2

Building of a Mathematical Theory

In addition to the physical reality (see Sect. 1.2), the mathematical theory, denoted by MT, is the second most significant part of a physical theory. As briefly as possible, we will describe the elements necessary for the construction of a mathematical theory. For a more detailed description, we refer, e.g., to [6].

"Mathematics deals with imagined objects and imagined relations between these objects." In order to clarify this assertion, one tries to formalize the methods and the results of mathematics, i.e., one formally establishes the structure of a mathematical text in order to clearly indicate what one understands by "terms," "relations," "axioms," "proofs," "theorems," etc. Since the unfolding of these formal methods is very close to what our intuition suggests, we will restrict ourselves in order to be able to give a concrete meaning to concepts such as "structure," "partial structure," "relation," etc. Without this mathematical construction, assertions such as "In a PT, a partial structure of an MT gives us a picture of a real structure of the reality" would only have a very vague significance.

It is for this reason that we now take the time to describe the formal construction of an MT. Since in mathematics this cannot be done in an entirely homogeneous manner, it is still even less possible to give an overall picture of the various possibilities of such a formal construction. We have chosen a possibility that seems the most appropriate to our ends [6], namely the application of an MT in a PT (see Chap. 3).

2.1 Formal Language

A mathematical theory MT is defined as an assembly of signs comprising certain rules. That one can define an MT results from the fact that the formulation of all mathematical expressions, i.e., the mathematical language, is possible with a few very simple rules. Thus, conversely, in a formal manner, one can define an MT by these rules of language, i.e., the rules for the signs.

One could call these rules, using a linguistical term, the *syntax* of the mathematical language. Our first task will be thus to describe this "syntax." We will describe it as the *rules of the game* with *signs*.

The signs that form the mathematical text are on the one hand letters, and on the other hand, recognizable signs such as, e.g., $\vee$, $\neg$, $\in$, $\subset$. The signs are joined together under the form of assemblies of signs, where an assembly is a succession of signs written next to one another. First of all, one introduces rules that must characterize the "well-formed" assemblies according to the syntax of the mathematical language, and which make it possible, among these, to distinguish the assemblies that represent "objects" and the assemblies that represent "relations."

This manner of considering a mathematical text is particularly significant to physics. On the one hand, one wants to denote the facts of the application domain A_p into "well-formed relations" formulated in the mathematical language of MT (see Chap. 3). On the other hand, one also wants the relations of MT to lead to assertions about the reality domain W (see Chap. 6). Consequently, it is clearly necessary to lay down the rules according to which well-formed relations must be formulated in an MT.

To this end, we divide the signs into three categories:

1. The logical signs: $\vee$, $\neg$, τ. The sign $\vee$ means "or," the sign $\neg$ means "not," and the sign τ means "an object which ...." The sense of the sign τ will be explained more precisely hereafter. These logical signs are sufficient. Given that a precise development of all the rules is not significant for us, we will write from now on more concretely: instead of $\neg A$ always "not A;" instead of $\vee AB$ always "A or B;" instead of $\vee\neg AB$, i.e., instead of "(not A) or B" always "$A \Rightarrow B$," or in words "A implies B;" and for "not [(not A) or (not B)]" simply "A and B."
2. The letters; they always represent objects. Assemblies can also represent objects.
3. The specific signs of the MT considered as, e.g., the sign $\in$ of the set theory.

Only the assemblies that result from the following rules are allowed in an MT and are concretely accepted as well-formed assemblies. A characteristic must be attributed to each specific sign belonging to the third category. This is a sign, either substantific or relational, i.e., a sign that determines a term (concretely, an object) or a relation (concretely, an assertion about object(s)). Each one of these specific signs must still receive a weight, an integer n.

One designates by "terms" of the MT all assemblies that begin with a τ or a substantific sign, or which consist only of one letter, and one designates by "relations" of the MT all the other assemblies.

One designates by "formative construction" in an MT a sequence of assemblies that have the following property, i.e., for each assembly A of the sequence, one of the following conditions is satisfied:

(a) A is a letter.
(b) A is identical to "not B," B being a relation preceding A in the sequence.
(c) A is identical to "B or C," B and C being relations preceding A in the sequence.
(d) A is identical to $\tau_x(B)$, B being a relation preceding A in the sequence and containing the letter x. To indicate this, we often write $B(x)$ instead of B. One can concretely interpret this by "x is an object in an assertion $B(x)$" (e.g., $x \in M$, i.e., x is an element of the set M). The term $\tau_x(B)$ is a "privileged" term which, inserted in $B(x)$, satisfies the relation B (e.g., $\tau_x(x \in M)$ is a privileged element of the set M).
(e) A is identical to $sA_1 \cdots A_n$, s being a specific sign of the third category, of weight n, and A_1 to A_n being terms preceding A in the sequence. If s is a substantific sign, then $sA_1 \cdots A_n$ is a new object constituted of objects (an assertion over the objects) $A_1 \cdots A_n$.

