Solutions for Chapter 2

Exercise 2.1.

(i) Jacobian determinant:

$$\det\left(\frac{\partial x_i}{\partial a_i}\right) = \det\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & e^t + e^{-t} & e^t - e^{-t} \\ 0 & e^t - e^{-t} & e^t + e^{-t} \end{bmatrix} = 1$$

Inverse transformation:

$$\begin{aligned} a_1 &= 4x_1 \\ a_2 &= x_2(e^t + e^{-t})/4 - x_3(e^t - e^{-t})/4 \\ a_3 &= x_3(e^t + e^{-t})/4 - x_2(e^t - e^{-t})/4 \end{aligned}$$

(ii) Eulerian displacements:

$$\begin{aligned} u_1(x_1, x_2, x_3) &= -3x_1 \\ u_2(x_1, x_2, x_3) &= x_2 - x_2(e^t + e^{-t})/4 - x_3(e^t - e^{-t})/4 \\ u_3(x_1, x_2, x_3) &= x_3 - x_3(e^t + e^{-t})/4 - x_2(e^t - e^{-t})/4 \end{aligned}$$

Lagrangian displacements:

$$\begin{aligned} u_1(a_1, a_2, a_3) &= -3a_1/4 \\ u_2(a_1, a_2, a_3) &= a_2(e^t + e^{-t}) + a_3(e^t - e^{-t}) - a_2 \\ u_3(a_1, a_2, a_3) &= a_3(e^t + e^{-t}) + a_2(e^t - e^{-t}) - a_3 \end{aligned}$$

Eulerian velocity components:

$$\begin{aligned} v_1(x_1, x_2, x_3) &= 0 \\ v_2(x_1, x_2, x_3) &= x_2(e^t - e^{-t})/4 - x_3(e^t + e^{-t})/4 \\ v_3(x_1, x_2, x_3) &= x_3(e^t - e^{-t})/4 - x_2(e^t + e^{-t})/4 \end{aligned}$$

Lagrangian velocity components:

$$\begin{aligned} v_1(a_1, a_2, a_3) &= 0 \\ v_2(a_1, a_2, a_3) &= a_2(e^t - e^{-t}) + a_3(e^t + e^{-t}) \\ v_3(a_1, a_2, a_3) &= a_3(e^t - e^{-t}) + a_2(e^t + e^{-t}) \end{aligned}$$

(iii) Green-Lagrange strain tensor:

$$G_{ij} = \frac{1}{2}\begin{bmatrix} -7/16 & 0 & 0 \\ 0 & e^{2t} + e^{-2t} + 1 & (e^t - e^{-t})(1 + e^t) \\ 0 & (e^t - e^{-t})(1 + e^t) & e^{2t} + e^{-2t} + 1 \end{bmatrix}$$

(Linear) Green-Cauchy strain tensor:

$$\varepsilon_{ij} = \frac{1}{2}\begin{bmatrix} -3/2 & 0 & 0 \\ 0 & 2(e^t + e^{-t}) & 2(e^t - e^{-t}) \\ 0 & 2(e^t - e^{-t}) & 2(e^t + e^{-t}) \end{bmatrix}$$

Exercise 2.2.

Balance of moment of momentum:

$$\frac{D}{Dt}\int\limits_V (\rho x_i v_j \varepsilon_{ijk})\,\mathrm{d}V = \int\limits_V (\rho x_i f_j \varepsilon_{ijk})\,\mathrm{d}V + \int\limits_S (x_j T_{li} n_l \varepsilon_{ijk})\,\mathrm{d}S$$

For first term:

$$\begin{aligned} \frac{D}{Dt}\int\limits_V (\rho x_i v_j \varepsilon_{ijk})\,\mathrm{d}V &= \int\limits_V (x_i v_j \varepsilon_{ijk})\frac{D}{Dt}(\rho \mathrm{d}V) + \int\limits_V (\rho v_i v_j \varepsilon_{ijk})\,\mathrm{d}V + \\ &\quad \int\limits_V (\rho x_i \frac{Dv_j}{Dt}\varepsilon_{ijk})\,\mathrm{d}V \\ &= \int\limits_V (\rho x_i \frac{Dv_j}{Dt}\varepsilon_{ijk})\,\mathrm{d}V \end{aligned}$$

because $\dfrac{D}{Dt}(\rho \mathrm{d}V) = 0$ (mass conservation) and $v_i v_j \varepsilon_{ijk} = 0$.
For third term:

$$\int\limits_S (x_j T_{li} n_l \varepsilon_{ijk})\,\mathrm{d}S = \int\limits_V (x_i \frac{\partial T_{il}}{\partial x_l}\varepsilon_{ijk} + T_{ji}\varepsilon_{ijk})\,\mathrm{d}V$$

Thus, the balance of moment of momentum becomes:

$$\int\limits_V x_i(\rho\frac{Dv_j}{Dt} - \frac{\partial T_{il}}{\partial x_l} - \rho f_j)\varepsilon_{ijk} + T_{ji}\varepsilon_{ijk}\,\mathrm{d}V = \int\limits_V T_{ji}\varepsilon_{ijk}\,\mathrm{d}V = 0$$

using the balance of momentum $\rho\dfrac{Dv_j}{Dt} - \dfrac{\partial T_{il}}{\partial x_l} - \rho f_j = 0$.

Thus, $T_{ji}\varepsilon_{ijk} = 0$ which means $T_{ij} = T_{ji}$.

Exercise 2.3.

Not available

Exercise 2.4.

Partial integration gives:

$$\int_1^3 (3v' + 2v'' - \sin x)\varphi \mathrm{d}x = (2v'(3) + 1)\varphi(3)$$

using $\varphi(1) = 0$.

Differential equation: $3v' + 2v'' = \sin x$

Boundary conditions: $v(1) = 4$ and $v'(3) = -1/2$

Solutions for Chapter 3

Exercise 3.1.

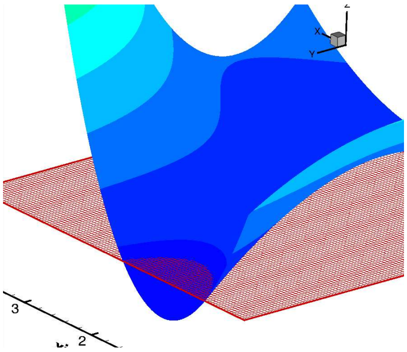
(i) Coordinate transformation:

$$x(\xi,\eta) = \frac{\xi}{4}\left(1 + \frac{\eta}{2} - \frac{\eta^2}{8}\right) \quad \text{and} \quad y(\xi,\eta) = \frac{\eta}{4}\left(1 - \frac{3\xi}{4} + \frac{3\xi^2}{16}\right)$$

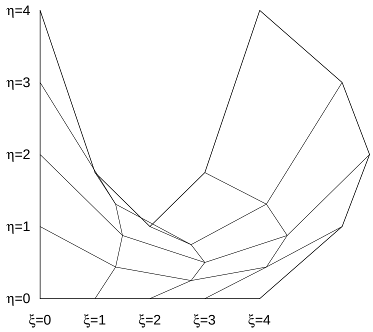
(ii) Jacobian determinant:

$$\begin{aligned} J = \det \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} &= \det \begin{bmatrix} 1/4 + \eta/8 - \eta^2/32 & \xi/8 - \xi\eta/16 \\ -3\eta/16 + 3\xi\eta/32 & 1/4 - 3\xi/16 + 3\xi^2/64 \end{bmatrix} \\ &= \left(256 - 192\xi + 128\eta + 48\xi^2 - 32\eta^2 - 24\xi^2\eta - 24\xi\eta^2 + 18\xi^2\eta^2\right)/4096 \end{aligned}$$

Evaluation gives $J = 0$ for some (ξ, η):



Mapping is not unique, grid lines intersect:



