
On Numerical Approximation of Stochastic Burgers' Equation

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Summary. We present a finite difference scheme for stochastic Burgers' equation driven by space-time white noise. We estimate the rate of convergence of the numerical scheme to the solution of stochastic Burgers's equation.

Key words: SPDE, Burgers' equation

Mathematics Subject Classification (2000): 60H15, 65M10, 65M15, 93E11

1 Introduction

We consider stochastic Burgers' equation

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + f(u(t, x)) + u(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{\partial W}{\partial t \partial x}(t, x), \quad (1.1)$$

for $t \in [0, T]$, $x \in [0, 1]$, with Dirichlet boundary condition

$$u(t, 0) = u(t, 1) = 0, \quad t > 0, \quad (1.2)$$

and initial condition

$$u(0, x) = u_0(x), \quad x \in [0, 1]. \quad (1.3)$$

Here f is a Lipschitz continuous function on the real line, u_0 is a square-integrable function over $[0, 1]$, and $\frac{\partial W}{\partial t \partial x}(t, x)$ is a space-time white noise. This

equation is very often viewed as a model equation of the motion of turbulent fluid. The solvability and the properties of its solution have been intensively studied in the literature, see, e.g., [1], [2], [7] and the references therein. Our aim is to investigate a numerical scheme for this equation. We study the following space-discretization of problem (1.1)–(1.2):

$$\begin{aligned} du^n(t, x_k^n) = & \left(\Delta_n u^n(t, x_k^n) + f(u(t, x_k^n)) + \frac{1}{2} \partial_n^- [[u^n(t)]](x_k^n) \right) dt \\ & + d\partial_n W(t, x_k^n), \quad k = 1, \dots, n-1, \end{aligned} \quad (1.4)$$

$$u^n(t, x_0^n) = u^n(t, x_n^n) = 0, \quad t \geq 0, \quad (1.5)$$

over the grid $\mathcal{G}^n := \{x_k^n = k/n : k = 0, 1, 2, \dots, n\}$, where d stands for the differential in t , and

$$\begin{aligned} \Delta_n h(x_k^n) &:= n^2 \left(h(x_{k+1}^n) - 2h(x_k^n) + h(x_{k-1}^n) \right), \\ \partial_n h(x_k^n) &:= n \left(h(x_{k+1}^n) - h(x_k^n) \right), \\ \partial_n^- h(x_k^n) &:= \left(h(x_k^n) - h(x_{k-1}^n) \right), \\ [[h]](x_k^n) &:= \frac{1}{3} \left(h^2(x_{k+1}^n) + h^2(x_k^n) + h(x_{k+1}^n)h(x_k^n) \right), \\ h(x_0^n) &= h(x_n^n) := 0, \end{aligned}$$

for functions h defined on the grid. For fixed $n \geq 2$ system (1.4) is a stochastic differential equation for the $(n-1)$ -dimensional process

$$u^n(t) = (u_k^n(t)) := (u^n(t, x_k^n)).$$

We show that for every initial condition $u^n(0) = (a_k^n) \in \mathbb{R}^{n-1}$ equation (1.4) has a unique solution $\{u^n(t) : t \in [0, T]\}$. We extend $u^n(t)$ from the grid onto $[0, 1]$ by $u^n(t, x) := u^n(t, [nx]/n)$, and show that this extension converges to u , the solution of stochastic Burgers' equation, provided that the initial condition $u^n(0)$ converges to u_0 . Moreover, we estimate the rate of convergence.

Numerical schemes for parabolic stochastic PDEs driven by space-time white noise have been investigated thoroughly in the literature, see, e.g., [3], [6], [10], [11] and the references therein. The class of equations considered in these papers does not contain stochastic Burgers' equation. A semi-discretization in time of stochastic Burgers' equation is studied in [9].

2 Formulation of the main result

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ be a filtered probability space carrying an \mathcal{F}_t -Brownian sheet $W = (W(t, x))$ on $[0, T] \times [0, 1]$. This means W is a Gaussian

field, $EW(t, x) = 0$, $E(W(t, x)W(s, y)) = (t \wedge s)(x \wedge y)$, $W(t, x)$ is \mathcal{F}_t -measurable, and $W(t, x) - W(s, x) + W(s, y) - W(t, y)$ is independent of \mathcal{F}_s for all $0 \leq s \leq t$ and $x, y \in [0, 1]$.

