1

Infinitesimal deformations

The purpose of this chapter is to introduce the reader to deformation theory in an elementary and direct fashion. We will be especially interested in *first-order deformations* and *obstructions* and in giving them appropriate interpretation mostly by elementary Čech cohomology computations. We will start by introducing some algebraic tools needed. For other notions used we refer the reader to the appendices.

1.1 Extensions

1.1.1 Generalities

Let $A \to R$ be a ring homomorphism. An A-*extension* of R (or of R by I) is an exact sequence

$$(R', \varphi): \quad 0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

where R' is an A-algebra and φ is a homomorphism of A-algebras whose kernel I is an ideal of R' satisfying $I^2 = (0)$. This condition implies that I has a structure of R-module. (R', φ) is also called *an extension of A-algebras*.

If (R', φ) and (R'', ψ) are A-extensions of R by I, an A-homomorphism $\xi : R' \to R''$ is called an *isomorphism of extensions* if the following diagram commutes:

$$\begin{array}{ccccccc} 0 \to & I & \to & R' & \to & R & \to 0 \\ & \| & & \downarrow \xi & & \| & \\ 0 \to & I & \to & R'' & \to & R & \to 0 \end{array}$$

Such a ξ is necessarily an isomorphism of A-algebras. More generally, given A-extensions (R', φ) and (R'', ψ) of R, not necessarily having the same kernel, a homomorphism of A-algebras $r : R' \to R''$ such that $\psi r = \varphi$ is called a *homomorphism of extensions*.

The following lemma is immediate.

Lemma 1.1.1. *Let (R', φ) be an extension as above. Given an A-algebra B and two A-homomorphisms $f_1, f_2 : B \to R'$ such that $\varphi f_1 = \varphi f_2$ the induced map*

$f_2 - f_1 : B \to I$ is an A-derivation. In particular, given two homomorphisms of extensions

$$r_1, r_2 : (R', \varphi) \to (R'', \psi)$$

the induced map $r_2 - r_1 : R' \to \ker(\psi)$ is an A-derivation.

The A-extension (R', φ) is called *trivial* if it has a *section*, that is, if there exists a homomorphism of A-algebras $\sigma : R \to R'$ such that $\varphi\sigma = 1_R$. We also say that (R', φ) *splits*, and we call σ a *splitting*.

Given an R-module I, a trivial A-extension of R by I can be constructed by considering the A-algebra $R\tilde{\oplus}I$ whose underlying A-module is $R \oplus I$ and with multiplication defined by:

$$(r, i)(s, j) = (rs, rj + si)$$

The first projection

$$p : R\tilde{\oplus}I \to R$$

defines an A-extension of R by I which is trivial: a section q is given by $q(r) = (r, 0)$.

The sections of p can be identified with the A-derivations $d : R \to I$. Indeed, if we have a section $\sigma : R \to R\tilde{\oplus}I$ with $\sigma(r) = (r, d(r))$ then for all $r, r' \in R$:

$$\sigma(rr') = (rr', d(rr')) = \sigma(r)\sigma(r') = (r, d(r))(r', d(r')) = (rr', rd(r') + r'd(r))$$

and if $a \in A$ then:

$$\sigma(ar) = (ar, d(ar)) = a\sigma(r) = a(r, d(r)) = (ar, ad(r))$$

hence $d : R \to I$ is an A-derivation. Conversely, every A-derivation $d : R \to I$ defines a section $\sigma_d : R \to R\tilde{\oplus}I$ by $\sigma_d(r) = (r, d(r))$.

Every trivial A-extension (R', φ) of R by I is isomorphic to $(R\tilde{\oplus}I, p)$. If $\sigma : R \to R'$ is a section an isomorphism $\xi : R\tilde{\oplus}I \to R'$ is given by:

$$\xi((r, i)) = \sigma(r) + i$$

and its inverse is

$$\xi^{-1}(r') = (\varphi(r'), r' - \sigma\varphi(r'))$$

An A-extension (P, f) of R will be called *versal* if for every other A-extension (R', φ) of R there is a homomorphism of extensions $r : (P, f) \to (R', \varphi)$. If $R = P/I$ where P is a polynomial algebra over A then

$$0 \to I/I^2 \to P/I^2 \to R \to 0$$

is a versal A-extension of R. Therefore, since such a P always exists, we see that every A-algebra R has a versal extension.

Examples 1.1.2. (i) Every A-extension of A is trivial because by definition it has a section. Therefore it is of the form $A\tilde{\oplus}V$ for an A-module V. In particular, if t is an indeterminate the A-extension $A[t]/(t^2)$ of A is trivial, and is denoted by $A[\epsilon]$ (where $\epsilon = t \bmod (t^2)$ satisfies $\epsilon^2 = 0$). The corresponding exact sequence is:

$$0 \to (\epsilon) \to A[\epsilon] \to A \to 0$$

$A[\epsilon]$ is called the *algebra of dual numbers* over A.

(ii) Assume that K is a field. If R is a local K-algebra with residue field K a K-extension of R by K is called a *small extension* of R. Let

$$(R', f): \quad 0 \to (t) \to R' \xrightarrow{f} R \to 0$$

be a small K-extension; in other words $t \in m_{R'}$ is annihilated by $m_{R'}$ so that (t) is a K-vector space of dimension one.

(R', f) is trivial if and only if the surjective linear map induced by f:

$$f_1 : \frac{m_{R'}}{m_{R'}^2} \to \frac{m_R}{m_R^2}$$

is not bijective.
Indeed for the trivial K-extension

$$0 \to (t) \to R\tilde{\oplus}(t) \to R \to 0$$

we have $t \in m_{R\tilde{\oplus}(t)} \setminus m^2_{R\tilde{\oplus}(t)}$, hence the map f_1 is not injective because $f_1(\bar{t}) = 0$. Conversely, if f_1 is not injective then $f_1(\bar{t}) = \bar{0}$; choose a vector subspace $U \subset m_{R'}/m^2_{R'}$ such that $m_{R'}/m^2_{R'} = U \oplus (\bar{t})$ and let $V \subset R'$ be the subring generated by U. Then V is a subring mapped isomorphically onto R by f. The inverse of $f_{|V}$ is a section of f, therefore (R', f) is trivial.

For example, it follows from this criterion that the extension of K-algebras

$$0 \to \frac{(t^n)}{(t^{n+1})} \to \frac{K[t]}{(t^{n+1})} \to \frac{K[t]}{(t^n)} \to 0$$

$n \geq 2$, is nontrivial.

(iii) Let K be a field. The K-algebra

$$K[\epsilon, \epsilon'] := K[t, t']/(t, t')^2$$

is a K-extension of $K[\epsilon]$ by K in two different ways. The first

$$0 \to (\epsilon') \to K[\epsilon, \epsilon'] \xrightarrow{p_\epsilon} K[\epsilon] \to 0$$

is a trivial extension, isomorphic to $p^*((K[\epsilon'], p'))$:

$$\begin{array}{ccccccccc} 0 \to & (\epsilon') & \to & K[\epsilon]\times_K K[\epsilon'] & \to & K[\epsilon] & \to 0 \\ & \| & & \downarrow & & \downarrow p & \\ 0 \to & (\epsilon') & \to & K[\epsilon'] & \xrightarrow{p'} & K & \to 0 \end{array}$$

The isomorphism is given by

$$\begin{array}{ccc} K[\epsilon,\epsilon'] & \longrightarrow & K[\epsilon]\times_K K[\epsilon'] \\ a+b\epsilon+b'\epsilon' & \longmapsto & (a+b\epsilon, a+b'\epsilon') \end{array}$$

The second way is by "sum":

$$\begin{array}{ccccc} 0 \to \quad (\epsilon-\epsilon') & \to K[\epsilon,\epsilon'] & \xrightarrow{+} & K[\epsilon] & \to 0 \\ & a+b\epsilon+b'\epsilon' & \mapsto & a+(b+b')\epsilon & \end{array}$$

We leave it as an exercise to show that $(K[\epsilon,\epsilon'],+)$ is isomorphic to $(K[\epsilon,\epsilon'], p_\epsilon)$.

1.1.2 The module $\mathrm{Ex}_A(R, I)$

Let $A \to R$ be a ring homomorphism. In this subsection we will show how to give an R-module structure to the set of isomorphism classes of extensions of an A-algebra R by a module I, closely following the analogous theory of extensions in an abelian category as explained, for example, in Chapter III of [123].

Let (R',φ) be an A-extension of R by I and $f : S \to R$ a homomorphism of A-algebras. We can define an A-extension $f^*(R',\varphi)$ of S by I, called the *pullback* of (R',φ) by f, in the following way:

$$\begin{array}{lcccccccc} f^*(R',\varphi): & 0\to & I & \to & R'\times_R S & \to & S & \to 0 \\ & & \| & & \downarrow & & \downarrow f & \\ (R',\varphi): & 0\to & I & \to & R' & \to & R & \to 0 \end{array}$$

where $R'\times_R S$ denotes the fibred product defined in the usual way.

Let $\lambda : I \to J$ be a homomorphism of R-modules. The *pushout* of (R',φ) by λ is the A-extension $\lambda_*(R',\varphi)$ of R by J defined by the following commutative diagram:

$$\begin{array}{ccccccc} 0\to & I & \xrightarrow{\alpha} & R' & \xrightarrow{\varphi} & R & \to 0 \\ & \downarrow\lambda & & \downarrow & & \| & \\ 0\to & J & \to & R'\coprod_I J & \to & R & \to 0 \end{array}$$

where

$$R'\coprod_I J = \frac{R'\tilde{\oplus} J}{\{(-\alpha(i),\lambda(i)), i\in I\}}$$

Definition 1.1.3. *For every A-algebra R and for every R-module I we define $\mathrm{Ex}_A(R,I)$ to be the set of isomorphism classes of A-extensions of R by I. If (R',φ) is such an extension we will denote by $[R',\varphi]\in\mathrm{Ex}_A(R,I)$ its class.*

Using the operations of pullback and pushout it is possible to define an R-module structure on $\mathrm{Ex}_A(R, I)$.