(iii) Stretching function:

$$s_i = 1 - \frac{(2/3)^{4-i} - 1}{(2/3)^4 - 1} \quad \text{for} \quad i = 0, \ldots, 4$$

Thus: $s_0 = 0$, $s_1 = 8/65$, $s_2 = 4/13$, $s_3 = 38/65$, $s_4 = 1$

(iv) Transformation formula:

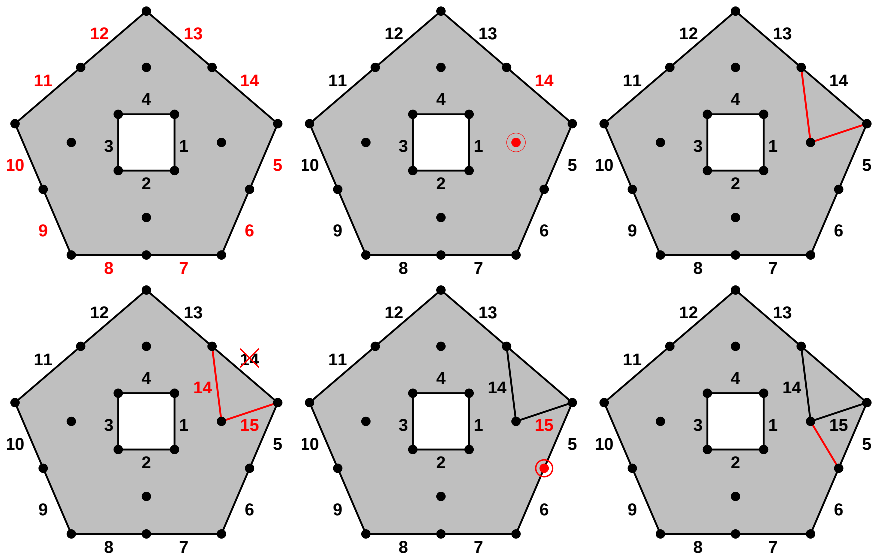
$$\phi_{xx} + \phi_{yy} = \left(a\phi_{\xi\xi} - 2b\phi_{\xi\eta} + c\phi_{\eta\eta} + d\phi_{\xi} + e\phi_{\eta}\right)/J^2$$

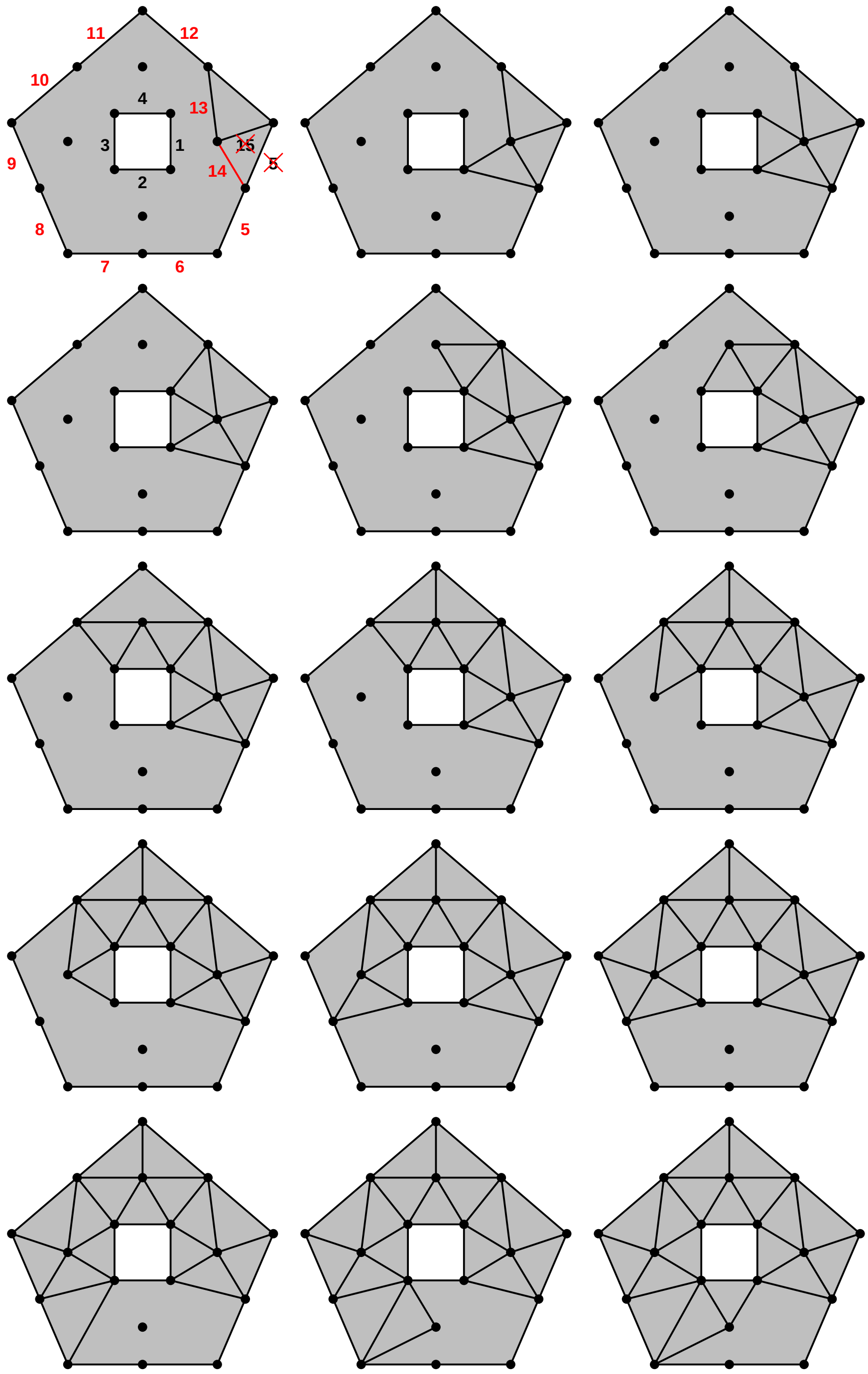
with

$$\begin{aligned}
J &= x_\xi y_\eta - x_\eta y_\xi \\
a &= x_\eta^2 + y_\eta^2 \\
b &= x_\xi x_\eta + y_\xi y_\eta \\
c &= x_\xi^2 + y_\xi^2 \\
\alpha &= a x_{\xi\xi} - 2b x_{\xi\eta} + c x_{\eta\eta} \\
\beta &= a y_{\xi\xi} - 2b y_{\xi\eta} + c y_{\eta\eta} \\
d &= \left(x_\eta \beta - y_\eta \alpha\right)/J \\
e &= \left(y_\xi \alpha - x_\xi \beta\right)/J
\end{aligned}$$

Exercise 3.2.