Let $f := f(z)$ be a locally bounded Borel function on \mathbb{R} , and let $u_0 = u_0(x)$ be an \mathcal{F}_0 -measurable random field such that almost surely $u_0 \in L^2([0, 1])$. We say that an $L^2([0, 1])$ -valued continuous \mathcal{F}_t -adapted random process is a solution of problem (1.1), (1.2), (1.3), if almost surely

$$\begin{aligned} \int_0^1 u(t, x) \varphi(x) dx &= \int_0^1 u_0(x) \varphi(x) dx + \int_0^t \int_0^1 u(s, x) \varphi''(x) dx ds \\ &+ \int_0^t \int_0^1 f(u(s, x)) \varphi(x) dx ds - \frac{1}{2} \int_0^t \int_0^1 u^2(s, x) \varphi'(x) dx ds \\ &+ \int_0^t \int_0^1 \varphi(x) dW(s, x) \end{aligned}$$

for all $t \in [0, T]$ and $\varphi \in C^2([0, 1])$, $\varphi(0) = \varphi(1) = 0$, where the last integral in the right-hand side of this equality is understood as Itô's integral, and φ' , φ'' denote the first and second derivatives of φ . We assume the following condition.

Assumption 2.1 *The force term f is Lipschitz continuous, i.e., there is a constant L such that*

$$|f(y) - f(z)| \leq L|y - z|$$

for all $y, z \in \mathbb{R}$.

It is well-known that under this condition problem (1.1), (1.2), (1.3) has a unique solution u , which satisfies also the integral equation

$$\begin{aligned} u(t, x) &= \int_0^1 G(t, x, y) u_0(y) dy + \int_0^t \int_0^1 G(t - s, x, y) f(u(s, y)) dy ds \\ &- \int_0^t \int_0^1 G_y(t - s, x, y) u^2(s, y) dy ds + \int_0^t \int_0^1 G(t - s, x, y) dW(s, y), \end{aligned} \quad (2.6)$$

where

$$G(t, x, y) := \sum_{j=1}^{\infty} \exp\{-j^2 \pi^2 t\} \varphi_j(x) \varphi_j(y), \quad \varphi_j(x) := \sqrt{2} \sin(j\pi x), \quad (2.7)$$

is the heat kernel, and

$$G_y(t, x, y) = \sum_{j=1}^{\infty} j\pi \exp\{-j^2 \pi^2 t\} \varphi_j(x) \psi_j(y), \quad \psi_j(x) := \sqrt{2} \cos(j\pi x). \quad (2.8)$$

Moreover, if u_0 is a continuous random field, then the solution u has a modification which is continuous in (t, x) , see [1], [2] and [7].

First we formulate our result for problem (1.4)–(1.5).

Theorem 2.1. *Let Assumption 2.1 hold. Let $n \geq 2$ be an integer, and let $(a_k^n)_{k=1}^{n-1}$ be an \mathcal{F}_0 -measurable random vector in \mathbb{R}^{d-1} . Then system (1.4)–(1.5) with the initial condition*

$$u^n(0, x_k^n) = a_k^n, \quad k = 1, 2, \dots, n-1, \quad (2.9)$$

admits a unique solution $u^n = \{u^n(t, x_k^n) : k = 0, 1, 2, \dots, n; t \geq 0\}$, which is continuous in $t \geq 0$. Moreover, for every $T > 0$, there is a finite random variable ξ such that

$$\sup_{t \leq T} \frac{1}{n} \sum_{j=1}^{n-1} |u^n(t, x_j^n)|^2 \leq \xi \left(\frac{1}{n} \sum_{j=1}^{n-1} |a_j^n|^2 + 1 \right) \quad (a.s.) \quad (2.10)$$

for all $n \geq 2$.

In order to formulate the main result of the paper we extend $(u^n(t, x_k^n))$, the solution of system (1.4)–(1.5) with initial condition $u^n(0, x_k^n) = u_0(x_k^n)$, $k = 0, 1, 2, \dots, n$, as follows:

$$u^n(t, x) := u^n(t, \kappa_n(x)), \quad x \in [0, 1], \quad t \geq 0,$$

where $\kappa_n(x) := [nx]/n$, and $[z]$ denotes the integer part of z . The main result of the present paper is the following.

Theorem 2.2. *Let Assumption 2.1 hold. Assume that $u_0 \in C([0, 1])$ almost surely. Then $u^n(t)$ almost surely converges in $L_2([0, 1])$ to $u(t)$, the solution of problem (1.1)–(1.3), uniformly in t in bounded intervals. Moreover, if almost surely $u_0 \in C^3([0, 1])$, then for each $\alpha < 1/2$, $T > 0$ there exists a finite random variable ζ_α such that*

$$\sup_{t \leq T} \int_0^1 |u^n(t, x) - u(t, x)|^2 dx \leq \zeta_\alpha n^{-\alpha} \quad (a.s.) \quad (2.11)$$

for all integers $n \geq 2$.

We prove Theorem 2.1 in the next section, and after presenting some preliminary estimates in Section 4, we prove Theorem 2.2 in Section 5.