If $r \in R$ and $[R', \varphi] \in \mathrm{Ex}_A(R, I)$ we define

$$r[R', \varphi] = [r_*(R', \varphi)]$$

where $r : I \to I$ is the multiplication by r.

Given $[R', \varphi], [R'', \psi] \in \mathrm{Ex}_A(R, I)$, to define their sum we use the following diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[d] & 0 \arrow[d] & \\
& I \oplus I \arrow[dr] & I \arrow[d] \arrow[r, equal] & I \arrow[d] & \\
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & R' \times_R R'' \arrow[r] \arrow[d] \arrow[dr] & R' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & R'' \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] \arrow[dr] & 0 \\
& & 0 & 0 & 0
\end{tikzcd}$$

which defines an A-extension:

$$(R' \times_R R'', \zeta) : \quad 0 \to I \oplus I \to R' \times_R R'' \xrightarrow{\zeta} R \to 0$$

We define

$$[R', \varphi] + [R'', \psi] := [\delta_*(R' \times_R R'', \zeta)]$$

where $\delta : I \oplus I \to I$ is the "sum homomorphism": $\delta(i \oplus j) = i + j$.

Proposition 1.1.4. *Let $A \to R$ be a ring homomorphism and I an R-module. With the operations defined above $\mathrm{Ex}_A(R, I)$ is an R-module whose zero element is $[R\tilde{\oplus}I, p]$. This construction defines a covariant functor:*

$$\begin{array}{ccc}
(R\text{-modules}) & \longrightarrow & (R\text{-modules}) \\
I & \longmapsto & \mathrm{Ex}_A(R, I) \\
(f : I \to J) & \longmapsto & (f_* : \mathrm{Ex}_A(R, I) \to \mathrm{Ex}_A(R, J))
\end{array}$$

Proof. Straightforward. □

It is likewise straightforward to check that if $f : R \to S$ is a homomorphism of A-algebras and I is an S-module, then the operation of pullback induces an application:

$$f^* : \mathrm{Ex}_A(S, I) \to \mathrm{Ex}_A(R, I)$$

which is a homomorphism of R-modules.

We have the following useful result.

Proposition 1.1.5. *Let A be a ring, $f : S \to R$ a homomorphism of A-algebras and let I be an R-module. Then there is an exact sequence of R-modules:*

$$\begin{aligned}
0 \to \mathrm{Der}_S(R, I) \to \mathrm{Der}_A(R, I) \to \mathrm{Der}_A(S, I) \otimes_S R \xrightarrow{\rho} \\
\to \mathrm{Ex}_S(R, I) \xrightarrow{\upsilon} \mathrm{Ex}_A(R, I) \xrightarrow{f^*} \mathrm{Ex}_A(S, I) \otimes_S R
\end{aligned}$$

Proof. υ is the obvious application sending an S-extension to itself considered as an A-extension. An A-extension

$$0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

is also an S-extension if and only if there exists $f' : S \to R$ such that the triangle

$$\begin{array}{ccc} R' & \to & R \\ & \nwarrow & \uparrow \\ & & S \end{array}$$

commutes, and this is equivalent to saying that $f^*(R', \varphi)$ is trivial. This proves the exactness in $\mathrm{Ex}_A(R, I)$.

The homomorphism ρ is defined by letting $\rho(d) = (R\tilde{\oplus}I, p)$ where the structure of S-algebra on $R\tilde{\oplus}I$ is given by the homomorphism

$$s \mapsto (f(s), d(s))$$

Clearly, $\upsilon\rho = 0$. On the other hand, for

$$\begin{array}{ccccccc} (R', \varphi): & 0 \to & I & \to & R' & \xrightarrow{\varphi} & R & \to 0 \\ & & & & \uparrow & & & \\ & & & & S & & & \end{array}$$

to define an element of $\ker(\upsilon)$ there must exist an isomorphism of A-algebras $R' \to R\tilde{\oplus}I$ inducing the identity on I and on R. Hence the composition $S \to R' \to R\tilde{\oplus}I$ is of the form

$$s \mapsto (f(s), d(s))$$

for some $d \in \mathrm{Der}_A(S, I)$: therefore the sequence is exact at $\mathrm{Ex}_S(R, I)$. To prove the exactness at $\mathrm{Der}_A(S, I)$ note that $\rho(d) = 0$ if and only if $p : R\tilde{\oplus}I \to R$ has a section as a homomorphism of S-algebras, if and only if there exists an A-derivation $R \to I$ whose restriction to S is d: this proves the assertion. The exactness at $\mathrm{Der}_S(R, I)$ and $\mathrm{Der}_A(R, I)$ is straightforward. □

Definition 1.1.6. *The R-module* $\mathrm{Ex}_A(R, R)$ *is called the* first cotangent module *of R over A and it is denoted by $T^1_{R/A}$. In the case $A = \mathbf{k}$ we will write T^1_R instead of $T^1_{R/\mathbf{k}}$.*

Proposition 1.1.7. *Let $A \to B$ be an e.f.t. ring homomorphism and let $B = P/J$ where P is a smooth A-algebra. Then for every B-module M we have an exact sequence:*

$$\mathrm{Der}_A(P, M) \to \mathrm{Hom}_B(J/J^2, M) \to \mathrm{Ex}_A(B, M) \to 0 \tag{1.1}$$

If $A \to B$ is a smooth homomorphism then $\mathrm{Ex}_A(B, M) = 0$ *for every B-module M.*

Proof. We have a natural surjective homomorphism

$$\begin{array}{ccc} \mathrm{Hom}_B(J/J^2, M) & \to & \mathrm{Ex}_A(B, M) \\ \lambda & \mapsto & \lambda_*(\eta) \end{array}$$

where

$$\eta : 0 \to J/J^2 \to P/J^2 \to B \to 0$$

The surjectivity follows from the fact that η is versal. The extension $\lambda_*(\eta)$ is trivial if and only if we have a commutative diagram

$$\begin{array}{ccccccc} 0 \to & J/J^2 & \to & P/J^2 & \to & B & \to 0 \\ & \downarrow \lambda & & \downarrow & & \| & \\ 0 \to & M & \to & B\tilde{\oplus}M & \to & B & \to 0 \end{array}$$

if and only if λ extends to an A-derivation $\bar{D} : P/J^2 \to M$, equivalently to an A-derivation $D : P \to M$. The last assertion is immediate (see Theorem C.9). □

Corollary 1.1.8. *If $A \to B$ is an e.f.t. ring homomorphism and M is a finitely generated B-module then $\mathrm{Ex}_A(B, M)$ is a finitely generated B-module. In particular $T^1_{B/A}$ is a finitely generated B-module and we have an exact sequence:*

$$\begin{aligned} 0 \to \mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, M) \to \\ \to \mathrm{Hom}_B(I/I^2, M) \to \mathrm{Ex}_A(B, M) \to 0 \end{aligned} \tag{1.2}$$

if $B = P/J$ for a smooth A-algebra P and an ideal $J \subset P$.

Proof. It is a direct consequence of the exact sequence (1.1). □

1.1.3 Extensions of schemes

Let $X \to S$ be a morphism of schemes. An *extension* of X/S is a closed immersion $X \subset X'$, where X' is an S-scheme, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ such that $\mathcal{I}^2 = 0$. It follows that $\mathcal{I}$ is, in a natural way, a sheaf of $\mathcal{O}_X$-modules, which coincides with the conormal sheaf of $X \subset X'$. To give an extension $X \subset X'$ of X/S is equivalent to giving an exact sequence on X:

$$\mathcal{E} : \quad 0 \to \mathcal{I} \to \mathcal{O}_{X'} \xrightarrow{\varphi} \mathcal{O}_X \to 0$$

where $\mathcal{I}$ is an $\mathcal{O}_X$-module, φ is a homomorphism of $\mathcal{O}_S$-algebras and $\mathcal{I}^2 = 0$ in $\mathcal{O}_{X'}$; we call $\mathcal{E}$ *an extension of X/S by $\mathcal{I}$* or *with kernel $\mathcal{I}$*. Two such extensions $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ are called *isomorphic* if there is an $\mathcal{O}_S$-homomorphism $\alpha : \mathcal{O}_{X'} \to \mathcal{O}_{X''}$ inducing the identity on both $\mathcal{I}$ and $\mathcal{O}_X$. It follows that α must necessarily be an S-isomorphism.

We denote by $\mathrm{Ex}(X/S, \mathcal{I})$ the set of isomorphism classes of extensions of X/S with kernel $\mathcal{I}$. In the case where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes and $\mathcal{I} = \tilde{M}$ we have an obvious identification:

$$\mathrm{Ex}_A(B, M) = \mathrm{Ex}(X/S, \mathcal{I})$$

If $S = \mathrm{Spec}(A)$ is affine we will sometimes write $\mathrm{Ex}_A(X, \mathcal{I})$ instead of $\mathrm{Ex}(X/\mathrm{Spec}(A), \mathcal{I})$. Exactly as in the affine case one proves that $\mathrm{Ex}(X/S, \mathcal{I})$ is a $\Gamma(X, \mathcal{O}_X)$-module with identity element the class of the *trivial extension*:

$$0 \to \mathcal{I} \to \mathcal{O}_X\tilde{\oplus}\mathcal{I} \to \mathcal{O}_X \to 0$$

where $\mathcal{O}_X \tilde{\oplus} \mathcal{I}$ is defined as in the affine case (see Section 1.1). The correspondence

$$\mathcal{I} \mapsto \mathrm{Ex}(X/S, \mathcal{I})$$

defines a covariant functor from $\mathcal{O}_X$-modules to $\Gamma(X, \mathcal{O}_X)$-modules.

