsymbol{V}_{xy}$ be the foot of the perpendicular from point $\boldsymbol{V}$ to the $\boldsymbol{XOY}$ plane. Then the coordinates of point $\boldsymbol{V}$ and the components of vector $\boldsymbol{v}$ are given by the triple $\left[v_x, v_y, v_z\right]$.

The components v_x, v_y, v_z are known as the *direction ratios* of vector $\boldsymbol{v}$. They allow us to express vector $\boldsymbol{v}$ as the sum of the $v_x\boldsymbol{i}$, $v_y\boldsymbol{j}$, $v_z\boldsymbol{k}$ vectors, which are collinear to the base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and have magnitudes v_x, v_y, v_z, respectively.

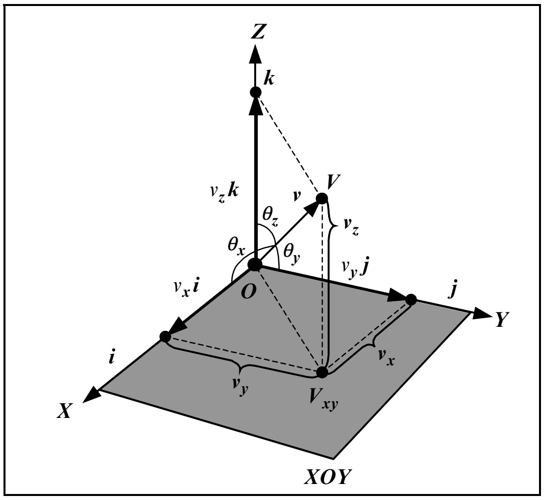

FIGURE 2.16. The direction ratios and direction cosines of a vector.

These vectors are projections of vector $\boldsymbol{v}$ onto the three base vectors. θ_x, θ_y, θ_z are the angles subtended by the vector $\boldsymbol{v}$ and each of the base vectors i, j, k, respectively.

If we compute the scalar product of vector $\boldsymbol{v}$ with each of the base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, respectively we obtain:

$$\begin{aligned} \boldsymbol{v} \odot i &= v_x \\ \boldsymbol{v} \odot j &= v_y \\ \boldsymbol{v} \odot k &= v_z \end{aligned} \tag{2.36}$$

Which are the lengths of the projection vectors $v_x\boldsymbol{i}$, $v_y\boldsymbol{j}$, $v_z\boldsymbol{k}$. Thus,

$$\begin{aligned} |v_x\boldsymbol{i}| &= v_x = |\boldsymbol{v}| \cos(\theta_x) \\ \left|v_y\boldsymbol{j}\right| &= v_y = |\boldsymbol{v}| \cos\left(\theta_y\right) \\ |v_z\boldsymbol{k}| &= v_z = |\boldsymbol{v}| \cos(\theta_z) \end{aligned} \tag{2.37}$$

Solving the above equation for the cosines of the angles we obtain:

$$\begin{aligned} \cos(\theta_x) &= \frac{v_x}{|\boldsymbol{v}|} = \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} \\ \cos\left(\theta_y\right) &= \frac{v_y}{|\boldsymbol{v}|} = \frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}} \\ \cos(\theta_z) &= \frac{v_z}{|\boldsymbol{v}|} = \frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}} \end{aligned} \tag{2.38}$$

These three quantities are known as the *direction cosines* of the vector $\boldsymbol{v}$. Thus, the direction cosines of a vector $\boldsymbol{v}$ are the components of the normalised vector $\hat{\boldsymbol{v}}$.

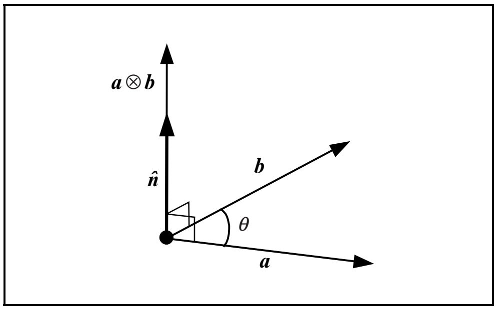

FIGURE 2.17. The vector product of two vectors.

2.16 The Vector Product of two Vectors

Let $\boldsymbol{a}, \boldsymbol{b}$ be two non-collinear vectors. Let $\hat{\boldsymbol{n}}$ be a unit vector which is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, which, when taken together with $\boldsymbol{a}$ and $\boldsymbol{b}$, forms a right-handed system $\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{n}}$ in this order. The vector $\hat{\boldsymbol{n}}$ is uniquely determined by $\boldsymbol{a}$ and $\boldsymbol{b}$, as it is the only vector that satisfies all the above constraints. Let θ be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, as shown in Fig. 2.17.

The *vector product* or *cross product* or *outer product* $\boldsymbol{a} \otimes \boldsymbol{b}$ of two vectors $\boldsymbol{a}, \boldsymbol{b}$ is defined as:

$$\boldsymbol{a} \otimes \boldsymbol{b} = (|\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta) \cdot \hat{\boldsymbol{n}} \tag{2.39}$$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear vectors, then their vector product is defined to be the zero vector. This includes the cases where one or both vectors are the zero vector.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-collinear vectors, then any vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ is collinear to $\boldsymbol{a} \otimes \boldsymbol{b}$.

From the above discussion it should be apparent that the vector product of two vectors is defined in three-dimensional space $\mathcal{E}^3$.

The following axioms and vector algebra rules apply to the vector product for all vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and all scalars α:

Existence of the vector product:
$\boldsymbol{v}_1 \otimes \boldsymbol{v}_2$ is a vector with magnitude $|\boldsymbol{v}_1|\,|\boldsymbol{v}_2| \sin(\theta)$ (A2.9)

Vector product of collinear vectors:
$$\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 = \vec{\boldsymbol{0}} \quad \Leftrightarrow \quad \boldsymbol{v}_1 || \boldsymbol{v}_2 \tag{A2.10}$$

Associative law (does not apply):
$$\boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) \neq (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 \tag{R2.10}$$

Anti-commutative law:
$$\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 = -(\boldsymbol{v}_2 \otimes \boldsymbol{v}_1) \tag{R2.11}$$

Distributive laws:
$$(\alpha \boldsymbol{v}_1) \otimes \boldsymbol{v}_2 = \boldsymbol{v}_1 \otimes (\alpha \boldsymbol{v}_2) = \alpha (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \tag{R2.12}$$
$$\boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 + \boldsymbol{v}_3) = \boldsymbol{v}_1 \otimes \boldsymbol{v}_2 + \boldsymbol{v}_1 \otimes \boldsymbol{v}_3 \tag{R2.13}$$

Given a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, then the vector products of combinations of its base vectors are given by:

$$\boldsymbol{i} \otimes \boldsymbol{i} = \boldsymbol{j} \otimes \boldsymbol{j} = \boldsymbol{k} \otimes \boldsymbol{k} = \vec{\boldsymbol{0}} \quad \text{(N.B. This result is also true for any basis)} \tag{2.40}$$

$$\boldsymbol{i} \otimes \boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j} \otimes \boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k} \otimes \boldsymbol{i} = \boldsymbol{j} \tag{2.41}$$

$$\boldsymbol{j} \otimes \boldsymbol{i} = -\boldsymbol{k}, \quad \boldsymbol{k} \otimes \boldsymbol{j} = -\boldsymbol{i}, \quad \boldsymbol{i} \otimes \boldsymbol{k} = -\boldsymbol{j} \tag{2.42}$$

2.17 The Vector Product Expressed in Terms of its Components

If $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is a right-handed basis and if $\boldsymbol{a}, \boldsymbol{b}$ are vectors with components $\left[a_x, a_y, a_z\right]$ and $\left[b_x, b_y, b_z\right]$, respectively, relative to this basis, then the vector product $\boldsymbol{a} \otimes \boldsymbol{b}$ (in terms of its components) is defined as:

$$\begin{aligned}
\boldsymbol{a} \otimes \boldsymbol{b} &= \left(a_x\boldsymbol{p} + a_y\boldsymbol{q} + a_z\boldsymbol{r}\right) \otimes \left(b_x\boldsymbol{p} + b_y\boldsymbol{q} + b_z\boldsymbol{r}\right) \\
&= a_x\boldsymbol{p} \otimes b_x\boldsymbol{p} + a_x\boldsymbol{p} \otimes b_y\boldsymbol{q} + a_x\boldsymbol{p} \otimes b_z\boldsymbol{r} \\
&\quad + a_y\boldsymbol{q} \otimes b_x\boldsymbol{p} + a_y\boldsymbol{q} \otimes b_y\boldsymbol{q} + a_y\boldsymbol{q} \otimes b_z\boldsymbol{r} \\
&\quad + a_z\boldsymbol{r} \otimes b_x\boldsymbol{p} + a_z\boldsymbol{r} \otimes b_y\boldsymbol{q} + a_z\boldsymbol{r} \otimes b_z\boldsymbol{r} \\
&= a_x b_x\boldsymbol{p} \otimes \boldsymbol{p} + a_y b_y\boldsymbol{q} \otimes \boldsymbol{q} + a_z b_z\boldsymbol{r} \otimes \boldsymbol{r} \\
&\quad + \left(a_y b_z\boldsymbol{q} \otimes \boldsymbol{r} + a_z b_y\boldsymbol{r} \otimes \boldsymbol{q}\right) + \left(a_z b_x\boldsymbol{r} \otimes \boldsymbol{p} + a_x b_z\boldsymbol{p} \otimes \boldsymbol{r}\right) \\
&\quad + \left(a_x b_y\boldsymbol{p} \otimes \boldsymbol{q} + a_y b_x\boldsymbol{q} \otimes \boldsymbol{p}\right)
\end{aligned}$$

Using the anti-commutative law (R2.11) and since $\boldsymbol{p} \otimes \boldsymbol{p} = \vec{\boldsymbol{0}}$, $\boldsymbol{q} \otimes \boldsymbol{q} = \vec{\boldsymbol{0}}$ and $\boldsymbol{r} \otimes \boldsymbol{r} = \vec{\boldsymbol{0}}$ we can simplify the above expression to:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \left(a_y b_z - a_z b_y\right) \boldsymbol{q} \otimes \boldsymbol{r} + \left(a_z b_x - a_x b_z\right) \boldsymbol{r} \otimes \boldsymbol{p} + \left(a_x b_y - a_y b_x\right) \boldsymbol{p} \otimes \boldsymbol{q} \tag{2.43}$$

The above definition of the vector product applies to any right-handed basis $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$. When dealing with a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ using Eq. (2.41) we can further simplify the above definition to:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \left(a_y b_z - a_z b_y\right) \boldsymbol{i} + \left(a_z b_x - a_x b_z\right) \boldsymbol{j} + \left(a_x b_y - a_y b_x\right) \boldsymbol{k} \tag{2.44}$$

Thus, the vector product $\boldsymbol{a} \otimes \boldsymbol{b}$ is a vector with components:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \left[\left(a_y b_z - a_z b_y\right), \left(a_z b_x - a_x b_z\right), \left(a_x b_y - a_y b_x\right)\right] \tag{2.45}$$

Alternatively:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \otimes \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \tag{2.46}$$

Finally, the vector product can be represented as the *determinant* of a *matrix* whose first column consists of the base *vectors*:

$$\begin{aligned}\boldsymbol{a}\otimes\boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & a_x & b_x \\ \boldsymbol{j} & a_y & b_y \\ \boldsymbol{k} & a_z & b_z \end{vmatrix} \\ &= \boldsymbol{i}\cdot\begin{vmatrix} a_y & b_y \\ a_z & b_z \end{vmatrix} - \boldsymbol{j}\cdot\begin{vmatrix} a_x & b_x \\ a_z & b_z \end{vmatrix} + \boldsymbol{k}\cdot\begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix} \\ &= \boldsymbol{i}\cdot\left(a_y b_z - a_z b_y\right) - \boldsymbol{j}\cdot\left(a_x b_z - a_z b_x\right) + \boldsymbol{k}\cdot\left(a_x b_y - a_y b_x\right) \\ &= \boldsymbol{i}\cdot\left(a_y b_z - a_z b_y\right) + \boldsymbol{j}\cdot\left(a_z b_x - a_x b_z\right) + \boldsymbol{k}\cdot\left(a_x b_y - a_y b_x\right) \end{aligned} \tag{2.47}$$

Such a determinant can only be evaluated symbolically, as we can not compute the value of a determinant that contains symbols. An identical result would be arrived at by transposing the rows and columns of the above determinant.