3 Proof of Theorem 2.1

Using the notation

$$u_k^n(t) := u^n(t, x_k^n) = u^n\left(t, \frac{k}{n}\right)$$

$$W_k^n(t) := \sqrt{n} \left(W(t, x_{k+1}^n) - W(t, x_k^n) \right)$$

for $k = 1, 2, \dots, n-1$, we can write equations (1.4)–(1.5) as

$$\begin{aligned}
du_k^n(t) &= n^2 \sum_{i=1}^{n-1} D_{ki} u_i^n(t) dt + f(u_k^n(t)) dt \\
&\quad + \frac{n}{6} \left(|u_{k+1}^n|^2(t) - |u_{k-1}^n|^2(t) + u_{k+1}^n(t) u_k^n(t) - u_k^n(t) u_{k-1}^n(t) \right) dt \\
&\quad + \sqrt{n} dW_k^n(t), \quad k = 1, 2, \dots, n-1,
\end{aligned} \tag{3.12}$$

$$u_k^n(0) = a_k^n, \quad k = 1, 2, \dots, n-1, \tag{3.13}$$

where $u_0^n = u_n^n := 0$, and $D_{kk} = -2$, $D_{ki} = 1$ for $|k - i| = 1$, $D_{ki} = 0$ for $|k - i| > 1$. Notice that $W^n(t) := (W_k^n(t))$ is an $(n-1)$ -dimensional Wiener process. Fix $n \geq 2$ and define the vector field

$$A(x) := n^2 Dx + F(x) + nH(x), \quad x \in \mathbb{R}^{n-1},$$

where $D = (D_{ij})$ is the $(n-1) \times (n-1)$ matrix given above, and

$$\begin{aligned}
F_k(x_1, x_2, \dots, x_{n-1}) &:= f(x_k), \\
H_k(x_1, x_2, \dots, x_{n-1}) &:= \frac{1}{6} (x_{k+1}^2 - x_{k-1}^2 + x_{k+1}x_k - x_kx_{k-1}),
\end{aligned}$$

for $k = 1, 2, \dots, n-1$, with $x_0 = x_n := 0$. Then equations (3.12)–(3.13) can be written as

$$du^n(t) = A(u^n(t)) dt + \sqrt{n} dW^n(t), \tag{3.14}$$

$$u^n(0) = a^n, \tag{3.15}$$

where $u^n(t) := (u_k^n(t))$ and $a^n := (a_k^n)$ are column vectors in \mathbb{R}^{n-1} . Notice that

$$(x, Dx) = -x_1^2 - x_{n-1}^2 - \sum_{k=1}^{n-2} (x_{k+1} - x_k)^2, \tag{3.16}$$

$$(x, H(x)) = 0, \tag{3.17}$$

$$(x, F(x)) = \sum_{k=1}^{n-1} x_k f(x_k) \leq C \left(n + \sum_{k=1}^{n-1} x_k^2 \right) \tag{3.18}$$

for all $x \in \mathbb{R}^{n-1}$, where $(x, y) := \sum_{k=1}^{n-1} x_k y_k$ is the inner product of vectors $x, y \in \mathbb{R}^{n-1}$, $C := L + f^2(0)$, and L is the Lipschitz constant from Assumption 2.1. Hence A satisfies the following growth condition:

$$(x, A(x)) = n^2(x, Dx) + (x, F(x)) \leq C \left(n + \sum_{k=1}^{n-1} x_k^2 \right)$$

for all $x \in \mathbb{R}^{n-1}$ and for every integer $n \geq 2$. Clearly, A is locally Lipschitz in $x \in \mathbb{R}^{n-1}$. This and the above growth condition imply that equation (3.14)

with initial condition (3.15) admits a unique solution u^n , which is an \mathcal{F}_t -adapted \mathbb{R}^{n-1} -valued continuous process. (See the general result, Theorem 1 in [4], or Theorem 3.1 in [8], for example.)

It remains to show estimate (2.10). To this end we rewrite equation (3.14) for the solution u^n in the form

$$\begin{aligned} u^n(t) &= e^{n^2 t D} a^n + \int_0^t e^{n^2(t-s)D} \left(F(u^n(s)) + nH(u^n(s)) \right) ds \\ &\quad + \sqrt{n} \int_0^t e^{n^2(t-s)D} dW^n(s), \end{aligned} \quad (3.19)$$

and consider the \mathbb{R}^{n-1} -valued random processes

$$\eta^n(t) := \sqrt{n} \int_0^t e^{n^2(t-s)D} dW^n(s), \quad v(t) := v^n(t) := u^n(t) - \eta^n(t).$$

Then from equation (3.19) we get that v satisfies

$$\begin{aligned} dv(t) &= \left(n^2 Dv(t) + F(v(t) + \eta(t)) + nH(v(t) + \eta^n(t)) \right) dt, \\ v(0) &= a^n. \end{aligned}$$