2.2 Axioms and Proofs

The rules described up to now are only used to characterize the well-formed assemblies. Now we must indicate the methods according to which one decides if an assertion (to speak concretely) is "true." This is done by the posing of axioms and by the building of proofs. In mathematics the axioms are, so to speak, true assertions by definition. If these axioms become assertions on the reality domain W (see Chap. 6), then the truth of an axiom in PT takes a new sense with respect to MT. For this reason, we do not want to speak about "true" and "false" in an MT, as is often the case, since the axioms are posed and are not the result of an act of knowledge. The "truth" of many axioms cannot be perceived, since such a truth does often not exist because in an MT it is possible to pose, instead of the axiom A, the axiom "not A" (as a "physical" example, see the "axiom of simultaneity," in [2, Chap. VII], and its "nonvalidity" in the theory of special relativity, in [2, Chap. IX]).

The posing of axioms is a process of decisive importance for mathematics, as well as for physics, which requires a general explanation. We distinguish the explicit axioms and the axiomatic rules below.

An *explicit axiom* is a relation written according to the rules of Sect. 2.1. More than one such axiom can be written. In these explicit axioms, some letters (concretely, indefinite basic objects of the MT) can appear. They are called constants of the MT. These explicit axioms concretely represent true assertions about these basic objects. But one can also say that the basic objects are implicitly defined by the axioms. In brief, one often gives names to these basic objects (as an abbreviation for the totality of the axioms posed for them).

Thus, e.g., a term x (where x is a basic object) denotes an "ordered set" if an ordering relation is defined on this term with corresponding axioms.

The *axiomatic rules* are not relations in the sense of Sect. 2.1, but rules from which one can obtain new relations starting from relations already present in the text. Intuitively it must provide "identically true" relations, i.e., whatever the relations used in the application of an axiomatic rule, one obtains (intuitively) a "true" relation. We will meet such axiomatic rules, e.g., in Sect. 2.3, as logical rules (i.e., intuitively as a combination of logically identical true relations).

The simplest way in which to express the axiomatic rules is to use abbreviations for the assemblies. An axiomatic rule can be written as a symbolic relation constituted of these "abbreviations." These symbolic relations are also called *implicit axioms*. Letters used as abbreviations do not really appear in the theory since "any" relations resulting from the theory can be used in their place. A mathematical theory MT then consists of a text of distinct relations (concretely, "true" or "valid" assertions) that can be obtained using the following three rules:

1. the explicit axioms themselves;
2. the implicit axioms, if they contain terms and relations built according to the rules of Sect. 2.1;
3. of a relation B, in the case where the two relations A and "$A \Rightarrow B$" appear previously in the text of the MT.

All of the relations resulting from (1) to (3) (concretely, "true" assertions compared to the only well-formed assertions in the sense of Sect. 2.1) are called *theorems* of MT. For greater practicality, we include all the explicit axioms to the theorems of MT.

If a well-formed relation (i.e., formed in the sense of Sect. 2.1) cannot be obtained with the three preceding rules, then it is *not a theorem* in MT. Let us note that the fact that the relation A is not a theorem in MT *does not imply* that "not A" must be a theorem in MT. This will also be significant in the development of physical theories and, in particular, in the transition to more extended theories, and in the appreciation of the "physical reality" of facts not observed (Chap. 6).

One already introduces here another concept which will become very significant in a PT. This concerns the comparison of two MTs. A theory MT_2 is said to be "stronger" than a theory MT_1 if all of the signs of MT_1 are signs of MT_2, all of the explicit axioms of MT_1 are theorems in MT_2, and all of the implicit axioms of MT_1 are implicit axioms of MT_2. It follows that all the theorems of MT_1 are theorems in MT_2.

The transition from an MT_1 to a stronger MT_2 will become of paramount importance during the construction and the extension of a PT, because the stronger the MT becomes, the more the PT on which it depends becomes more expressive.

2.3 Logics

The first implicit axioms to be introduced are related to logic. Here one decides to use "normal," "bivalent" logic and not a polyvalent or other form of logic. Given that particular relations of an MT become in a PT assertions on the reality, we thus presuppose this logic for all PT, which will become clearer in Chaps. 3 and 6.

Attempts to modify logic were tried in mathematics as well as in physics. Quantum mechanics was used in physics as an argument for the need of a polyvalent logic, a probability logic with a continuous scale of values having "true" and "false" as limiting extremes. The fact that we build a quantum theory with normal logic shows that such a necessity does not exist.