2.18 Properties of the Vector Product

2.18.1 The Geometric Interpretation of the Vector Product

Let $\boldsymbol{a}$, $\boldsymbol{b}$ be two non-collinear vectors that are represented by the directed segments $\overrightarrow{\boldsymbol{OA}}, \overrightarrow{\boldsymbol{OB}}$, respectively. Let θ be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Let $\boldsymbol{C}$ be the fourth corner of the parallelogram that has $\boldsymbol{OA}$ and $\boldsymbol{OB}$ as two adjacent sides, as shown in Fig. 2.18. Using simple trigonometry, the area of the triangle $\boldsymbol{AOB}$ is given by $\frac{1}{2}\,|\boldsymbol{a}|\cdot h = \frac{1}{2}\,|\boldsymbol{a}|\,|\boldsymbol{b}|\,|\sin\theta|$, thus the area of the parallelogram $\boldsymbol{AOBC}$ is $|\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta$. But the magnitude of the vector product $\boldsymbol{a}\otimes\boldsymbol{b}$ is $|\boldsymbol{a}\otimes\boldsymbol{b}| = ||\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta|$. Thus, the magnitude of the vector product $\boldsymbol{a}\otimes\boldsymbol{b}$ is equal to the area of the parallelogram spanned by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Extra care should be taken with the sign of angle θ, which is positive on the left diagram but negative on the right diagram of Fig. 2.18.

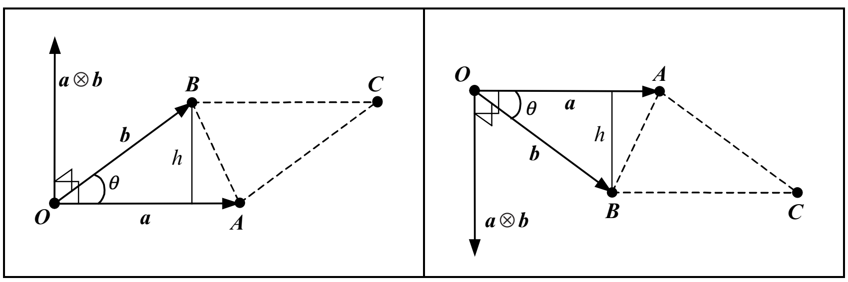

FIGURE 2.18. The magnitude of the vector product.

2.18.2 The Magnitude of the Vector Product in Terms of its Components

As we have seen in Eq. (2.45), given two non-collinear vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with components $\left[a_x, a_y, a_z\right]$ and $\left[b_x, b_y, b_z\right]$ relative to a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, the components of their vector product are:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \left[\left(a_y b_z - a_z b_y\right), \left(a_z b_x - a_x b_z\right), \left(a_x b_y - a_y b_x\right)\right]$$

Thus the magnitude of the vector product $\boldsymbol{a} \otimes \boldsymbol{b}$, in terms of the components of $\boldsymbol{a}$ and $\boldsymbol{b}$, is given by:

$$|\boldsymbol{a} \otimes \boldsymbol{b}| = \sqrt{\left(a_y b_z - a_z b_y\right)^2 + \left(a_z b_x - a_x b_z\right)^2 + \left(a_x b_y - a_y b_x\right)^2} \tag{2.48}$$

2.18.3 The Square of the Magnitude of the Vector Product

Given two non-collinear vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the square of the magnitude of their vector product $\boldsymbol{a} \otimes \boldsymbol{b}$ is given by:

$$\begin{aligned} |\boldsymbol{a} \otimes \boldsymbol{b}|^2 &= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \sin^2\theta \\ &= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \left(1 - \cos^2\theta\right) \\ &= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - \left(|\boldsymbol{a}| |\boldsymbol{b}| \cos\theta\right)^2 \\ &= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \odot \boldsymbol{b})^2 \\ \therefore \quad |\boldsymbol{a} \otimes \boldsymbol{b}|^2 &= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \odot \boldsymbol{b})^2 \end{aligned} \tag{2.49}$$

2.18.4 The Magnitude of the Sine of the Angle between Two Vectors

Given two non-zero and non-collinear vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the magnitude of their vector product $\boldsymbol{a} \otimes \boldsymbol{b}$ is given by:

$$|\boldsymbol{a} \otimes \boldsymbol{b}| = \left||\boldsymbol{a}| \, |\boldsymbol{b}\,| \sin\theta\right|$$

Solving this equation for $|\sin\theta|$ we obtain:

$$|\sin\theta| = \frac{|\boldsymbol{a} \otimes \boldsymbol{b}|}{|\boldsymbol{a}| \, |\boldsymbol{b}|} \tag{2.50}$$

If the components of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined relative to a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, then $|\sin(\theta)|$ is given by:

$$|\sin\theta| = \frac{|\boldsymbol{a} \otimes \boldsymbol{b}|}{|\boldsymbol{a}\,| \, |\boldsymbol{b}|} = \frac{\sqrt{\left(a_y b_z - a_z b_y\right)^2 + \left(a_z b_x - a_x b_z\right)^2 + \left(a_x b_y - a_y b_x\right)^2}}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}} \tag{2.51}$$

2.19 Triple Products of Vectors

Triple products of vectors combine the operations of scalar multiplication and/or vector multiplication and define products involving three or four vectors.

2.19.1 *The Triple Scalar Product*

Given any three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, the products $(\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c}$ and $\boldsymbol{a} \odot (\boldsymbol{b} \otimes \boldsymbol{c})$ (which are equal) are known as the *triple scalar products* of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in this particular order. They are denoted by $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ or $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ or $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle$ or $\det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. Thus the triple scalar product of three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is defined as:

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = (\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c} = \boldsymbol{a} \odot (\boldsymbol{b} \otimes \boldsymbol{c}) \tag{2.52}$$

From this definition we see that the triple scalar product is in reality the scalar product of two vectors (one of which is the vector product of two other vectors) and thus it produces a scalar result.

If vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ have components $\left[a_x, a_y, a_z\right]$,$\left[b_x, b_y, b_z\right]$,$\left[c_x, c_y, c_z\right]$ relative to a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, then the triple scalar product is defined in terms of these components as:

$$\begin{aligned}
(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) &= \boldsymbol{a} \odot (\boldsymbol{b} \otimes \boldsymbol{c}) \\
&= \left[a_x, a_y, a_z\right] \odot \left(\left[b_x, b_y, b_z\right] \otimes \left[c_x, c_y, c_z\right]\right) \\
&= \left[a_x, a_y, a_z\right] \odot \left[\left(b_y c_z - b_z c_y\right), \left(b_z c_x - b_x c_z\right), \left(b_x c_y - b_y c_x\right)\right] \\
\therefore \quad (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) &= a_x \left(b_y c_z - b_z c_y\right) + a_y \left(b_z c_x - b_x c_z\right) + a_z \left(b_x c_y - b_y c_x\right)
\end{aligned} \tag{2.53}$$

Alternatively we can define the triple scalar product using the determinant of a matrix whose rows or columns are the components of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$:

$$\begin{aligned}
(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) &= \boldsymbol{a} \odot (\boldsymbol{b} \otimes \boldsymbol{c}) \\
&= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \\
&= a_x \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - a_y \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + a_z \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \\
&= a_x \left(b_y c_z - b_z c_y\right) - a_y \left(b_x c_z - b_z c_x\right) + a_z \left(b_x c_y - b_y c_x\right) \\
\therefore \quad (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) &= a_x \left(b_y c_z - b_z c_y\right) + a_y \left(b_z c_x - b_x c_z\right) + a_z \left(b_x c_y - b_y c_x\right)
\end{aligned} \tag{2.54}$$

The reason why the triple scalar product is of interest is because of its geometric interpretation. The triple scalar product $(\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c}$ is equal in magnitude to the volume of the parallelepiped having vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as concurrent sides.

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be three non-coplanar vectors, which are represented by the directed segments $\overrightarrow{\boldsymbol{OA}}, \overrightarrow{\boldsymbol{OB}}, \overrightarrow{\boldsymbol{OC}}$, respectively. Let $\boldsymbol{AOBD}$ be the parallelogram spanning

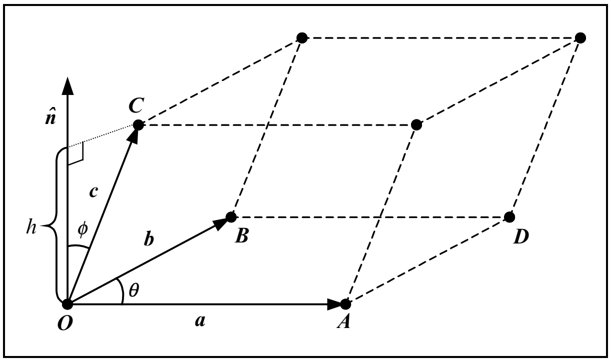

FIGURE 2.19. The parallelepiped whose volume is defined by the triple scalar product.

the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, as shown in Fig. 2.19. Let $\hat{\boldsymbol{n}}$ be the unit vector normal to the plane defined by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e.

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{a} \otimes \boldsymbol{b}}{|\boldsymbol{a} \otimes \boldsymbol{b}|}$$

Finally, let θ be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and let ϕ be the angle between vectors $\hat{\boldsymbol{n}}$ and $\boldsymbol{c}$.

By definition the vector product $\boldsymbol{a} \otimes \boldsymbol{b}$ is given by:

$$\boldsymbol{a} \otimes \boldsymbol{b} = (|\boldsymbol{a}|\, |\boldsymbol{b}| \sin\theta) \cdot \hat{\boldsymbol{n}}$$

As we have seen in Subsection 2.18.1, the area α of the parallelogram of the base is given by:

$$\alpha = ||\boldsymbol{a}|\, |\boldsymbol{b}| \sin\theta|$$

The magnitude of the vector product can now be rewritten as:

$$|\boldsymbol{a} \otimes \boldsymbol{b}| = \left|\alpha \cdot \hat{\boldsymbol{n}}\right|$$

and the magnitude of the triple scalar product $(\boldsymbol{a} \otimes \boldsymbol{b}) \odot c$ can be rewritten as:

$$|(\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c}| = \left|\left(\alpha \cdot \hat{\boldsymbol{n}}\right) \odot \boldsymbol{c}\right| = \alpha \cdot \left|\left(\hat{\boldsymbol{n}} \odot \boldsymbol{c}\right)\right| = \alpha \cdot \left|\hat{\boldsymbol{n}}\right| \cdot |\boldsymbol{c}| \cdot |\cos\phi| = \alpha \cdot h = \nu$$

Where $h = |c| \cdot |\cos\phi|$ represents the height of the parallelepiped and $\nu = \alpha \cdot h$ represents the volume of the parallelepiped.

Looking at the triple scalar product $(\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c}$ from this geometric point of view has a number of useful consequences. If any two of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are collinear, then the parallelepiped collapses to a plane and has zero volume, i.e. $(\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c} = 0$. Similarly, if any of the vectors is the zero vector or if all the vectors are coplanar, then the parallelepiped collapses and has zero volume.

Another important application of the triple scalar product is that it allows us to determine the handedness of a basis. Suppose that vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ form a basis.

This implies that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \neq 0$. If $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) > 0$, then this basis is *right-handed*. Alternatively, if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) < 0$, then it is *left-handed*. Thus, the basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is right-handed, while the basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is left-handed.

The following axioms and vector algebra rules apply to the triple scalar product for all vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$:

Existence of the triple scalar product:

$$(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \text{ is a scalar} \tag{A2.11}$$

Triple scalar product of coplanar vectors:

$$if\ all\ \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\ are\ coplanar\ \Rightarrow (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0 \tag{A2.12}$$

Triple scalar product of zero vectors:

$$\boldsymbol{v}_1 = \vec{\boldsymbol{0}} \vee \boldsymbol{v}_2 = \vec{\boldsymbol{0}} \vee \boldsymbol{v}_3 = \vec{\boldsymbol{0}} \Rightarrow (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0 \tag{A2.13}$$

Triple scalar product of collinear vectors:

$$\boldsymbol{v}_1 || \boldsymbol{v}_2 \vee \boldsymbol{v}_2 || \boldsymbol{v}_3 \vee \boldsymbol{v}_3 || \boldsymbol{v}_1 \Rightarrow (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0 \tag{A2.14}$$

Cyclic permutation rule:

$$(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = (\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_1) = (\boldsymbol{v}_3, \boldsymbol{v}_1, \boldsymbol{v}_2) \tag{R2.14}$$

i.e., $\boldsymbol{v}_1 \odot (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) = \boldsymbol{v}_2 \odot (\boldsymbol{v}_3 \otimes \boldsymbol{v}_1) = \boldsymbol{v}_3 \odot (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2)$

and $(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \odot \boldsymbol{v}_3 = (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) \odot \boldsymbol{v}_1 = (\boldsymbol{v}_3 \otimes \boldsymbol{v}_1) \odot \boldsymbol{v}_2$

Non-cyclic permutation rule:

$$(\boldsymbol{v}_2, \boldsymbol{v}_1, \boldsymbol{v}_3) = (\boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_1) = (\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_2) = -(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \tag{R2.15}$$

2.19.2 *The Triple Vector Product*

Given any three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, the products $(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes \boldsymbol{c}$ and $\boldsymbol{a} \otimes (\boldsymbol{b} \otimes \boldsymbol{c})$ (which are not equal in general) are known as the *triple vector products* of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in this particular order. If vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ have components $[a_x, a_y, a_z]$, $[b_x, b_y, b_z]$, $[c_x, c_y, c_z]$ relative to a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, then the triple vector product is defined in terms of these components as:

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes \boldsymbol{c} &= \left(\left[a_x, a_y, a_z\right] \otimes \left[b_x, b_y, b_z\right]\right) \otimes \left[c_x, c_y, c_z\right] \\ &= \left[\left(a_y b_z - a_z b_y\right), \left(a_z b_x - a_x b_z\right), \left(a_x b_y - a_y b_x\right)\right] \otimes \left[c_x, c_y, c_z\right]\end{aligned}$$

$$\therefore\ (\boldsymbol{a} \otimes \boldsymbol{b}) \otimes \boldsymbol{c} = \begin{bmatrix} (a_z b_x - a_x b_z) c_z - \left(a_x b_y - a_y b_x\right) c_y \\ \left(a_x b_y - a_y b_x\right) c_x - \left(a_y b_z - a_z b_y\right) c_z \\ \left(a_y b_z - a_z b_y\right) c_y - (a_z b_x - a_x b_z) c_x \end{bmatrix}^T \tag{2.55}$$

Expanding the first component of this vector we get:

$$\begin{aligned}(a_z b_x - a_x b_z) c_z - \left(a_x b_y - a_y b_x\right) c_y &= a_z c_z b_x - b_z c_z a_x - b_y c_y a_x + a_y c_y b_x \\ &= a_z c_z b_x - b_z c_z a_x - b_y c_y a_x + a_y c_y b_x \\ &\quad + a_x c_x b_x - b_x c_x a_x \\ &= \left(a_x c_x + a_y c_y + a_z c_z\right) b_x \\ &\quad - \left(b_x c_x + b_y c_y + b_z c_z\right) a_x \\ &= (\boldsymbol{a} \odot \boldsymbol{c}) b_x - (\boldsymbol{b} \odot \boldsymbol{c}) a_x\end{aligned}$$

Expanding its second component we get:

$$\begin{aligned}\left(a_x b_y - a_y b_x\right) c_x - \left(a_y b_z - a_z b_y\right) c_z &= a_x c_x b_y - b_x c_x a_y - b_z c_z a_y + a_z c_z b_y \\ &= a_x c_x b_y - b_x c_x a_y - b_z c_z a_y + a_z c_z b_y \\ &\quad + a_y c_y b_y - b_y c_y a_y \\ &= \left(a_x c_x + a_y c_y + a_z c_z\right) b_y \\ &\quad - \left(b_x c_x + b_y c_y + b_z c_z\right) a_y \\ &= (\boldsymbol{a} \odot \boldsymbol{c})\, b_y - (\boldsymbol{b} \odot \boldsymbol{c})\, a_y\end{aligned}$$

Expanding its third component we get:

$$\begin{aligned}\left(a_y b_z - a_z b_y\right) c_y - (a_z b_x - a_x b_z)\, c_x &= a_y c_y b_z - b_y c_y a_z - b_x c_x a_z + a_x c_x b_z \\ &= a_y c_y b_z - b_y c_y a_z - b_x c_x a_z + a_x c_x b_z \\ &\quad + a_z c_z b_z - b_z c_z a_z \\ &= \left(a_x c_x + a_y c_y + a_z c_z\right) b_z \\ &\quad - \left(b_x c_x + b_y c_y + b_z c_z\right) a_z \\ &= (\boldsymbol{a} \odot \boldsymbol{c})\, b_z - (\boldsymbol{b} \odot \boldsymbol{c})\, a_z\end{aligned}$$

Thus:

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes \boldsymbol{c} &= [((\boldsymbol{a} \odot \boldsymbol{c})b_x - (\boldsymbol{b} \odot \boldsymbol{c})a_x)\,, \\ &\qquad \left((\boldsymbol{a} \odot \boldsymbol{c}) \odot b_y - (\boldsymbol{b} \odot \boldsymbol{c})a_y\right), ((\boldsymbol{a} \odot \boldsymbol{c})b_z - (\boldsymbol{b} \odot \boldsymbol{c})a_z)] \\ \therefore\quad (\boldsymbol{a} \otimes \boldsymbol{b}) \otimes \boldsymbol{c} &= (\boldsymbol{a} \odot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \odot \boldsymbol{c})\boldsymbol{a} \qquad (2.56)\end{aligned}$$

Similarly we can prove that:

$$\boldsymbol{a} \otimes (\boldsymbol{b} \otimes \boldsymbol{c}) = (\boldsymbol{a} \odot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \odot \boldsymbol{b})\boldsymbol{c}$$

The following axioms and vector algebra rules apply to the triple vector product for all vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$:

Existence of the triple vector product:
$(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3$ is a vector (A2.15)
Associative law (does not apply):
$(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 \neq \boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3)$ (R2.16)
Permutation rule:
$(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 = \boldsymbol{v}_3 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_1)$ (R2.17)
Expansion rules:
$(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 = (\boldsymbol{v}_1 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_2 - (\boldsymbol{v}_2 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_1$ (R2.18)
$\boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) = (\boldsymbol{v}_1 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_2 - (\boldsymbol{v}_1 \odot \boldsymbol{v}_2)\, \boldsymbol{v}_3$ (R2.19)

2.19.3 The Scalar Product of Two Vector Products

Given any four vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, the product $(\boldsymbol{a} \otimes \boldsymbol{b}) \odot (\boldsymbol{c} \otimes \boldsymbol{d})$ is known as the *scalar product of two vector products* or the *scalar product of four vectors*. Such a product results in a scalar.

First we use the cyclic permutation rule (R2.14) to rewrite this product:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \odot (\boldsymbol{c} \otimes \boldsymbol{d}) = \boldsymbol{c} \odot (\boldsymbol{d} \otimes (\boldsymbol{a} \otimes \boldsymbol{b}))$$

Where for the cyclic permutation $\boldsymbol{v}_1 \odot (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) = \boldsymbol{v}_2 \odot (\boldsymbol{v}_3 \otimes \boldsymbol{v}_1)$ we used mappings $(\boldsymbol{a} \otimes \boldsymbol{b}) \to \boldsymbol{v}_1, \boldsymbol{c} \to \boldsymbol{v}_2, \boldsymbol{d} \to \boldsymbol{v}_3$.

Next we apply the expansion rule (R2.19) to rewrite this result:

$$c \odot (\boldsymbol{d} \otimes (\boldsymbol{a} \otimes \boldsymbol{b})) = \boldsymbol{c} \odot (\boldsymbol{a}(\boldsymbol{b} \odot d) - b(\boldsymbol{d} \odot \boldsymbol{a})) = (\boldsymbol{c} \odot \boldsymbol{a})(\boldsymbol{b} \odot \boldsymbol{d}) - (\boldsymbol{c} \odot \boldsymbol{b})(\boldsymbol{d} \odot \boldsymbol{a})$$

Where for the expansion $\boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) = (\boldsymbol{v}_1 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_2 - (\boldsymbol{v}_1 \odot \boldsymbol{v}_2)\, \boldsymbol{v}_3$, we used mappings $\boldsymbol{d} \to \boldsymbol{v}_1, \boldsymbol{a} \to \boldsymbol{v}_2, \boldsymbol{b} \to \boldsymbol{v}_3$.

$$\therefore\ (\boldsymbol{a} \otimes \boldsymbol{b}) \odot (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{c} \odot \boldsymbol{a})(\boldsymbol{b} \odot \boldsymbol{d}) - (\boldsymbol{c} \odot \boldsymbol{b})(\boldsymbol{d} \odot \boldsymbol{a}) \tag{2.57}$$

Using this result we can calculate the square of the vector product as:

$$(\boldsymbol{a} \otimes \boldsymbol{b})^2 = (\boldsymbol{a} \odot \boldsymbol{a})(\boldsymbol{b} \odot \boldsymbol{b}) - (\boldsymbol{a} \odot \boldsymbol{b})(\boldsymbol{a} \odot \boldsymbol{b}) = |\boldsymbol{a}|^2\, |\boldsymbol{b}|^2 - (\boldsymbol{a} \odot \boldsymbol{b})^2 \tag{2.58}$$

2.19.4 The Vector Product of Two Vector Products

Given any four vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, the product $(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes (\boldsymbol{c} \otimes \boldsymbol{d})$ is known as the *vector product of two vector products* or the *vector product of four vectors*. Such a product results in a vector.

Using the expansion rule (R2.18) we can rewrite this product as:

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes (\boldsymbol{c} \otimes \boldsymbol{d}) &= (\boldsymbol{a} \odot (\boldsymbol{c} \otimes \boldsymbol{d}))\, \boldsymbol{b} - (\boldsymbol{b} \odot (\boldsymbol{c} \otimes \boldsymbol{d}))\, a \\ &= (\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d})\, \boldsymbol{b} - (\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})\, \boldsymbol{a}\end{aligned} \tag{2.59}$$

Where for the expansion $(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 = (\boldsymbol{v}_1 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_2 - (\boldsymbol{v}_2 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_1$, we used mappings $\boldsymbol{a} \to \boldsymbol{v}_1, \boldsymbol{b} \to \boldsymbol{v}_2, (\boldsymbol{c} \otimes \boldsymbol{d}) \to \boldsymbol{v}_3$.

Similarly, using the expansion rule (R2.19) we can rewrite this product as:

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes (\boldsymbol{c} \otimes \boldsymbol{d}) &= ((\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{d})\, \boldsymbol{c} - ((\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c})\, \boldsymbol{d} \\ &= (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})\, \boldsymbol{c} - (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})\, \boldsymbol{d}\end{aligned} \tag{2.60}$$

Where for the expansion $\boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) = (\boldsymbol{v}_1 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_2 - (\boldsymbol{v}_1 \odot \boldsymbol{v}_2)\, \boldsymbol{v}_3$, we used mappings $(\boldsymbol{a} \otimes \boldsymbol{b}) \to \boldsymbol{v}_1, \boldsymbol{c} \to \boldsymbol{v}_2, \boldsymbol{d} \to \boldsymbol{v}_3$.

Thus, the vector product of two vector products is defined as:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d})\, \boldsymbol{b} - (\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})\, \boldsymbol{a} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})\, \boldsymbol{c} - (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})\, \boldsymbol{d} \tag{2.61}$$

2.20 The Components of a Vector Relative to a Non-orthogonal Basis

Let $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}$ be three non-coplanar and non-zero vectors that form a right-handed basis. Let the directed segments $\overrightarrow{\boldsymbol{OS}}, \overrightarrow{\boldsymbol{OT}}, \overrightarrow{\boldsymbol{OU}}$ represent the vectors $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}$, respectively, as shown in Fig. 2.20. Let $\boldsymbol{v}$ be any vector and let the directed segment

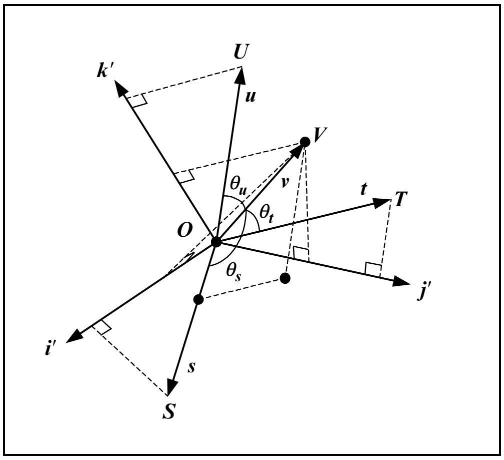

FIGURE 2.20. The non-orthonormal right-handed $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ basis.

$\overrightarrow{OV}$ represent vector $\boldsymbol{v}$. The components of the $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$, $\boldsymbol{v}$ vectors with respect to a Cartesian basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are given by $[s_x, s_y, s_z]$,$[t_x, t_y, t_z]$,$[u_x, u_y, u_z]$,$[v_x, v_y, v_z]$, respectively.

Suppose we wish to express the components of $\boldsymbol{v}$ with respect to the right-handed basis $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$. That is, we wish to determine the triple of components $[v_s, v_t, v_u]$.

Had the **s, t, u** basis been a Cartesian basis like the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ basis, this would be simple. As we have seen in Section 2.14.3, we would project $\boldsymbol{v}$ onto each of the $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ base vectors of this basis. The magnitudes of these projection vectors would be equal to the components v_s, v_t, v_u. Thus, when $\boldsymbol{s} = \boldsymbol{i}$, $\boldsymbol{t} = \boldsymbol{j}$ and $\boldsymbol{u} = \boldsymbol{k}$ these components are given by:

$$\begin{aligned} v_s &= |v| \cdot \cos(\theta_s) = \boldsymbol{s} \odot \boldsymbol{v} \\ v_t &= |v| \cdot \cos(\theta_t) = \boldsymbol{t} \odot \boldsymbol{v} \\ v_u &= |v| \cdot \cos(\theta_u) = \boldsymbol{u} \odot \boldsymbol{v} \end{aligned}$$

where $\theta_s, \theta_t, \theta_u$ are the angles subtended by $\boldsymbol{v}$ and $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$, respectively.