Hence for $|v(t)|^2 := \sum_{k=1}^{n-1} |v_k(t)|^2$ we get

$$\begin{aligned} d|v(t)|^2 &= 2n^2(v(t), Dv(t)) dt + 2(v(t), F(v(t) + \eta^n(t))) dt \\ &\quad + 2n(v(t), H(v(t) + \eta^n(t))) dt \\ &\leq -2n^2 \sum_{k=1}^n (v_{k+1}(t) - v_k(t))^2 dt + 4C(n + |v(t)|^2) \\ &\quad + 2n(v(t), H(v(t) + \eta^n(t)) - H(v(t))) dt \end{aligned} \quad (3.20)$$

with $v_0(t) := v_n(t) := 0$, by virtue of (3.16), (3.17), (3.18), where C is the constant from inequality (3.18). Taking into account that for $x \in \mathbb{R}^{n-1}$

$$H_k(x) = [[x]]_k - [[x]]_{k-1}, \quad k = 1, \dots, n-1$$

with

$$[[x]]_j := \frac{1}{6}(x_{j+1}^2 + x_j^2 + x_{j+1}x_j), \quad j = 0, 1, \dots, n-1, \quad x_0 := x_n := 0,$$

we have

$$\begin{aligned} 2|(v(t), H(v(t) + \eta(t)) - H(v(t)))| &= \\ 2 \left| \sum_{k=0}^{n-1} (v_{k+1}(t) - v_k(t)) \{ [[v(t) + \eta^n(t)]]_k - [[v(t)]]_k \} \right| \end{aligned}$$

$$\begin{aligned}
&\leq n \sum_{k=0}^{n-1} (v_{k+1}(t) - v_k(t))^2 + n^{-1} \sum_{k=0}^{n-1} \{[[v(t) + \eta^n(t)]]_k - [[v(t)]]_k\}^2 \\
&\leq n \sum_{k=0}^{n-1} (v_{k+1}(t) - v_k(t))^2 + 100n^{-1} \sum_{k=1}^{n-1} (|\bar{\eta}_n|^2 |v_k|^2(t) + |\bar{\eta}_n|^4), \quad (3.21)
\end{aligned}$$

where

$$\bar{\eta}^n := \max_{0 < k < n} \sup_{t \leq T} |\eta_k^n(t)|.$$

Thus from (3.20) and (3.21) we get

$$\frac{1}{n} |v(t)|^2 \leq \frac{1}{n} |v(0)|^2 + 100|\bar{\eta}^n|^4 + 4Ct + (100|\bar{\eta}^n|^2 + 4C) \int_0^t \frac{1}{n} |v(s)|^2 ds.$$

Hence by Gronwall's inequality

$$\sup_{t \leq T} \frac{1}{n} |v(t)|^2 \leq e^{(100|\bar{\eta}^n|^2 + 4C)T} \left(\frac{1}{n} |v(0)|^2 + 100|\bar{\eta}^n|^4 + 4CT \right),$$

which implies

$$\sup_{t \leq T} \frac{1}{n} \sum_{k=1}^{n-1} |u_k^n(t)|^2 \leq \xi_n \left(\frac{1}{n} \sum_{k=1}^{n-1} |a_k^n|^2 + 1 \right) \quad (3.22)$$

with

$$\xi_n := e^{(100|\bar{\eta}^n|^2 + 4C)T} + 100|\bar{\eta}^n|^4 + 4CT + 2|\bar{\eta}^n|^2.$$

We are going to show that $\xi := \sup_{n \geq 2} \xi_n$ is a finite random variable. To this end note that the vectors e_1, \dots, e_{n-1} defined by

$$e_j = (e_j(k)) = \left(\sqrt{\frac{2}{n}} \sin \left(j \frac{k}{n} \pi \right) \right), \quad k = 1, 2, \dots, n-1,$$

form an orthonormal basis in \mathbb{R}^{n-1} , and that they are eigenvectors of the matrix $n^2 D$, with eigenvalues

$$\lambda_j^n := -4 \sin^2 \left(\frac{j}{2n} \pi \right) n^2 = -j^2 \pi^2 c_j^n,$$

where

$$\frac{4}{\pi^2} \leq c_j^n := \sin^2 \left(\frac{j\pi}{2n} \right) / \left(\frac{j\pi}{2n} \right)^2 \leq 1 \quad (3.23)$$

for $j = 1, 2, \dots, n-1$ and every $n \geq 1$. Therefore, for the random field $\{\eta^n(t, x) : t \geq 0, x \in [0, 1]\}$ defined by