Triangulation with advancing front method:




11
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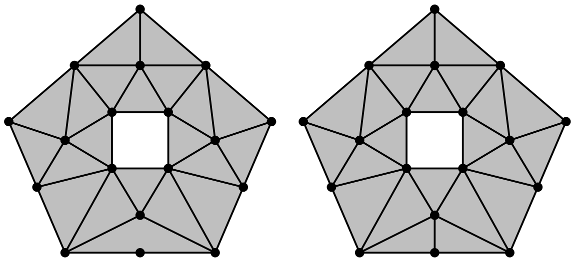
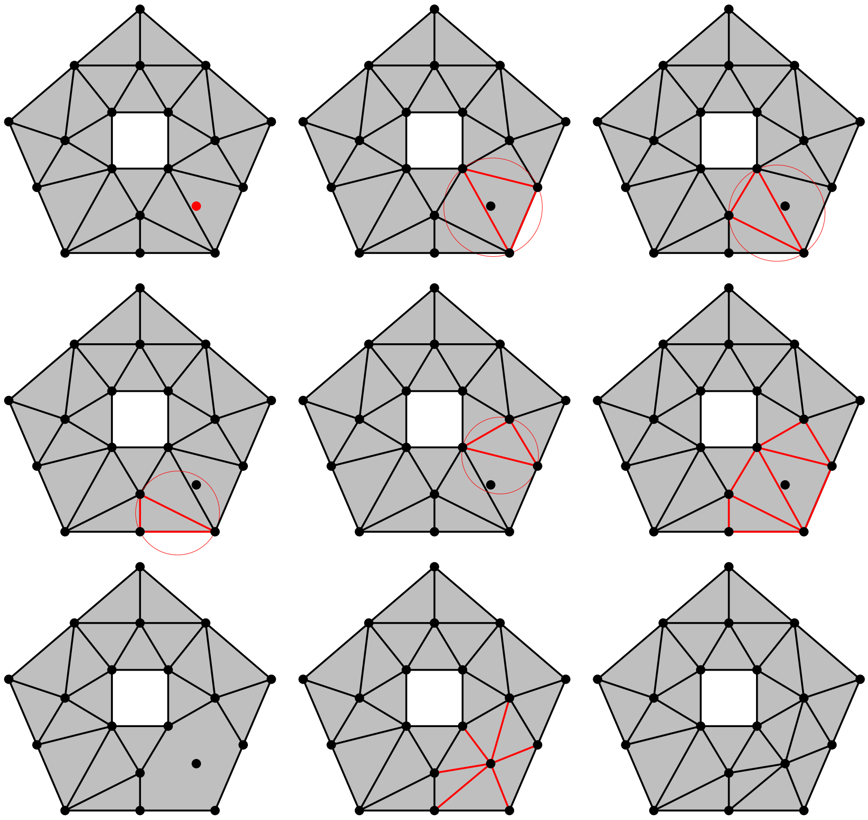
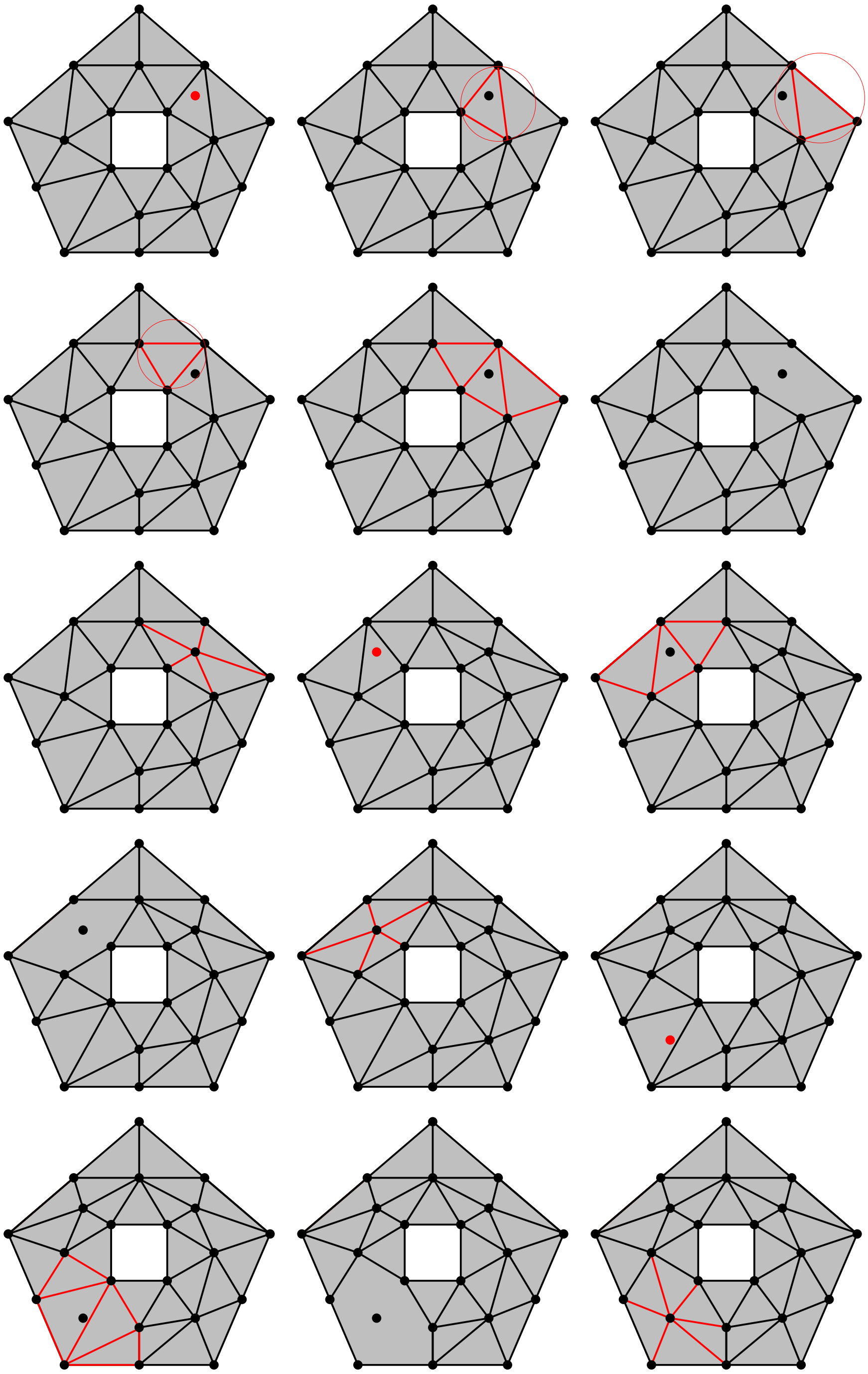


Insertion of point by Bowyer-Watson method:





Solutions for Chapter 4

Exercise 4.1.

Midpoint rule:

$$\int\limits_{x_\mathrm{w}}^{x_\mathrm{e}} \phi \,\mathrm{d}x \approx (x_\mathrm{e} - x_\mathrm{w})\phi_\mathrm{P}$$

Taylor:

$$\int\limits_{x_\mathrm{w}}^{x_\mathrm{e}} \left[\phi_\mathrm{P} + (x - x_\mathrm{P})\phi'_\mathrm{P} + \frac{(x - x_\mathrm{P})^2}{2}\,\phi''_\mathrm{P} + T_\mathrm{H}\right] \mathrm{d}x$$

$$= (x_\mathrm{e} - x_\mathrm{w})\phi_\mathrm{P} + \frac{(x_\mathrm{e} - x_\mathrm{w})^3}{12}\,\phi''_\mathrm{P} + T_\mathrm{H}$$

Error:

$$\tau = \frac{(x_\mathrm{e} - x_\mathrm{w})^3}{12}\,\phi''_\mathrm{P} + T_\mathrm{H}$$

Trapezoidal rule:

$$\int\limits_{x_\mathrm{w}}^{x_\mathrm{e}} \phi \,\mathrm{d}x \approx (x_\mathrm{e} - x_\mathrm{w})(\phi_\mathrm{w} + \phi_\mathrm{e})/2$$

Taylor:

$$\phi_\mathrm{P} = \frac{1}{2}(\phi_\mathrm{e} + \phi_\mathrm{w}) - \frac{(x_\mathrm{e} - x_\mathrm{w})^2}{8} + T_\mathrm{H}$$

Error:

$$\tau = -\frac{(x_\mathrm{e} - x_\mathrm{w})^3}{24}\,\phi''_\mathrm{P} + T_\mathrm{H}$$

Exercise 4.2.

