Erratum

Spectral Methods

Fundamentals in Single Domains

C. Canuto · M.Y. Hussaini · A. Quarteroni · T.A. Zang

Due to a technical error the caption of Figure 1.6 on page 29 and the content of pages 311 and 312 were reproduced in non-final form. Please find the corrected pages below. On pages 311 and 312 the changes are highlighted in red.

many results obtained from their high-resolution simulations was convincing evidence that the scaled energy spectrum (where the wavenumber is scaled by the inverse of the Kolmogorov length scale $\eta = (\nu^3/\bar{\epsilon})^{1/4}$, with ν the viscosity and $\bar{\epsilon}$ the average dissipation rate) is not the classical Kolmogorov result of $k^{-5/3}$, but rather k^{-m} with $m \simeq 5/3 - 0.10$.

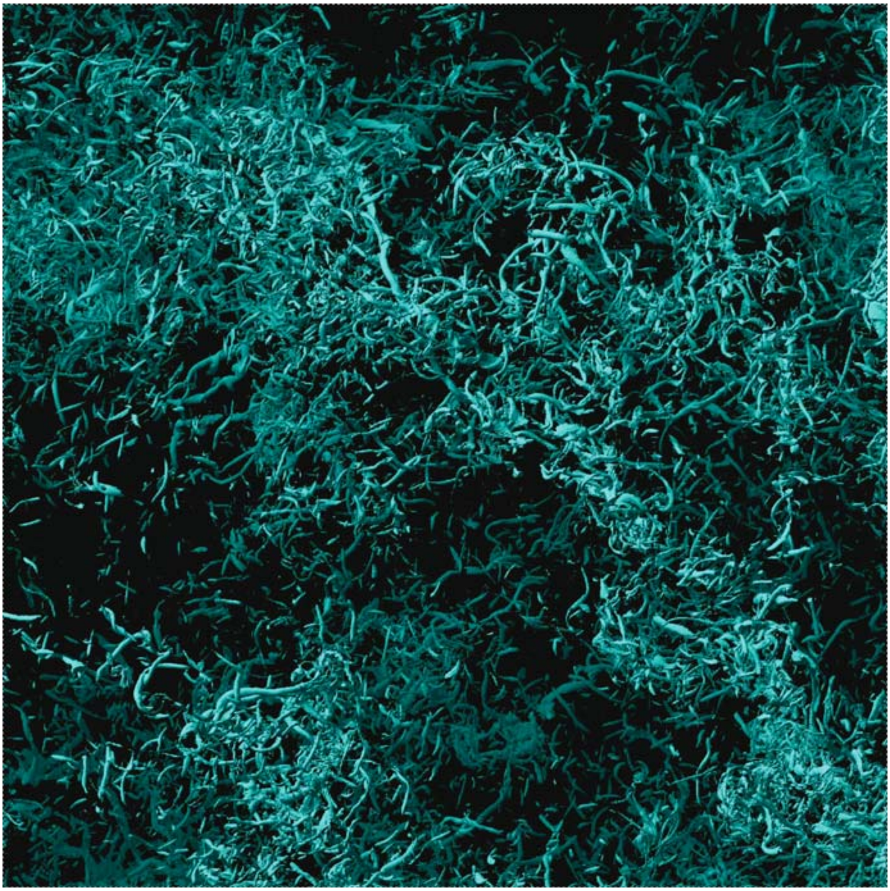

Fig. 1.6. Direct numerical simulation of incompressible isotropic turbulence on a 2048^3 grid by Y. Kaneda and T. Ishihara (2006): High-Resolution Direct Numerical Simulation of Turbulence. Journal of Turbulence **7**(20), 1–17. The figure shows the regions of intense vorticity in a subdomain with 1/4 the length in each coordinate direction of the full domain [Reprinted with kind permission by the authors and the publisher Taylor & Francis Ltd., http://www.tandf.co.uk/journals]

Rogallo (1977) developed a transformation that permits Fourier spectral methods to be used for homogeneous turbulence flows, such as flows with uniform shear. Blaisdell, Mansour and Reynolds (1993) used the extension of this transformation to the compressible case to simulate compressible, homoge-

$$\|v\|_{H^m_w(\mathbb{R}_+)} = \left(\sum_{j=0}^{m} \|v^{(j)}\|^2_{L^2_w(\mathbb{R}_+)} \right)^{1/2} .$$

A related family of weighted Sobolev spaces is useful, namely,

$$H^m_{w;\alpha}(\mathbb{R}_+) = \left\{ v \in L^2_w(\mathbb{R}_+) \mid (1+x)^{\alpha/2} v \in H^m_w(\mathbb{R}_+) \right\} , \qquad m \geq 0 , \tag{5.7.3}$$

equipped with the natural norm $\|v\|_{H^m_{w;\alpha}(\mathbb{R}_+)} = \|(1+x)^{\alpha/2} v\|_{H^m_w(\mathbb{R}_+)}$.