$$\eta^n(t, x_k) := \eta_k^n := \sqrt{n} \int_0^t e^{n^2(t-s)D} dW^n(s)$$

for $x_k := k/n$, $n = 1, 2, \dots, n-1$, and

$$\eta^n(t, 0) = \eta^n(t, 1) = 0,$$

$$\eta^n(t, x) := \eta^n(t, \kappa_n(x)), \quad x \in (0, 1),$$

we have

$$\eta^n(t, x) = \int_0^t \int_0^1 G^n(t, x, y) dW(t, y),$$

for all $t \geq 0$, $x \in [0, 1]$, where

$$G^n(t, x, y) := \sum_{j=1}^{n-1} \exp(\lambda_j^n t) \varphi_j^n(\kappa_n(x)) \varphi_j(\kappa_n(y)), \quad (3.24)$$

$$\varphi_j(x) := \sqrt{2} \sin(jx\pi).$$

(Recall that $\kappa_n(y) := [ny]/n$.) Thus considering the special case $f = 0$, $\sigma = 1$, u_0 in Theorem 3.1 of [5], we get that almost surely

$$\sup_{n \geq 2} \bar{\eta}^n \leq \sup_{x \in [0, 1]} \sup_{t \leq T} |\eta^n(t, x)| < \infty,$$

which obviously implies that $\xi := \sup_{n \geq 2} \xi_n$ is a finite random variable. The proof of Theorem 2.2 is now complete. \square

4 Preliminary estimates

Define

$$\begin{aligned} G_y^n(t, x, y) &:= \partial_n G^n(t, x, y) := n(G^n(t, x, y + \frac{1}{n}) - G^n(t, x, y)) \\ &= \sum_{j=1}^{n-1} \exp\{-j^2 \pi^2 c_j^n t\} \varphi_j(\kappa_n(x)) n(\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y))), \end{aligned} \quad (4.25)$$

for $t \geq 0$, $x, y \in [0, 1]$, where $\kappa_n^+(y) =: \kappa_n(y) + \frac{1}{n}$.

Lemma 4.1. *For each $T > 0$ there exists a constant $K > 0$ such that*

$$\int_0^1 (G_y^n - G_y)^2(s, x, y) dx = K n^{-2} s^{-5/2}$$

for all $y \in [0, 1]$, $s \in (0, T]$ and all integers $n \geq 2$.

Proof. Clearly,

$$G_y^n - G_y = A_1 + A_2 + A_3 + A_4, \quad (4.26)$$

where

$$\begin{aligned} A_1 &:= \sum_{j=1}^{\infty} \exp\{-j^2 \pi^2 s\} [\varphi_j(x) - \varphi_j(\kappa_n(x))] j \pi \psi_j(y), \\ A_2 &:= \sum_{j=n}^{\infty} \exp\{-j^2 \pi^2 s\} \varphi_j(\kappa_n(x)) j \pi \psi_j(y), \\ A_3 &:= \sum_{j=1}^{n-1} \exp\{-j^2 \pi^2 s\} \varphi_j(\kappa_n(x)) [j \pi \psi_j(y) - n(\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y)))], \\ A_4 &:= \sum_{j=1}^{n-1} \{ [\exp(-j^2 \pi^2 s) - \exp(-j^2 \pi^2 c_j^n s)] \\ &\quad \times \varphi_j(\kappa_n(x)) n(\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y))) \}. \end{aligned}$$

Let $\|A_i\|$ denote the $L_2([0, 1])$ -norm of A_i in the x -variable. Fix $T > 0$, and let K denote constants, which are independent of $t \in [0, T]$, $x, y \in [0, 1]$, $s \in (0, T]$, $n \geq 2$, but can be different even if they appear in the same line. Then notice that

$$\begin{aligned} \|A_1\|^2 &= \int_0^1 |G_y(s, x, y) - G_y(s, x, y)|^2 dx \\ &\leq K n^{-2} \int_0^1 |G_{yx}(s, x, y)|^2 dx = K n^{-2} s^{-5/2}, \end{aligned} \quad (4.27)$$

by the well-known estimate

$$|G_{yx}(s, x, y)| \leq K s^{-3/2} e^{-(x-y)^2/s}, \quad s \in [0, T], \quad x, y \in [0, 1],$$

on the heat kernel. By the orthogonality of $\{\varphi_j\}$ in $L_2([0, 1])$,

$$\begin{aligned} \|A_2\|^2 &= \sum_{j=n}^{\infty} \exp\{-2j^2 \pi^2 s\} j^2 \pi^2 \psi_j(y)^2 \\ &\leq \sum_{j=n}^{\infty} j^2 \exp\{-j^2 s\} \leq 32 \sum_{j=n}^{\infty} j^2 \frac{1}{(j s^{1/2})^5} \leq K n^{-2} s^{-5/2}. \end{aligned} \quad (4.28)$$