In deformation theory the case $\mathcal{I} = \mathcal{O}_X$ is the most important one, being related to first-order deformations. If, more generally, $\mathcal{I}$ is a locally free sheaf we get the notions of *ribbon, carpet* etc. (see [17]).

Using the fact that the exact sequence (1.2) of page 15 localizes, it is immediate to check that the cotangent module localizes. More specifically, it is straightforward to show that given a morphism of finite type of schemes $f : X \to S$ one can define a quasi-coherent sheaf $T^1_{X/S}$ on X with the following properties. If $U = \mathrm{Spec}(A)$ is an affine open subset of S and $V = \mathrm{Spec}(B)$ is an affine open subset of $f^{-1}(U)$, then

$$\Gamma(V, T^1_{X/S}) = T^1_{B/A}$$

It follows from the properties of the cotangent modules that $T^1_{X/S}$ is coherent. $T^1_{X/S}$ is called the *first cotangent sheaf* of X/S. We will write T^1_X if $S = \mathrm{Spec}(\mathbf{k})$. For future reference it will be convenient to state the following:

Proposition 1.1.9. *(i) If X is an algebraic scheme then T^1_X is supported on the singular locus of X. More generally, if $X \to S$ is a morphism of finite type of algebraic schemes then $T^1_{X/S}$ is supported on the locus where X is not smooth over S.*

(ii) If we have a closed embedding $X \subset Y$ with Y nonsingular, then we have an exact sequence of coherent sheaves on X:

$$0 \to T_X \to T_{Y|X} \to N_{X/Y} \to T^1_X \to 0 \tag{1.3}$$

so that, letting $N'_{X/Y} = \ker[N_{X/Y} \to T^1_X]$, we have the short exact sequence

$$0 \to T_X \to T_{Y|X} \to N'_{X/Y} \to 0 \tag{1.4}$$

$N'_{X/Y}$ is called the equisingular normal sheaf of X in Y.

(iii) For every scheme S and morphism of S-schemes $f : X \to Y$ we have an exact sequence of sheaves

$$0 \to \mathrm{T}_{X/Y} \to T_{X/S} \to \mathcal{H}om(f^*\Omega^1_{Y/S}, \mathcal{O}_X) \to T^1_{X/Y} \to T^1_{X/S} \to f^*T^1_{Y/S} \tag{1.5}$$

(iv) When $S = \mathrm{Spec}(\mathbf{k})$ and f is a closed embedding of algebraic schemes, with Y nonsingular, we have $\mathrm{T}_{X/Y} = 0$ and $N_{X/Y} = T^1_{X/Y}$. Moreover, (1.3) is a special case of (1.5) in this case.

Proof. (i) Use Proposition 1.1.7.

(ii) (1.3) globalizes the exact sequence (1.2).

(iii) (1.5) globalizes the exact sequence of Proposition 1.1.5.

(iv) Follows from (1.2) and 1.1.5. □

Note that the first half of the exact sequence (1.5) is the dual of the cotangent sequence of f. For more about (1.5) see also (3.43), page 162. The following is a basic result:

Theorem 1.1.10. *Let $X \to S$ be a morphism of finite type of algebraic schemes and $\mathcal{I}$ a coherent locally free sheaf on X. Assume that X is reduced and S-smooth on a dense open subset. Then there is a canonical identification*

$$\mathrm{Ex}(X/S, \mathcal{I}) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$$

which to the isomorphism class of an extension of X/S:

$$\mathcal{E} : 0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

associates the isomorphism class of the relative conormal sequence of $X \subset X'$:

$$c_{\mathcal{E}} : 0 \to \mathcal{I} \xrightarrow{\delta} (\Omega^1_{X'/S})_{|X} \to \Omega^1_{X/S} \to 0$$

(which is exact also on the left).

Proof. Suppose given an extension $\mathcal{E}$. Since $\mathcal{I}$ is locally free and X is reduced in order to show that $c_{\mathcal{E}}$ is exact on the left it suffices to prove that $\ker(\delta)$ is torsion, equivalently that $c_{\mathcal{E}}$ is exact at every general closed point x of any irreducible component of X. Since X is smooth over S at x it follows from 1.1.7 that there is an affine open neighbourhood U of x such that $\mathcal{E}_{|U}$ is trivial. From Theorem B.3 we deduce that the relative conormal sequence of $\mathcal{E}_{|U}$ is split exact. Since it coincides with the restriction of $c_{X'}$ to U we see that $\delta_{|U}$ is injective; this shows that $\ker(\delta)$ is torsion and $c_{\mathcal{E}}$ is exact. Since isomorphic extensions have isomorphic relative cotangent sequences we have a well-defined map

$$c_{-} : \mathrm{Ex}(X/S, \mathcal{I}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$$

Now let

$$\eta : 0 \to \mathcal{I} \to \mathcal{A} \xrightarrow{p} \Omega^1_{X/S} \to 0$$

define an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$. Letting $d : \mathcal{O}_X \to \Omega^1_{X/S}$ be the canonical derivation, consider the sheaf of $\mathcal{O}_S$ algebras $\mathcal{O} = \mathcal{A} \times_{\Omega^1_{X/S}} \mathcal{O}_X$: over an open subset $U \subset X$ we have $\Gamma(U, \mathcal{O}) = \{(a, f) : p(a) = d(f)\}$ and the multiplication rule is

$$(a, f)(a', f') = (fa' + f'a, ff')$$

Then we have an exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathcal{I} & \to & \mathcal{O} & \to & \mathcal{O}_X & \to 0 \\ & \| & & \downarrow \bar{d} & & \downarrow d & \\ 0 \to & \mathcal{I} & \to & \mathcal{A} & \to & \Omega^1_{X/S} & \to 0 \end{array}$$

where one immediately checks that the projection $\bar{d}$ is an $\mathcal{O}_S$-derivation and therefore it must factor as

$$\mathcal{O} \to \Omega^1_{\mathcal{O}/\mathcal{O}_S} \otimes_{\mathcal{O}} \mathcal{O}_X \to \mathcal{A}$$

and we have an exact commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{I} & \to & \mathcal{O} & \to & \mathcal{O}_X & \to 0 \\ & \| & & \downarrow & & \downarrow d & \\ & \mathcal{I} & \to & \Omega^1_{\mathcal{O}/\mathcal{O}_S} \otimes_{\mathcal{O}} \mathcal{O}_X & \to & \Omega^1_{X/S} & \to 0 \\ & \| & & \downarrow & & \| & \\ 0 \to & \mathcal{I} & \to & \mathcal{A} & \to & \Omega^1_{X/S} & \to 0 \end{array} \tag{1.6}$$

which implies $\Omega^1_{\mathcal{O}/\mathcal{O}_S} \otimes_{\mathcal{O}} \mathcal{O}_X \cong \mathcal{A}$. Therefore, letting e_η be the extension given by the first row of (1.6), we see that $c_{e_\eta} = \eta$. Similarly, one shows that $e_{c_\mathcal{E}} = \mathcal{E}$ for any $[\mathcal{E}] \in \mathrm{Ex}(X/S, \mathcal{I})$. Therefore c_- and e_- are inverse of each other and the conclusion follows. □

Corollary 1.1.11. *Let $X \to S$ be a morphism of finite type of algebraic schemes, smooth on a dense open subset of X. Assume X reduced. Then there is a canonical isomorphism of coherent sheaves on X:*

$$T^1_{X/S} \cong Ext^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X)$$

In particular, if X is a reduced algebraic scheme then

$$T^1_X \cong Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

and if, moreover, $X = \mathrm{Spec}(B_0)$ then

$$T^1_{B_0} \cong \mathrm{Ext}^1_{\mathbf{k}}(\Omega_{B_0/\mathbf{k}}, B_0)$$

Proof. An immediate consequence of the above theorem. □

A closer analysis of the proof of Theorem 1.1.10 shows that without assuming X reduced we only have an inclusion

$$Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \subset T^1_X$$

NOTES

1. An alternative approach to the topics treated in this section can be obtained by means of the so-called "truncated cotangent complex", first introduced in [76]. A more general version of the cotangent complex was introduced in [120] and later incorporated in general theories of André [4], Quillen [146] and Tate [181]. Here we will just recall the main facts about the truncated cotangent complex, without entering into any details, with the only purpose of showing the relation to the notions introduced in this section. For details we refer to [94].

Let A be a ring and R an A-algebra. To every A-extension

$$\eta : 0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

we associate a complex $c_\bullet(\eta)$ of R-modules (also denoted $c_\bullet(\varphi)$) defined as follows:

$$\begin{array}{lcll} c_0(\eta) & = & \Omega_{R'/A} \otimes_{R'} R & \\ c_1(\eta) & = & I & \\ c_n(\eta) & = & (0) & n \neq 0, 1 \end{array}$$

$d_1 : c_1(\eta) \to c_0(\eta)$ is the map $x \mapsto d(x) \otimes 1$. In other words $c_\bullet(\eta)$ consists of the first map in the conormal sequence of φ. If $r : (R', \varphi) \to (R'', \psi)$ is a homomorphism of A-extensions then r induces a homomorphism of complexes

$$c_\bullet(r) : c_\bullet(\varphi) \to c_\bullet(\psi)$$

in an obvious way. The following is easy to establish:

- Let $r_1, r_2 : (R', \varphi) \to (R'', \psi)$ be two homomorphisms of A-extensions of R. Then $c_\bullet(r_1)$ and $c_\bullet(r_2)$ are homotopic. As an immediate consequence we have:
- if (E, p) and (F, q) are two versal A-extensions of R then the complexes $c_\bullet(p)$ and $c_\bullet(q)$ are homotopically equivalent.

