1 Fourier Series

Mapping of a *Periodic* Function $f(t)$ to a Series of Fourier Coefficients C_k

1.1 Fourier Series

This section serves as a starter. Many readers may think it too easy; but it should be read and taken seriously all the same. Some preliminary remarks are in order:

i. To make things easier to understand, the whole book will only be concerned with functions in the time domain and their Fourier transforms in the frequency domain. This represents the most common application, and porting it to other pairings, such as space/momentum, for example, is pretty straightforward indeed.
ii. We use the angular frequency ω when we refer to the frequency domain. The unit of the angular frequency is radians/second (or simpler s^{-1}). It is easily converted to the frequency ν of radio-stations – for example FM 105.4 MHz – using the following equation:

$$\omega = 2\pi\nu. \tag{1.1}$$

The unit of ν is Hz, short for Hertz.

By the way, in case someone wants to do like H.J. Weaver, my much appreciated role-model, and use different notations to avoid having the tedious factors 2π crop up everywhere, do not buy into that. For each 2π you save somewhere, there will be more factors of 2π somewhere else. However, there are valid reasons, as detailed for example in "Numerical Recipes", to use t and ν.

In this book I will stick to the use of t and ω, cutting down on the cavalier use of 2π that is in vogue elsewhere.

1.1.1 Even and Odd Functions

All functions are either

$$f(-t) = f(t) : \text{even} \tag{1.2}$$

or

$$f(-t) = -f(t) : \text{odd} \tag{1.3}$$

or a "mixture" of both, i.e. even and odd parts superimposed. The decomposition gives:

$$\begin{aligned} f_{\text{even}}(t) &= (f(t) + f(-t))/2 \\ f_{\text{odd}}(t) &= (f(t) - f(-t))/2. \end{aligned}$$

See examples in Fig. 1.1.

1.1.2 Definition of the Fourier Series

Fourier analysis is often also called harmonic analysis, as it uses the trigonometric functions sine – an odd function – and cosine – an even function – as basis functions that play a pivotal part in harmonic oscillations.

Similar to expanding a function into a power series, especially periodic functions may be expanded into a series of the trigonometric functions sine and cosine.

Definition 1.1 (Fourier Series).

$$f(t) = \sum_{k=0}^{\infty} (A_k \cos \omega_k t + B_k \sin \omega_k t) \tag{1.4}$$

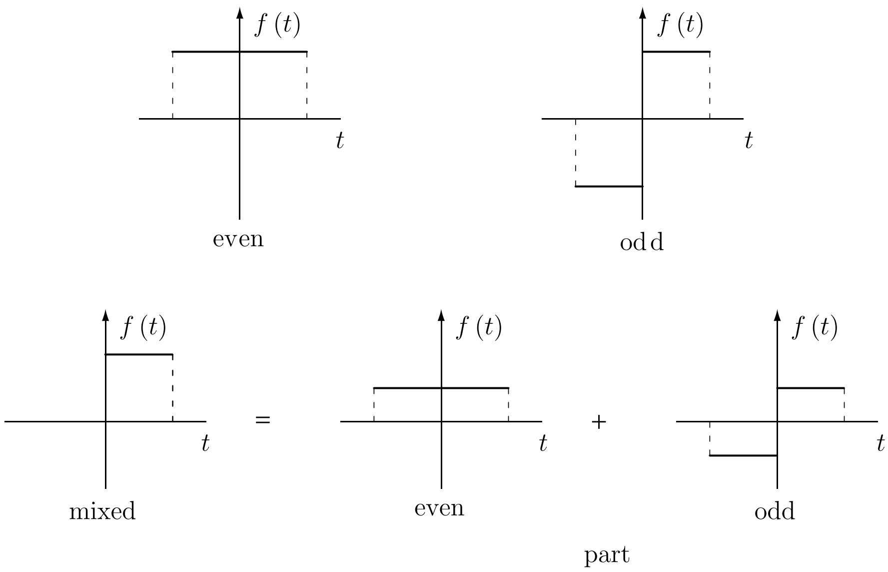

Fig. 1.1. Examples of even, odd and mixed functions

$$\text{with } \omega_k = \frac{2\pi k}{T} \text{ and } B_0 = 0.$$

Here T means the period of the function $f(t)$. The amplitudes or Fourier coefficients A_k and B_k are determined in such a way – as we'll see in a moment – that the infinite series is identical with the function $f(t)$. Equation (1.4) therefore tells us that any periodic function can be represented as a superposition of sine-function and cosine-function with appropriate amplitudes – with an infinite number of terms, if need be – yet using only precisely determined frequencies:

$$\omega = 0, \frac{2\pi}{T}, \frac{4\pi}{T}, \frac{6\pi}{T}, \ldots$$

Figure 1.2 shows the basis functions for $k = 0, 1, 2, 3$.