Discrete equation for a CV:

$$-3(\phi_\mathrm{e} - \phi_\mathrm{w}) - 2(\phi'_\mathrm{e} - \phi'_\mathrm{w}) = (x_\mathrm{e} - x_\mathrm{w})x_\mathrm{P}\cos(\pi x_\mathrm{P})$$

CV1: $11\phi_1 - 11\phi_2 = 36$

CV2: $-2\phi_1 + 17\phi_2 = 0$

Solution: $\phi_1 = 204/55$ and $\phi_2 = 24/55$

Exercise 4.3.

a) Non-equidistant grid

For CV1:

$$\begin{aligned}&-\lambda\left(\frac{T_3-T_1}{x_{\mathrm{E}}-x_{\mathrm{P}}}\right)(y_{\mathrm{n}}-y_{\mathrm{s}})-\lambda\left(\frac{T_2-T_1}{y_{\mathrm{N}}-y_{\mathrm{P}}}\right)(x_{\mathrm{e}}-x_{\mathrm{w}})\\&-2y_{\mathrm{w}}^3(y_{\mathrm{n}}-y_s)+\lambda\left(\frac{T_1-T_{\mathrm{S}}}{y_{\mathrm{P}}-y_{\mathrm{S}}}\right)(x_{\mathrm{e}}-x_{\mathrm{w}})=q(x_{\mathrm{e}}-x_{\mathrm{w}})(y_{\mathrm{n}}-y_{\mathrm{s}})\end{aligned}$$

$$\Rightarrow\quad 10T_1-3T_2-3T_3=99.2656$$

For CV2:

$$\begin{aligned}&-\lambda\left(\frac{T_4-T_2}{x_{\mathrm{E}}-x_{\mathrm{P}}}\right)(y_{\mathrm{n}}-y_{\mathrm{s}})-\lambda\left(\frac{T_N-T_2}{y_{\mathrm{N}}-y_{\mathrm{P}}}\right)(x_{\mathrm{e}}-x_{\mathrm{w}})\\&-2y_{\mathrm{w}}^3(y_{\mathrm{n}}-y_{\mathrm{s}})+\lambda\left(\frac{T_2-T_1}{y_{\mathrm{P}}-y_{\mathrm{S}}}\right)(x_{\mathrm{e}}-x_{\mathrm{w}})=q(x_{\mathrm{e}}-x_{\mathrm{w}})(y_{\mathrm{n}}-y_{\mathrm{s}})\end{aligned}$$

$$\Rightarrow\quad -3T_1+16T_2-T_4=93.4844$$

For CV3:

$$\begin{aligned}&-(24y_{\mathrm{e}}-2y_e^3)(y_{\mathrm{n}}-y_{\mathrm{s}})-\lambda\left(\frac{T_4-T_3}{y_{\mathrm{N}}-y_{\mathrm{P}}}\right)(x_{\mathrm{e}}-x_{\mathrm{w}})\\&+\lambda\left(\frac{T_3-T_1}{x_{\mathrm{P}}-x_{\mathrm{W}}}\right)(y_{\mathrm{n}}-y_{\mathrm{s}})+\lambda\left(\frac{T_3-T_{\mathrm{S}}}{y_{\mathrm{P}}-y_{\mathrm{S}}}\right)(x_{\mathrm{e}}-x_{\mathrm{w}})=q(x_{\mathrm{e}}-x_{\mathrm{w}})(y_{\mathrm{n}}-y_{\mathrm{s}})\end{aligned}$$

$$\Rightarrow\quad -3T_1+16T_3/3-T_4=58.4010$$

For CV4:

$$\begin{aligned}&-(24y_{\mathrm{e}}-2y_{\mathrm{e}}^3)(y_{\mathrm{n}}-y_{\mathrm{s}})-\lambda\left(\frac{T_{\mathrm{N}}-T_4}{y_{\mathrm{N}}-y_{\mathrm{P}}}\right)(x_{\mathrm{e}}-x_{\mathrm{w}})\\&+\lambda\left(\frac{T_4-T_2}{x_{\mathrm{P}}-x_{\mathrm{W}}}\right)(y_{\mathrm{n}}-y_{\mathrm{s}})+\lambda\left(\frac{T_4-T_3}{y_{\mathrm{P}}-y_{\mathrm{S}}}\right)(x_{\mathrm{e}}-x_{\mathrm{w}})=q(x_{\mathrm{e}}-x_{\mathrm{w}})(y_{\mathrm{n}}-y_{\mathrm{s}})\end{aligned}$$

$$\Rightarrow\quad -T_2-T_3+6T_4=52.5156$$

Equation system:

$$\begin{bmatrix}10&-3&-3&0\\-3&16&0&-1\\-3&0&16/3&-1\\0&-1&-1&6\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\end{bmatrix}=\begin{bmatrix}99.2656\\93.4844\\58.4010\\52.5156\end{bmatrix}$$

Solution:

	T_1	T_2	T_3	T_4
Numerical	20.74	10.65	25.38	14.76
Analytic	18.88	10.59	22.16	13.88
Rel. error [%]	9.9	0.6	14.5	6.3

b) Solution for equidistant grid:

	T_1	T_2	T_3	T_4
Numerical	20.35	15.40	21.90	17.35
Analytic	19.50	14.00	21.00	15.50
Rel. error [%]	4.4	4.3	10.0	11.9

Exercise 4.4.

Not available

Exercise 4.5.

Not available

Exercise 4.6.

(i) Taylor series expansion:

$$I = \phi(3,\alpha)\Delta y + (y_\mathrm{e} - \alpha)\Delta y \, \frac{\partial \phi}{\partial y}(3,\alpha) + T_\mathrm{H}$$

Leading error term:

$$(y_\mathrm{e} - \alpha)\Delta y \, \frac{\partial \phi}{\partial y}(3,\alpha)$$

First order for $\alpha \neq y_\mathrm{e} = 2$, second order for $\alpha = y_\mathrm{e} = 2$

(ii) Analytically: $I = 6534/5 = 1306.8$

Numerically: $I \approx 864$

Exercise 4.7.

(i) Simpson rule:

$$\begin{aligned}
I &= -\int_1^2 (\cos\pi y)\mathrm{d}y + \int_1^2 (2\cos\pi y)\mathrm{d}y - \int_1^2 (x^4)\mathrm{d}x + \int_1^2 (2x^4)\mathrm{d}x \\
&\approx \frac{1}{6}(-\cos\pi - 4\cos 3\pi/2 - \cos 2\pi) + \\
&\quad \frac{2}{6}(\cos\pi + 4\cos y 3\pi/2 + \cos 2\pi) + \\
&\quad \frac{1}{6}(-1 - 4(3/2)^4 - 2^4) + \frac{2}{6}(1 + 4(3/2)^4 + 2^4) \\
&\approx 149/24 \approx 6.2083
\end{aligned}$$

(ii) Gauß integral theorem:

$$I = \int_V (\cos\pi y + x^4)\mathrm{d}V$$

Midpoint rule:

$$I \approx \cos 3\pi/2 + (3/2)^4 = 81/16 = 5.0625$$

Solutions for Chapter 5

Exercise 5.1.

Not available

Exercise 5.2.

a) Biquadratic parallelogram element:

$N_1^{\mathrm{e}} = (1-\xi)(1-\eta)\xi\eta/4$

$N_2^{\mathrm{e}} = -(1-\xi^2)\eta(1-\eta)/2$

$N_3^{\mathrm{e}} = -(1+\xi)(1-\eta)\xi\eta/4$

$N_4^{\mathrm{e}} = -(1-\xi)\xi(1-\eta^2)/2$

$N_5^{\mathrm{e}} = (1-\xi^2)(1-\eta^2)$

$N_6^{\mathrm{e}} = (1+\xi)\xi(1-\eta^2)/2$

$N_7^{\mathrm{e}} = -(1-\xi)(1+\eta)\xi\eta/4$

$N_8^{\mathrm{e}} = (1-\xi^2)\eta(1+\eta)/2$

$N_9^{\mathrm{e}} = (1+\xi)(1+\eta)\xi\eta/4$

b) Cubic triangular element:

Not available

Exercise 5.4.