The above result can be generalised as follows:

$$v_s = \frac{the\ magnitude\ of\ the\ projection\ of\ \mathbf{v}\ onto\ \mathbf{i}}{the\ magnitude\ of\ the\ projection\ of\ \mathbf{s}\ onto\ \mathbf{i}} = \frac{\boldsymbol{i} \odot \boldsymbol{v}}{\boldsymbol{i} \odot \boldsymbol{s}}$$

$$v_t = \frac{the\ magnitude\ of\ the\ projection\ of\ \boldsymbol{v}\ onto\ \boldsymbol{j}}{the\ magnitude\ of\ the\ projection\ of\ \boldsymbol{t}\ onto\ \boldsymbol{j}} = \frac{\boldsymbol{j} \odot \boldsymbol{v}}{\boldsymbol{j} \odot \boldsymbol{t}}$$

$$v_u = \frac{the\ magnitude\ of\ the\ projection\ of\ \boldsymbol{v}\ onto\ k}{the\ magnitude\ of\ the\ projection\ of\ \boldsymbol{u}\ onto\ \boldsymbol{k}} = \frac{\boldsymbol{k} \odot \boldsymbol{v}}{\boldsymbol{k} \odot \boldsymbol{u}}$$

Which when $\boldsymbol{s} = \boldsymbol{i}$, $\boldsymbol{t} = \boldsymbol{j}$ and $\boldsymbol{u} = \boldsymbol{k}$ reduces to:

$$
\begin{aligned}
v_s &= \frac{\boldsymbol{i} \odot \boldsymbol{v}}{\boldsymbol{i} \odot \boldsymbol{i}} = \frac{\boldsymbol{s} \odot \boldsymbol{v}}{1} = \boldsymbol{s} \odot \boldsymbol{v} \\
v_t &= \frac{\boldsymbol{j} \odot \boldsymbol{v}}{\boldsymbol{j} \odot \boldsymbol{j}} = \frac{\boldsymbol{t} \odot \boldsymbol{v}}{1} = \boldsymbol{t} \odot \boldsymbol{v} \\
v_u &= \frac{\boldsymbol{k} \odot \boldsymbol{v}}{\boldsymbol{k} \odot \boldsymbol{k}} = \frac{\boldsymbol{u} \odot \boldsymbol{v}}{1} = \boldsymbol{u} \odot \boldsymbol{v}
\end{aligned}
$$

In the general case however $\boldsymbol{s} \neq \boldsymbol{i}$, $\boldsymbol{t} \neq \boldsymbol{j}$ and $\boldsymbol{u} \neq \boldsymbol{k}$, we must compute a new set of vectors $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ that correspond to base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. As vector $\boldsymbol{i}$ is normal to the plane defined by the other two vectors $\boldsymbol{j}$, $\boldsymbol{k}$ (of the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ basis), so $\boldsymbol{i}'$ must be normal to the plane defined by vectors $\boldsymbol{t}$, $\boldsymbol{u}$ (of the $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}$ basis). Similarly $\boldsymbol{j}'$ must be normal to the plane defined by $\boldsymbol{u}, \boldsymbol{s}$, and $\boldsymbol{k}'$ must be normal to the plane defined by $\boldsymbol{s}$, $\mathbf{t}$.

Thus:

$$
\begin{aligned}
\boldsymbol{i}' &= \boldsymbol{t} \otimes \boldsymbol{u} \\
\boldsymbol{j}' &= \boldsymbol{u} \otimes \boldsymbol{s} \\
\boldsymbol{k}' &= \boldsymbol{s} \otimes \boldsymbol{t}
\end{aligned}
$$

Using this new set of vectors $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ we can compute the components of vector $\boldsymbol{v}$ with respect to the right-handed basis $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}$:

$$
\begin{aligned}
v_s &= \frac{(\boldsymbol{t} \otimes \boldsymbol{u}) \odot \boldsymbol{v}}{(\boldsymbol{t} \otimes \boldsymbol{u}) \odot \boldsymbol{s}} = \frac{(\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v})}{(t, u, s)} \\
v_t &= \frac{(\boldsymbol{u} \otimes \boldsymbol{s}) \odot \boldsymbol{v}}{(\boldsymbol{u} \otimes \boldsymbol{s}) \odot \boldsymbol{t}} = \frac{(\boldsymbol{u}, \boldsymbol{s}, \boldsymbol{v})}{(\boldsymbol{u}, \boldsymbol{s}, \boldsymbol{t})} \\
v_u &= \frac{(\boldsymbol{s} \otimes \boldsymbol{t}) \odot \boldsymbol{v}}{(\boldsymbol{s} \otimes \boldsymbol{t}) \odot \boldsymbol{u}} = \frac{(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{v})}{(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u})}
\end{aligned}
\tag{2.62}
$$

Expanding the triple scalar products we obtain:

$$
\begin{aligned}
v_s &= \frac{(\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v})}{(\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{s})} = \frac{\left(t_y u_z - t_z u_y\right) v_x + \left(t_z u_x - t_x u_z\right) v_y + \left(t_x u_y - t_y u_x\right) v_z}{\left(t_y u_z - t_z u_y\right) s_x + \left(t_z u_x - t_x u_z\right) s_y + \left(t_x u_y - t_y u_x\right) s_z} \\
v_t &= \frac{(\boldsymbol{u}, \boldsymbol{s}, \boldsymbol{v})}{(\boldsymbol{u}, \boldsymbol{s}, \boldsymbol{t})} = \frac{\left(u_y s_z - u_z s_y\right) v_x + \left(u_z s_x - u_x s_z\right) v_y + \left(u_x s_y - u_y s_x\right) v_z}{\left(u_y s_z - u_z s_y\right) t_x + \left(u_z s_x - u_x s_z\right) t_y + \left(u_x s_y - u_y s_x\right) t_z} \\
v_u &= \frac{(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{v})}{(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u})} = \frac{\left(s_y t_z - s_z t_y\right) v_x + \left(s_z t_x - s_x t_z\right) v_y + \left(s_x t_y - s_y t_x\right) v_z}{\left(s_y t_z - s_z t_y\right) u_x + \left(s_z t_x - s_x t_z\right) u_y + \left(s_x t_y - s_y t_x\right) u_z}
\end{aligned}
\tag{2.63}
$$

The above discussion was useful in illuminating the geometric aspects of our problem. It illustrated how geometric arguments are helpful in reasoning and solving problems in vector algebra, but it did not provide a conclusive mathematical proof of the correctness of our arguments. At some stage of our discussion we claimed that a *result could be generalised* without offering any evidence to back our claim. We will do so in the next section.

2.21 The Decomposition of a Vector According to a Basis

In this section we revisit the problem dealt with in the previous section, but we do so in a more mathematically rigorous way and provide a vector algebraic proof of our argument.

We start by restating the problem. Given four non-coplanar and non-zero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ decompose vector $\boldsymbol{d}$ according to a basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, i.e. express the components of $\boldsymbol{d}$ with respect to a right-handed basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ (which is not required to be orthonormal or even orthogonal). From Section 2.7 we know that this is possible. Indeed, given any four non-coplanar and non-zero vectors in $\mathcal{E}^3$ we can express one of the vectors as a linear combination of the remaining three vectors.

By combining Eqs. (2.59) and (2.60) we obtain:

$$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\,\boldsymbol{c}-(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\,\boldsymbol{d}=(\boldsymbol{a},\boldsymbol{c},\boldsymbol{d})\,\boldsymbol{b}-(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\,\boldsymbol{a}$$

$$\therefore\;(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\,\boldsymbol{a}-(\boldsymbol{a},\boldsymbol{c},\boldsymbol{d})\,\boldsymbol{b}+(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\,\boldsymbol{c}-(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\,\boldsymbol{d}=\overrightarrow{\mathbf{0}}$$

Using the non-cyclic permutation rule (R2.15) we obtain:

$$(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\,\boldsymbol{a}+(\boldsymbol{a},\boldsymbol{d},\boldsymbol{c})\,\boldsymbol{b}+(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\,\boldsymbol{c}-(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\,\boldsymbol{d}=\overrightarrow{\mathbf{0}}$$

But given that any four vectors in $\mathcal{E}^3$ are always linearly dependent (i.e. we can decompose one in terms of the remaining three), we can solve the above equation for $\boldsymbol{d}$:

$$(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\,\boldsymbol{a}+(\boldsymbol{a},\boldsymbol{d},\boldsymbol{c})\,\boldsymbol{b}+(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\,\boldsymbol{c}=(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\,\boldsymbol{d}$$

$$\therefore\;\boldsymbol{d}=\frac{(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\,\boldsymbol{a}+(\boldsymbol{a},\boldsymbol{d},\boldsymbol{c})\,\boldsymbol{b}+(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})\,\boldsymbol{c}}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}=\frac{(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}\boldsymbol{a}+\frac{(\boldsymbol{a},\boldsymbol{d},\boldsymbol{c})}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}\boldsymbol{b}+\frac{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}\boldsymbol{c}$$

Expressing $\boldsymbol{d}$ in component form we obtain:

$$\boldsymbol{d}=\left[\frac{(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})},\frac{(\boldsymbol{a},\boldsymbol{d},\boldsymbol{c})}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})},\frac{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}\right]$$

Using the cyclic permutation rule (R2.14) we obtain:

$$\boldsymbol{d}=\left[\frac{(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}{(\boldsymbol{b},\boldsymbol{c},\boldsymbol{a})},\frac{(\boldsymbol{c},\boldsymbol{a},\boldsymbol{d})}{(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b})},\frac{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d})}{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})}\right]$$

Finally, expanding the triple scalar products we obtain:

$$\boldsymbol{d}=\left[\frac{(\boldsymbol{b}\otimes\boldsymbol{c})\odot\boldsymbol{d}}{(\boldsymbol{b}\otimes\boldsymbol{c})\odot\boldsymbol{a}},\frac{(\boldsymbol{c}\otimes\boldsymbol{a})\odot\boldsymbol{d}}{(\boldsymbol{c}\otimes\boldsymbol{a})\odot\boldsymbol{b}},\frac{(\boldsymbol{a}\otimes\boldsymbol{b})\odot\boldsymbol{d}}{(\boldsymbol{a}\otimes\boldsymbol{b})\odot\boldsymbol{c}}\right]$$

The individual components of $\boldsymbol{d}$ are given as:

$$
\begin{aligned}
d_a &= \frac{(\boldsymbol{b} \otimes \boldsymbol{c}) \odot \boldsymbol{d}}{(\boldsymbol{b} \otimes \boldsymbol{c}) \odot \boldsymbol{a}} = \frac{(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})}{(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a})} \\
d_b &= \frac{(\boldsymbol{c} \otimes \boldsymbol{a}) \odot \boldsymbol{d}}{(\boldsymbol{c} \otimes \boldsymbol{a}) \odot \boldsymbol{b}} = \frac{(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{d})}{(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})} \\
d_c &= \frac{(\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{d}}{(\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c}} = \frac{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})}{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})}
\end{aligned}
\tag{2.64}
$$

Which accords with the result we obtained in Eq. (2.62). QED

If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ have components $[a_x, a_y, a_z]$,$[b_x, b_y, b_z]$,$[c_x, c_y, c_z]$,$[d_x, d_y, d_z]$, respectively, then by expanding the triple scalar products we obtain:

$$
\begin{aligned}
d_a &= \frac{(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})}{(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a})} = \frac{\left(b_y c_z - b_z c_y\right) d_x + (b_z c_x - b_x c_z)\, d_y + \left(b_x c_y - b_y c_x\right) d_z}{\left(b_y c_z - b_z c_y\right) a_x + (b_z c_x - b_x c_z)\, a_y + \left(b_x c_y - b_y c_x\right) a_z} \\
d_b &= \frac{(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{d})}{(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})} = \frac{\left(c_y a_z - c_z a_y\right) d_x + (c_z a_x - c_x a_z)\, d_y + \left(c_x a_y - c_y a_x\right) d_z}{\left(c_y a_z - c_z a_y\right) b_x + (c_z a_x - c_x a_z)\, b_y + \left(c_x a_y - c_y a_x\right) b_z} \\
d_c &= \frac{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})}{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})} = \frac{\left(a_y b_z - a_z b_y\right) d_x + (a_z b_x - a_x b_z)\, d_y + \left(a_x b_y - a_y b_x\right) d_z}{\left(a_y b_z - a_z b_y\right) c_x + (a_z b_x - a_x b_z)\, c_y + \left(a_x b_y - a_y b_x\right) c_z}
\end{aligned}
\tag{2.65}
$$

2.22 The Vector Equation of the Line Revisited

2.22.1 The Line Defined by Two Position Vectors

In Section 2.5 we have seen that the vector equation of a line $\boldsymbol{\Lambda}$ passing through two points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ with position vectors $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ relative to some origin $\boldsymbol{O}$ is given by:

$$\boldsymbol{p} = (1 - t) \cdot \boldsymbol{p}_1 + t \cdot \boldsymbol{p}_2 \tag{2.66}$$

$$\text{or} \quad \boldsymbol{p} = \boldsymbol{p}_1 + t \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_1) \tag{2.67}$$

where t is a scalar parameter and $\boldsymbol{p}$ is the position vector of the general point $\boldsymbol{P}$ on the line, as shown in Fig. 2.21.