Example 1.1 ("Trigonometric identity").

$$f(t) = \cos^2 \omega t = \frac{1}{2} + \frac{1}{2}\cos 2\omega t \ . \tag{1.5}$$

Trigonometric manipulation in (1.5) already determined the Fourier coefficients A_0 and A_2: $A_0 = 1/2$, $A_2 = 1/2$ (see Fig. 1.3). As function $\cos^2 \omega t$ is an even function, we need no B_k. Generally speaking, all "smooth" functions without steps (i.e. without discontinuities) and without kinks (i.e. without discontinuities in their first derivative) – and strictly speaking without discontinuities in all their derivatives – are limited as far as their bandwidth is concerned. This means that a *finite* number of terms in the series will do for practical purposes. Often data gets recorded using a device with limited bandwidth, which puts a limit on how quickly $f(t)$ can vary over time anyway.

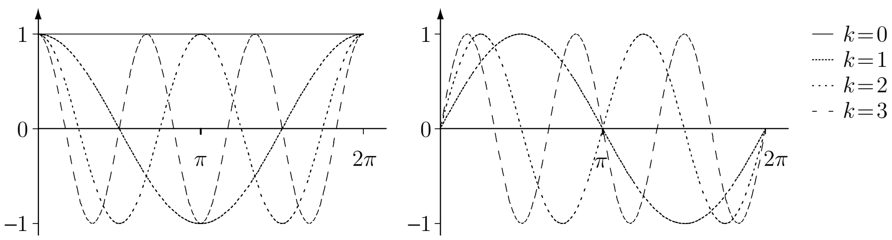

Fig. 1.2. Basis functions of Fourier transformation: cosine (*left*); sine (*right*)

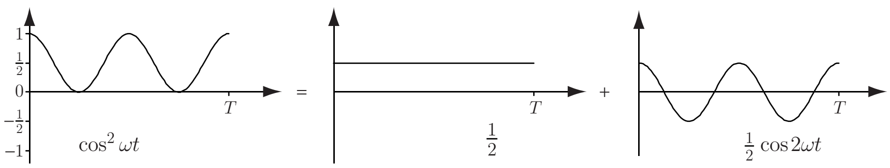

Fig. 1.3. Decomposition of $\cos^2 \omega t$ into the average 1/2 and an oscillation with amplitude 1/2 and frequency 2ω

1.1.3 Calculation of the Fourier Coefficients

Before we dig into the calculation of the Fourier coefficients, we need some tools.

In all following integrals we integrate from $-T/2$ to $+T/2$, meaning over an interval with the period T that is *symmetrical* to $t = 0$. We could also pick any other interval, as long as the integrand is periodic with period T and gets integrated over a *whole* period. The letters n and m in the formulas below are natural numbers $0, 1, 2, \ldots$ Let's have a look at the following:

$$\int_{-T/2}^{+T/2} \cos \frac{2\pi nt}{T} \mathrm{d}t = \begin{cases} 0 & \text{for } n \neq 0 \\ T & \text{for } n = 0 \end{cases}, \tag{1.6}$$

$$\int_{-T/2}^{+T/2} \sin \frac{2\pi nt}{T} \mathrm{d}t = 0 \qquad \text{for all } n. \tag{1.7}$$

This results from the fact that the areas on the positive half-plane and the ones on the negative one cancel out each other, provided we integrate over a whole number of periods. Cosine integral for $n = 0$ requires special treatment, as it lacks oscillations and therefore areas can't cancel out each other: there the integrand is 1, and the area under the horizontal line is equal to the width of the interval T.