Definition 1.1.12. *Let A be a ring, R an A-algebra and (E, p) a versal A-extension of R. The homotopy class of the complex $c_\bullet(p)$ is called the* (truncated) cotangent complex *of R over A and denoted by $\check{T}(R/A)$.*

If $R = P/I$ for a polynomial A-algebra P the A-extension

$$0 \to I/I^2 \to P/I^2 \to R \to 0 \tag{1.7}$$

is versal and therefore the complex

$$I/I^2 \xrightarrow{\delta} \Omega_{P/A} \otimes_P R \tag{1.8}$$

where δ is the map appearing in the conormal sequence of $A \to P \to R$, represents $\check{T}(R/A)$. If R is e.f.t. and P is replaced by a smooth A-algebra then (1.7) is again a versal A-extension and (1.8) again represents $\check{T}(R/A)$. From the fact that every A-algebra R can be obtained as the quotient of a polynomial A-algebra P it follows that the cotangent complex $\check{T}(R/A)$ exists for every A-algebra R.

The cotangent complex can be used to define "upper and lower cotangent functors", as follows.

Definition 1.1.13. *Let $A \to B$ be a ring homomorphism, M a B-module, and let $c_\bullet = \{C_1 \xrightarrow{d_1} C_0\}$ represent $\check{T}(B/A)$. Then for $i = 0, 1$ the* lower cotangent module *of B over A relative to M is:*

$$\check{T}_i(B/A, M) = H_i(c_\bullet \otimes_B M)$$

and the upper cotangent module *of B over A relative to M is:*

$$\check{T}^i(B/A, M) = H^i(\mathrm{Hom}(c_\bullet, M))$$

Because of the definition of cotangent complex it follows that the cotangent modules are independent on the choice of the complex $c_\bullet$ representing $\check{T}(B/A)$, but only depend on

A, B, M. Moreover, one immediately checks that the definition is functorial in M and therefore we have covariant functors:

$$\check{T}_i(B/A,-) : (B\text{-modules}) \to (B\text{-modules}) \qquad i = 0, 1$$

and

$$\check{T}^i(B/A,-) : (B\text{-modules}) \to (B\text{-modules}) \qquad i = 0, 1$$

One immediately sees that for $i = 0$ the cotangent functors are:

$$\check{T}_0(B/A, M) = \Omega_{B/A} \otimes_B M$$

and

$$\check{T}^0(B/A, M) = \mathrm{Der}_A(B, M)$$

From the extension (1.7) we obtain the exact sequences:

$$0 \to \check{T}_1(B/A, M) \to I/I^2 \otimes_B M \to \Omega_{P/A} \otimes_P M \to \Omega_{B/A} \otimes_B M \to 0$$

and

$$0 \to \mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, M) \to$$

$$\mathrm{Hom}_B(I/I^2, M) \to \check{T}^1(B/A, M) \to 0$$

If B is e.f.t. then in (1.7) P can be chosen to be a smooth A-algebra; in this case it follows that $\check{T}_i(B/A, M)$ and $\check{T}^i(B/A, M)$ are finitely generated B-modules if M is finitely generated. Moreover, recalling Corollary 1.1.8, we see that we have an identification

$$\check{T}^1(B/A, M) = \mathrm{Ex}_A(B, M)$$

2. The topics of this section originate from [76]. See also [1], Ch. 0_{IV}, § 18. The proof of Theorem 1.1.10 has been taken from [17]; see also [66].

1.2 Locally trivial deformations

1.2.1 Generalities on deformations

Let X be an algebraic scheme. A cartesian diagram of morphisms of schemes

$$\eta : \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

where π is flat and surjective, and S is connected, is called a *family of deformations*, or simply a *deformation*, of X parametrized by S, or over S; we call S and $\mathcal{X}$ respectively the *parameter scheme* and the *total scheme* of the deformation. If S is algebraic, for each $\mathbf{k}$-rational point $t \in S$ the scheme-theoretic fibre $\mathcal{X}(t)$ is also called a *deformation* of X. When $S = \mathrm{Spec}(A)$ with A in ob$(\mathcal{A}^*)$ and $s \in S$ is the closed point we have a *local family of deformations* (shortly a *local deformation*) of X over A. The deformation η will be also denoted by (S, η) or (A, η) when $S = \mathrm{Spec}(A)$.

The local deformation (A, η) is *infinitesimal* (resp. *first-order*) if $A \in \mathrm{ob}(\mathcal{A})$ (resp. $A = \mathbf{k}[\epsilon]$). Given another deformation

$$\xi : \begin{array}{ccc} X & \to & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & S \end{array}$$

of X over S, an isomorphism of η with ξ is an S-isomorphism $\phi : \mathcal{X} \to \mathcal{Y}$ inducing the identity on X, i.e. such that the following diagram is commutative:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & \xrightarrow{\phi} & & \mathcal{Y} \\ & \searrow & & \swarrow & \\ & & S & & \end{array}$$

By a *pointed scheme* we will mean a pair (S, s) where S is a scheme and $s \in S$. If K is a field we call (S, s) a K*-pointed scheme* if $K \cong \mathbf{k}(s)$.

Observe that for every X and for every $\mathbf{k}$-pointed scheme (S, s) there exists at least one family of deformation of X over S, namely the *product family*:

$$\begin{array}{ccc} X & \to & X \times S \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

A deformation of X over S is called *trivial* if it is isomorphic to the product family. It will be also called a *trivial family with fibre* X. All fibres over $\mathbf{k}$-rational points of a trivial deformation of X parametrized by an algebraic scheme are isomorphic to X. The converse is not true: there are deformations which are not trivial but have isomorphic fibres over all the $\mathbf{k}$-rational points (see Example 1.2.2(ii) below). The scheme X is called *rigid* if every infinitesimal deformation of X over A is trivial for every A in $\mathrm{ob}(\mathcal{A})$.

Given a deformation η of X over S as above and a morphism $(S', s') \to (S, s)$ of $\mathbf{k}$-pointed schemes there is induced a commutative diagram by base change

$$\begin{array}{ccc} X & \to & \mathcal{X} \times_S S' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & S' \end{array}$$

which is clearly a deformation of X over S'. This operation is functorial, in the sense that it commutes with composition of morphisms and the identity morphism does not change η. Moreover, it carries isomorphic deformations to isomorphic ones.

An infinitesimal deformation η of X is called *locally trivial* if every point $x \in X$ has an open neighbourhood $U_x \subset X$ such that

$$\begin{array}{ccc} U_x & \to & \mathcal{X}_{|U_x} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & S \end{array}$$

is a trivial deformation of U_x.

Remark 1.2.1. Let

$$\begin{array}{cccc} & X & \xrightarrow{j} & \mathcal{X} \\ \eta: & \downarrow & & \downarrow \pi \\ & \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

be a family of deformations of an algebraic scheme X parametrized by an algebraic scheme S and let $Z \subset X$ be a proper closed subset. Then

$$\begin{array}{ccc} X\backslash Z & \xrightarrow{j} & \mathcal{X}\backslash j(Z) \\ \downarrow & & \downarrow \pi' \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

is a family of deformations of $X\backslash Z$ having the same fibres as π over $t \in S$ for $t \neq s$: thus such fibres are deformations both of X and of $X\backslash Z$. This shows that the definition of family of deformations given above is somewhat ambiguous unless we assume that π is projective or that the deformation is infinitesimal. In what follows we will restrict to the consideration of deformations of projective schemes and/or of infinitesimal deformations when discussing the general theory, so that such ambiguity will be removed; only occasionally will we consider non-infinitesimal deformations of affine schemes.

Examples 1.2.2. (i) The quadric $Q \subset \mathbf{A}^3$ of equation $xy - t = 0$ defines, via the projection

$$\begin{array}{ccc} \mathbf{A}^3 & \rightarrow & \mathbf{A}^1 \\ (x, y, t) & \mapsto & t \end{array}$$

a flat family $Q \rightarrow \mathbf{A}^1$ whose fibres are affine conics. This family is not trivial since the fibre $Q(0)$ is singular, hence not isomorphic to the fibres $Q(t)$, $t \neq 0$, which are nonsingular.

(ii) Consider, for a given integer $m \geq 0$, the rational ruled surface

$$F_m = I\!P(\mathcal{O}_{I\!P^1}(m) \oplus \mathcal{O}_{I\!P^1})$$

The structural morphism $\pi : F_m \rightarrow I\!P^1$ defines a flat family whose fibres are all isomorphic to $I\!P^1$; but if $m > 0$ then π is not a trivial family because $F_m \ncong F_0 = I\!P^1 \times I\!P^1$ (see Example B.11(iii)).