If points $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}$ have coordinates $[x_1, y_1, z_1]$,$[x_2, y_2, z_2]$,$[x, y, z]$, respectively, then the line equation can be rewritten as:

$$
\begin{aligned}
x &= (1 - t) \cdot x_1 + t \cdot x_2 \\
y &= (1 - t) \cdot y_1 + t \cdot y_2 \\
z &= (1 - t) \cdot z_1 + t \cdot z_2
\end{aligned}
\tag{2.68}
$$

$$
\text{or} \quad
\begin{aligned}
x &= x_1 + t \cdot (x_2 - x_1) \\
y &= y_1 + t \cdot (y_2 - y_1) \\
z &= z_1 + t \cdot (z_2 - z_1)
\end{aligned}
\tag{2.69}
$$

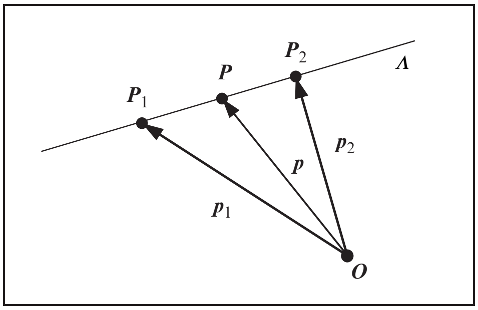

FIGURE 2.21. A line defined by two position vectors.

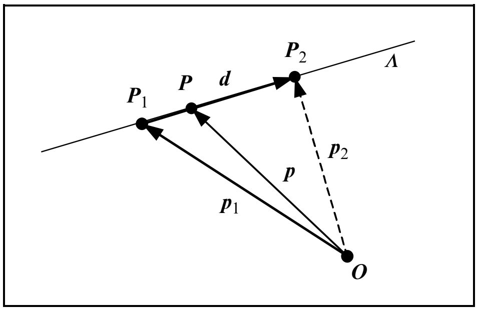

FIGURE 2.22. A line defined by a position vector and a direction vector.

2.22.2 *The Line Defined by a Position Vector and Direction Vector*

An alternative but equivalent representation of the line $\boldsymbol{\Lambda}$ is derived as follows. Let $\boldsymbol{\Lambda}$ be a line that passes through a point $\boldsymbol{P}_1$ (having position vector $\boldsymbol{p}_1$) in the direction of a direction unit vector $\boldsymbol{d}$ represented by the directed segment $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}$ in Fig. 2.22.

Then the line equation can be written as:

$$\boldsymbol{p} = \boldsymbol{p}_1 + t \cdot \boldsymbol{d} \tag{2.70}$$

where, as above, t is a scalar parameter and $\boldsymbol{p}$ is the position vector of the general point $\boldsymbol{P}$ on the line. This equation is reminiscent of the Eq. (2.67). This is not surprising as $\boldsymbol{d} = (\boldsymbol{p}_2 - \boldsymbol{p}_1)$.

If points $\boldsymbol{P}_1, \boldsymbol{P}$ have coordinates $[x_1, y_1, z_1]$, $[x, y, z]$, respectively and vector $\boldsymbol{d}$ has components $\left[d_x, d_y, d_z\right]$, then the line equation can be rewritten as:

$$\begin{aligned} x &= x_1 + t \cdot d_x \\ y &= y_1 + t \cdot d_y \\ z &= z_1 + t \cdot d_z \end{aligned} \tag{2.71}$$

Which is reminiscent of Eq. (2.69).

Sometimes it is convenient to rewrite Eq. (2.71) in what is known as the *standard form* of the line equation:

$$\frac{(x - x_1)}{d_x} = \frac{(y - y_1)}{d_y} = \frac{(z - z_1)}{d_z} = t \tag{2.72}$$

Here it is understood that if a denominator vanishes, then so does the corresponding numerator. Thus, if $d_x = 0$, then the above equation becomes:

$$x = x_1, \frac{(y - y_1)}{d_y} = \frac{(z - z_1)}{d_z} = t$$

Since the direction cosines of the direction vector $\boldsymbol{d}$ are $d_x = x_2 - x_1$, $d_y = y_2 - y_1$ and $d_z = z_2 - z_1$, respectively, an alternative form of Eq. (2.72) is:

$$\frac{(x - x_1)}{(x_2 - x_1)} = \frac{(y - y_1)}{(y_2 - y_1)} = \frac{(z - z_1)}{(z_2 - z_1)} = t \tag{2.73}$$

Here again it is understood that if a denominator vanishes, then so does the corresponding numerator.

2.23 The Vector Equation of the Plane

2.23.1 The Plane Defined by a Position Vector and a Normal Vector

To derive a vector equation of the plane we use the fact that any line defined by any two points on a plane $\boldsymbol{\Pi}$ is perpendicular to the vector normal to the plane, as seen in Fig. 2.23.

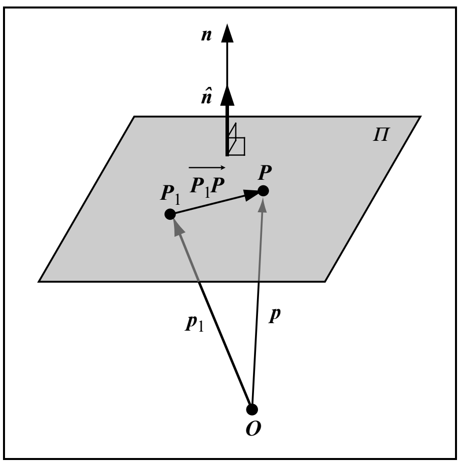

FIGURE 2.23. A plane defined by a position vector and a normal direction vector.

Let $\boldsymbol{P}_1$ be a point on the plane with position vector $\boldsymbol{p}_1$ and let $\boldsymbol{P}$ be the general point on the plane with position vector $\boldsymbol{p}$. Let $\boldsymbol{n}$ be a vector that is normal to the plane and let $\hat{\boldsymbol{n}}$ be the unit normal of the plane. Let vectors $\boldsymbol{p}_1$, $\boldsymbol{p}$, $\boldsymbol{n}$ have components $[x_1, y_1, z_1]$,$[x, y, z]$,$[a, b, c]$, respectively. The directed segment $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}}$ represents vector $(\boldsymbol{p} - \boldsymbol{p}_1)$, which is perpendicular to $\boldsymbol{n}$. Thus:

$$(\boldsymbol{p} - \boldsymbol{p}_1) \odot \boldsymbol{n} = 0 \tag{2.74}$$

$$\therefore\ \boldsymbol{p} \odot \boldsymbol{n} - \boldsymbol{p}_1 \odot \boldsymbol{n} = 0$$

$$\therefore\ \boldsymbol{p} \odot \boldsymbol{n} = \boldsymbol{p}_1 \odot \boldsymbol{n} \tag{2.75}$$

$$\text{or}\quad \boldsymbol{p} \odot \boldsymbol{n} = \boldsymbol{d}_o \tag{2.76}$$

Equations (2.74), (2.75) and (2.76) are all alternative forms of the vector equation of the plane. In the special case where $\boldsymbol{n} = \hat{\boldsymbol{n}}$, then d_o represents the distance of the plane from the origin $\boldsymbol{O}$.

Expressing Eq. (2.74) in component form we obtain:

$$a \cdot (x - x_1) + b \cdot (y - y_1) + c \cdot (z - z_1) = 0 \tag{2.77}$$

This equation provides both *the necessary and sufficient condition* for a point $[x, y, z]$ to lie on the plane containing the point $[x_1, y_1, z_1]$ and being perpendicular to the direction $[a, b, c]$.

Similarly Eq. (2.76) in component form gives:

$$a \cdot x + b \cdot y + c \cdot z + d = 0 \tag{2.78}$$

where $d = -d_o$ is the negative distance of the plane from the origin. Thus the plane is represented by a linear equation of x, y, z.

2.23.2 The Plane Defined by Three Position Vectors

Let $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$ be three non-collinear points lying on a plane $\boldsymbol{\Pi}$ with position vectors $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ and components $[x_1, y_1, z_1]$, $[x_2, y_2, z_2]$, $[x_3, y_3, z_3]$, respectively. Directed segments $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}, \overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_3}$ represent the vectors $(p_2 - p_1)$, $(p_3 - p_1)$, respectively, as shown in Fig. 2.24.

Both directed segments lie on the plane Π and are therefore perpendicular to the normal of the plane. Thus:

$$\boldsymbol{n} = \left(\boldsymbol{p}_2 - \boldsymbol{p}_1\right) \otimes \left(\boldsymbol{p}_3 - \boldsymbol{p}_1\right) \tag{2.79}$$

Using the distributive rule (R2.13):

$$\boldsymbol{n} = \boldsymbol{p}_2 \otimes \boldsymbol{p}_3 - \boldsymbol{p}_2 \otimes \boldsymbol{p}_1 - \boldsymbol{p}_1 \otimes \boldsymbol{p}_3 + \boldsymbol{p}_1 \otimes \boldsymbol{p}_1$$

Using axiom (A2.10) and the anti-commutative rule (R2.11):

$$\boldsymbol{n} = \boldsymbol{p}_2 \otimes \boldsymbol{p}_3 + \boldsymbol{p}_1 \otimes \boldsymbol{p}_2 + \boldsymbol{p}_3 \otimes \boldsymbol{p}_1$$

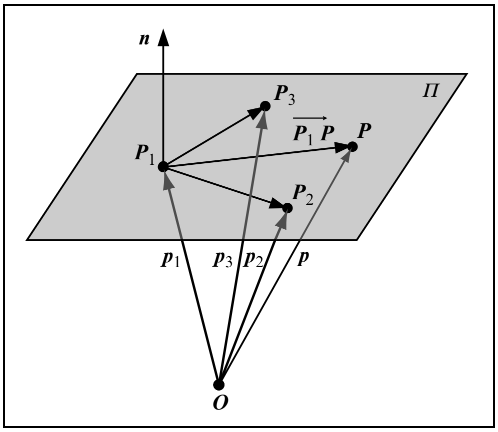

FIGURE 2.24. A plane defined by three position vectors.

Substituting this normal into Eq. (2.74) we obtain:

$$(\boldsymbol{p} - \boldsymbol{p}_1) \odot ((\boldsymbol{p}_2 - \boldsymbol{p}_1) \otimes (\boldsymbol{p}_3 - \boldsymbol{p}_1)) = 0 \tag{2.80}$$

$$\therefore \quad (\boldsymbol{p} - \boldsymbol{p}_1) \odot (\boldsymbol{p}_2 \otimes \boldsymbol{p}_3 + \boldsymbol{p}_3 \otimes \boldsymbol{p}_1 + \boldsymbol{p}_1 \otimes \boldsymbol{p}_2) = 0 \tag{2.81}$$

Rewriting Eq. (2.81) in component form we obtain:

$$\begin{aligned}
&(x - x_1) \cdot ((y_2z_3 - y_3z_2) + (y_3z_1 - y_1z_3) + (y_1z_2 - y_2z_1)) \\
&+ (y - y_1) \cdot ((z_2x_3 - z_3x_2) + (z_3x_1 - z_1x_3) + (z_1x_2 - z_2x_1)) \\
&+ (z - z_1) \cdot ((x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)) = 0
\end{aligned} \tag{2.82}$$

To simplify this expression we label the bracketed vector products as a, b, c. Thus:

$$\begin{aligned}
&(x - x_1) \cdot a + (y - y_1) \cdot b + (z - z_1) \cdot c = 0 \\
&\therefore (a \cdot x + b \cdot y + c \cdot z) - (a \cdot x_1 + b \cdot y_1 + c \cdot z_1) = 0 \\
&\therefore a \cdot x + b \cdot y + c \cdot z + d = 0
\end{aligned} \tag{2.83}$$

where:

$$\begin{aligned}
a &= (y_2z_3 - y_3z_2) + (y_3z_1 - y_1z_3) + (y_1z_2 - y_2z_1) \\
b &= (z_2x_3 - z_3x_2) + (z_3x_1 - z_1x_3) + (z_1x_2 - z_2x_1) \\
c &= (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1) \\
d &= -(a \cdot x_1 + b \cdot y_1 + c \cdot z_1)
\end{aligned}$$

Here a, b, c are the direction ratios of the plane normal and d is equal to the negative distance of the plane from the origin scaled by the magnitude of the plane normal.