Element matrices and element load vectors (for $i = 1, \dots, 4$):

$$\mathbf{S}^i = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} - \frac{1}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad , \quad \mathbf{b}^i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Global system (without boundary conditions):

$$\frac{1}{24}\begin{bmatrix} -14 & 11 & 0 & 0 & 0 \\ -13 & -4 & 11 & 0 & 0 \\ 0 & -13 & -4 & 11 & 0 \\ 0 & 0 & -13 & -4 & 11 \\ 0 & 0 & 0 & -13 & 10 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Global system after consideration of boundary condition $\phi_1 = \phi(0) = 1$:

$$\frac{1}{24}\begin{bmatrix} -4 & 11 & 0 & 0 \\ -13 & -4 & 11 & 0 \\ 0 & -13 & -4 & 11 \\ 0 & 0 & -13 & 10 \end{bmatrix}\begin{bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 13 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution (analytic solution: $\phi(x) = e^x$):

	ϕ_2	ϕ_3	ϕ_4	ϕ_5
Numerical	1.242	1.633	2.062	2.680
Analytic	1.284	1.649	2.117	2.718
Rel. error	3.4	1.0	2.6	1.4

Exercise 5.5.

Bilinear ansatz:

$$\phi(x, y) = a_1 + a_2 x + a_3 y + a_4 xy$$

Inserting nodal variables gives:

$$\begin{aligned} a_1 &= \phi_1 \\ a_2 &= \phi_2 - \phi_1 \\ a_3 &= \phi_3/2 - \phi_1/2 \\ a_4 &= \phi_4/2 - \phi_2/2 + \phi_1/2. \end{aligned}$$

Thus:

$$\phi(x, y) = (1 - x - y/2 + xy/2)\phi_1 + (x - xy/2)\phi_2 + (y/2)\phi_3 + (xy/2)\phi_4$$

Shape functions:

$$\begin{aligned} N_1(x, y) &= 1 - x - y/2 + xy/2 \\ N_2(x, y) &= x - xy/2 \\ N_3(x, y) &= y/2 \\ N_4(x, y) &= xy/2. \end{aligned}$$

Exercise 5.6.

(i) Coincidence matrix:

Local	Element			
nodal variable	1	2	3	4
1	5	10	9	7
2	3	8	7	5
3	4	1	8	6
4	6	2	10	8

(ii) $b_2^2 = \int\limits_1^2 \int\limits_2^3 xy(6-2x-3y+xy)\mathrm{d}x\mathrm{d}y = 7/9$

(iii) $S_{87} = S_{41}^4 + S_{32}^3$ and $b_5 = b_1^1 + b_2^4$

Exercise 5.7.

Element stiffness matrices and element load vectors:

$$\boldsymbol{S}^e = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} \quad , \quad \boldsymbol{b}^e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Coincidence matrix:

Local nodal variable	Element 1	2	3	4
1	1	2	4	5
2	2	3	5	6
3	5	6	8	9
4	4	5	7	8

Global matrix (without boundary conditions):

$$\tilde{\mathbf{S}} = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
S_{11}^1 & S_{12}^1 & & S_{14}^1 & S_{13}^1 & & & & \\
S_{21}^1 & S_{22}^1+S_{11}^2 & S_{12}^2 & S_{24}^1 & S_{23}^1+S_{14}^2 & S_{13}^2 & & & \\
 & S_{21}^2 & S_{22}^2 & & S_{24}^2 & S_{23}^2 & & & \\
S_{41}^1 & S_{42}^1 & & S_{44}^1+S_{11}^3 & S_{43}^1+S_{12}^3 & & S_{14}^3 & S_{13}^3 & \\
S_{31}^1 & S_{32}^1+S_{41}^2 & S_{42}^2 & S_{34}^1+S_{21}^3 & S_{33}^1+S_{44}^2+S_{22}^3+S_{11}^4 & S_{43}^2+S_{12}^4 & S_{24}^3 & S_{23}^3+S_{14}^4 & S_{13}^4 \\
 & S_{31}^2 & S_{32}^2 & & S_{34}^2+S_{21}^4 & S_{33}^2+S_{22}^4 & & S_{24}^4 & S_{23}^4 \\
 & & & S_{41}^3 & S_{42}^3 & & S_{44}^3 & S_{43}^3 & \\
 & & & S_{31}^3 & S_{32}^3+S_{41}^4 & S_{42}^4 & S_{34}^3 & S_{33}^3+S_{44}^4 & S_{43}^4 \\
 & & & & S_{31}^4 & S_{32}^4 & & S_{34}^4 & S_{33}^4
\end{array}\right]$$

Load vector (without boundary conditions):

$$\tilde{\mathbf{b}} = [b_1^1,\ b_2^1+b_1^2,\ b_2^2,\ b_4^1+b_1^3,\ b_3^1+b_4^2+b_2^3+b_1^4,\ b_3^2+b_2^4,\ b_4^3,\ b_3^3+b_4^4,\ b_3^4]^T$$

Inserting values:

$$\mathbf{S} = \frac{1}{6}\begin{bmatrix} 4 & -1 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\ -1 & 8 & -1 & -2 & -2 & -2 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & -2 & -1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 8 & -2 & 0 & -1 & -2 & 0 \\ -2 & -2 & -2 & -2 & 16 & -2 & -2 & -2 & -2 \\ 0 & -2 & -1 & 0 & -2 & 8 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 & -2 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 & -1 & 8 & -1 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & -1 & 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 4 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Boundary shape functions:

$$\begin{aligned} N_1^1(y) &= N_1^2(y) = 1 - y \\ N_2^1(y) &= N_2^2(y) = y \\ N_1^3(y) &= N_1^4(y) = 2 - y \\ N_2^3(y) &= N_2^4(y) = y - 1 \end{aligned}$$

Boundary coincidence matrix:

Local	Element			
nodal variable	1	2	3	4
1	1	3	4	6
2	4	6	7	9

Boundary integral contributions:

$$\mathbf{r} = \begin{bmatrix} \int_0^1 y^3 N_1^1(y)\ \mathrm{d}y \\ 0 \\ \int_0^1 (12y - y^3) N_1^2(y)\ \mathrm{d}y \\ \int_0^1 y^3 N_2^1(y)\ \mathrm{d}y + \int_1^2 y^3 N_1^3(y)\ \mathrm{d}y \\ 0 \\ \int_0^1 (12y - y^3) N_2^2(y)\ \mathrm{d}y + \int_1^2 (12y - y^3) N_1^4(y)\ \mathrm{d}y \\ \int_1^2 y^3 N_2^3(y)\ \mathrm{d}y \\ 0 \\ \int_1^2 (12y - y^3) N_2^4(y)\ \mathrm{d}y \end{bmatrix}$$

Modified load vector:

$$\tilde{\mathbf{b}} + \mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 4 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{20} \begin{bmatrix} 1 \\ 0 \\ 39 \\ 30 \\ 0 \\ 210 \\ 49 \\ 0 \\ 151 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 21 \\ 40 \\ 59 \\ 70 \\ 80 \\ 250 \\ 69 \\ 40 \\ 171 \end{bmatrix}$$

Considering the Dirichlet conditions $T_1 = T_2 = T_3 = 20$, $T_7 = 12$, $T_8 = 6$, $T_9 = 12$ gives:

$$\begin{bmatrix} 8 & -2 & 0 \\ -2 & 16 & -2 \\ 0 & -2 & 8 \end{bmatrix} \begin{bmatrix} T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 105 \\ 204 \\ 159 \end{bmatrix}$$

Solution: $T_4 = 17.625$, $T_4 = 18$, $T_4 = 24.375$

Comparison:

	T_4	T_5	T_6	Relative error
Triangles	18.750	18.000	23.250	0.028
Quadrilaterals	17.625	18.000	24.375	0.014
Analytic	18.000	18.000	24.000	

Exercise 5.9.