By the mean-value theorem

$$\|A_3\|^2 = \sum_{j=1}^{n-1} \exp\{-2j^2 \pi^2 s\} [j \pi \psi_j(y) - n(\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y)))]^2$$

$$= \sum_{j=1}^{n-1} \exp\{-2j^2\pi^2 s\} [j\pi\psi_j(y) - j\pi\psi_j(\theta_n(y))]^2,$$

where $\theta_n(y) \in [\kappa_n(y), \kappa_n^+(y)]$. Hence

$$\begin{aligned} \|A_3\|^2 &\leq K n^{-2} \sum_{j=1}^{n-1} j^4 \exp\{-j^2 s\} \leq K n^{-2} s^{-2} \sum_{j=1}^{n-1} j^4 s^2 \exp\{-j^2 s\} \\ &\leq K n^{-2} s^{-2} \int_0^{n\sqrt{s}} x^4 \exp\{-x^2\} s^{-1/2} dx \leq K n^{-2} s^{-5/2}. \end{aligned} \quad (4.29)$$

Finally,

$$\begin{aligned} \|A_4\|^2 &= \sum_{j=1}^{n-1} [\exp\{-j^2\pi^2 s\} - \exp\{-j^2\pi^2 c_j^n s\}]^2 n^2 [\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y))]^2 \\ &\leq K \sum_{j=1}^{n-1} j^2 [\exp\{-j^2\pi^2 s\} - \exp\{-j^2\pi^2 c_j^n s\}]^2 \\ &\leq K \sum_{j=1}^{n-1} j^2 [j^2\pi^2 \exp\{-j^2\pi^2 c_j^n s\} (1 - c_j^n) s]^2 \\ &\leq K \sum_{j=1}^{n-1} j^6 (1 - c_j^n)^2 s^2 \exp\{-j^2 s\} \end{aligned}$$

by the mean-value theorem and the fact that $c_j^n \leq 1$. Hence by the definition of c_j^n in (3.23), using $\sin x = x + O(x^3)$, we have

$$\begin{aligned} \|A_4\|^2 &\leq K \sum_{j=1}^{n-1} j^6 (j\pi/2n)^4 s^2 \exp\{-j^2 s\} \leq K \sum_{j=1}^{n-1} j^6 (j/n)^4 s^2 \exp\{-j^2 s\} \\ &\leq K n^{-4} \sum_{j=1}^{n-1} j^{10} s^2 \exp\{-j^2 s\} \leq K n^{-2} s^{-2} \sum_{j=1}^{n-1} j^8 s^4 \exp\{-j^2 s\} \\ &\leq K n^{-2} s^{-2} \int_0^{s\sqrt{n}} x^8 \exp\{-x^2\} s^{-1/2} dx \leq K n^{-2} s^{-5/2}. \end{aligned} \quad (4.30)$$

Thus by virtue of equality (4.26) and inequalities (4.27), (4.28), (4.29) and (4.30) the proof is complete. \square

Lemma 4.2. *For each $T > 0$ there exists a constant K such that*

$$I := \int_0^T \left(\int_0^1 |G_y^n - G_y|^2(s, x, y) dx \right)^{1/2} ds \leq K n^{-1/2} \quad (4.31)$$

for all $y \in [0, 1]$.

Proof. Clearly, $I \leq I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &:= \int_0^\varepsilon \left(\int_0^1 G_y(s, x, y)^2 dx \right)^{1/2} ds, \\ I_2 &:= \int_0^\varepsilon \left(\int_0^1 G_y^n(s, x, y)^2 dx \right)^{1/2} ds dy, \\ I_3 &:= \int_\varepsilon^T \int_0^1 (G_y^n - G_y)^2(s, x, y) dx \Big)^{1/2} ds dy. \end{aligned}$$

From

$$G_y(s, x, y) = \sum_{j=1}^{\infty} \exp(-j^2 \pi^2 s) \varphi_j(x) j \pi \psi_j(y),$$

using the orthogonality of $\{\varphi_j\}$, we get

$$\begin{aligned} \int_0^1 G_y(s, x, y)^2 dx &\leq \sum_{j=1}^{\infty} \exp(-2j^2 \pi^2 s) j^2 \pi^2 \psi_j^2(y) \\ &\leq 20 \sum_{j=1}^{\infty} \exp(-j^2 s) j^2 \leq C s^{-3/2} \end{aligned}$$

for some constant C . Therefore,

$$I_1 \leq \int_0^\varepsilon C s^{-3/4} ds \leq 4C \varepsilon^{1/4}.$$

In exactly the same way, we obtain a constant C such that $I_2 \leq C \varepsilon^{1/4}$. By the estimate in Lemma 4.1, there is a constant C such that

$$I_3 \leq C n^{-1} \int_\varepsilon^T s^{-5/4} ds dy \leq C n^{-1} \varepsilon^{-1/4}.$$