An alternative representation of the plane can be arrived at by expressing the left-hand side of Eq. (2.80) as a triple scalar product:

$$\begin{aligned}&(\boldsymbol{p}-\boldsymbol{p}_1)\odot((\boldsymbol{p}_2-\boldsymbol{p}_1)\otimes(\boldsymbol{p}_3-\boldsymbol{p}_1)) = ((\boldsymbol{p}-\boldsymbol{p}_1),(\boldsymbol{p}_2-\boldsymbol{p}_1),(\boldsymbol{p}_3-\boldsymbol{p}_1))\\ &\therefore ((\boldsymbol{p}-\boldsymbol{p}_1),(\boldsymbol{p}_2-\boldsymbol{p}_1),(\boldsymbol{p}_3-\boldsymbol{p}_1)) = 0 \end{aligned}\tag{2.84}$$

Expanding the left-hand side of Eq. (2.81) we obtain:

$$\begin{aligned}\therefore\ &\boldsymbol{p}\odot(\boldsymbol{p}_2\otimes\boldsymbol{p}_3)+\boldsymbol{p}\odot(\boldsymbol{p}_3\otimes\boldsymbol{p}_1)+\boldsymbol{p}\odot(\boldsymbol{p}_1\otimes\boldsymbol{p}_2)-\boldsymbol{p}_1\odot(\boldsymbol{p}_2\otimes\boldsymbol{p}_3)\\ &-\boldsymbol{p}_1\odot(\boldsymbol{p}_3\otimes\boldsymbol{p}_1)-\boldsymbol{p}_1\odot(\boldsymbol{p}_1\otimes\boldsymbol{p}_2)=0\end{aligned}$$

The last two terms of the above equation cancel out by Axiom (A2.14) giving:

$$\boldsymbol{p}\odot(\boldsymbol{p}_2\otimes\boldsymbol{p}_3)+\boldsymbol{p}\odot(\boldsymbol{p}_3\otimes\boldsymbol{p}_1)+\boldsymbol{p}\odot(\boldsymbol{p}_1\otimes\boldsymbol{p}_2)-\boldsymbol{p}_1\odot(\boldsymbol{p}_2\otimes\boldsymbol{p}_3)=0$$

Which in triple scalar product form is:

$$\begin{aligned}&(\boldsymbol{p},\boldsymbol{p}_2,\boldsymbol{p}_3)+(\boldsymbol{p},\boldsymbol{p}_3,\boldsymbol{p}_1)+(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2)-(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)=0\\ &\therefore (\boldsymbol{p},\boldsymbol{p}_2,\boldsymbol{p}_3)+(\boldsymbol{p},\boldsymbol{p}_3,\boldsymbol{p}_1)+(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2)=(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)\end{aligned}\tag{2.85}$$

2.24 Some Applications of Vector Algebra in Analytical Geometry

2.24.1 The Distance Between Two Points in Space

Given two points $\boldsymbol{P}_1, \boldsymbol{P}_2$ with position vectors $\boldsymbol{p}_1 = [x_1, y_1, z_1], \boldsymbol{p}_2 = [x_2, y_2, z_2]$, the distance between these points is given by the magnitude of the directed segment $\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}$, which represents the vector $(\boldsymbol{p}_2 - \boldsymbol{p}_1)$, as shown in Fig. 2.25. Thus,

$$|\overrightarrow{\boldsymbol{P}_1\boldsymbol{P}_2}| = |\boldsymbol{p}_2-\boldsymbol{p}_1| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\tag{2.86}$$

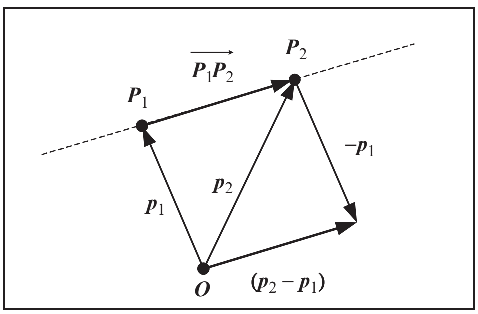

FIGURE 2.25. The distance between two points.

2.24.2 The Perpendicular Distance from a Point to a Line

A line $\boldsymbol{\Lambda}$ is defined by a position vector $\boldsymbol{a} = \left[a_x, a_y, a_z\right]$ and a direction unit vector $\boldsymbol{b} = \left[b_x, b_y, b_z\right]$. Let $\boldsymbol{P}$ be a point not on the line $\boldsymbol{\Lambda}$ with position vector $\boldsymbol{p} = \left[p_x, p_y, p_z\right]$. Suppose we wish to find the foot $\boldsymbol{Q}$ of the perpendicular from point $\boldsymbol{P}$ to the line as well as the perpendicular distance from this point to the line, as seen in Fig. 2.26.

The general point $\boldsymbol{x}$ on the line is given by its parametric equation:

$$\boldsymbol{x} = \boldsymbol{a} + t \cdot \boldsymbol{b}$$

Hence the position vector of point $\boldsymbol{Q}$ is given by:

$$\boldsymbol{q} = \boldsymbol{a} + t_q \cdot \boldsymbol{b} \tag{2.87}$$

for some scalar value t_q. Therefore the directed segment $\overrightarrow{\boldsymbol{PQ}}$ represents the vector:

$$\boldsymbol{v} = \boldsymbol{q} - \boldsymbol{p} = \boldsymbol{a} + t_q \cdot \boldsymbol{b} - \boldsymbol{p} \tag{2.88}$$

and since $\overrightarrow{\boldsymbol{PQ}}$ is perpendicular to $\boldsymbol{\Lambda}$ we have:

$$\left(\boldsymbol{a} + t_q \cdot \boldsymbol{b} - \boldsymbol{p}\right) \odot \boldsymbol{b} = 0$$

Expanding this scalar product and recalling that $\boldsymbol{b}$ is a unit vector, we get:

$$\boldsymbol{a} \odot \boldsymbol{b} + t_q \cdot \boldsymbol{b} \odot \boldsymbol{b} - \boldsymbol{p} \odot \boldsymbol{b} = 0$$

$$\therefore \ \boldsymbol{a} \odot \boldsymbol{b} + t_q \cdot 1 - \boldsymbol{p} \odot \boldsymbol{b} = 0$$

Solving for t_q we get:

$$t_q = (\boldsymbol{p} - \boldsymbol{a}) \odot \boldsymbol{b}$$

which in component form is:

$$t_q = (p_x - a_x) \cdot b_x + \left(p_y - a_y\right) \cdot b_y + (p_z - a_z) \cdot b_z$$

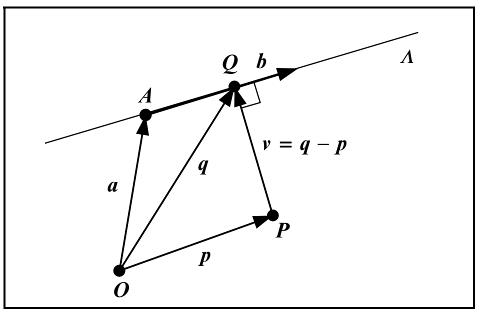

FIGURE 2.26. The foot of the perpendicular from a point to a line.

Substituting this value in Eq. (2.87) we obtain:

$$\boldsymbol{q} = \boldsymbol{a} + ((\boldsymbol{p} - \boldsymbol{a}) \odot \boldsymbol{b}) \cdot \boldsymbol{b} \tag{2.89}$$

which is the required foot of the perpendicular. Rewriting the above equation in component form we get:

$$\begin{aligned} q_x &= a_x + t_q \cdot b_x \\ q_y &= a_y + t_q \cdot b_y \text{ where } t_q = (p_x - a_x) \cdot b_x + \left(p_y - a_y\right) \cdot b_y + (p_z - a_z) \cdot b_z \\ q_z &= a_z + t_q \cdot b_z \end{aligned} \tag{2.90}$$

Now we can calculate the length of the perpendicular distance quite simply as:

$$|\boldsymbol{v}| = |\boldsymbol{q} - \boldsymbol{p}| = \sqrt{(q_x - p_x)^2 + \left(q_y - p_y\right)^2 + (q_z - p_z)^2} \tag{2.91}$$

2.24.3 *The Distance of a Point from a Line*

This is essentially the same problem as in the previous section, but here we do not calculate the foot of the perpendicular. A line $\boldsymbol{\Lambda}$ is defined by a position vector $\boldsymbol{a} = \left[a_x, a_y, a_z\right]$ and a direction unit vector $\boldsymbol{b} = \left[b_x, b_y, b_z\right]$, as shown in Fig. 2.27.

As before, the general point $\boldsymbol{x}$ on the line is given by its parametric equation:

$$\boldsymbol{x} = \boldsymbol{a} + t \cdot \boldsymbol{b}$$

Let the general point $\boldsymbol{P}$ have position vector $\boldsymbol{p} = \left[p_x, p_y, p_z\right]$. The directed segment $\overrightarrow{\boldsymbol{AP}}$ represents the vector $\boldsymbol{c} = (\boldsymbol{p} - \boldsymbol{a})$. The perpendicular distance of the point from the line is the magnitude of the directed segment $\overrightarrow{\boldsymbol{PQ}}$:

$$\left|\overrightarrow{\boldsymbol{PQ}}\right| = |\boldsymbol{p} - \boldsymbol{a}| \cdot \sin\theta = |\boldsymbol{b} \otimes (\boldsymbol{p} - \boldsymbol{a})| \tag{2.92}$$

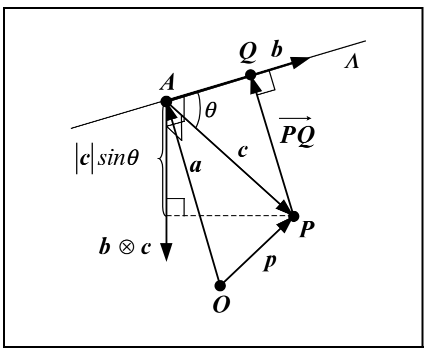

FIGURE 2.27. The perpendicular distance of a point from a line.

The components of the vector product are given by:

$$\begin{aligned}\boldsymbol{b} \otimes (\boldsymbol{p} - \boldsymbol{a}) = \big[&\left(b_y\left(p_z - a_z\right) - b_z\left(p_y - a_y\right)\right), \\ &\left(b_z\left(p_x - a_x\right) - b_x\left(p_z - a_z\right)\right), \left(b_x\left(p_y - a_y\right) - b_y\left(p_x - a_x\right)\right)\big]\end{aligned}$$

and its magnitude is given by:

$$|\boldsymbol{b} \otimes (\boldsymbol{p} - \boldsymbol{a})| = \sqrt{\begin{array}{l}\left(b_y\left(p_z - a_z\right) - b_z\left(p_y - a_y\right)\right)^2 + \\ \left(b_z\left(p_x - a_x\right) - b_x\left(p_z - a_z\right)\right)^2 + \\ \left(b_x\left(p_y - a_y\right) - b_y\left(p_x - a_x\right)\right)^2\end{array}} \tag{2.93}$$

2.24.4 The Distance Between Two Parallel Lines

Let $\boldsymbol{\Lambda}_1$,$\boldsymbol{\Lambda}_2$ be two parallel lines. Where $\boldsymbol{a}_1 = \left[a_{1x}, a_{1y}, a_{1z}\right]$, $\boldsymbol{a}_2 = \left[a_{2x}, a_{2y}, a_{2z}\right]$ are position vectors of lines $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2$, respectively and $\boldsymbol{b} = \left[b_x, b_y, b_z\right]$ is the direction unit vector of both lines, as shown in Fig. 2.28.

The perpendicular distance between the two parallel lines is equal to the distance of point $\boldsymbol{A}_2$ from line $\boldsymbol{\Lambda}_1$ (or analogously the distance of point $\boldsymbol{A}_1$ from line $\boldsymbol{\Lambda}_2$). As in the previous section, the directed segment $\overrightarrow{\boldsymbol{A}_1\boldsymbol{A}_2}$ represents the vector $\boldsymbol{c} = (\boldsymbol{a}_2 - \boldsymbol{a}_1)$. The perpendicular distance of point $\boldsymbol{A}_2$ from line $\boldsymbol{\Lambda}_1$ is the magnitude of the directed segment $\overrightarrow{\boldsymbol{A}_2\boldsymbol{Q}}$:

$$\left|\overrightarrow{\boldsymbol{A}_2\boldsymbol{Q}}\right| = |\boldsymbol{a}_2 - \boldsymbol{a}_1| \cdot \sin\theta = |\boldsymbol{b} \otimes (\boldsymbol{a}_2 - \boldsymbol{a}_1)| \tag{2.94}$$

The components of the vector product are given by:

$$\begin{aligned}\boldsymbol{b} \otimes (\boldsymbol{a}_2 - \boldsymbol{a}_1) = \big[&\left(b_y\left(a_{2z} - a_{1z}\right) - b_z\left(a_{2y} - a_{1y}\right)\right), \\ &\left(b_z\left(a_{2x} - a_{1x}\right) - b_x\left(a_{2z} - a_{1z}\right)\right), \\ &\left(b_x\left(a_{2y} - a_{1y}\right) - b_y\left(a_{2x} - a_{1x}\right)\right)\big]\end{aligned}$$

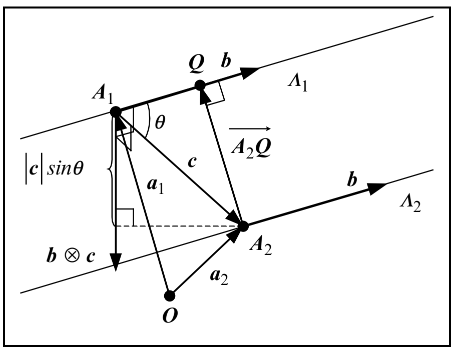

FIGURE 2.28. The perpendicular distance between two parallel lines.

and its magnitude is given by:

$$|\boldsymbol{b} \otimes (\boldsymbol{a}_2 - \boldsymbol{a}_1)| = \sqrt{\begin{matrix} \left(b_y\left(a_{2z} - a_{1z}\right) - b_z\left(a_{2y} - a_{1y}\right)\right)^2 + \\ \left(b_z\left(a_{2x} - a_{1x}\right) - b_x\left(a_{2z} - a_{1z}\right)\right)^2 + \\ \left(b_x\left(a_{2y} - a_{1y}\right) - b_y\left(a_{2x} - a_{1x}\right)\right)^2 \end{matrix}} \tag{2.95}$$

2.24.5 The Distance Between Two Non-Parallel Lines

Let $\boldsymbol{\Lambda}_1$,$\boldsymbol{\Lambda}_2$ be two non-parallel lines. Where line $\boldsymbol{\Lambda}_1$ is defined by a position vector $\boldsymbol{a}_1 = \left[a_{1x}, a_{1y}, a_{1z}\right]$ and a direction unit vector $\boldsymbol{b}_1 = \left[b_{1x}, b_{1y}, b_{1z}\right]$ and line is defined by a position vector $\boldsymbol{a}_2 = \left[a_{2x}, a_{2y}, a_{2z}\right]$ and a direction unit vector $\boldsymbol{b}_2 = \left[b_{2x}, b_{2y}, b_{2z}\right]$, as shown in Fig. 2.29.