Furthermore, we need the following trigonometric identities:

$$\begin{aligned} \cos\alpha\cos\beta &= 1/2\ [\cos(\alpha+\beta) + \cos(\alpha-\beta)], \\ \sin\alpha\sin\beta &= 1/2\ [\cos(\alpha-\beta) - \cos(\alpha+\beta)], \\ \sin\alpha\cos\beta &= 1/2\ [\sin(\alpha+\beta) + \sin(\alpha-\beta)]. \end{aligned} \tag{1.8}$$

Using these tools we're able to prove, without further ado, that the system of basis functions consisting of:

$$1,\ \cos\frac{2\pi t}{T},\ \sin\frac{2\pi t}{T},\ \cos\frac{4\pi t}{T},\ \sin\frac{4\pi t}{T},\ \ldots \tag{1.9}$$

is an *orthogonal system*[1].

Put in formulas, this means:

$$\int_{-T/2}^{+T/2} \cos\frac{2\pi nt}{T} \cos\frac{2\pi mt}{T} \mathrm{d}t = \begin{cases} 0 & \text{for } n \neq m \\ T/2 & \text{for } n = m \neq 0 \\ T & \text{for } n = m = 0 \end{cases}, \tag{1.10}$$

[1] Similar to two vectors at right angles to each other whose dot product is 0, we call a set of basis functions an orthogonal system if the integral over the product of two different basis functions vanishes.

$$\int_{-T/2}^{+T/2} \sin\frac{2\pi nt}{T}\sin\frac{2\pi mt}{T}\mathrm{d}t = \begin{cases} 0 & \text{for} \quad \begin{array}{c} n \neq m,\ n=0 \\ \text{and/or } m=0 \end{array} \\ T/2 & \text{for } n=m\neq 0 \end{cases}, \tag{1.11}$$

$$\int_{-T/2}^{+T/2} \cos\frac{2\pi nt}{T}\sin\frac{2\pi mt}{T}\mathrm{d}t = 0\ . \tag{1.12}$$

The right-hand side of (1.10) and (1.11) shows that our basis system is not an *orthonormal system*, i.e. the integrals for $n = m$ are not normalised to 1. What's even worse, the special case of (1.10) for $n = m = 0$ is a nuisance, and will keep bugging us again and again.

Using the above orthogonality relations, we're able to calculate the Fourier coefficients straight away. We need to multiply both sides of (1.4) with $\cos\omega_k t$ and integrate from $-T/2$ to $+T/2$. Due to the orthogonality, only terms with $k = k'$ will remain; the second integral will always disappear.

This gives us:

$$A_k = \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\cos\omega_k t\,\mathrm{d}t \qquad \text{for} \qquad k \neq 0 \tag{1.13}$$

and for our "special" case:

$$A_0 = \frac{1}{T}\int_{-T/2}^{+T/2} f(t)\,\mathrm{d}t. \tag{1.14}$$

Please note the prefactors $2/T$ or $1/T$, respectively, in (1.13) and (1.14). Equation (1.14) simply is the average of the function $f(t)$. The "electricians" amongst us, who might think of $f(t)$ as current varying over time, would call A_0 the "DC"-component (DC = direct current, as opposed to AC = alternating current). Now let's multiply both sides of (1.4) with $\sin\omega_k t$ and integrate from $-T/2$ to $+T/2$.

We now have:

$$B_k = \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\sin\omega_k t\,\mathrm{d}t \qquad \text{for all } k. \tag{1.15}$$

Equations (1.13) and (1.15) may also be interpreted like: by weighting the function $f(t)$ with $\cos\omega_k t$ or $\sin\omega_k t$, respectively, we "pick" the spectral components from $f(t)$, when integrating, corresponding to the even or odd components, respectively, of the frequency ω_k. In the following examples, we'll only state the functions $f(t)$ in their basic interval $-T/2 \le t \le +T/2$. They have to be extended periodically, however, as the definition goes, beyond this basic interval.