(iii) Let $0 \leq n < m$ be two distinct nonnegative integers having the same parity and let $k = \frac{1}{2}(m - n)$. Consider two copies of $\mathbf{A}^2 \times I\!P^1$ given as $\mathrm{Proj}(\mathbf{k}[t, z, \xi_0, \xi_1]) =: W$ and $\mathrm{Proj}(\mathbf{k}[t, z', \xi_0', \xi_1']) =: W'$ (here the rings are graded with respect to the variables ξ_i and ξ_i'). Letting $\xi = \xi_1/\xi_0$ and $\xi' = \xi_1'/\xi_0'$ consider the open subsets

$$\mathrm{Spec}(\mathbf{k}[t, z, \xi]) \subset W, \quad \mathrm{Spec}(\mathbf{k}[t, z', \xi']) \subset W'$$

and glue them together along the open subsets

$$\mathrm{Spec}(\mathbf{k}[t, z, z^{-1}, \xi]) \subset \mathrm{Spec}(\mathbf{k}[t, z, \xi])$$

and

$$\mathrm{Spec}(\mathbf{k}[t, z', z'^{-1}, \xi']) \subset \mathrm{Spec}(\mathbf{k}[t, z', \xi'])$$

according to the following rules:

$$z' = z^{-1}, \quad \xi' = z^m \xi + t z^k \tag{1.9}$$

This induces a gluing of W and W' along

$$\mathrm{Proj}(\mathbf{k}[t, z, z^{-1}, \xi_0, \xi_1]) \quad \text{and} \quad \mathrm{Proj}(\mathbf{k}[t, z', z'^{-1}, \xi'_0, \xi'_1])$$

Call the resulting scheme $\mathcal{W}$ and $f : \mathcal{W} \to \mathbf{A}^1 = \mathrm{Spec}(\mathbf{k}[t])$ the morphism induced by the projections. Then f is a flat morphism because it is locally a projection; moreover,

$$\mathcal{W}(0) \cong F_m$$

Let $\mathcal{W}^\circ = f^{-1}(\mathbf{A}^1 \backslash \{0\})$ and $f^\circ : \mathcal{W}^\circ \to \mathbf{A}^1 \backslash \{0\}$ the restriction of f.

In $\mathbf{k}[t, t^{-1}, z, \xi]$ define

$$\zeta = \frac{z^k \xi - t}{t \xi}$$

and in $\mathbf{k}[t, t^{-1}, z', \xi']$

$$\zeta' = \frac{\xi'}{t z'^{m-k} \xi' + t^2}$$

It is straightforward to verify that the gluing (1.9) induces the relation

$$\zeta' = z^n \zeta$$

This means that we have an isomorphism

$$\mathcal{W}^\circ \cong F_n \times (\mathbf{A}^1 \backslash \{0\})$$

compatible with the projections to $\mathbf{A}^1 \backslash \{0\}$. Therefore the family f° is trivial, in particular all its fibres are isomorphic to F_n, but the family f is not trivial because $\mathcal{W}(0) \cong F_m$.

(iv) Let $f : X \to Y$ be a surjective morphism of algebraic schemes, with X integral and Y an irreducible and nonsingular curve. Then f is flat. This is a special case of Prop. III.9.7 of [84]. Therefore f defines a family of deformations of any of its closed fibres.

1.2.2 Infinitesimal deformations of nonsingular affine schemes

We will start by considering infinitesimal deformations of affine schemes. We need the following:

Lemma 1.2.3. *Let Z_0 be a closed subscheme of a scheme Z, defined by a sheaf of nilpotent ideals $N \subset \mathcal{O}_Z$. If Z_0 is affine then Z is affine as well.*

Proof. Let $r \geq 2$ be the smallest integer such that $N^r = (0)$. Since we have a chain of inclusions

$$Z \supset V(N^{r-1}) \supset V(N^{r-2}) \supset \cdots \supset V(N) = Z_0$$

it suffices to prove the assertion in the case $r = 2$. In this case N is a coherent $\mathcal{O}_{Z_0}$-module, and therefore

$$H^1(Z, N) = H^1(Z_0, N) = 0$$

Let R_0 be the $\mathbf{k}$-algebra such that $Z_0 = \operatorname{Spec}(R_0)$. We have the exact sequence:

$$0 \to H^0(Z, N) \to H^0(Z, \mathcal{O}_Z) \to R_0 \to 0$$

Put $R = H^0(Z, \mathcal{O}_Z)$ and let $Z' = \operatorname{Spec}(R)$. We have a commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\theta} & Z' \\ & \nwarrow\ \nearrow & \\ & Z_0 & \end{array}$$

The sheaf homomorphism $\theta^{-1}\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is clearly injective and θ is a homeomorphism. It will therefore suffice to prove that $\theta^{-1}\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is surjective.

Let $z \in Z$ and $f \in \Gamma(U, \mathcal{O}_Z)$ for some affine open neighbourhood U of z. Let $f_0 = f_{|U \cap Z_0}$. It is possible to find $\varphi_0, \psi_0 \in R_0$ such that $f_0 = \frac{\varphi_0}{\psi_0}$, $\psi_0(z) \neq 0$ and $\psi_0 = 0$ on $Z_0 \backslash U$, because Z_0 is affine. Let $\psi \in R$ be such that $\psi_{|Z_0} = \psi_0$ (it exists by the surjectivity of $R \to R_0$). Then $\psi(z) \neq 0$ and $\psi = 0$ on $Z \backslash U$. There exists $n \gg 0$ such that $\psi^n f =: g \in R$ (it suffices to cover Z with affines). Then $f = \frac{g}{\psi^n} \in \theta^{-1}\mathcal{O}_{Z'}$. □

Let B_0 be a $\mathbf{k}$-algebra, and let $X_0 = \operatorname{Spec}(B_0)$. Consider an infinitesimal deformation of X_0 parametrized by $\operatorname{Spec}(A)$, where A is in $\operatorname{ob}(\mathcal{A})$. By definition this is a cartesian diagram

$$\begin{array}{ccc} X_0 & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathbf{k}) & \to & \operatorname{Spec}(A) \end{array}$$

where $\mathcal{X}$ is a scheme flat over $\operatorname{Spec}(A)$. By Lemma 1.2.3 $\mathcal{X}$ is necessarily affine. Therefore, equivalently, we can talk about an *infinitesimal deformation of* B_0 *over* A as a cartesian diagram of $\mathbf{k}$-algebras:

$$\begin{array}{ccc} B & \to & B_0 \\ \uparrow & & \uparrow \\ A & \to & \mathbf{k} \end{array} \tag{1.10}$$

with $A \to B$ flat. Note that to give this diagram is the same as to give $A \to B$ flat and a $\mathbf{k}$-isomorphism $B \otimes_A \mathbf{k} \to B_0$. We will sometimes abbreviate by calling $A \to B$ the deformation.

Given another deformation $A \to B'$ of B_0 over A, an isomorphism of deformations of $A \to B$ to $A \to B'$ is a homomorphism $\varphi : B \to B'$ of A-algebras inducing a commutative diagram:

$$\begin{tikzcd}
 & B_0 & \\
B \arrow[ur] \arrow[rr, "\varphi"] & & B' \arrow[ul] \\
 & A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

It follows from Lemma A.4 that such a φ is an isomorphism.

An infinitesimal deformation of B_0 over A is trivial if it is isomorphic to the product deformation

$$\begin{array}{ccc} B_0 \otimes_{\mathbf{k}} A & \to & B_0 \\ \uparrow & & \uparrow \\ A & \to & \mathbf{k} \end{array}$$

The $\mathbf{k}$-algebra B_0 is called *rigid* if $\mathrm{Spec}(B_0)$ is rigid.

Theorem 1.2.4. *Every smooth* $\mathbf{k}$*-algebra is rigid. In particular, every affine nonsingular algebraic variety is rigid.*

Proof. Suppose $\mathbf{k} \to B_0$ is smooth, and suppose given a first-order deformation of B_0:

$$\eta_0 : \begin{array}{ccc} B & \to & B_0 \\ \uparrow f & & \uparrow \\ \mathbf{k}[\epsilon] & \to & \mathbf{k} \end{array}$$

Consider the commutative diagram:

$$\begin{array}{ccc} B & \to & B_0 \\ \uparrow f & & \uparrow \\ \mathbf{k}[\epsilon] & \to & B_0[\epsilon] \end{array}$$

where $B_0[\epsilon] = B_0 \otimes \mathbf{k}[\epsilon]$. Since f is smooth (because flat with smooth fibre, see [84], ch. III, Th. 10.2) and the right vertical morphism is a $\mathbf{k}[\epsilon]$-extension, by Theorem C.9 there exists a $\mathbf{k}[\epsilon]$-homomorphism $\phi : B \to B_0[\epsilon]$ making the diagram

$$\begin{array}{ccc} B & \to & B_0 \\ \uparrow f & \searrow & \uparrow \\ \mathbf{k}[\epsilon] & \to & B_0[\epsilon] \end{array}$$

commutative. Therefore ϕ is an isomorphism of deformations and η_0 is trivial.

Consider more generally a deformation of B_0

$$\eta : \begin{array}{ccc} B & \to & B_0 \\ \uparrow f & & \uparrow \\ A & \to & \mathbf{k} \end{array}$$

parametrized by A in $\mathrm{ob}(\mathcal{A})$. To show that η is trivial we proceed by induction on $d = \dim_{\mathbf{k}}(A)$. The case $d = 2$ has been already proved; assume $d \geq 3$ and let

$$0 \to (t) \to A \to A' \to 0$$

be a small extension. Consider the commutative diagram:

$$\begin{array}{ccc} B & \to & B\otimes_A A' \quad \cong B_0\otimes_{\mathbf{k}} A' \\ \uparrow f & & \uparrow \\ A & \to & B_0\otimes_{\mathbf{k}} A \end{array}$$

f is smooth, the upper right isomorphism is by the inductive hypothesis, and the right vertical homomorphism is an A-extension. By the smoothness of f and by Theorem C.9 we deduce the existence of an A-homomorphism $B \to B_0 \otimes_{\mathbf{k}} A$ which is an isomorphism of deformations. □

Example 1.2.5. Let $\lambda \in \mathbf{k}$ and $B_0 = \mathbf{k}[X, Y]/(Y^2 - X(X-1)(X-\lambda))$. If $\lambda \neq 0, 1$ then B_0 is a smooth $\mathbf{k}$-algebra, being the coordinate ring of a nonsingular plane cubic curve. By Theorem 1.2.4, B_0 is rigid. On the other hand, the elementary theory of elliptic curves (see [84]) shows that the following flat family of affine curves

$$\begin{array}{c} \mathrm{Spec}\mathbf{k}[X, Y]/(Y^2 - X(X-1)(X-(\lambda+t))) \\ \downarrow \\ \mathrm{Spec}(\mathbf{k}[t]) \end{array}$$

is not trivial around the origin $t = 0$ so that it defines a nontrivial (non-infinitesimal) deformation of B_0. This example shows that by studying infinitesimal deformations of affine schemes we are losing some information. In this specific case we will see that this information is recovered by considering the infinitesimal deformations of the projective closure of $\mathrm{Spec}(B_0)$ (see Corollary 2.6.6, page 94).