We introduce logic by the following axiomatic rules: If A, B, C are relations, then the relations

$$(A \text{ or } A) \Rightarrow A, \tag{2.3.1}$$

$$A \Rightarrow (A \text{ or } B), \tag{2.3.2}$$

$$(A \text{ or } B) \Rightarrow (B \text{ or } A), \tag{2.3.3}$$

$$(A \Rightarrow B) \Rightarrow ((C \text{ or } A) \Rightarrow (C \text{ or } B)) \tag{2.3.4}$$

are implicit axioms of MT.

If one considers a relation with *two* possible values, "true" or "false," and one attributes to the relation "A or B" the value true if at least one of the two relations A or B is true, otherwise the value false, and one attributes to the relation "not A" the value true if A is false, and conversely, then (2.3.1) to (2.3.4) represent true relations (because "$A \Rightarrow B$" means by definition "(not A) or B," i.e., "$A \Rightarrow B$" is true if A and B are true or A is false).

And yet one should not confuse the logical implicit axioms (2.3.1) to (2.3.4) with the intuitive association of "true" or "false" to "any" relations $A, B, \ldots$. However, in Sect. 2.2 we had not introduced the "values" true and false for relations, but only laid down the rules of proof deriving new relations from axioms. What one would express intuitively is as follows: In this MT, "A is a true relation" is replaced by the new expression "A is a theorem in MT." We have already outlined above that if A is not a theorem in MT, then it does not result inevitably that "not A" must be a theorem in MT. It may be thus that neither A nor "not A" are theorems in MT. It is only in the following way that the logic introduced by the implicit axioms described above is a normal logic:

(a) If a relation A, as well as "not A," is a theorem in MT, then each well-formed relation B (in the sense of Sect. 2.1) is a theorem in MT. Such an MT is contradictory and is unusable, since it does not in fact state anything. We will see in Chap. 3 that such a contradictory MT already leads to a *completely* "unusable" PT before one has *tested such an MT by* *experiment*. For this reason, we eliminate all the contradictory MT. Then only A or only "not A" can be a theorem in MT.

(b) The following principle of *proof by contradiction*, often used during proofs, is also valid: If one adds to MT, as an additional axiom, the relation "not A," then one obtains a theory MT' *stronger* than MT (in the sense of Sect. 2.2); and if MT' is contradictory, then A is a theorem in MT.

(c) If A is a theorem in MT, then "A or B" is a theorem in MT.

(d) If B is a theorem in MT, then "A or B" is a theorem in MT.

(e) If "not A" and "not B" are theorems in MT, then "not (A or B)" is a theorem in MT; and also if "A or B" is a theorem in MT, then "not [(not A) and (not B)]" is a theorem in MT, i.e., "not A" as well as "not B" cannot be theorems in MT.

The two criteria (a) and (b) fix the sense of "not," and what we briefly indicate in the *new form* by bivalent logic.

The criteria (c) to (e) fix the sense of "or," which replaces the sense of "or" introduced above in an intuitive way, using the values "true" and "false." In short, we say that (c) to (e) fix the "normal" sense of "or."

In this sense, on the basis of (a) to (e), we say finally that the bivalent logic is introduced by the implicit axioms (2.3.1) to (2.3.4).

Of course, we will not draw here all the significant consequences for the technique of mathematical proofs of the implicit axioms (2.3.1) to (2.3.4) mentioned above (in particular to prove the deductions (a) to (e)). This is not necessary because the deductions obtained are for the most part "intuitively" obvious, and the reader is certainly accustomed to applying such logic and the methods of proof in mathematics. For a detailed description, we refer to [6]. In particular, it is easy to establish the above deductions (a) to (e) using the deductions mentioned in [6, Chap. I, Sect. 3.]. (a) is proved in [6, Chap. I, Sect. 3.1], (b) is identical to C 15 of [6, Chap. I, Sect. 3.3], and (c) and (d) easily result from the implicit axioms (2.3.2) and (2.3.3) mentioned above and the rule of proof (3) of the preceding section of this book (i.e., Sect. 2.2). (e) is a consequence of the equivalent relations mentioned under C 24 of [6, Chap. I, Sect. 3.5], "not (not A) $\Leftrightarrow A$" and "(A or B) $\Leftrightarrow$ not [(not A) and (not B)]."

Because of their interest, particularly for the very significant reflection from the point of view of physics (see Chap. 3 and especially Chap. 6), two other relations which result from the implicit axioms (2.3.1) to (2.3.4) are added:

(f) Let A be a relation in MT, and let MT' be the theory obtained by adjoining A to the axioms of MT. If B is a theorem in MT, then "$A \Rightarrow B$" is a theorem in MT. (*Proof*: see C 14 of [6, Chap. I, Sect. 3.3]).