Example 1.2 ("Constant"). See Fig. 1.4(*left*):

$$\begin{aligned} f(t) &= 1 \\ A_0 &= 1 \text{ "Average"} \\ A_k &= 0 \text{ for all } k \neq 0 \\ B_k &= 0 \text{ for all } k \text{ (as } f \text{ is even).} \end{aligned}$$

Example 1.3 ("Triangular function"). See Fig. 1.4(*right*):

$$f(t) = \begin{cases} 1 + \dfrac{2t}{T} \text{ for } -T/2 \le t \le 0 \\ \\ 1 - \dfrac{2t}{T} \text{ for } 0 \le t \le +T/2 \end{cases}.$$

Let's recall: $\omega_k = \dfrac{2\pi k}{T}$ $A_0 = 1/2$ ("Average").
For $k \neq 0$ we get:

$$\begin{aligned} A_k &= \frac{2}{T}\left[\int\limits_{-T/2}^{0}\left(1+\frac{2t}{T}\right)\cos\frac{2\pi kt}{T}\mathrm{d}t + \int\limits_{0}^{+T/2}\left(1-\frac{2t}{T}\right)\cos\frac{2\pi kt}{T}\mathrm{d}t\right] \\ &= \underbrace{\frac{2}{T}\int\limits_{-T/2}^{0}\cos\frac{2\pi kt}{T}\mathrm{d}t + \frac{2}{T}\int\limits_{0}^{+T/2}\cos\frac{2\pi kt}{T}\mathrm{d}t}_{=0} \\ &\quad + \frac{4}{T^2}\int\limits_{-T/2}^{0} t\cos\frac{2\pi kt}{T}\mathrm{d}t - \frac{4}{T^2}\int\limits_{0}^{+T/2} t\cos\frac{2\pi kt}{T}\mathrm{d}t \end{aligned}$$

Fig. 1.4. "Constant" (*left*); "Triangular function" (*right*). We only show the basic intervals for both functions

$$= -\frac{8}{T^2} \int_0^{+T/2} t \cos \frac{2\pi kt}{T} \mathrm{d}t.$$

In a last step, we'll use $\int x \cos ax \, \mathrm{d}x = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax$ which finally gives us:

$$\begin{aligned} A_k &= \frac{2(1 - \cos \pi k)}{\pi^2 k^2} \qquad (k > 0), \\ B_k &= 0 \qquad (\text{as } f \text{ is even}). \end{aligned} \tag{1.16}$$

A few more comments on the expression for A_k are in order:

i. For all even k, A_k disappears.
ii. For all odd k we get $A_k = 4/(\pi^2 k^2)$.
iii. For $k = 0$ we better use the average A_0 instead of inserting $k = 0$ in (1.16).

We could make things even simpler:

$$A_k = \begin{cases} \frac{1}{2} & \text{for } k = 0 \\ \frac{4}{\pi^2 k^2} & \text{for k odd} \\ 0 & \text{for k even, } k \neq 0 \end{cases} . \tag{1.17}$$

The series' elements decrease rapidly while k rises (to the power of two in the case of odd k), but in principle we still have an infinite series. That's due to the "pointed roof" at $t = 0$ and the kink (continued periodically!) at $\pm T/2$ in our function $f(t)$. In order to describe these kinks, we need an infinite number of Fourier coefficients.

The following illustrations will show that things are never as bad as they seem to be:

Using $\omega = 2\pi/T$ (see Fig. 1.5) we get:

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2} \left(\cos \omega t + \frac{1}{9} \cos 3\omega t + \frac{1}{25} \cos 5\omega t + \ldots \right) . \tag{1.18}$$

We want to plot the frequencies of this Fourier series. Figure 1.6 shows the result as produced, for example, by a spectrum analyser,[2] if we would use our "triangular function" $f(t)$ as input signal.

[2] On offer by various companies – for example as a plug-in option for oscilloscopes – for a tidy sum of money.