Taking $\varepsilon = n^{-2}$, we obtain the statement of the lemma. \square

5 Proof of Theorem 2.2

We prove the theorem when $f = 0$. The proof in the general case of a Lipschitz function f goes in the same way, with some additional terms in the calculations, but without new difficulties. Notice that $u^n(t, x)$ satisfies

$$\begin{aligned} u^n(t, x) &= \int_0^1 G^n(t, x, y) u(0, \kappa_n(y)) dy \\ &\quad - \int_0^t \int_0^1 G_y^n(t-s, x, y) [[u^n(s)]](\kappa_n(y)) dy ds \\ &\quad + \int_0^t \int_0^1 G^n(t-s, x, y) dW(s, y), \end{aligned} \tag{5.32}$$

where G^n and G_y^n are defined by (4.25) and (4.25), respectively. From equations (2.6) and (5.32)

$$\|u^n(t, \cdot) - u(t, \cdot)\| \leq A(t) + B(t) + C(t), \quad (5.33)$$

with

$$A(t) := \left\| \int_0^1 G^n(t, \cdot, y) u_0^n(y) dy - \int_0^1 G(t, \cdot, y) u_0(y) dy \right\|, \quad (5.34)$$

$$B(t) := \left\| \int_0^t \int_0^1 G_y(t-s, \cdot, y) u(s, y)^2 dy ds - \int_0^t \int_0^1 G_y^n(t-s, \cdot, y) [[u^n(s)]](\kappa_n(y)) dy ds \right\|,$$

$$C(t) := \left\| \int_0^t \int_0^1 G^n(t-s, x, y) dW(s, y) - \int_0^t \int_0^1 G(t-s, x, y) dW(s, y) \right\|. \quad (5.35)$$

Clearly, $B \leq B_1 + B_2$, where

$$B_1^2(t) := \int_0^1 \left(\int_0^t \int_0^1 (G_y^n - G_y)(t-s, x, y) [[u^n(s)]](y) dy ds \right)^2 dx,$$

$$B_2^2(t) := \int_0^1 \left(\int_0^t \int_0^1 G_y(t-s, x, y) ([u^n(s)](y) - |u(s, y)|^2) dy ds \right)^2 dx.$$

By Minkowski's inequality, Lemma 4.2 and Theorem 2.1 we get

$$\begin{aligned} B_1^2(t) &\leq \left(\int_0^1 \int_0^t \left(\int_0^1 (G_y^n - G_y)^2(s, x, y) dx \right)^{1/2} [[u^n(t-s)]](y) ds dy \right)^2 \\ &\leq K n^{-1} \left(\int_0^t \int_0^1 [[u^n(s)]](y) dy ds \right)^2 \leq \xi n^{-1} \end{aligned} \quad (5.36)$$

for all $t \in [0, T]$, where K is a constant and ξ is a finite random variable, independent of t and n . By Lemma 3.1 (i) from [7], (take $q = 1$, $\rho = 2$, $\kappa = 1/2$ there), we have

$$B_2^2(t) \leq K \left(\int_0^t (t-s)^{-3/4} \|[[u^n(s, \cdot)]] - |u(s, \cdot)|^2\|_1 ds \right)^2 \quad (5.37)$$

for all $t \in [0, T]$, where $\|\cdot\|_1$ denotes the $L_1([0, 1])$ -norm. By simple calculations, using the Cauchy–Bunyakovskii inequality we get

$$\begin{aligned} \|[[u^n(s, \cdot)]] - |u(s, \cdot)|^2\|_1 &\leq K \|u^n(s, \cdot) - u(s, \cdot)\| (\|u^n(s, \cdot)\| + \|u(s, \cdot)\|) \\ &\quad + K \|u(s, \cdot) - u(s, \cdot + n^{-1})\| \|u^n(s, \cdot)\| \end{aligned} \quad (5.38)$$

for all $s \in [0, T]$ with a constant K . By Theorem 2.1 and Theorem 1 in [7], there is a finite random variable ξ such that almost surely

$$\|u^n(s, \cdot)\|^2 \leq \xi, \quad \|u(s, \cdot)\|^2 \leq \xi$$

for all $s \in [0, T]$ and integers $n \geq 2$. Thus from (5.38) and (5.37) by Jensen's inequality we obtain

$$|B_2(t)|^2 \leq \xi \int_0^t (t-s)^{-3/4} \|u^n(s, \cdot) - u(s, \cdot)\|^2 ds + \xi \zeta_n \quad (5.39)$$

for all $t \in [0, T]$ and $n \geq 2$, where

$$\zeta_n := \sup_{s \leq T} \|u(s, \cdot) - u(s, \cdot + n^{-1})\|^2, \quad (5.40)$$

