
Contents

I	Operators on Hilbert Space	1
I.1	Hilbert Space	1
I.1.1	Inner Products	1
I.1.2	Orthogonality	2
I.1.3	Dual Spaces and Weak Topology	3
I.1.4	Standard Constructions	4
I.1.5	Real Hilbert Spaces	5
I.2	Bounded Operators	5
I.2.1	Bounded Operators on Normed Spaces	5
I.2.2	Sesquilinear Forms	6
I.2.3	Adjoint	7
I.2.4	Self-Adjoint, Unitary, and Normal Operators	8
I.2.5	Amplifications and Commutants	9
I.2.6	Invertibility and Spectrum	10
I.3	Other Topologies on $\mathcal{L}(\mathcal{H})$	13
I.3.1	Strong and Weak Topologies	13
I.3.2	Properties of the Topologies	14
I.4	Functional Calculus	17
I.4.1	Functional Calculus for Continuous Functions	18
I.4.2	Square Roots of Positive Operators	19
I.4.3	Functional Calculus for Borel Functions	19
I.5	Projections	19
I.5.1	Definitions and Basic Properties	20
I.5.2	Support Projections and Polar Decomposition	21
I.6	The Spectral Theorem	23
I.6.1	Spectral Theorem for Bounded Self-Adjoint Operators	23
I.6.2	Spectral Theorem for Normal Operators	25
I.7	Unbounded Operators	27
I.7.1	Densely Defined Operators	27
I.7.2	Closed Operators and Adjoints	29

	I.7.3	Self-Adjoint Operators	30
	I.7.4	The Spectral Theorem and Functional Calculus for Unbounded Self-Adjoint Operators	32
I.8		Compact Operators	36
	I.8.1	Definitions and Basic Properties	36
	I.8.2	The Calkin Algebra	37
	I.8.3	Fredholm Theory	37
	I.8.4	Spectral Properties of Compact Operators	40
	I.8.5	Trace-Class and Hilbert-Schmidt Operators	41
	I.8.6	Duals and Preduals, σ -Topologies	43
	I.8.7	Ideals of $\mathcal{L}(\mathcal{H})$	44
I.9		Algebras of Operators	47
	I.9.1	Commutant and Bicommutant	47
	I.9.2	Other Properties	48
II		C*-Algebras	51
II.1		Definitions and Elementary Facts	51
	II.1.1	Basic Definitions	51
	II.1.2	Unitization	53
	II.1.3	Power series, Inverses, and Holomorphic Functions	54
	II.1.4	Spectrum	54
	II.1.5	Holomorphic Functional Calculus	55
	II.1.6	Norm and Spectrum	57
II.2		Commutative C*-Algebras and Continuous Functional Calculus	59
	II.2.1	Spectrum of a Commutative Banach Algebra	59
	II.2.2	Gelfand Transform	60
	II.2.3	Continuous Functional Calculus	61
II.3		Positivity, Order, and Comparison Theory	63
	II.3.1	Positive Elements	63
	II.3.2	Polar Decomposition	67
	II.3.3	Comparison Theory for Projections	72
	II.3.4	Hereditary C*-Subalgebras and General Comparison Theory	75
II.4		Approximate Units	79
	II.4.1	General Approximate Units	79
	II.4.2	Strictly Positive Elements and σ -Unital C*-Algebras	81
	II.4.3	Quasicentral Approximate Units	82
II.5		Ideals, Quotients, and Homomorphisms	82
	II.5.1	Closed Ideals	83
	II.5.2	Nonclosed Ideals	85
	II.5.3	Left Ideals and Hereditary Subalgebras	89
	II.5.4	Prime and Simple C*-Algebras	93
	II.5.5	Homomorphisms and Automorphisms	95

II.6	States and Representations	100
II.6.1	Representations	101
II.6.2	Positive Linear Functionals and States	103
II.6.3	Extension and Existence of States	106
II.6.4	The GNS Construction	107
II.6.5	Primitive Ideal Space and Spectrum	111
II.6.6	Matrix Algebras and Stable Algebras	116
II.6.7	Weights	118
II.6.8	Traces and Dimension Functions	121
II.6.9	Completely Positive Maps	124
II.6.10	Conditional Expectations	132
II.7	Hilbert Modules, Multiplier Algebras, and Morita Equivalence	137
II.7.1	Hilbert Modules	137
II.7.2	Operators	141
II.7.3	Multiplier Algebras	144
II.7.4	Tensor Products of Hilbert Modules	147
II.7.5	The Generalized Stinespring Theorem	149
II.7.6	Morita Equivalence	150
II.8	Examples and Constructions	154
II.8.1	Direct Sums, Products, and Ultraproducts	154
II.8.2	Inductive Limits	156
II.8.3	Universal C^* -Algebras and Free Products	158
II.8.4	Extensions and Pullbacks	167
II.8.5	C^* -Algebras with Prescribed Properties	176
II.9	Tensor Products and Nuclearity	179
II.9.1	Algebraic and Spatial Tensor Products	180
II.9.2	The Maximal Tensor Product	180
II.9.3	States on Tensor Products	182
II.9.4	Nuclear C^* -Algebras	184
II.9.5	Minimality of the Spatial Norm	186
II.9.6	Homomorphisms and Ideals	187
II.9.7	Tensor Products of Completely Positive Maps ...	190
II.9.8	Infinite Tensor Products	191
II.10	Group C^* -Algebras and Crossed Products	192
II.10.1	Locally Compact Groups	193
II.10.2	Group C^* -Algebras	197
II.10.3	Crossed products	199
II.10.4	Transformation Group C^* -Algebras	205
II.10.5	Takai Duality	211
II.10.6	Structure of Crossed Products	212
II.10.7	Generalizations of Crossed Product Algebras ...	212
II.10.8	Duality and Quantum Groups	214