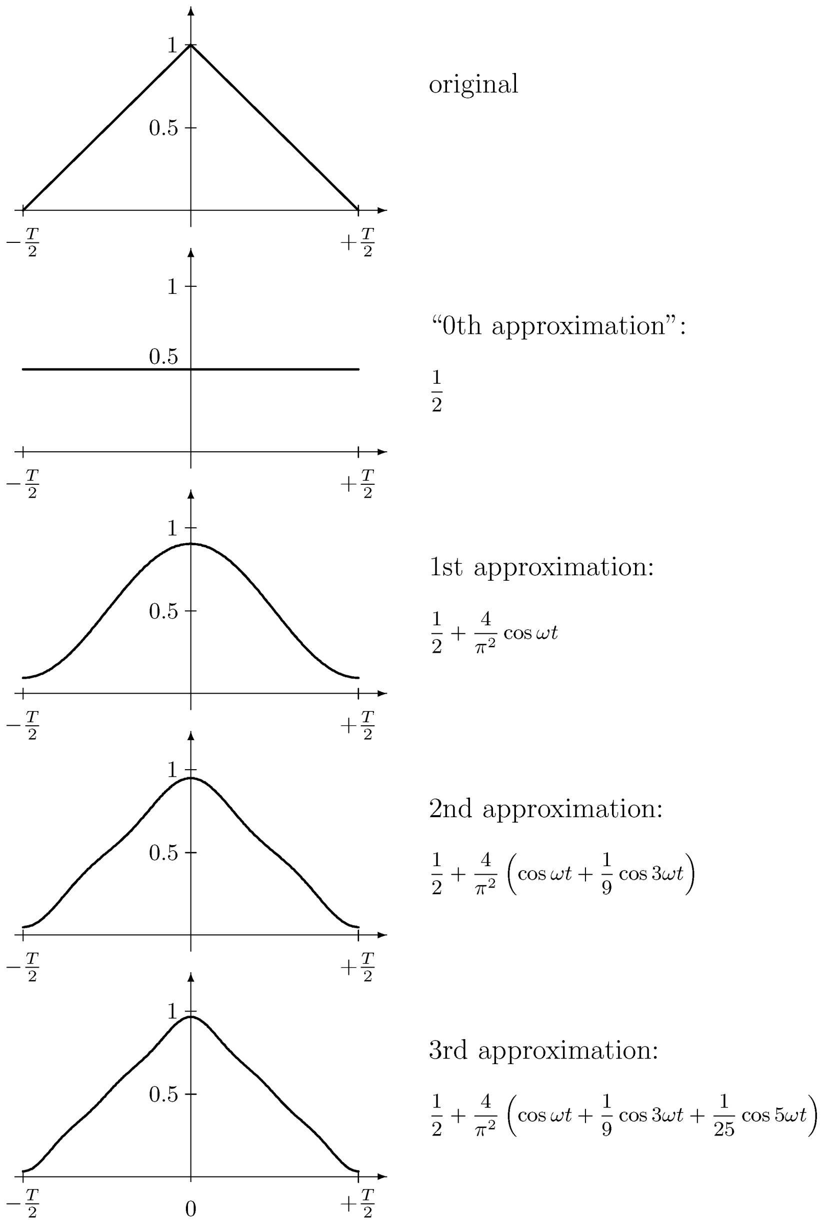

Fig. 1.5. The "triangular function" $f(t)$ and consecutive approximations by a Fourier series with more and more terms

Apart from the DC peak at $\omega = 0$ we can also see the fundamental frequency ω and all odd "harmonics". We may also use this frequency plot to get an idea about the margins of error resulting from discarding frequencies above, say, 7ω. We will cover this in more detail later on.

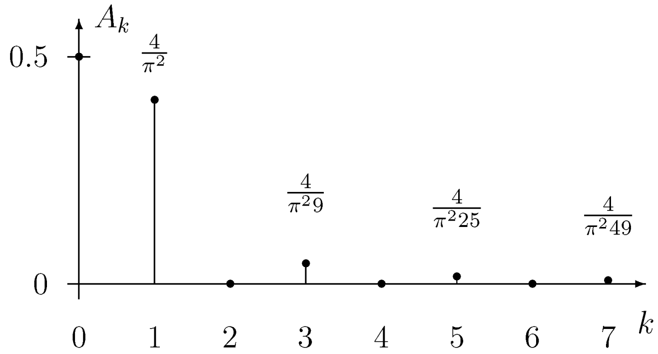

Fig. 1.6. Plot of the "triangular function's" frequencies

1.1.4 Fourier Series in Complex Notation

Let me give you a mild warning before we dig into this chapter: in (1.4) k starts from 0, meaning that we will rule out *negative* frequencies in our Fourier series.

The cosine terms didn't have a problem with negative frequencies. The sign of the cosine argument doesn't matter anyway, so we would be able to go halves, like between brothers, for example, as far as the spectral intensity at the positive frequency $k\omega$ was concerned: $-k\omega$ and $k\omega$ would get equal parts, as shown in Fig. 1.7.