1.2.3 Extending automorphisms of deformations

In deformation theory it is very important to have a good control of automorphisms of deformations and of their extendability properties. We will now begin to introduce such matters and to recall some terminology. In Section 2.6 we will consider these problems again in general, and we will relate them with the property of "prorepresentability". Let's start with a basic lemma.

Lemma 1.2.6. *Let B_0 be a* **k***-algebra, and*

$$e : 0 \to (t) \to \tilde{A} \to A \to 0$$

a small extension in $\mathcal{A}$. Then there is a canonical isomorphism of groups:

$$\left\{ \begin{array}{c} \text{automorphisms of the trivial deformation } B_0 \otimes_{\mathbf{k}} \tilde{A} \\ \text{inducing the identity on } B_0 \otimes_{\mathbf{k}} A \end{array} \right\} \to \mathrm{Der}_{\mathbf{k}}(B_0, B_0)$$

In particular the group on the left is abelian.

Proof. Every automorphism $\theta : B_0 \otimes_{\mathbf{k}} \tilde{A} \to B_0 \otimes_{\mathbf{k}} \tilde{A}$ belonging to the first group must be $\tilde{A}$-linear and induce the identity mod t. Therefore:

$$\theta(x) = x + t dx$$

where $d : B_0 \otimes_{\mathbf{k}} \tilde{A} \to B_0$ is a $\tilde{A}$-derivation (Lemma 1.1.1). But

$$\mathrm{Der}_{\tilde{A}}(B_0 \otimes_{\mathbf{k}} \tilde{A}, B_0) = \mathrm{Hom}_{B_0 \otimes_{\mathbf{k}} \tilde{A}}(\Omega_{B_0 \otimes_{\mathbf{k}} \tilde{A}/\tilde{A}}, B_0)$$

$$= \mathrm{Hom}_{B_0}(\Omega_{B_0/\mathbf{k}}, B_0) = \mathrm{Der}_{\mathbf{k}}(B_0, B_0)$$

By sending $\theta \mapsto d$ we define the correspondence of the statement. Since θ is determined by d the correspondence is one to one. Clearly the identity corresponds to the zero derivation. If we compose two automorphisms:

$$B_0 \otimes_{\mathbf{k}} \tilde{A} \xrightarrow{\theta} B_0 \otimes_{\mathbf{k}} \tilde{A} \xrightarrow{\sigma} B_0 \otimes_{\mathbf{k}} \tilde{A}$$

where $\theta(x) = x + t dx$, $\sigma(x) = x + t\delta x$, we obtain:

$$\sigma(\theta(x)) = \theta(x) + t\delta(\theta(x)) = x + t dx + t(\delta x + t\delta(dx)) = x + t(dx + \delta x)$$

therefore the correspondence is a group isomorphism. □

Recall the following well-known definition.

Definition 1.2.7. *Let G be a group acting on a set T and let*

$$\pi : G \times T \to T$$

be the map defining the action. T is called a homogeneous space *under (the action of) G if π is transitive, i.e. if*

$$\pi(G \times \{t\}) = T$$

for some (equivalently for any) $t \in T$ (i.e. if there is only one orbit). The action is called free *if for every point $t \in T$ the stabilizer $G_t = \{g \in G : gt = t\}$ is trivial, i.e. $gt = t$ implies $g = 1_G$ for all $t \in T$. If the action is both transitive and free then T is called a* principal homogeneous space *(or a* torsor*) under (the action of) G.*

To an action $\pi : G \times T \to T$ we can associate the map:

$$\begin{array}{rccc} p: & G \times T & \to & T \times T \\ & (g, t) & \mapsto & (gt, t) \end{array}$$

The condition that the action is transitive (resp. free) is equivalent to p being surjective (resp. injective); therefore T is a torsor under G if and only if p is bijective. Note that $\pi = \mathrm{pr}_1 p$ is determined by p.

More generally, suppose that we have a map of sets $f : T \to T'$. Then the map p factors through $T \times_{T'} T \subset T \times T$ if and only if the action π is compatible with f, i.e. if $f(t) = f(gt)$ for all $t \in T$, $g \in G$. As before, the map

$$p : G \times T \to T \times_{T'} T$$

is surjective (resp. injective) if and only if the action of G is transitive (resp. free) on all the non-empty fibres of f. In particular, p is bijective if and only if all the non-empty fibres of f are torsors under G. In this case one also says, according to [3],

p. 114, that T over T' is a *formal principal homogeneous space* (or a *pseudo-torsor*) under G.

Now we come back to deformations and we prove a generalization of Lemma 1.2.6.

Lemma 1.2.8. *Let B_0 be a $\mathbf{k}$-algebra,*

$$e : 0 \to (t) \to \tilde{A} \to A \to 0$$

a small extension in $\mathcal{A}$, $\tilde{A} \to \tilde{B}$ a deformation of B_0 and $A \to B = \tilde{B} \otimes_{\tilde{A}} A$ the induced deformation of B_0 over A. Let $\sigma : B \to B$ be an automorphism of the deformation. Then:

(i) If

$$\mathrm{Aut}_\sigma(\tilde{\mathrm{B}}) := \left\{\text{automorphisms } \tau : \tilde{B} \to \tilde{B} \text{ such that } \tau \otimes_{\tilde{A}} A = \sigma\right\} \neq \emptyset$$

then there is a free and transitive action

$$\mathrm{Der}_{\mathbf{k}}(B_0, B_0) \times \mathrm{Aut}_\sigma(\tilde{\mathrm{B}}) \to \mathrm{Aut}_\sigma(\tilde{\mathrm{B}})$$

defined by

$$(d, \tau) \mapsto \tau + td$$

(ii) If B_0 is a smooth $\mathbf{k}$-algebra then $\mathrm{Aut}_\sigma(\tilde{\mathrm{B}}) \neq \emptyset$ for any σ.

Proof. (i) Recall that we have a chain of natural identifications

$$\mathrm{Der}_{\tilde{A}}(\tilde{B}, B_0) = \mathrm{Hom}_{\tilde{B}}(\Omega_{\tilde{B}/\tilde{A}}, B_0) = \mathrm{Hom}_{B_0}(\Omega_{\tilde{B}/\tilde{A}} \otimes_{\tilde{A}} \mathbf{k}, B_0)$$

$$= \mathrm{Hom}_{B_0}(\Omega_{B_0/\mathbf{k}}, B_0) = \mathrm{Der}_{\mathbf{k}}(B_0, B_0)$$

Therefore the action in the statement is well defined once we consider a $d \in \mathrm{Der}_{\mathbf{k}}(B_0, B_0)$ as an $\tilde{A}$-derivation of $\tilde{B}$ into B_0. Given any two elements $\tau, \eta \in \mathrm{Aut}_\sigma(\tilde{\mathrm{B}})$ we have by definition:

$$q\tau = \sigma q = q\eta$$

where $q : \tilde{B} \to B$ is the projection; hence by Lemma 1.1.1,

$$\eta - \tau : \tilde{B} \to tB_0 = \ker(q)$$

is an $\tilde{A}$-derivation which is 0 if and only if $\eta = \tau$. This implies that the action is free and transitive.

(ii) Since B_0 is smooth the deformation $\tilde{A} \to \tilde{B}$ is trivial (Theorem 1.2.4), so that we have an $\tilde{A}$-isomorphism $\tilde{B} \cong B_0 \otimes_{\mathbf{k}} \tilde{A}$; moreover, $B_0 \otimes_{\mathbf{k}} \tilde{A}$ is a smooth $\tilde{A}$-algebra (Proposition C.2(iii)) hence $\tilde{B}$ is $\tilde{A}$-smooth. Let $\sigma : B \to B$ be any automorphism of the deformation and consider the diagram of $\tilde{A}$-algebras:

$$\begin{array}{ccccc} \tilde{B} & \xrightarrow{q} & B & \xrightarrow{\sigma} & B \\ & & & & \uparrow q \\ & & & & \tilde{B} \end{array}$$

Since $\ker(q) = tB_0$ is a square-zero ideal, and since $\tilde{B}$ is $\tilde{A}$-smooth, we deduce that there is $\tilde{\sigma} : \tilde{B} \to \tilde{B}$ such that $q\tilde{\sigma} = \sigma q$. It is immediate to check that $\tilde{\sigma}$ is an isomorphism and therefore $\tilde{\sigma} \in \mathrm{Aut}_\sigma(\tilde{\mathrm{B}})$. □

In (ii) the condition that B_0 is smooth cannot be removed. A simple example is given in 2.6.8(i). The extendability of automorphisms of deformations of not necessarily affine schemes will be studied in § 2.6.

1.2.4 First-order locally trivial deformations

We will now apply 1.2.6 to first-order deformations of any algebraic variety.