For each $u \in L^2_w(\mathbb{R}_+)$, let $P_N u \in \mathbb{P}_N$ be the truncation of its Laguerre series, i.e., the orthogonal projection of u upon $\mathbb{P}_N$ with respect to the inner product of $L^2_w(\mathbb{R}_+)$:

$$\int_{\mathbb{R}_+} (u - P_N u)\, \phi\, e^{-x}\, dx = 0 \quad \text{for all } \phi \in \mathbb{P}_N .$$

The following error estimate holds for any $m \geq 0$ and $0 \leq k \leq m$:

$$\|u - P_N u\|_{H^k_w(\mathbb{R}_+)} \leq C N^{k - \frac{m}{2}} \|u\|_{H^m_{w;m}(\mathbb{R}_+)} . \tag{5.7.4}$$

For the orthogonal projection P^1_N upon $\mathbb{P}_N$ in the norm of $H^1_w(\mathbb{R}_+)$, the following estimate holds for $m \geq 1$, $1 \leq k \leq m$:

$$\|u - P^1_N u\|_{H^k_w(\mathbb{R}_+)} \leq C N^{k + \frac{1}{2} - \frac{m}{2}} \|u\|_{H^m_{w;m-1}(\mathbb{R}_+)} ; \tag{5.7.5}$$

the same result holds for the projection $P^{1,0}_N$ upon $\mathbb{P}^0_N$ (Guo and Shen (2000)).

Concerning interpolation, let us consider the $N+1$ Gauss-Radau points x_j, $j = 0, \dots, N$, where $x_0 = 0$ and x_j, for $j = 1, \dots, N$, are the zeros of $l'_{N+1}(x)$, the derivative of the $(N+1)$-th Laguerre polynomial. For each continuous function u on $\mathbb{R}_+$, let $I_N u \in \mathbb{P}_N$ be the interpolant of u at the points x_j. Then, for any integer $m \geq 1$, $0 \leq k \leq m$ and $0 < \epsilon < 1$, one has

$$\|u - I_N u\|_{H^k_w(\mathbb{R}_+)} \leq C_\epsilon N^{k + \frac{1}{2} + \epsilon - \frac{m}{2}} \|u\|_{H^m_{w;m}(\mathbb{R}_+)} \tag{5.7.6}$$

(see Xu and Guo (2002), where additional approximation results can be found). The result stems from the error analysis given by Mastroianni and Monegato (1997) in the family of norms ($r \geq 0$ real)

$$\|v\|_{H^r_{w;*}(\mathbb{R}_+)} = \left(\sum_{k=0}^{\infty} (1+k)^r\, \hat{v}_k^2 \right)^{1/2} ,$$

where $\hat{v}_k = (v, l^{(0)}_k)_{L^2_w(\mathbb{R}_+)}$ are the Laguerre coefficients of v. For such norms, one has $\|v\|_{H^r_{w;*}(\mathbb{R}_+)} \leq c \|v\|_{H^r_{w;r}(\mathbb{R}_+)}$ for any integer r. Examples of applications to spectral Laguerre discretizations of boundary-value problems in $\mathbb{R}_+$ are provided in the above references. Usually, an appropriate change of unknown function is needed to cast the differential problem into the correct functional setting based on Laguerre-weighted Sobolev spaces.

Hermite approximations can be studied in a similar manner. The basic weighted space $L^2_w(\mathbb{R})$ involves the norm

$$\|v\|_{L^2_w(\mathbb{R})} = \left(\int_{\mathbb{R}} v^2(x)e^{-x^2}\,dx\right)^{1/2} .$$

The Sobolev spaces $H^m_w(\mathbb{R})$ are defined as above, with respect to this norm. The L^2_w-orthogonal projection operator P_N upon $\mathbb{P}_N$ satisfies the estimate

$$\|u - P_N u\|_{H^k_w(\mathbb{R})} \le CN^{\frac{k}{2}-\frac{m}{2}}\|u\|_{H^m_w(\mathbb{R})} \tag{5.7.7}$$

for all $m \ge 0$ and $0 \le k \le m$ (Guo (1999)). Interestingly, all H^ℓ_w-orthogonal projection operators P^ℓ_N upon $\mathbb{P}_N$, for $\ell \ge 0$, coincide with P_N, due to property (2.6.12) of Hermite polynomials. For the interpolation operator I_N at the Hermite-Gauss nodes in $\mathbb{R}$, Guo and Xu (2000) proved the estimate

$$\|u - I_N u\|_{H^k_w(\mathbb{R})} \le CN^{\frac{1}{3}+\frac{k}{2}-\frac{m}{2}}\|u\|_{H^m_w(\mathbb{R})} , \tag{5.7.8}$$

for $m \ge 1$ and $0 \le k \le m$.

When dealing with the unbounded intervals $\mathbb{R}_+$ and $\mathbb{R}$, an alternative to polynomials as approximating functions is given by functions that are the product of a polynomial times the natural weight for the interval. Thus, one uses the Laguerre functions $\psi(x) = \phi(x)e^{-x}$ in $\mathbb{R}_+$ or the Hermite functions $\psi(x) = \phi(x)e^{-x^2}$ in $\mathbb{R}$, where ϕ is any polynomial in $\mathbb{P}_N$. The behavior at infinity of the function to be approximated may suggest such a choice. We refer, e.g., to Funaro and Kavian (1990) and to Guo and Shen (2003) for the corresponding approximation results and for applications.

5.8 Approximation in Cartesian-Product Domains

We shall now extend to several space dimensions some of the approximation results we presented in the previous sections for a single spatial variable. The three expansions of Fourier, Legendre and Chebyshev will be considered. However, we will only be concerned with those Sobolev-type norms that are most frequently applied to the convergence analysis of spectral methods.

5.8.1 Fourier Approximations

Let us consider the domain $\Omega = (0, 2\pi)^d$ in $\mathbb{R}^d$, for $d = 2$ or 3, and denote an element of $\mathbb{R}^d$ by $\mathbf{x} = (x_1, \dots, x_d)$. The space $L^2(\Omega)$, as well as the Sobolev spaces $H^m_p(\Omega)$ of periodic functions, are defined in Appendix A (see (A.9.h)

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