The directed segment $\overrightarrow{A_1A_2}$ represents the vector $\boldsymbol{c} = (\boldsymbol{a}_2 - \boldsymbol{a}_1)$. The feet of the line that is mutually perpendicular to the lines $\boldsymbol{\Lambda}_1$,$\boldsymbol{\Lambda}_2$ are $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, respectively. The perpendicular distance between the two lines is the magnitude of the directed segment $\overrightarrow{Q_1Q_2}$. The magnitude of this segment is equal to the ratio of the volume of the parallelepiped having vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{c}$ as concurrent sides over the area of its base:

$$\begin{aligned} \left|\overrightarrow{Q_1Q_2}\right| &= \frac{\textit{volume of the parallelepiped}\ \left((\boldsymbol{a}_2 - \boldsymbol{a}_1), \boldsymbol{b}_1, \boldsymbol{b}_2\right)}{\textit{area of the base of the parallelepiped}\ \left((\boldsymbol{a}_2 - \boldsymbol{a}_1), \boldsymbol{b}_1, \boldsymbol{b}_2\right)} \\ &= \frac{|(\boldsymbol{a}_2 - \boldsymbol{a}_1) \odot (\boldsymbol{b}_1 \otimes \boldsymbol{b}_2)|}{|\boldsymbol{b}_1 \otimes \boldsymbol{b}_2|} \end{aligned} \tag{2.96}$$

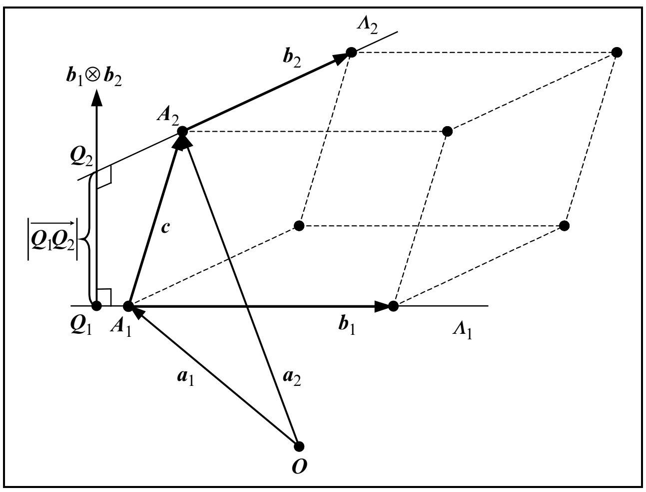

FIGURE 2.29. The distance between two non-parallel lines.

The perpendicular distance between two non-parallel lines in component form is given by:

$$\left|\overrightarrow{Q_1Q_2}\right| = \frac{|\boldsymbol{a} \odot \boldsymbol{b}|}{|\boldsymbol{b}|} = \frac{|a_x b_x + a_y b_y + a_z b_z|}{\sqrt{b_x^2 + b_y^2 + b_z^2}}$$

$$for \begin{cases} \boldsymbol{a} = \boldsymbol{a}_2 - \boldsymbol{a}_1 = \left[(a_{2x} - a_{1x}), (a_{2y} - a_{1y}), (a_{2z} - a_{1z})\right] \\ \boldsymbol{b} = \boldsymbol{b}_1 \otimes \boldsymbol{b}_2 = \left[(b_{1y}b_{2z} - b_{1z}b_{2y}), (b_{1z}b_{2x} - b_{1x}b_{2z}), (b_{1x}b_{2y} - b_{1y}b_{2x})\right] \end{cases} \tag{2.97}$$

2.24.6 The Cosine of the Angle between Two Lines

Let $\boldsymbol{\Lambda}_1$,$\boldsymbol{\Lambda}_2$ be two non-parallel lines. Where line $\boldsymbol{\Lambda}_1$ is defined by a position vector $\boldsymbol{a}_1 = \left[a_{1x}, a_{1y}, a_{1z}\right]$ and a direction unit vector $\boldsymbol{b}_1 = \left[b_{1x}, b_{1y}, b_{1z}\right]$ and line is defined by a position vector $\boldsymbol{a}_2 = \left[a_{2x}, a_{2y}, a_{2z}\right]$ and a direction unit vector $\boldsymbol{b}_2 = \left[b_{2x}, b_{2y}, b_{2z}\right]$, as shown in Fig. 2.30.

The cosine of the angle between the two lines is given by:

$$\cos\theta = \boldsymbol{b}_1 \odot \boldsymbol{b}_2 = b_{1x}b_{2x} + b_{1y}b_{2y} + b_{1z}b_{2z} \tag{2.98}$$

2.24.7 The Cosine of the Angle between Two Planes

Let $\boldsymbol{\Pi}_1$, $\boldsymbol{\Pi}_2$ be two non-parallel planes with unit normal vectors $\hat{\boldsymbol{n}}_1 = \left[\hat{n}_{1x}, \hat{n}_{1y}, \hat{n}_{1z}\right]$, $\hat{\boldsymbol{n}}_2 = \left[\hat{n}_{2x}, \hat{n}_{2y}, \hat{n}_{2z}\right]$, respectively, as seen in Fig. 2.31.

The angle between the planes is by definition the angle between their unit normals, thus:

$$\cos\theta = \hat{\boldsymbol{n}}_1 \odot \hat{\boldsymbol{n}}_2 = \hat{n}_{1x}\hat{n}_{2x} + \hat{n}_{1y}\hat{n}_{2y} + \hat{n}_{1z}\hat{n}_{2z} \tag{2.99}$$

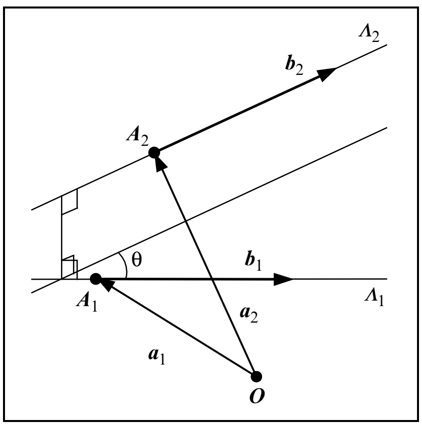

FIGURE 2.30. The cosine of the angle between two non-parallel lines.

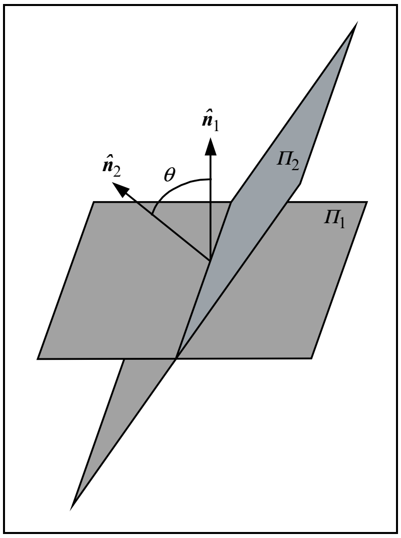

FIGURE 2.31. The cosine of the angle between two planes.

2.24.8 The Distance of a Point from a Plane

From Eq. (2.76) we know that the plane $\boldsymbol{\Pi}$ is defined by:

$$\boldsymbol{x} \odot \hat{\boldsymbol{n}} = d_o$$

where $\boldsymbol{x}$ is the position vector of the general point $\boldsymbol{X}$ on the plane, $\hat{\boldsymbol{n}}$ is the unit normal of the plane and d_o is the distance of the plane from the origin $\boldsymbol{O}$. A point $\boldsymbol{P}$ off the plane has position vector $\boldsymbol{p}$, as shown in Fig. 2.32.

The magnitude of the projection of $\boldsymbol{p}$ onto $\hat{n}$ is given by:

$$\boldsymbol{p} \odot \hat{\boldsymbol{n}} = m_p$$

The signed distance of point $\boldsymbol{P}$ from the plane is given by:

$$d_p = m_p - d_o = \boldsymbol{p} \odot \hat{\boldsymbol{n}} - \boldsymbol{x} \odot \hat{\boldsymbol{n}} = (\boldsymbol{p} - \boldsymbol{x}) \odot \hat{\boldsymbol{n}} \tag{2.100}$$

$$\therefore \; d_p = (p_x - x_x) \cdot \hat{n}_x + \left(p_y - x_y\right) \cdot \hat{n}_y + (p_z - x_z) \cdot \hat{n}_z \tag{2.101}$$

The observant reader would have noticed that the distance of point $\boldsymbol{P}$ from plane $\boldsymbol{\Pi}$ should be $d_p = d_o - m_p$ and not $d_p = m_p - d_o$. This change of sign was done deliberately to satisfy the following convention. Plane $\boldsymbol{\Pi}$ divides three-dimensional space into two *half-spaces*. A *positive half-space* that lies *in front of the plane*, i.e. in the direction that the plane normal points, and a *negative half-space* that lies *behind the plane*. By convention the origin lies in the negative half-space and any point lying on the same half-space as the origin is assumed to have a negative distance from the plane. For example, in CG a cube defined

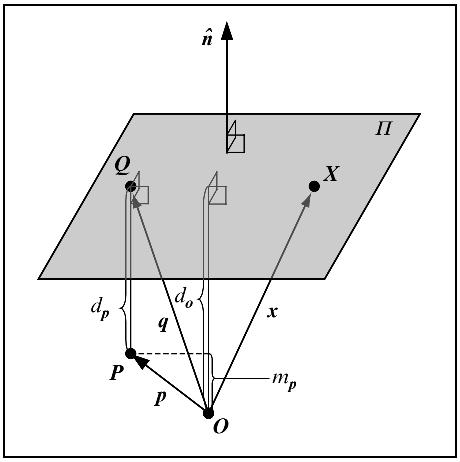

FIGURE 2.32. The distance of a point from a plane.

around the origin has the normals of its faces pointing outwards and the origin is behind the planes that form its faces. Thus:

If $d_p > 0$, then the point $\boldsymbol{P}$ lies on the opposite side of the plane to the origin (i.e. $\boldsymbol{P}$ is in front of the plane).

If $d_p < 0$, then the point $\boldsymbol{P}$ lies on the same side of the plane as the origin (i.e. $\boldsymbol{P}$ is behind of the plane).

If $d_p = 0$, then the point $\boldsymbol{P}$ lies on the plane (i.e. $\boldsymbol{P}$ is on the plane).

Let $\boldsymbol{Q}$ be the foot of the perpendicular from point $\boldsymbol{P}$ to plane $\boldsymbol{\Pi}$. Point $\boldsymbol{Q}$ lies on the plane and thus satisfies the equation of the plane, i.e.

$$\boldsymbol{q} \odot \hat{\boldsymbol{n}} = d_o$$

Point $\boldsymbol{Q}$ also lies on the perpendicular line whose equation is given by:

$$\boldsymbol{x}_l = \boldsymbol{p} + t \cdot \hat{\boldsymbol{n}}$$

Thus, point $\boldsymbol{Q}$ satisfies the perpendicular line equation:

$$\boldsymbol{q} = \boldsymbol{p} + t_q \cdot \hat{\boldsymbol{n}}$$

The parameter t_q represents the distance travelled along $\hat{\boldsymbol{n}}$, starting from $\boldsymbol{P}$ and ending at $\boldsymbol{Q}$, i.e.

$$t_q = d_o - m_p$$

Thus, the position vector of the foot $\boldsymbol{Q}$ of the perpendicular from point $\boldsymbol{P}$ to plane $\boldsymbol{\Pi}$ is given by:

$$\boldsymbol{q} = \boldsymbol{p} + \left(d_o - m_p\right) \cdot \hat{\boldsymbol{n}}$$

2.24.9 The Point of Intersection of a Line and a Plane

A line $\boldsymbol{\Lambda}$ is defined by a position vector $\boldsymbol{a} = \left[a_x, a_y, a_z\right]$ and a direction unit vector $\boldsymbol{b} = \left[b_x, b_y, b_z\right]$. The general point $\boldsymbol{x}$ on the line is given by its parametric equation:

$$\boldsymbol{x} = \boldsymbol{a} + t \cdot \boldsymbol{b}$$

A plane $\boldsymbol{\Pi}$ is defined by:

$$\hat{\boldsymbol{n}} \odot \boldsymbol{x} = d$$

where $\boldsymbol{x}$ is the position vector of the general point $\boldsymbol{X}$ on the plane, $\hat{\boldsymbol{n}} = \left[\hat{n}_x, \hat{n}_y, \hat{n}_z\right]$ is the unit normal of the plane and d is the distance of the plane from the origin $\boldsymbol{O}$.