Residual:

$$R = 5\left[\sum_{i=1}^{2} c_i \varphi_i'(x)\right] - \left[\sum_{i=1}^{2} c_i \varphi_i''(x)\right] - 4x$$

Galerkin method:

$$\int_0^1 \left[5\left(\sum_{i=1}^{2} c_i \varphi_i'(x)\right) - \left(\sum_{i=1}^{2} c_i \varphi_i''(x)\right) - 4x\right] \varphi_j(x)\,\mathrm{d}x = 0.$$

Weak form:

$$5\sum_{i=1}^{2} c_i \int_0^1 \varphi_i' \varphi_j\,\mathrm{d}x + \sum_{i=1}^{2} c_i \int_0^1 \varphi_i' \varphi_j'\,\mathrm{d}x - 4\int_0^1 x\varphi_j\,\mathrm{d}x = -\varphi_j(0).$$

Equation system:

$$\begin{aligned} -9c_1 - 4c_2 &= 2 \\ -2c_1 - c_2 &= 0 \end{aligned}$$

Solution: $c_1 = -2$, $c_2 = 4$. Thus: $\phi(x) \approx 4x^2 - 2x - 2$

Solutions for Chapter 6

Exercise 6.1.

Not available

Exercise 6.2.

Ansatz:

$$\phi(x_i, t) = \sum_k \phi_k(t) N_k(x_i)$$

Residual:

$$R = \sum_k \rho \frac{\partial \phi_k}{\partial t} N_k + \sum_k \phi_k \left(\rho v_i \frac{\partial N_k}{\partial x_i} - \alpha \frac{\partial^2 N_k}{\partial x_i^2} \right) - f$$

Galerkin method:

$$\sum_k \underbrace{\frac{\partial \phi_k}{\partial t}}_{\dot{\boldsymbol{\phi}}} \underbrace{\int_V \rho N_k N_j \, \mathrm{d}V}_{\mathbf{M}} + \sum_k \underbrace{\phi_k}_{\boldsymbol{\phi}} \underbrace{\int_V \left(\rho v_i \frac{\partial N_k}{\partial x_i} + \alpha \frac{\partial N_k}{\partial x_i} \frac{\partial N_j}{\partial x_i} \right) \mathrm{d}V}_{\mathbf{S}}$$

$$= \underbrace{\int_V f N_j \, \mathrm{d}V + \int_S \gamma N_j \, \mathrm{d}S}_{\mathbf{b}}$$

$\Rightarrow \mathbf{M}\dot{\boldsymbol{\phi}} + \mathbf{S}\boldsymbol{\phi} = \mathbf{b}$

θ-method:

$$[\mathbf{M} + (1-\theta)\Delta t \mathbf{S}]\phi^{n+1} = (\mathbf{M} - \theta \Delta t \mathbf{S})\boldsymbol{\phi}^n + (1-\theta)\mathbf{b}^{n+1} + \theta \mathbf{b}^n$$

Exercise 6.3.

Not available

Exercise 6.4.

Not available

Exercise 6.5.

(i) Discrete system:

$$\begin{bmatrix} 1-2\Delta t\Theta & 0 \\ -t\Delta t\Theta & 1-\sqrt{t}\Delta t\Theta \end{bmatrix}^{n+1} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}^{n+1} + \Theta \begin{bmatrix} \Delta t \sin(\pi t) \\ \Delta t\sqrt{t} \end{bmatrix}^{n+1}$$

$$= \begin{bmatrix} 1+2\Delta t - 2\Theta\Delta t & 0 \\ t\Delta t - t\Delta t\Theta & 1+\sqrt{t}\Delta t - \sqrt{t}\Delta t\Theta \end{bmatrix}^{n} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}^{n} + (1-\Theta) \begin{bmatrix} \Delta t \sin(\pi t) \\ \Delta t\sqrt{t} \end{bmatrix}^{n}$$

(ii) For $\phi_1^1 = \phi_1(\Delta t)$ (**not** $\phi_2(\Delta t)$ **as written in book!**) with $\Delta t = 2$:

$$\begin{bmatrix} 1-4\Theta & 0 \\ -4\Theta & 1-2\sqrt{2}\Theta \end{bmatrix} \begin{bmatrix} \phi_1^1 \\ \phi_2^1 \end{bmatrix} + \Theta \begin{bmatrix} 2\sin(2\pi) \\ 2\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 5-4\Theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (1-\Theta) \begin{bmatrix} 2\sin(0) \\ 0 \end{bmatrix}$$

Thus: $\phi_1^1 = (10-8\Theta)/(1-4\Theta)$

Solutions for Chapter 7

Exercise 7.1.

(i) Gauß elimination gives:

$$\begin{bmatrix} 1 & -1/4 & 0 \\ 0 & 1 & -4/15 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 11/5 \\ 1 \end{bmatrix}$$

Backward substitution gives the solution: $\phi_1 = \phi_2 = \phi_3 = 1$

(ii) Eigenvalues of system matrix:

$\lambda_1 = 4$, $\lambda_2 = \lambda_{\max} = 4 + \sqrt{2}$, $\lambda_3 = \lambda_{\min} = 4 - \sqrt{2}$

Condition number: $\kappa = \lambda_{\max}/\lambda_{\min} = (9 + 4\sqrt{2})/7$

Jacobi method:

$$\mathbf{C} = \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} , \quad \lambda_{\max} = \sqrt{2}/4$$

Gauß-Seidel method:

$$\mathbf{C} = \begin{bmatrix} 0 & -1/4 & 0 \\ 0 & 1/16 & 1/4 \\ 0 & 1/64 & 1/16 \end{bmatrix} , \quad \lambda_{\max} = 1/8$$

SOR method with $\omega_{\text{opt}} = 16 - 4\sqrt{14}$:

$$\mathbf{C} = \begin{bmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{bmatrix} , \quad \lambda_{\max} = 1 - \omega_{\text{opt}} + \omega_{\text{opt}}^2/16$$

(iii) Jacobi method: $\boldsymbol{\phi}^1 = [1/4, 11/5, 1]^T, \ldots$

Gauß-Seidel method: $\boldsymbol{\phi}^1 = [3/4, 11/32, 107/128]^T, \ldots$

CG method: $\boldsymbol{\phi}^1 = [33/32, 11/16, 33/32]^T, \ldots$

Exercise 7.2.

$(\phi_1^1, \phi_2^1) = (14/15, 11/15)$, $\phi_3^1 = 14/15$,

$(\phi_1^2, \phi_2^2) = (224/225, 221/225)$, $\phi_3^2 = 896/900, \ldots$

Exercise 7.3.

(i) LU decomposition:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4/15 & 1 & 0 & 0 & 0 \\ 0 & 0 & -15/56 & 1 & 0 & 0 \\ -1/4 & -1/15 & -1/56 & -1/209 & 1 & 0 \\ 0 & -4/15 & -1/14 & -4/209 & -56/195 & 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 4 & -1 & 0 & 0 & -1 & 0 \\ 0 & 15/4 & -1 & 0 & -1/4 & -1 \\ 0 & 0 & 56/16 & -1 & -1/15 & -4/15 \\ 0 & 0 & 0 & 209/56 & -1/56 & -1/14 \\ 0 & 0 & 0 & 0 & 780/109 & -224/209 \\ 0 & 0 & 0 & 0 & 0 & 664/195 \end{bmatrix}$$

(ii) Not available

Exercise 7.4.

Not available

Solutions for Chapter 8

Exercise 8.1.