and ξ is a finite random variable independent of t and n . By Burkholder's inequality for every $p \geq 1$ there exists a constant K_p such that

$$E \left[\sup_{t \leq T} |C(t)|^{2p} \right] \leq K_p \left\| \int_0^t \int_0^1 (G^n - G)^2(t-s, \cdot, y) dy ds \right\|_p,$$

where $\|\cdot\|_p$ stands for the $L_p([0, 1])$ norm. Consequently, for each $p \geq 1$ there exists a constant C_p such that

$$E \left[\sup_{t \leq T} |C(t)|^{2p} \right] \leq C_p n^{-p},$$

since

$$\sup_{x \in [0, 1]} \int_0^\infty \int_0^1 |G^n - G|^2(t, x, y) dy dt \leq \frac{c}{n}$$

with a universal constant c by Lemma 3.2 part (i) in [5]. Hence, by standard arguments, for any $\alpha \in (0, 1)$, one gets a finite random variable ξ_α such that almost surely

$$\sup_{t \leq T} |C(t)|^2 \leq \xi_\alpha n^{-\alpha} \quad (5.41)$$

for all $n \geq 2$. From (5.33) (5.36), (5.39) and (5.41) we get that almost surely

$$\begin{aligned} \|u^n(t, \cdot) - u(t, \cdot)\|^2 &\leq \xi \int_0^t (t-s)^{-3/4} \|u^n(s, \cdot) - u(s, \cdot)\|^2 ds \\ &\quad + \xi(\zeta_n + |A(t)|^2 + n^{-1}) + \xi_\alpha n^{-\alpha} \end{aligned}$$

for all $t \in [0, T]$, and integers $n \geq 2$, with a finite random variable ξ , where $A(t)$, ζ_n and ξ_α are defined in (5.34), (5.40) and (5.41), respectively. Hence, applying a Gronwall-type lemma (e.g. Lemma 3.4 from [5]), we obtain that almost surely

$$\sup_{t \leq T} \|u^n(t, \cdot) - u(t, \cdot)\|^2 \leq \xi \left(\zeta_n + \sup_{t \leq T} |A(t)|^2 + n^{-1} + \xi_\alpha n^{-\alpha} \right) \quad (5.42)$$

Now we are going to investigate the behaviour of $A(t)$ and ζ_n as $n \rightarrow \infty$. Set

$$\begin{aligned} v^n(t, x) &:= \int_0^1 G^n(t, x, y) u_0(\kappa_n(y)) \, dy \\ v(t, x) &:= \int_0^1 G(t, x, y) u_0(y) \, dy. \end{aligned}$$

Assume that $u_0 \in C^3([0, 1])$. Then by Proposition 3.8 in [5] we have a finite random variable ξ such that almost surely

$$\sup_{t \in [0, T]} \sup_{x \in [0, 1]} |v^n(t, x) - v(t, x)| \leq \xi n^{-1}$$

for all $n \geq 2$. Hence almost surely

$$\sup_{t \in [0, T]} |A(t)|^2 = \int_0^1 |v^n(t, x) - v(t, x)|^2 \, dx \leq \xi^2 n^{-2} \quad (5.43)$$

for all $t \in [0, T]$ and integers $n \geq 2$. Moreover, using Lemma 3.1 (iii) from [7] (with $\rho = 2$, $q = 1$ and $\kappa = 1/2$ there), we get a finite random variable ξ , such that almost surely

$$\zeta_n := \sup_{s \leq T} \|u(s, \cdot) - u(s, \cdot + n^{-1})\|^2 \leq \xi n^{-1} \quad (5.44)$$

for all $n \geq 2$. Consequently, inequalities (5.42), (5.43) and (5.44) imply estimate (2.11) of Theorem 2.2. Assume now that $u_0 \in C([0, 1])$. Then by Lemma 3.1 (iii) from [7] and Proposition 3.8 in [5] we have that almost surely

$$\sup_{t \in [0, T]} A(t) + \zeta_n \rightarrow 0,$$

as $n \rightarrow \infty$. Hence as $n \rightarrow \infty$,

$$\sup_{t \leq T} \|u^n(t, \cdot) - u(t, \cdot)\|^2 \rightarrow 0 \quad (a.s.).$$

The proof of Theorem 2.2 is complete. \square

Acknowledgements

The authors thank Jessica Gaines for her help in the preparation of the first version of this paper and for computer simulations. They also thank Jordan Stoyanov for improvements in the presentation of the paper.

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From Stochastic Calculus to Mathematical Finance

The Shiryaev Festschrift

Kabanov, Y.; Liptser, R.; Stoyanov, J. (Eds.)

2006, XXXVII, 633 p., Hardcover

ISBN: 978-3-540-30782-2