III Von Neumann Algebras	221
III.1 Projections and Type Classification	222
III.1.1 Projections and Equivalence	222
III.1.2 Cyclic and Countably Decomposable Projections	225
III.1.3 Finite, Infinite, and Abelian Projections	227
III.1.4 Type Classification	231
III.1.5 Tensor Products and Type I von Neumann Algebras	232
III.1.6 Direct Integral Decompositions	237
III.1.7 Dimension Functions and Comparison Theory	240
III.1.8 Algebraic Versions	243
III.2 Normal Linear Functionals and Spatial Theory	244
III.2.1 Normal and Completely Additive States	245
III.2.2 Normal Maps and Isomorphisms of von Neumann Algebras	248
III.2.3 Polar Decomposition for Normal Linear Functionals and the Radon-Nikodym Theorem	257
III.2.4 Uniqueness of the Predual and Characterizations of W^* -Algebras	259
III.2.5 Traces on von Neumann Algebras	260
III.2.6 Spatial Isomorphisms and Standard Forms	269
III.3 Examples and Constructions of Factors	275
III.3.1 Infinite Tensor Products	275
III.3.2 Crossed Products and the Group Measure Space Construction	280
III.3.3 Regular Representations of Discrete Groups	288
III.3.4 Uniqueness of the Hyperfinite II_1 Factor	291
III.4 Modular Theory	293
III.4.1 Notation and Basic Constructions	293
III.4.2 Approach using Bounded Operators	295
III.4.3 The Main Theorem	295
III.4.4 Left Hilbert Algebras	296
III.4.5 Corollaries of the Main Theorems	299
III.4.6 The Canonical Group of Outer Automorphisms and Connes' Invariants	302
III.4.7 The KMS Condition and the Radon-Nikodym Theorem for Weights	306
III.4.8 The Continuous and Discrete Decompositions of a von Neumann Algebra	310
III.4.8.1 The Flow of Weights	312
III.5 Applications to Representation Theory of C^* -Algebras	313
III.5.1 Decomposition Theory for Representations	313
III.5.2 The Universal Representation and Second Dual	318

IV Further Structure	323
IV.1 Type I C^* -Algebras	323
IV.1.1 First Definitions	323
IV.1.2 Elementary C^* -Algebras	326
IV.1.3 Liminal and Postliminal C^* -Algebras	327
IV.1.4 Continuous Trace, Homogeneous, and Subhomogeneous C^* -Algebras	329
IV.1.5 Characterization of Type I C^* -Algebras	337
IV.1.6 Continuous Fields of C^* -Algebras	340
IV.1.7 Structure of Continuous Trace C^* -Algebras	344
IV.2 Classification of Injective Factors	350
IV.2.1 Injective C^* -Algebras	352
IV.2.2 Injective von Neumann Algebras	353
IV.2.3 Normal Cross Norms	360
IV.2.4 Semidiscrete Factors	362
IV.2.5 Amenable von Neumann Algebras	365
IV.2.6 Approximate Finite Dimensionality	367
IV.2.7 Invariants and the Classification of Injective Factors	367
IV.3 Nuclear and Exact C^* -Algebras	368
IV.3.1 Nuclear C^* -Algebras	368
IV.3.2 Completely Positive Liftings	374
IV.3.3 Amenability for C^* -Algebras	378
IV.3.4 Exactness and Subnuclearity	383
IV.3.5 Group C^* -Algebras and Crossed Products	391
V K-Theory and Finiteness	395
V.1 K -Theory for C^* -Algebras	395
V.1.1 K_0 -Theory	396
V.1.2 K_1 -Theory and Exact Sequences	402
V.1.3 Further Topics	408
V.1.4 Bivariant Theories	411
V.1.5 Axiomatic K -Theory and the Universal Coefficient Theorem	413
V.2 Finiteness	418
V.2.1 Finite and Properly Infinite Unital C^* -Algebras	418
V.2.2 Nonunital C^* -Algebras	423
V.2.3 Finiteness in Simple C^* -Algebras	430
V.2.4 Ordered K -Theory	434
V.3 Stable Rank and Real Rank	444
V.3.1 Stable Rank	445
V.3.2 Real Rank	452
V.4 Quasidiagonality	457
V.4.1 Quasidiagonal Sets of Operators	457
V.4.2 Quasidiagonal C^* -Algebras	460

XX Contents

V.4.3 Generalized Inductive Limits	464
References	479
Index	505

Operator Algebras

Theory of C*-Algebras and von Neumann Algebras

Blackadar, B.

2006, XX, 528 p., Hardcover

ISBN: 978-3-540-28486-4