Trucation error:

$$\begin{aligned}\tau^n &= \frac{2\phi^{n+1} + a\phi^n + b\phi^{n-1}}{\Delta t} - (\phi')^n = \\ &= \frac{2+a+b}{\Delta t}\phi^n + (1-b)(\phi')^n + [\Delta t(1+b/2)](\phi'')^n + O(\Delta t^2)\end{aligned}$$

Scheme is consistent if:

$$\begin{aligned}2+a+b &= 0 \\ 1-b &= 0\end{aligned}$$

Thus: $a = -3$ and $b = 1$

Leading term in truncation error in this case: $\tau^n = 3\Delta t(\phi'')^n/2$.

Exercise 8.2.

Not available

Exercise 8.3.

All coefficients positive if $-1 < \alpha < 1$, otherwise scheme might be unstable.

Exercise 8.4.

Not available

Exercise 8.5.

Not available

Exercise 8.6.

(i) CDS2/CDS2 approximation:

$$\rho u \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} - \alpha \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} = 0$$

Inserting $\phi_i = b^i$ yields:

$$b^{i-1}(b-1)\left[\rho v \Delta x(b+1) - 2\alpha(b-1)\right] = 0$$

Non-trivial solution: $b = (1+\sigma)/(1-\sigma)$ with $\sigma = (\rho v \Delta x)/(2\alpha)$

From Boundary conditions:

$$C_1 = \frac{1}{1 - \left(\dfrac{1+\sigma}{1-\sigma}\right)^{N+1}} \quad \text{and} \quad C_2 = -C_1$$

Solution:

$$\phi_i^{\mathrm{CDS}} = \frac{1 - \left(\dfrac{1+\sigma}{1-\sigma}\right)^{i}}{1 - \left(\dfrac{1+\sigma}{1-\sigma}\right)^{N+1}}$$

Insbesondere bedeutet das, da"s fr $\alpha \to 0 \Rightarrow \Delta x \downarrow$

$$\lim_{\substack{\alpha \to 0 \\ \Delta x = \mathrm{const.}}} \phi_i = \frac{1-(-1)^i}{1-(-1)^{n+1}} \quad \Rightarrow \quad \text{alternierend um } 0$$

UDS/CDS2 approximation:

$$\rho u \frac{\phi_i - \phi_{i-1}}{\Delta x} - \alpha \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} = 0$$

$b = \rho v \Delta x/\alpha + 1 = 2\sigma + 1$

$$C_1 = \frac{1}{1-(1+2\sigma)^{N+1}} \quad \text{and} \quad C_2 = -C_1$$

Solution:

$$\phi_i^{\mathrm{UDS}} = \frac{1-(1+2\sigma)^i}{1-(1+2\sigma)^{N+1}}$$

Da $\sigma = \rho v \Delta x/\alpha$ im betrachteten Fall ($u > 0$) per Definition immer positiv ist, kommt es zu keinen Oszillationen von ϕ_i. Ein Alternieren ist nicht m"oglich.

(ii) Analytic solution of continuous problem:

$$\phi(x) = \frac{1 - \mathrm{e}^{\rho v x/\alpha}}{1 - \mathrm{e}^{\rho v/\alpha}}$$

ϕ_i^{CDS} alternates if $\sigma > 1$, i.e., $\Delta x > 2\alpha/\rho v$ d. h. f"ur jedes Verh"altnis von Konvektion zu Diffusion u/α mu"s Δx so abgesch"atzt werden, da"s es nicht zu gro"s wird.

Exercise 8.7.

It is $|-1+\alpha| = |1-\alpha|$ for all α.

Thus, sufficient condition for boundedness: $|1-2\alpha| > |5\alpha|$, i.e., $0 < \alpha < 1/7$

Exercise 8.8.

(i) Taylor series expansion:

$$I = \phi(3,\alpha)\Delta y + (y_\mathrm{e} - \alpha)\Delta y \,\frac{\partial \phi}{\partial y}(3,\alpha) + T_\mathrm{H}$$

Leading error term:

$$(y_\mathrm{e} - \alpha)\Delta y \,\frac{\partial \phi}{\partial y}(3,\alpha)$$

First order for $\alpha \neq y_\mathrm{e} = 2$, second order for $\alpha = y_\mathrm{e} = 2$

(ii) Analytically: $I = 6534/5 = 1306.8$

Numerically: $I \approx 864$

Exercise 8.9.

(i) Not conservative for $\alpha \neq 1 + \beta$

(ii) Bounded for $|\alpha| > 5 + |\beta|$

Exercise 8.10.

Order:

$$2^p = \frac{\phi_{4h} - \phi_{2h}}{\phi_{2h} - \phi_h} = 16 \quad \Rightarrow \quad p = 4.$$

Grid independent solution:

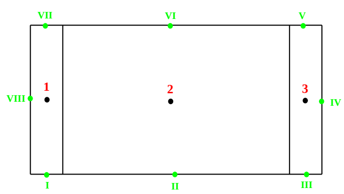
$$\phi = \phi_h + \frac{\phi_h - \phi_{2h}}{2^p - 1} \quad \Rightarrow \quad \phi = 276.$$

Exercise 8.11.

For all grids:

$$\left(\frac{\partial \phi}{\partial x}\right)_\mathrm{w} \delta S_\mathrm{w} - \left(\frac{\partial \phi}{\partial x}\right)_\mathrm{e} \delta S_\mathrm{e} + \left(\frac{\partial \phi}{\partial y}\right)_\mathrm{s} \delta S_\mathrm{s} - \left(\frac{\partial \phi}{\partial y}\right)_\mathrm{n} \delta S_\mathrm{n} = 0$$

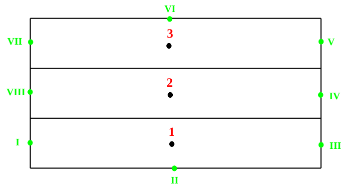
Grid 1:



System matrix and Gauß-Seidel iteration matrix:

$$\mathbf{A} = \begin{bmatrix} 97/12 & -3/4 & 0 \\ -3/4 & 65/6 & -3/4 \\ 0 & -3/4 & 97/12 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & 0.09278 & 0 \\ 0 & 0.00642 & 0.06923 \\ 0 & 0.00059 & 0.00642 \end{bmatrix}$$

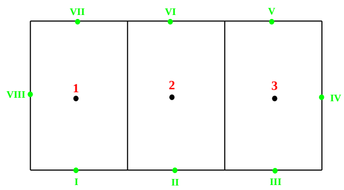
Grid 2:



System matrix and Gauß-Seidel iteration matrix:

$$\mathbf{A} = \begin{bmatrix} 247/9 & -9 & 0 \\ -9 & 166/9 & -9 \\ 0 & -9 & 247/9 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & 0.32793 & 0 \\ 0 & 0.16001 & 0.48795 \\ 0 & 0.00437 & 0.01333 \end{bmatrix}$$

Grid 3:



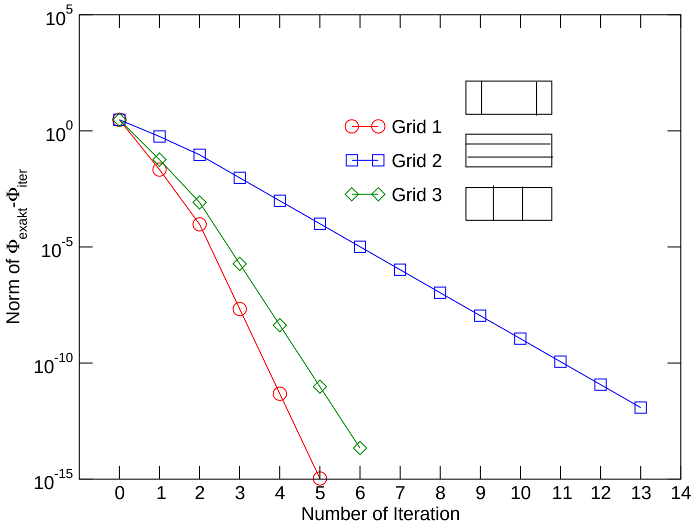
System matrix and Gauß-Seidel iteration matrix:

$$\mathbf{A} = \begin{bmatrix} 7 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & -1 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & 0.14285 & 0 \\ 0 & 0.02380 & 0.16666 \\ 0 & 0.00340 & 0.02380 \end{bmatrix}$$

Eigenvalues of system matrix, condition number, and spectral radius of Gauß-Seidel iteration matrix:

	λ_1	λ_2	λ_3	κ	$\lambda_{\max}$
Grid 1:	11.16	8.083	7.75	1.44	0.01156
Grid 2:	36.444	27.444	9.4444	3.859	0.32
Grid 3:	8	7	5	1.6	0.04762

Convergence of Gauß-Seidel method:



Solutions for Chapter 9

Exercise 9.1.