(g) Let $A(x)$ and B be relations in MT (x is not a constant of MT), let T be a term such that $A(T)$ is a theorem in MT, and let MT' be the theory obtained by adjoining $A(x)$ to the axioms of MT (x is thus a constant of MT'). If B is a theorem in MT', then B is a theorem in MT. (*Proof*: see C 19 of [6, Chap. I, Sect. 3.3]).

From the two relations (a) and (b) above, for an MT without contradiction, it results that if neither A nor "not A" are theorems in MT, then one can add to MT the relations A as well as "not A" as axioms, and in this manner thus obtain two theories, MT_1 and respectively MT_2, which are both stronger than MT and without contradiction. As already mentioned above, this situation is very significant for physics. In particular, we return to the discussion of the relation between the Galileo–Newton space-time theory and the special relativity theory in [2, Chap. IX].

We will not examine here the problems of the "proof" of noncontradiction in an MT. We adopt the point of view that the MTs used are without contradiction as long as a contradiction is not discovered. In the case where there would be a contradiction in MT, we would have to change the axioms in order to eliminate it.

If A and B are relations, we briefly write for the relation "$(A \Rightarrow B)$ and $(B \Rightarrow A)$": "$A \Leftrightarrow B$" and we say that "A is *equivalent* to B." For any relations, because of the axioms introduced above, the following equivalences (C) are valid (as theorems in MT, see [6, Chap. I, Sect. 3.3]):

$$\begin{aligned}
&(A \text{ and } (B \text{ or } C)) \Leftrightarrow ((A \text{ and } B) \text{ or } (A \text{ and } C)),\\
&(A \text{ or } (B \text{ and } C)) \Leftrightarrow ((A \text{ or } B) \text{ and } (A \text{ or } C)),\\
&(\text{not } (A \text{ and } B)) \Leftrightarrow ((\text{not } A) \text{ or } (\text{not } B)),\\
&(\text{not } (A \text{ or } B)) \Leftrightarrow ((\text{not } A) \text{ and } (\text{not } B)),\\
&(\text{not } (\text{not } A)) \Leftrightarrow A.
\end{aligned} \tag{C}$$

If we "formally" regard the sign $\Leftrightarrow$ as a sign of equality, and if we put the sign "$\wedge$" instead of "and" and the sign "$\vee$" instead of "or," then the logical theorems written above enter *formally* into the rules of calculation for a complemented distributive lattice. The fact that

$$(A \Rightarrow B) \Leftrightarrow [(A \text{ or } B) \Rightarrow B]$$

is a theorem can formally be interpreted so that the sign $\Rightarrow$ is the order determined by the lattice operations $\wedge$, $\vee$, so that also conversely the sign $\Leftrightarrow$ becomes formally a sign of equality.

However, there is a completely *decisive* difference between a complemented distributive lattice and the logical theorems above. The letters that appear in the logical relations are not elements of a set, and can in no way be applied directly to the logical relationship between relations. Relations in MT and elements of a set are essentially different objects and should not be confused.

In spite of this, the five relations above (C) can be considered as the most concrete formulation used to introduce the classical logic in MT by the implicit axioms (2.3.1) to (2.3.4). In what follows, we will always presuppose the validity of the implicit axioms (2.3.1) to (2.3.4).

Having introduced the logical sense of "or" and "not" in MT, one must emphasize that nothing has been said about the manner of logically binding relations related to facts of reality, because some such relations are first of all not combinations of signs appearing as relations $A, B, \dots$ in MT (see Sect. 2.1). It is only in Chap. 6 that we will speak of the problem of a relation between an MT and the reality domain W, and by this also of the problem of the interpretation of the signs $\vee$ and $\neg$ of Sect. 2.1. With regard to the logical signs, the mathematization process (cor) is not simple; this is seen particularly with the sign τ introduced in Sect. 2.1, which will be further outlined in this chapter.

Whereas $\vee$ and $\neg$ obtained their sense by the implicit axioms (2.3.1) to (2.3.4) (concretely as "or" and "not"), we must now give a sense to τ by axiomatic rules. Previously, we introduced some abbreviations which have an obvious concrete sense. If R is an assembly containing the letter x, then one can form the assembly $\tau_x(R)$ which does not contain any x (see Sect. 2.1 and [6, Chap. I, Sect. 1.1]). If one substitutes x by the assembly $\tau_x(R)$ in R (i.e., everywhere where x appears in R), then one obtains a new assembly which we denote by $(\exists x)R$. The assembly $(\exists x)R$ does not contain any x. The assembly $\tau_x(R)$ is concretely an object which satisfies R. The assembly $(\exists x)R$ is thus R with a "particular object which satisfies R" put in the place of x. We can also say that "there exists an object which satisfies R." If R is a relation, then $(\exists x)R$ is a relation ($\tau_x(R)$ is a term), i.e., according to Sect. 2.1, $(\exists x)R$ can appear in MT only if R is a relation. The fact that "there is not an object which satisfies (not R)" is concretely expressed by "R is valid for all the objects." For this reason we shorten "not $((\exists x)(\text{not } R))$" by $(\forall x)R$. If R is a relation, then $(\forall x)R$ is a relation, and it is meaningful in MT.