If the line $\boldsymbol{\Lambda}$ intersects the plane $\boldsymbol{\Pi}$, then let X_0 be their point of intersection, as shown in Fig. 2.33.

The point of intersection $\boldsymbol{X}_0$ has a position vector given by:

$$\boldsymbol{x}_0 = \boldsymbol{a} + t_0 \cdot \boldsymbol{b}$$

where t_0 is the value of the line parameter at point $\boldsymbol{X}_0$. To determine this value we proceed as follows. First, we determine the magnitude m_a of the projection of position vector $\boldsymbol{a}$ onto the plane normal:

$$\boldsymbol{m}_a = \hat{\boldsymbol{n}} \odot \boldsymbol{a}$$

Then, we find the length of the projection of the directed segment $\overrightarrow{\boldsymbol{AX}_0}$ onto the plane normal (i.e. the value of the line parameter at point $\boldsymbol{X}_0$) by subtracting m_a from the distance of the plane from the origin. See the right-hand diagram of the Fig. Thus:

$$t_0 = d - m_a = d - (\hat{\boldsymbol{n}} \odot \boldsymbol{a}) = d - \left(n_x a_x + n_y a_y + n_z a_z\right)$$

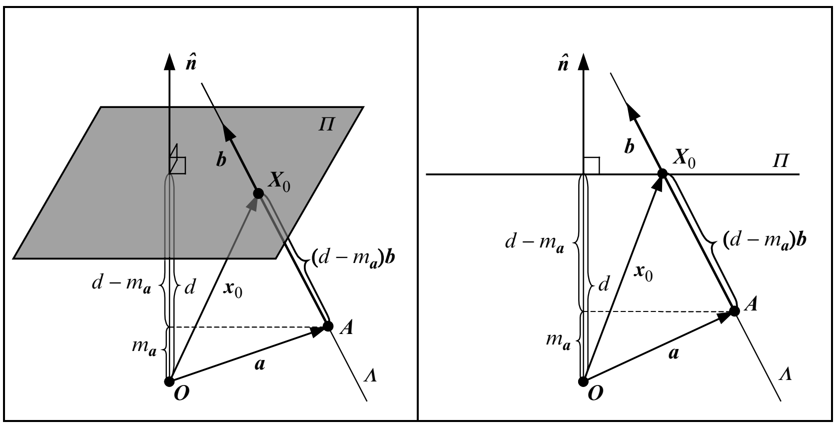

FIGURE 2.33. The distance of a point from a plane.

Now, the position vector of the point of intersection in component form is given by:

$$x_0 = \left[(a_x + t_0 b_x), \left(a_y + t_0 b_y\right), (a_z + t_0 b_z)\right] \quad \text{where}$$
$$t_0 = d - \left(\hat{n}_x a_x + \hat{n}_y a_y + \hat{n}_z a_z\right) \tag{2.102}$$

2.25 Summary of Vector Algebra Axioms and Rules

In this section we collect together, for ease of reference, all the axioms and vector algebra rules that apply to all the vector operations we have examined.

Multiplication of a Vector by a Scalar
The following axioms and rules of vector algebra apply to the vector by scalar product for all vectors $\boldsymbol{v}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and all scalars α, β:

Existence of the vector by scalar product:
$\alpha \boldsymbol{v} = \left[\alpha \boldsymbol{v}_x, \alpha \boldsymbol{v}_y, \alpha \boldsymbol{v}_z\right]$ (A2.1)
Existence of the zero element:
$0\boldsymbol{v} = \boldsymbol{v}0 = \vec{\boldsymbol{0}}$ (A2.2)
Existence of the neutral element:
$1\boldsymbol{v} = \boldsymbol{v}1 = \boldsymbol{v}$ (A2.3)
Associative law:
$\alpha(\beta \boldsymbol{v}) = (\alpha\beta)\boldsymbol{v}$ (R2.1)
Distributive laws:
$(\alpha + \beta)\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\boldsymbol{v}$ (R2.2)
$\alpha(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \alpha\boldsymbol{v}_1 + \alpha\boldsymbol{v}_2$ (R2.3)

Vector Addition
The following axioms and rules of vector algebra apply to vector addition hold for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{v}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$:

Existence of the vector sum:
$\boldsymbol{a} + \boldsymbol{b} = \left[(a_x + b_x), \left(a_y + b_y\right), (a_z + b_z)\right]$ (A2.4)
Existence of the neutral element:
$\vec{\boldsymbol{0}} + \boldsymbol{v} = \boldsymbol{v} + \vec{\boldsymbol{0}} = \boldsymbol{v}$ (A2.5)
Existence of the inverse element:
$\boldsymbol{v} + (-\boldsymbol{v}) = (-\boldsymbol{v}) + \boldsymbol{v} = \vec{\boldsymbol{0}}$ (A2.6)
Commutative law:
$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_2 + \boldsymbol{v}_1$ (R2.4)
Associative law:
$(\boldsymbol{v}_1 + \boldsymbol{v}_2) + \boldsymbol{v}_3 = \boldsymbol{v}_1 + (\boldsymbol{v}_2 + \boldsymbol{v}_3)$ (R2.5)
Change of detection rule:
$-(\boldsymbol{v}_1 - \boldsymbol{v}_2) = (\boldsymbol{v}_2 - \boldsymbol{v}_1)$ (R2.6)

The Scalar Product of Vectors

The following axioms and vector algebra rules apply to the scalar product of vectors for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}$ and all scalars α:

Existence of the scalar product:

$$\boldsymbol{a} \odot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\theta \\ a_x b_x + a_y b_y + a_z b_z \end{cases} \quad \text{(Cartesian basis case only)} \tag{A2.7}$$

Powers of a vector:

$$\boldsymbol{v}^2 = \boldsymbol{v} \odot \boldsymbol{v} = |\boldsymbol{v}|^2 \tag{A2.8}$$

Commutative law:

$$\boldsymbol{v}_1 \odot \boldsymbol{v}_2 = \boldsymbol{v}_2 \odot \boldsymbol{v}_1 \tag{R2.7}$$

Distributive laws:

$$(\alpha \boldsymbol{v}_1) \odot \boldsymbol{v}_2 = \boldsymbol{v}_1 \odot (\alpha \boldsymbol{v}_2) = \alpha\, (\boldsymbol{v}_1 \odot \boldsymbol{v}_2) \tag{R2.8}$$

$$\boldsymbol{v}_1 \odot (\boldsymbol{v}_2 + \boldsymbol{v}_3) = \boldsymbol{v}_1 \odot \boldsymbol{v}_2 + \boldsymbol{v}_1 \odot \boldsymbol{v}_3 \tag{R2.9}$$

The Vector Product of Two Vectors

The following axioms and vector algebra rules apply to the vector product for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and all scalars α:

Existence of the vector product:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \begin{cases} (|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \sin\theta)\, \hat{\boldsymbol{n}} \\ \left[\left(a_y b_z - a_z b_y\right), (a_z b_x - a_x b_z), \quad \text{(Cartesian basis case only)} \right. \\ \left. \left(a_x b_y - a_y b_x\right)\right] \end{cases} \tag{A2.9}$$

Vector product of collinear vectors:

$$\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 = \vec{\boldsymbol{0}} \quad \Leftrightarrow \quad \boldsymbol{v}_1 || \boldsymbol{v}_2 \tag{A2.10}$$

Associative law:

$$\boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) \neq (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 \tag{R2.10}$$

(does not apply)

Anti-commutative law:

$$\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 = -\,(\boldsymbol{v}_2 \otimes \boldsymbol{v}_1) \tag{R2.11}$$

Distributive laws:

$$(\alpha \boldsymbol{v}_1) \otimes \boldsymbol{v}_2 = \boldsymbol{v}_1 \otimes (\alpha \boldsymbol{v}_2) = \alpha\, (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \tag{R2.12}$$

$$\boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 + \boldsymbol{v}_3) = \boldsymbol{v}_1 \otimes \boldsymbol{v}_2 + \boldsymbol{v}_1 \otimes \boldsymbol{v}_3 \tag{R2.13}$$

The Triple Scalar Product

The following axioms and vector algebra rules apply to the triple scalar product for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$:

Existence of the triple scalar product:

$$(a, b, c) = \begin{cases} (\boldsymbol{a} \otimes \boldsymbol{b}) \odot \boldsymbol{c} = \boldsymbol{a} \odot (\boldsymbol{b} \otimes \boldsymbol{c}) \\ a_x \left(b_y c_z - b_z c_y\right) + a_y \left(b_z c_x - b_x c_z\right) \quad \text{(Cartesian basis case only)} \\ + a_z \left(b_x c_y - b_y c_x\right) \end{cases} \tag{A2.11}$$

Triple scalar product of coplanar vectors:

$$\textit{if all } \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \textit{ are coplanar} \Rightarrow (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0 \tag{A2.12}$$

Triple scalar product of zero vectors:

$$\boldsymbol{v}_1 = \vec{\mathbf{0}} \vee \boldsymbol{v}_2 = \vec{\mathbf{0}} \vee \boldsymbol{v}_3 = \vec{\mathbf{0}} \Rightarrow (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0 \tag{A2.13}$$

Triple scalar product of collinear vectors:

$$\boldsymbol{v}_1 || \boldsymbol{v}_2 \vee \boldsymbol{v}_2 || \boldsymbol{v}_3 \vee \boldsymbol{v}_3 || \boldsymbol{v}_1 \Rightarrow (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = 0 \tag{A2.14}$$

Cyclic permutation rule:

$$(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = (\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_1) = (\boldsymbol{v}_3, \boldsymbol{v}_1, \boldsymbol{v}_2) \tag{R2.14}$$

$$\textit{i.e.}, \boldsymbol{v}_1 \odot (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) = \boldsymbol{v}_2 \odot (\boldsymbol{v}_3 \otimes \boldsymbol{v}_1) = \boldsymbol{v}_3 \odot (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2)$$

$$\textit{and } (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \odot \boldsymbol{v}_3 = (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) \odot \boldsymbol{v}_1 = (\boldsymbol{v}_3 \otimes \boldsymbol{v}_1) \odot \boldsymbol{v}_2$$

Non-cyclic permutation rule:

$$(\boldsymbol{v}_2, \boldsymbol{v}_1, \boldsymbol{v}_3) = (\boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_1) = (\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_2) = -(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \tag{R2.15}$$

The Triple Vector Product

The following axioms and vector algebra rules apply to the triple vector product for all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$:

Existence of the triple vector product:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes \boldsymbol{c} = (\boldsymbol{a} \odot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{b} \odot \boldsymbol{c})\, \boldsymbol{a} \tag{A2.15}$$

Associative law (does not apply):

$$(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 \neq \boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) \tag{R2.16}$$

Permutation rule:

$$(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 = \boldsymbol{v}_3 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_1) \tag{R2.17}$$

Expansion rules:

$$(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \otimes \boldsymbol{v}_3 = (\boldsymbol{v}_1 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_2 - (\boldsymbol{v}_2 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_1 \tag{R2.18}$$

$$\boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_3) = (\boldsymbol{v}_1 \odot \boldsymbol{v}_3)\, \boldsymbol{v}_2 - (\boldsymbol{v}_1 \odot \boldsymbol{v}_2)\, \boldsymbol{v}_3 \tag{R2.19}$$

The Scalar Product of Two Vector Products

The following axiom applies to the scalar product of two vector products for all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$:

Existence of the scalar product of two vector products:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \odot (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{c} \odot \boldsymbol{a})\, (\boldsymbol{b} \odot \boldsymbol{d}) - (\boldsymbol{c} \odot \boldsymbol{b})\, (\boldsymbol{d} \odot \boldsymbol{a}) \tag{A2.16}$$

The Vector Product of Two Vector Products

The following axiom applies to the vector product of two vector products for all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$:

Existence of the vector product of two vector products:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes (\boldsymbol{c} \otimes \boldsymbol{d}) = \begin{cases} (\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d})\, \boldsymbol{b} - (\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})\, \boldsymbol{a} \\ (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})\, \boldsymbol{c} - (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})\, \boldsymbol{d} \end{cases} \tag{A2.17}$$

2.26 A Simple Vector Algebra C Library

See *Appendix 1*.

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Mathematical and Computer Programming Techniques
for Computer Graphics
Comninos, P.
2006, XX, 548 p., Hardcover
ISBN: 978-1-85233-902-9