Proposition 1.2.9. *Let X be an algebraic variety. There is a 1–1 correspondence:*

$$\kappa : \left\{ \begin{array}{c} \text{isomorphism classes of first order} \\ \text{locally trivial deformations of } X \end{array} \right\} \to H^1(X, T_X)$$

called the Kodaira–Spencer correspondence*, where $T_X = Hom(\Omega^1_X, \mathcal{O}_X) = Der_{\mathbf{k}}(\mathcal{O}_X, \mathcal{O}_X)$, such that $\kappa(\xi) = 0$ if and only if ξ is the trivial deformation class. In particular if X is nonsingular then κ is a 1–1 correspondence*

$$\kappa : \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{first-order deformations of } X \end{array} \right\} \to H^1(X, T_X)$$

Proof. Given a first-order locally trivial deformation

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

choose an affine open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\mathcal{X}_{|U_i}$ is trivial for all i. For each index i we therefore have an isomorphism of deformations:

$$\theta_i : U_i \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \to \mathcal{X}_{|U_i}$$

by 1.2.4. Then for each $i, j \in I$

$$\theta_{ij} := \theta_i^{-1}\theta_j : U_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \to U_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon])$$

is an automorphism of the trivial deformation $U_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon])$. By Lemma 1.2.6, θ_{ij} corresponds to a $d_{ij} \in \Gamma(U_{ij}, T_X)$. Since on each U_{ijk} we have

$$\theta_{ij}\theta_{jk}\theta_{ik}^{-1} = 1_{U_{ijk} \times \mathrm{Spec}(\mathbf{k}[\epsilon])} \tag{1.11}$$

it follows that

$$d_{ij} + d_{jk} - d_{ik} = 0$$

i.e. $\{d_{ij}\}$ is a Čech 1-cocycle and therefore defines an element of $H^1(X, T_X)$. It is easy to check that this element does not depend on the choice of the open cover $\mathcal{U}$. If we have another deformation

$$\begin{array}{ccc} X & \rightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \rightarrow & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

and $\Phi : \mathcal{X} \to \mathcal{X}'$ is an isomorphism of deformations then for each $i \in I$ there is induced an automorphism:

$$\alpha_i : U_i \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \xrightarrow{\theta_i} \mathcal{X}_{|U_i} \xrightarrow{\Phi_{|U_i}} \mathcal{X}'_{|U_i} \xrightarrow{\theta_i'^{-1}} U_i \times \mathrm{Spec}(\mathbf{k}[\epsilon])$$

and therefore a corresponding $a_i \in \Gamma(U_i, T_X)$. We have $\theta'_i \alpha_i = \Phi_{|U_i} \theta_i$ and therefore

$$(\theta'_i \alpha_i)^{-1} (\theta'_j \alpha_j) = \theta_i^{-1} \Phi^{-1}_{|U_{ij}} \Phi_{|U_{ij}} \theta_j = \theta_i^{-1} \theta_j$$

thus

$$\alpha_i^{-1} \theta'_{ij} \alpha_j = \theta_{ij}$$

Equivalently:

$$d'_{ij} + a_j - a_i = d_{ij}$$

namely, $\{d_{ij}\}$ and $\{d'_{ij}\}$ are cohomologous, and therefore define the same element of $H^1(X, T_X)$.

Conversely, given $\theta \in H^1(X, T_X)$ we can represent it by a Čech 1-cocycle $\{d_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, T_X)$ with respect to some affine open cover $\mathcal{U}$. To each d_{ij} we can associate an automorphism θ_{ij} of the trivial deformation $U_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon])$ by Lemma 1.2.6. They satisfy the identities (1.11). We can therefore use these automorphisms to patch the schemes $U_i \times \mathrm{Spec}(\mathbf{k}[\epsilon])$ by the well-known procedure (see [84], p. 69). We obtain a $\mathrm{Spec}(\mathbf{k}[\epsilon])$-scheme $\mathcal{X}$ which, by construction, defines a locally trivial first-order deformation of X. The equivalence between $\kappa(\xi) = 0$ and the triviality of ξ is easily proved. The last assertion follows from the first one because all deformations of a nonsingular variety are locally trivial by Theorem 1.2.4. □

Definition 1.2.10. *For every locally trivial first-order deformation ξ of a variety X the cohomology class $\kappa(\xi) \in H^1(X, T_X)$ is called the* Kodaira–Spencer class *of ξ.*

Let

$$\xi : \begin{array}{ccc} X & \rightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array} \tag{1.12}$$

be a family of deformations of a nonsingular variety X. By pulling back this family by morphisms $\mathrm{Spec}(\mathbf{k}[\epsilon]) \to S$ with image s and applying the Kodaira–Spencer correspondence (Proposition 1.2.9) we define a linear map

$$\kappa_\xi : T_{S,s} \to H^1(X, T_X)$$

also denoted by $\kappa_{f,s}$ or $\kappa_{\mathcal{X}/S,s}$, which is called the *Kodaira–Spencer map* of the family ξ.

Examples 1.2.11. (i) Let $m \geq 1$ and let $\pi : F_m \to \mathbb{P}^1$ be the structural morphism of the rational ruled surface F_m (see B.11(iii)). Then π is not a trivial family but has a trivial restriction around each closed point $s \in \mathbb{P}^1$, thus $\kappa_{\pi,s} = 0$.

(ii) Consider an unramified covering $\pi : X \to S$ of degree $n \geq 2$ where X and S are projective nonsingular and irreducible algebraic curves. All fibres of π over the closed points consist of n distinct points, hence they are all isomorphic. Moreover, each such fibre is rigid and unobstructed as an abstract variety. In particular the Kodaira–Spencer map is zero at each closed point $s \in S$. On the other hand, $\pi^{-1}(U)$ is irreducible for each open subset $U \subset S$ and therefore the restriction $\pi_U : \pi^{-1}(U) \to U$ is a nontrivial family; this follows also from the fact that π does not have rational sections.

This example exhibits a phenomenon which is not detected by infinitesimal considerations and in some sense opposite to the one described in Example 1.2.5: we can have a flat projective family of deformations, all of whose geometric fibres are isomorphic, but which is nevertheless nontrivial over every Zariski open subset of the base. Note that this is different from what happens with the projections $F_m \to \mathbb{P}^1$, $m \geq 1$ of Example (i), which are nontrivial but have trivial restriction to a Zariski open neighbourhood of every point of $\mathbb{P}^1$. See Subsection 2.6.2 for more about this.

(iii) Let $0 \leq n < m$ be integers having the same parity, and let $k = \frac{1}{2}(m-n)$. Consider the smooth proper morphism $f : \mathcal{W} \to \mathbf{A}^1$ introduced in Example 1.2.2(iii), whose fibres are $\mathcal{W}(0) \cong F_m$, and $\mathcal{W}(t) \cong F_n$ for $t \neq 0$. Recall that the family f is given as the gluing of two copies of $\mathbf{A}^2 \times \mathbb{P}^1$:

$$W = \mathrm{Proj}(\mathbf{k}[t, z, \xi_0, \xi_1]), \qquad W' = \mathrm{Proj}(\mathbf{k}[t, z', \xi_0', \xi_1'])$$

along $\mathrm{Proj}(\mathbf{k}[t, z, z^{-1}, \xi_0, \xi_1])$ and $\mathrm{Proj}(\mathbf{k}[t, z', z'^{-1}, \xi_0', \xi_1'])$ according to the rules:

$$z' = z^{-1}, \;\; \xi' = z^m \xi + t z^k$$

where $\xi = \xi_1/\xi_0$ and $\xi' = \xi_1'/\xi_0'$ are nonhomogeneous coordinates on the corresponding copies of $\mathbb{P}^1$.

Let's compute the local Kodaira–Spencer map $\kappa_{f,0}$ of f at 0. The image $\kappa_{f,0}(\frac{d}{dt})$ is the element of $H^1(F_m, T_{F_m})$ corresponding to the first-order deformation of F_m obtained by gluing

$$W_0 := \mathrm{Proj}(\mathbf{k}[\epsilon, z, \xi_0, \xi_1]) \qquad W_0' := \mathrm{Proj}(\mathbf{k}[\epsilon, z', \xi_0', \xi_1'])$$

along $\mathrm{Proj}(\mathbf{k}[\epsilon, z, z^{-1}, \xi_0, \xi_1])$ and $\mathrm{Proj}(\mathbf{k}[\epsilon, z', z'^{-1}, \xi_0', \xi_1'])$ according to the rules

$$z' = z^{-1}, \;\; \xi' = z^m \xi + \epsilon z^k$$

By definition we have that $\kappa_{f,0}(\frac{d}{dt})$ is the element of $H^1(\mathcal{U}, T_{F_m})$, where $\mathcal{U} = \{W_0, W_0'\}$, defined by the 1-cocycle corresponding to the vector field on $W_0 \cap W_0'$

$$\left\{z^k \frac{\partial}{\partial \xi}\right\}$$

According to Example B.11(iii) this element is nonzero; therefore $\kappa_{f,0}$ is injective.