We now introduce the meaning, corresponding to the intuitive sense, of $(\exists x)R$ by an axiomatic rule.

If $R(x)$ is a relation containing the letter x and if T is a term, then

$$R(T) \Rightarrow (\exists x)R(x) \tag{2.3.5}$$

is an implicit axiom. $R(T)$ is the relation which results from $R(x)$ if x is replaced by T. The implicit axiom (2.3.5) thus expresses the fact that there exists an object which satisfies R if there is a T which satisfies R.

For the details and consequences of the implicit axioms (2.3.1) to (2.3.5) introduced until now, one can refer to [6, Chap. I, Sect. 4], or one can follow the method exerted intuitively by using the expressions "there exists" and "for all."

However, some theorems will be mentioned without proof, since they will play a role in Chap. 3.

(α) If $R(x)$ is a theorem in MT and if x is not a constant of MT, then $(\forall x)R$ is a theorem in MT.

(β) If $A(x)$ and $R(x)$ are relations of MT (x is not a constant of MT), and if $A(x) \Rightarrow R(x)$ is a theorem in MT, then "$(\exists x)A(x) \Rightarrow (\exists x)R(x)$" is a theorem in MT.

(γ) If $A(x)$ and $R(x)$ are relations of MT, then the relations "$(\exists x)$ ($A(x)$ and not $R(x)$)" and "$(\forall x)(A(x) \Rightarrow R(x))$" are equivalent.

By introducing the sign τ, the relation $(\exists x)R$,the relation $(\forall x)R$, and the implicit axiom (2.3.5), we were not so concerned with the establishment of such theorems as such, but only with showing where these relations intervene in MT; however, one does not speak of physics or facts of reality. For this reason we warn, expressly and with insistence, not to identify blindly the expressions "there exists" and "for all" with "any forms of everyday assertion" on reality. To be able to express warnings here, we briefly give some examples of "common factual assertions" to which one will not give a sense during the construction of a PT (at least not a directly obvious sense). It must be outlined that such factual assertions are not used as a basis for the construction of a PT as is presented in this book.

Such meaningless assertions (at least for the moment) are, e.g., expressions such as "all ravens are black," "all electrons have the same mass m," "all men are mortal," etc. Let us take as an example the first expression: "all ravens are black." One can easily formalize it into the mathematical shape of Sect. 2.1: r is a relational sign of weight 1 with the sense "to be raven," s is a relational sign of weight 1 with the sense "to be black." The expression "all ravens are black" would then be written

$$(\forall x)(r(x) \Rightarrow s(x)).$$

But contrary to the everyday expression above, the relational signs r and s do not have significance, as for the contents (which is the intention) one cannot introduce the sign $\forall$ only "formally" as above in MT. Should "all ravens" also have a factual reference (and not only a formal one)? But what kind of a reference? What do we mean by "all ravens"? and how can "all ravens" be shown? In fact, where can "all ravens" be found?

We will never use in the construction of PT expressions such as those mentioned above, except in *forms of shortened expressions*, but on which the logical rules are *not* applicable!

After this *warning*, "to not apply blindly forms of logical assertions, and logical rules of MT to reality," we continue with the logical construction of an MT by introducing the relation "to be identical" by a sign in MT, and by giving it a meaning in MT using axiomatic rules.

Let us introduce as additional sign (for all the MTs which will be used later), the equality sign "=," a relational sign of weight 2 with the prescription, in the sense of Sect. 2.1, that "$= AB$" is a relation between two terms (i.e., two objects) A,B. Instead of "$= AB$" we write "$A = B$." For "not $(A = B)$"

we write "$A \neq B$." We fix the sense of the sign "=" by the following axiomatic rules:

If $R(x)$ is a relation, and if A and B are terms, then

$$(A = B) \Rightarrow (R(A) \Leftrightarrow R(B)) \tag{2.3.6}$$

is an implicit axiom.

If $R(x)$ and $S(x)$ are relations, then

$$[(\forall x)(R(x) \Leftrightarrow S(x))] \Rightarrow [\tau_x(R) = \tau_x(S)] \tag{2.3.7}$$

is an implicit axiom.