Not available

Exercise 9.2.

Not available

Exercise 9.3.

$N_1^{\mathrm{e}} = (1 - \xi)(1 - \eta)(1 - 2\xi - 2\eta)$
$N_2^{\mathrm{e}} = -\xi(1 - \eta)(1 - 2\xi - 2\eta)$
$N_3^{\mathrm{e}} = \xi\eta(3 - 2\xi - 2\eta)$
$N_4^{\mathrm{e}} = -\eta(1 - \xi)(1 - 2\xi - 2\eta)$
$N_5^{\mathrm{e}} = 4\xi(1 - \xi)(1 - \eta)$
$N_6^{\mathrm{e}} = 4\xi\eta(1 - \eta)$
$N_7^{\mathrm{e}} = 4\xi\eta(1 - \xi)$
$N_8^{\mathrm{e}} = 4\xi(1 - \xi)(1 - \eta)$

Exercise 9.4.

Not available

Solutions for Chapter 10

Exercise 10.1.

Not available

Exercise 10.2.

Not available

Solutions for Chapter 12

Exercise 12.1.

For all control volumes:

$$\frac{u_{i+1}-u_i}{\Delta x}-\frac{u_i-u_{i-1}}{\Delta x}=f\,\Delta x$$

Discrete system with 2 CVs (coarse grid):

$$\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} -8 \\ -4 \end{bmatrix}$$

Discrete system with 4 CVs (fine grid):

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}=\begin{bmatrix} -2 \\ -2 \\ -2 \\ 0 \end{bmatrix}$$

Damped Jacobi method:

$$\mathbf{u}^{k+1}=\mathbf{u}^k-\frac{1}{2}\mathbf{A}_D^{-1}\underbrace{\left(\mathbf{A}\mathbf{u}^k-\mathbf{b}\right)}_{\mathbf{r}^k}$$

On fine grid:

$$\mathbf{u}_h^0=\begin{bmatrix} 10 \\ -10 \\ 10 \\ -10 \end{bmatrix}, \quad \mathbf{r}_h^0=\begin{bmatrix} 42 \\ -38 \\ 42 \\ -20 \end{bmatrix}, \quad \mathbf{u}_h^1=\begin{bmatrix} 3 \\ -0.5 \\ -0.5 \\ 0 \end{bmatrix}, \quad \mathbf{r}_h^1=\begin{bmatrix} -11.5 \\ 1.5 \\ -1.5 \\ -0.5 \end{bmatrix}$$

Coarse grid equation:

$$\mathbf{A}_{2h}\mathbf{e}_{2h}=\mathbf{r}_{2h}, \quad \text{starting value: } \mathbf{e}_{2h}^0=\mathbf{0}$$

Restriction:

$$\begin{aligned}(r_{2h})_1 &= \left(r_h^1\right)_1+\left(r_h^1\right)_1=-10\\ (r_{2h})_2 &= \left(r_h^1\right)_3+\left(r_h^1\right)_4=-2\end{aligned}$$

After one damped Jacobi smoothing iteration:

$$\mathbf{e}_{2h}^1=\begin{bmatrix} -1.6667 \\ -1 \end{bmatrix}$$

Interpolation of corrections:

$$
\begin{aligned}
(\tilde{e_h})_1 &= 0.5\left(e^1_{2h}\right)_1 + 0.5\varDelta\left(e^1_{2h}\right)_{\mathrm{b}} = -0.8333 \\
(\tilde{e_h})_2 &= 0.75\left(e^1_{2h}\right)_1 + 0.25\varDelta\left(e^1_{2h}\right)_2 = -1.5 \\
(\tilde{e_h})_3 &= 0.25\left(e^1_{2h}\right)_1 + 0.75\varDelta\left(e^1_{2h}\right)_2 = -1.1667 \\
(\tilde{e_h})_4 &= 0.5\left(e^1_{2h}\right)_2 + 0.5\varDelta\left(e^1_{2h}\right)_{\mathrm{b}} = -0.5
\end{aligned}
$$

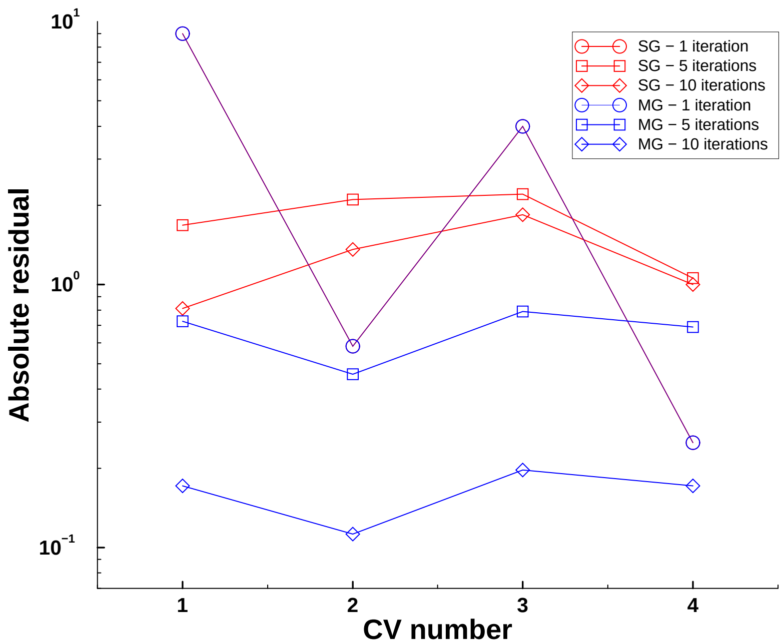
with boundary value $\left(e^1_{2h}\right)_{\mathrm{b}} = 0$

Correction on fine grid:

$$
\begin{aligned}
(u^*_h)_1 &= 3 - 0.8333 = 2.1667 \\
(u^*_h)_2 &= -0.5 - 1.5 = -2 \\
(u^*_h)_3 &= -0.5 - 1.6667 = -1.6667 \\
(u^*_h)_4 &= 0 - 0.5 = -0.5
\end{aligned}
$$

and so on.

Residuals for single grid and multigrid methods:



Iteration numbers for different grid sizes with single grid and multigrid methods with different numbers of grid levels:

N	SG	2G	3G	4g	5g	6g	7g	8g	9g	10g	FMG	MSG
2	105										105	105
4	480	87									76	435
8	2146	404	86								70	1877
16	9014	1744	392	88							61	7588
32	36647	7208	1665	400	91						69	29620
64	146618	29078	6818	1670	411	95					69	113477
128	581886	115887	27381	6765	1712	424	99				68	429904
256	>10E6	458988	108864	26952	6899	1761	436	102			66	>10E6
512	>10E6	>10E6	430620	106586	27314	7095	1812	449	106		61	>10E6
1024	>10E6	>10E6	>10E6	420219	107201	28068	7304	1863	461	109	61	>10E6

Exercise 12.2.

Not available

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