As frequency $\omega = 0$ – a frequency as good as any other frequency $\omega \neq 0$ – has no "brother", it will not have to go halves. A change of sign for the sine-terms' arguments would result in a change of sign for the corresponding series' term. The splitting of spectral intensity like "between brothers" – equal parts of $-\omega_k$ and $+\omega_k$ now will have to be like "between sisters": the sister for $-\omega_k$ also gets 50%, but hers is *minus* 50%!

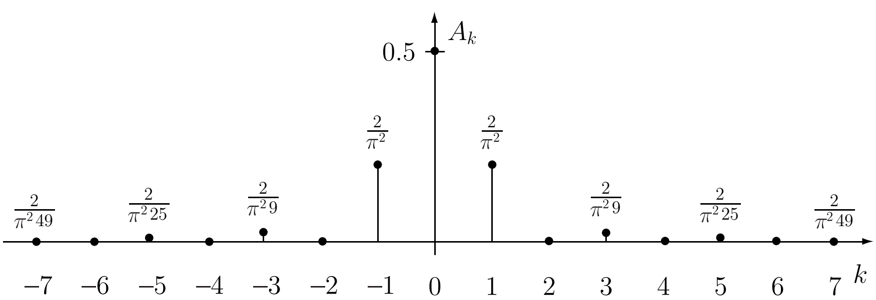

Fig. 1.7. Like Fig. 1.6, yet with positive and negative frequencies

Instead of using (1.4) we might as well use:

$$f(t) = \sum_{k=-\infty}^{+\infty} (A_k' \cos \omega_k t + B_k' \sin \omega_k t), \tag{1.19}$$

where, of course, the following is true: $A_{-k}' = A_k'$, $B_{-k}' = -B_k'$. The formulas for the calculation of A_k' and B_k' for $k > 0$ are identical to (1.13) and (1.15), though they lack the extra factor 2! Equation (1.14) for A_0 stays unaffected by this. This helps us avoid to provide a special treatment for the DC-component.

Instead of (1.16) we could have used:

$$A_k' = \frac{(1 - \cos \pi k)}{\pi^2 k^2}, \tag{1.20}$$

which would also be valid for $k = 0$! To prove it, we'll use a "dirty trick" or commit a "venial" sin: we'll assume, for the time being, that k is a continuous variable that may steadily decrease towards 0. Then we apply l'Hospital's rule to the expression of type "0:0", stating that numerator and denominator may be differentiated separately with respect to k until $\lim_{k \to 0}$ does not result in an expression of type "0:0" any more. Like:

$$\lim_{k \to 0} \frac{1 - \cos \pi k}{\pi^2 k^2} = \lim_{k \to 0} \frac{\pi \sin \pi k}{2\pi^2 k} = \lim_{k \to 0} \frac{\pi^2 \cos \pi k}{2\pi^2} = \frac{1}{2}. \tag{1.21}$$

If you're no sinner, go for the "average" $A_0 = 1/2$ straight away!

Hint: In many standard Fourier transformation programs a factor 2 between A_0 and $A_{k \neq 0}$ is wrong. This could be mainly due to the fact that frequencies were permitted to be positive only for the basis functions, or positive and negative – like in (1.4). The calculation of the average A_0 is easy as pie, and therefore always recommended as a first test in case of a poorly documented program. As $B_0 = 0$, according to the definition, B_k is a bit harder to check out. Later on we'll deal with simpler checks (for example Parseval's theorem).

Now we're set and ready for the introduction of complex notation. In the following we'll always assume that $f(t)$ is a real function. Generalising this for complex $f(t)$ is no problem. Our most important tool is Euler's identity:

$$\mathrm{e}^{\mathrm{i}\alpha t} = \cos \alpha t + \mathrm{i} \sin \alpha t. \tag{1.22}$$

Here, we use i as the imaginary unit that results in -1 when raised to the power of two.