Similarly, we can consider a smooth proper family $F : \mathcal{Y} \to \mathbf{A}^{m-1}$ defined as follows. $\mathcal{Y}$ is the gluing of

$$Y := \mathrm{Proj}(\mathbf{k}[t_1, \ldots, t_{m-1}, z, \xi_0, \xi_1])$$

and

$$Y' := \mathrm{Proj}(\mathbf{k}[t_1, \ldots, t_{m-1}, z', \xi_0', \xi_1'])$$

along $\mathrm{Proj}(\mathbf{k}[t_1, \ldots, t_{m-1}, z, z^{-1}, \xi_0, \xi_1])$ and $\mathrm{Proj}(\mathbf{k}[t_1, \ldots, t_{m-1}, z', z'^{-1}, \xi_0', \xi_1'])$ according to the rules:

$$z' = z^{-1}, \quad \xi' = z^m \xi + \sum_{j=1}^{m-1} t_j z^j$$

The morphism F is defined by the projections onto $\mathrm{Spec}(\mathbf{k}[t_1, \ldots, t_{m-1}])$; the fibre of F over $\underline{0}$ is $\mathcal{Y}(\underline{0}) \cong F_m$. The computation we just did immediately implies that the local Kodaira–Spencer map

$$\kappa_{F,\underline{0}} : T_{\underline{0}}\mathbf{A}^{m-1} \to H^1(F_m, T_{F_m})$$

is an isomorphism.

1.2.5 Higher-order deformations – obstructions

Let X be a nonsingular algebraic variety. Consider a small extension

$$e : 0 \to (t) \to \tilde{A} \to A \to 0$$

in $\mathcal{A}$ and let

$$\xi : \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

be an infinitesimal deformation of X. A *lifting of* ξ *to* $\tilde{A}$ consists in a deformation

$$\tilde{\xi} : \begin{array}{ccc} X & \to & \tilde{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\tilde{A}) \end{array}$$

and an isomorphism of deformations

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathcal{X} \arrow[rr, "\phi"] \arrow[dr] & & \tilde{\mathcal{X}} \times_{\mathrm{Spec}(\tilde{A})} \mathrm{Spec}(A) \arrow[dl] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

If we want to study arbitrary infinitesimal deformations, and not only first-order ones, it is important to know whether, given ξ and e, a lifting of ξ to $\tilde{A}$ exists, and how many there are. Such information can then be used to build an inductive procedure for the description of infinitesimal deformations. The following proposition addresses this question.

Proposition 1.2.12. *Given A in* ob$(\mathcal{A})$ *and an infinitesimal deformation ξ of X over A:*

(i) To every small extension e of A there is associated an element $o_\xi(e) \in H^2(X, T_X)$ called the obstruction to lifting ξ to $\tilde{A}$, *which is* 0 *if and only if a lifting of ξ to $\tilde{A}$ exists.*

(ii) If $o_\xi(e) = 0$ then there is a natural transitive action of $H^1(X, T_X)$ on the set of isomorphism classes of liftings of ξ to $\tilde{A}$.

(iii) The correspondence $e \mapsto o_\xi(e)$ defines a **k***-linear map*

$$o_\xi : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to H^2(X, T_X)$$

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an affine open cover of X. We have isomorphisms

$$\theta_i : U_i \times \mathrm{Spec}(A) \to \mathcal{X}_{|U_i}$$

and consequently, $\theta_{ij} := \theta_i^{-1}\theta_j$ is an automorphism of the trivial deformation $U_{ij} \times \mathrm{Spec}(A)$. Moreover,

$$\theta_{ij}\theta_{jk} = \theta_{ik} \tag{1.13}$$

on $U_{ijk} \times \mathrm{Spec}(A)$. To give a lifting $\tilde{\xi}$ of ξ to $\tilde{A}$ it is necessary and sufficient to give a collection of automorphisms $\{\tilde{\theta}_{ij}\}$ of the trivial deformations $U_{ij} \times \mathrm{Spec}(\tilde{A})$ such that

(a) $\tilde{\theta}_{ij}\tilde{\theta}_{jk} = \tilde{\theta}_{ik}$
(b) $\tilde{\theta}_{ij}$ restricts to θ_{ij} on $U_{ij} \times \mathrm{Spec}(A)$

In fact from such data we will be able to define $\tilde{\mathcal{X}}$ by patching the local pieces $U_i \times \mathrm{Spec}(\tilde{A})$ along the open subsets $U_{ij} \times \mathrm{Spec}(\tilde{A})$ in the usual way. To establish the existence of the collection $\{\tilde{\theta}_{ij}\}$ let's choose arbitrarily automorphisms $\{\tilde{\theta}_{ij}\}$ satisfying condition (b); they exist by Lemma 1.2.8(ii). Let

$$\tilde{\theta}_{ijk} = \tilde{\theta}_{ij}\tilde{\theta}_{jk}\tilde{\theta}_{ik}^{-1}$$

This is an automorphism of the trivial deformation $U_{ijk} \times \mathrm{Spec}(\tilde{A})$. Since by (1.13) it restricts to the identity on $U_{ijk} \times \mathrm{Spec}(A)$, by Lemma 1.2.6 we can identify each $\tilde{\theta}_{ijk}$

with a $\tilde{d}_{ijk} \in \Gamma(U_{ijk}, T_X)$ and it is immediate to check that $\{\tilde{d}_{ijk}\} \in \mathcal{Z}^2(\mathcal{U}, T_X)$. If we choose different automorphisms $\{\Phi_{ij}\}$ of the trivial deformations $U_{ij} \times \mathrm{Spec}(\tilde{A})$ satisfying the analogue of condition (b) then

$$\Phi_{ij} = \tilde{\theta}_{ij} + t d_{ij} \tag{1.14}$$

for some $d_{ij} \in \Gamma(U_{ij}, T_X)$, by Lemma 1.2.8(i). For each i, j, k the automorphism

$$\Phi_{ij}\Phi_{jk}\Phi_{ik}^{-1}$$

corresponds to the derivation

$$\delta_{ijk} = \tilde{d}_{ijk} + (d_{ij} + d_{jk} - d_{ik})$$

and therefore we see that the 2-cocycles $\{\tilde{d}_{ijk}\}$ and $\{\delta_{ijk}\}$ are cohomologous. Their cohomology class

$$o_\xi(e) \in H^2(X, T_X)$$

depends only on ξ and e and is 0 if and only if we can find a collection of automorphisms $\{\Phi_{ij}\}$ such that $\delta_{ijk} = 0$ for all $i, j, k \in I$. In such a case $\{\Phi_{ij}\}$ defines a lifting $\tilde{\xi}$ of ξ. This proves (i).

Assume that $o_\xi(e) = 0$, i.e. that the lifting $\tilde{\xi}$ of ξ exists. Then we can choose the collection $\{\tilde{\theta}_{ij}\}$ of automorphisms satisfying conditions (a) and (b) as above, in particular $\tilde{d}_{ijk} = 0$, all i, j, k. Any other choice of a lifting $\bar{\xi}$ of ξ to $\tilde{A}$ corresponds to a choice of automorphisms $\{\Phi_{ij}\}$ satisfying (1.14) and the analogue of condition (b). Therefore, for all i, j, k, we have

$$0 = \delta_{ijk} = d_{ij} + d_{jk} - d_{ik}$$

so that $\{d_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, T_X)$ defines an element $\bar{d} \in H^1(X, T_X)$. As before, one checks that this element depends only on the isomorphism class of $\bar{\xi}$; it follows in a straightforward way that the correspondence $(\tilde{\xi}, \bar{d}) \mapsto \bar{\xi}$ defines a transitive action of $H^1(X, T_X)$ on the set of isomorphism classes of liftings of ξ to $\tilde{A}$. This proves (ii).

(iii) is left to the reader. □

Definition 1.2.13. *The deformation ξ is called* unobstructed *if o_ξ is the zero map; otherwise ξ is called* obstructed*. X is* unobstructed *if every infinitesimal deformation of X is unobstructed; otherwise X is* obstructed*.*

Corollary 1.2.14. *A nonsingular variety X is unobstructed if*

$$H^2(X, T_X) = (0)$$

The proof is obvious.

Corollary 1.2.15. *A nonsingular variety X is rigid if and only if*

$$H^1(X, T_X) = (0)$$

Proof. The hypothesis implies, by Proposition 1.2.9, that all first-order deformations of X are trivial; moreover, by Proposition 1.2.12(ii), it implies that every infinitesimal deformation of X over any A in ob($\mathcal{A}$) has at most one lifting to any small extension of A. These two facts together easily give the conclusion. □

Examples 1.2.16. (i) If X is a projective nonsingular curve of genus g then from the Riemann–Roch theorem it follows that

$$h^1(X, T_X) = \begin{cases} 0 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 3g - 3 & \text{if } g \geq 2 \end{cases}$$

and $h^2(X, T_X) = 0$. In particular, projective nonsingular curves are unobstructed.

(ii) If X is a projective, irreducible and nonsingular surface X then

$$H^2(X, T_X) \cong H^0(X, \Omega^1_X \otimes K_X)^\vee$$

by Serre duality, and this rarely vanishes. For example, a nonsingular surface of degree ≥ 5 in $\mathbb{P}^3$ satisfies $H^2(X, T_X) \neq (0)$, but it is nevertheless unobstructed (see Example 3.2.11(i)); therefore the sufficient condition of Corollary 1.2.14 is not necessary. In general a surface such that $H^2(X, T_X) \neq (0)$ can be obstructed, but explicit examples are not elementary (see [96], [24], [87]). We will describe a class of such examples in Theorem 3.4.26, page 185. In § 2.4 we will show how to construct examples of obstructed 3-folds (see remarks following Proposition 3.4.25, page 185). The first examples of obstructed compact complex manifolds where given in Kodaira–Spencer [107], § 16: they are of the form $T \times \mathbb{P}^1$, where T is a two-dimensional complex torus.

(iii) The projective space $\mathbb{P}^n$ is rigid for every $n \geq 1$. In fact it follows immediately from the Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

that $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$. Similarly, one shows that finite products

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

of projective spaces are rigid.

(iv) The ruled surfaces F_m are unobstructed because

$$h^2(F_m, T_{F_m}) = 0$$

(see (B.13), page 292).

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