The implicit axiom (2.3.6) expresses the fact that it is "equal" if, in a relation $R(x)$ which includes the letter x, one replaces x by A or a B identical to A, i.e., that the relations $R(A)$ and $R(B)$ are "the same" or – more precisely – they are equivalent. One can also say: Two identical terms A, B also have the same "property" R. The implicit axiom (2.3.7) is not perceived in so intuitive a way, since the sign τ (concretely, "an object which...") is less easy to grasp in its intuitive content. The implicit axiom (2.3.7) says that for any x, two identical properties R and S imply that the term (concretely, object) determined by τ is identical for R as well as for S, i.e., that the process of selection τ chooses "in the same manner" the identical properties R and S.

For a detailed description of the consequences of this axiom we refer to [6] I, Sect. 5. Two theorems will be given, without proof, since they will often be used later and they will also have an importance from the physical point of view in Chap. 3. Let us begin with a definition:

If the relation

$$(\forall y)(\forall x)((R(y) \text{ and } R(x)) \Rightarrow (x = y))$$

is a theorem in MT (it is often said that there exists at most one x such that R), then $R(x)$ is said to be "single-valued in x" in MT. For each MT which satisfies the axioms (2.3.1) to (2.3.7) there is the relation

(δ) If $R(x)$ is single-valued in x in MT,

$$R(x) \Rightarrow (x = \tau_x(R))$$

is a theorem in MT.

And so, conversely, for a term T, the relation

$$R(x) \Rightarrow (x = T)$$

is a theorem in MT, then R is single-valued in x in MT.

Let us introduce another definition: If $R(x)$ is single-valued in x and if

$$(\exists x)R(x)$$

is a theorem in MT, one says "there exists one and only one x such that $R(x)$," and that $R(x)$ is "functional" in x in MT. One has then

(ε) If $R(x)$ is functional in x in MT,

$$R(x) \Leftrightarrow (x = \tau_x(R))$$

is a theorem in MT.

And if, conversely, for a term T,

$$R(x) \Leftrightarrow (x = T)$$

is a theorem in MT, then $R(x)$ is functional in x in MT.

In conclusion, we repeat once again that the fundamental logical axioms of an MT (as summarized in Sect. 2.3) were not described to deduce as theorems of an MT the known conclusions in mathematics, but were described to better distinguish later the sense of these axioms of an MT in a PT (e.g., in Chaps. 3 and 6).

2.4 Set Theory

Given that we presuppose also the set theory, we briefly turn our attention to the problem of the pose of axioms. We want especially to indicate some elements which will be significant for the use of these axioms in a PT. The problems of the use of the set theory in the representation MT of a PT can only be dealt with later; for this reason, almost no indication on the physical meaning will be given here for the moment.

In the set theory there appears as a new relational sign: "$z \in y$" (concretely, "z is an element of y"). As an abbreviation for "$(\forall z)((z \in x) \Rightarrow (z \in y))$," i.e., for the relation "all the elements z of x are also elements of y," we briefly write "$x \subset y$" (concretely, "x is part of y;" or "y contains x" or similar expressions). For "not $(z \in y)$" respectively "not $(x \subset y)$," we often write "$z \notin y$" respectively "$x \not\subset y$."

The relational sign $\in$ will become of decisive importance for the use of an MT in a PT. Concretely, a PT includes assertions about the facts of reality as elements of a set (see Chap. 3).

For the set theory, it is decisive that the set is intuitively seen as the collective whole of all its elements. But, it is precisely this "whole of all" which is doubtful in physics, as we have already outlined in the introduction of the logical sign $\forall$ in MT. For example, the statement "set of all electrons" is not regarded as meaningful during the construction of a PT, because it is doubtful that this totality of all electrons exists.

In mathematics, "to collect in a set" is a significant notion of the set theory. If the set is a collection of its elements, then two sets must be identical if they have the same elements; for this reason, one requires as a first explicit axiom

$$(\forall x)(\forall y)((x \subset y \text{ and } y \subset x) \Rightarrow (x = y)). \tag{2.4.1}$$

But it is precisely this "collection in a set," intuitively so obvious, which leads to contradictions in mathematics when one neglects to take certain precautionary measures.

If we try to join together formally all the x of a determined kind in a set, this can be carried out as follows: Let $R(x)$ be a relation, we shorten the relation "$(\exists y)(\forall x)((x \in y) \Leftrightarrow R(x))$" by "$\mathrm{Coll}_x R$." If $\mathrm{Coll}_x R$ is a theorem in MT (the relation R is said to be collectivizing in x in MT), one says that the relation $R(x)$ determines a set. y is the "set of all x which satisfies $R(x)$," and because of "$(\forall x)((x \in y) \Leftrightarrow R)$" and "$(\forall x)((x \in z) \Leftrightarrow R)$," it follows the equality "$z = y$." For the relation "$S(y) \mid (\forall x)((x \in y) \Leftrightarrow R)$," there exists therefore at most one y such that $S(y)$, i.e., according to Sect. 2.3, is single-valued in y. If $(\exists y)S(y)$ is a theorem in MT (i.e. if $S(y)$ is functional in y), according to Sect. 2.3, then "$S(y) \Leftrightarrow (y = \tau_y(S))$" is also true. Consequently, if $\mathrm{Coll}_x R$, i.e., $(\exists y)S(y)$, is a theorem in MT, we can denote the set of y such that $S(y)$ by "$\tau_y[(\forall x)((x \in y) \Leftrightarrow R(x))]$," for that we write $\mathcal{E}_x(r)$ concretely "$\mathcal{E}_x(r)$ is the set of x such that $R(x)$." The relation "$(\forall x)((x \in \mathcal{E}_x(R)) \Leftrightarrow R)$" is thus equivalent to $\mathrm{Coll}_x R$, and the relation $R(x)$ is equivalent to "$x \in \mathcal{E}_x(R)$." Later on, the set $\mathcal{E}_x(r)$ will often be written in the usual form "$\{x \mid R(x)\}$."

But the set $\mathcal{E}_x(r)$ "exists" only if $\mathrm{Coll}_x R$ is a theorem in MT. In no case, for all $R(x)$, is the relation $\mathrm{Coll}_x R$ a theorem in MT. This seems curious, since there should "always" be, i.e., for each $R(x)$, the "set of x such that $R(x)$." Wouldn't it be easy to conceive of $\mathrm{Coll}_x R$ as an axiom for all $R(x)$? All those who have dealt with problems of an "intuitive" set theory know that such a general condition contains problems. For this reason we will proceed in a more careful way.

If one considers only the x for which $R(x)$ is true and which are elements of a set z (z being susceptible to contain elements that do not satisfy R), then one expects that the x which satisfy R form a subset of z, i.e., $\mathrm{Coll}_x R$ becomes a theorem in MT. If the relation R still depends on an object y, and if all x that satisfy the relation R are elements of a set z (which possibly depend on y), y being fixed, then all x that satisfy R, for at least an element y of a set u, must form a set, which we require in the implicit axiom

$$((\forall y)(\exists z)(\forall x)(R \Rightarrow (x \in z)) \Rightarrow (\forall u)\mathrm{Coll}_x((\exists y)((y \in u) \text{ and } R)). \tag{2.4.2}$$

This will make it possible to obtain, starting from sets, new sets using relations. But to be able to produce sets we pose the following axioms:

$$(\forall x)(\forall y)\mathrm{Coll}_z(z = x \text{ or } z = y). \tag{2.4.3}$$

This means that if x and y are objects, then there is a set whose only elements are x and y. We indicate this by $\{x, y\}$. This axiom is very easy to interpret in a PT as any finite set in which a finite number of $x_1 \cdots x_n$ are collected together. But for the MT, the infinite sets (which will be defined later) will be of great importance. These infinite sets made necessary a concrete axiomatization of the set theory; but it is not possible to interpret them physically, which we have already mentioned above, and which we will more precisely discuss in Sect. 3.2.4.

To continue to develop the set theory, we still need the possibility of introducing a pair (x, y) of terms (objects) as a new term, i.e., a new object made up of two individual objects x, y

$$(\forall x)(\forall x')(\forall y)(\forall y')((x, y) = (x', y') \Rightarrow (x = x' \text{ and } y = y')). \tag{2.4.4}$$

The pair (x, y) is different from the set $\{x, y\}$! In the pair, according to (2.3.4), the components x and y are ordered.

$$(\forall x)\mathrm{Coll}_y(y \subset x) \tag{2.4.5}$$

means that "the set of all the subsets of a set x exists." The last axiom

postulates the existence of an infinite set. (2.4.6)

An infinite set is precisely a set not finite. A finite set is defined by the fact that the cardinality changes if one adds one element to the set.

Each MT used in a PT is stronger than the set theory, i.e., all the axioms indicated until now are valid in MT. *In what follows, we suppose that each MT is stronger than the set theory.*

With regard to physics – as outlined in Sect. 3.2.4. – one could add to the set theory a seventh axiom that postulates that "there is no set whose size is strictly between that of the integers and that of the continuum." One could show that this seventh axiom is independent of the precedents. Otherwise, one could require that each set of an MT is either a set at most countable, or a subset of an echelon on sets at most countable.

In an MT (stronger than the set theory), starting from n sets (terms) $E_1, \ldots, E_n$, one can build step by step new sets. We denote by $\mathcal{P}(E)$ the set of all subsets of E, and we denote by $E_1 \times E_2$ the set of all pairs (x, y) with $x \in E_1$ and $y \in E_2$. When one applies a finite number of times the operations $\mathcal{P}$ and $\times$ to $E_1, \ldots, E_n$, one obtains new sets. Such a process, applicable in a finite number of steps, is called an *echelon construction* and the set thus obtained is called an *echelon.* The sets $E_1, \ldots, E_n$ are the base sets of the echelon construction. We denote an echelon by $S(E_1, \ldots, E_n)$, where the letter S denotes the *echelon construction scheme*, whereby one obtained the echelon. If $E'_1, \ldots, E'_n$ are n different sets, then $S(E'_1, \ldots, E'_n)$ is also an echelon of scheme S, but on the base sets $E'_1, \ldots, E'_n$.

In MT, let f_i be mappings of the sets E_i onto the sets E'_i, i.e., for any $x \in E_i$ one has $f_i(x) \in E'_i$, where $f_i(x)$ is defined for any $x \in E_i$. From the mappings f_i, one can then very easily build by *canonical extension* mappings of $E_1, \ldots, E_n$ onto $E'_1, \ldots, E'_n$. This is carried out step by step:

1. By defining a mapping g of $\mathcal{P}(E)$ onto $\mathcal{P}(E')$ starting from a mapping f of E onto E' such that, for a subset $e \subset E$, $g(e)$ is defined as the subset of all $f(x)$ such that $x \in e$.
2. By defining a mapping g of $E_1 \times E_2$ onto $E'_1 \times E'_2$, starting from the mappings f_1 of E_1 onto E'_1 and f_2 of E_2 onto E'_2, by $g(x, y) = (f_1(x), f_2(y))$.

The application of $S(E_1, \ldots, E_n)$ onto $S(E'_1, \ldots, E'_n)$ thus obtained is denoted by $\langle f_1, \ldots, f_n \rangle^S$.

If all f_i are injective (respectively surjective), then $\langle f_1, \ldots, f_n \rangle^S$ is also injective (respectively surjective), which one can easily show because this is valid for each step $\mathcal{P}$ or $\times$ of the echelon construction scheme S. If f_i are mappings of E_i onto E'_i and g_i of E'_i onto E''_i, one denotes the mapping of E_i onto E''_i by $g_i f_i$. One has then

$$\langle g_1 f_1, \ldots, g_n f_n \rangle^S = \langle g_1, \ldots, g_n \rangle^S \langle f_1, \ldots, f_n \rangle^S.$$

If all f_i are bijective (i.e., injective and surjective), with $g_i = f_i^{-1}$, then $\langle f_1, \ldots, f_n \rangle^S$ is also bijective and

$$(\langle f_1, \ldots, f_n \rangle^S)^{-1} = \langle f_1^{-1}, \ldots, f_n^{-1} \rangle^S,$$

where f^{-1} is the inverse bijection of f.

If there are several elements $s_1, \ldots, s_p$ of any echelons $G_1, \ldots, G_p$, then one can give oneself *an* element $s = (s_1, \ldots, s_p)$ of the set $G_1 \times \cdots \times G_p$, which is also an echelon. If there is a relation $R(x_1, \ldots, x_p)$, one can consider the relation

$$R(x_1, \ldots, x_p) \text{ and } x_1 \in G_1 \text{ and } \ldots \text{ and } x_p \in G_p.$$

Otherwise, one can also take R as a relation of only one x of $G = G_1 \times \ldots \times G_p$.

There is the theorem: $\mathrm{Coll}_x \Big(R(x) \text{ and } x \in G \Big)$, i.e., $R(x)$ determines in G a subset $H \subset G$ such that $\{x \in H \subset R(x) \text{ and } x \in G\}$.

This set H was previously denoted by $E_x(R(x)$ and $x \in G)$. Later we will denote this set H by $\{x \mid x \in G$ and $R(x)\}$ (as already mentioned). The set H is still an element of $\mathcal{P}(G)$, i.e., a relation R can be characterized by a *subset* of an echelon or as an element of an echelon. In the same way, functions, applications, etc. can be characterized by an element of an echelon. In particular the relations of representation R_μ (see Chap. 3) can thus be described by a subset r_μ of an echelon S_μ on the terms of representation as base sets or as elements $r_\mu \in \mathcal{P}(S_\mu)$. One can also naturally consider for all R_μ the element $(r_1, r_2, \ldots) = s$ of $\mathcal{P}(S_1) \times \mathcal{P}(S_2) \times \cdots$; and if, conversely, $s \in \mathcal{P}(S_1) \times \mathcal{P}(S_2) \times \cdots$, then $s = (r_1, r_2, \ldots)$ is equivalent to

$$r_1 \in \mathcal{P}(S_1) \text{ and } r_2 \in \mathcal{P}(S_2) \text{ and } \ldots.$$

Axioms or theorems, expressible only by the R_μ, transform themselves into a relation P of s into which enter the base sets.

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