This allows us to rewrite the trigonometric functions as follows:

$$\begin{aligned} \cos \alpha t &= \frac{1}{2}(\mathrm{e}^{\mathrm{i}\alpha t} + \mathrm{e}^{-\mathrm{i}\alpha t}), \\ \sin \alpha t &= \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\alpha t} - \mathrm{e}^{-\mathrm{i}\alpha t}). \end{aligned} \tag{1.23}$$

Inserting into (1.4) gives:

$$f(t) = A_0 + \sum_{k=1}^{\infty} \left(\frac{A_k - \mathrm{i}B_k}{2} \mathrm{e}^{\mathrm{i}\omega_k t} + \frac{A_k + \mathrm{i}B_k}{2} \mathrm{e}^{-\mathrm{i}\omega_k t} \right). \tag{1.24}$$

Using the short-cuts:

$$\begin{aligned} C_0 &= A_0, \\ C_k &= \frac{A_k - \mathrm{i}B_k}{2}, \\ C_{-k} &= \frac{A_k + \mathrm{i}B_k}{2}, \qquad k = 1, 2, 3, \ldots, \end{aligned} \tag{1.25}$$

we finally get:

$$f(t) = \sum_{k=-\infty}^{+\infty} C_k \mathrm{e}^{\mathrm{i}\omega_k t}, \qquad \omega_k = \frac{2\pi k}{T}. \tag{1.26}$$

Caution: For $k < 0$ there will be *negative* frequencies. (No worries, according to our above digression!) Pretty handy that C_k and C_{-k} are conjugated complex to each other (see “brother and sister”). Now C_k can be formulated just as easily:

$$C_k = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) \mathrm{e}^{-\mathrm{i}\omega_k t} \mathrm{d}t \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots \tag{1.27}$$

Please note that there is a negative sign in the exponent. It will stay with us till the end of this book. Please also note that the index k runs from $-\infty$ to $+\infty$ for C_k, whereas it runs from 0 to $+\infty$ for A_k and B_k.

1.2 Theorems and Rules

1.2.1 Linearity Theorem

Expanding a periodic function into a Fourier series is a linear operation. This means that we may use the two Fourier pairs:

$$\begin{aligned} f(t) &\leftrightarrow \{C_k; \omega_k\} \text{ and} \\ g(t) &\leftrightarrow \{C'_k; \omega_k\} \end{aligned}$$

to form the following linear combination:

$$h(t) = af(t) + bg(t) \leftrightarrow \{aC_k + bC'_k; \omega_k\}. \tag{1.28}$$

Thus, we may easily determine the Fourier series of a function by splitting it into items whose Fourier series we already know.

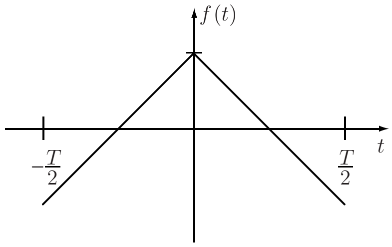

Fig. 1.8. "Triangular function" with average 0

Example 1.4 (Lowered "triangular function"). The simplest example is our "triangular function" from Example 1.3, though this time it is symmetrical to its base line (see Fig. 1.8): we only have to subtract 1/2 from our original function. That means that the Fourier series remained unchanged while only the average A_0 now turned to 0.

The linearity theorem appears to be so trivial that you may accept it at face-value even when you have "strayed from the path of virtue". Straying from the path of virtue is, for example, something as elementary as squaring.

1.2.2 The First Shifting Rule (Shifting within the Time Domain)

Often, we want to know how the Fourier series changes if we shift the function $f(t)$ along the time axis. This, for example, happens on a regular basis if we use a different interval, e.g. from 0 to T, instead of the symmetrical one from $-T/2$ to $T/2$ we have used so far. In this situation, the First Shifting Rule comes in very handy:

$$\begin{aligned} f(t) &\leftrightarrow \{C_k; \omega_k\}, \\ f(t-a) &\leftrightarrow \left\{C_k \mathrm{e}^{-\mathrm{i}\omega_k a}; \omega_k\right\}. \end{aligned} \tag{1.29}$$

Proof (First Shifting Rule).

$$\begin{aligned} C_k^{\text{new}} &= \frac{1}{T} \int\limits_{-T/2}^{+T/2} f(t-a)\mathrm{e}^{-\mathrm{i}\omega_k t}\mathrm{d}t = \frac{1}{T} \int\limits_{-T/2-a}^{+T/2-a} f(t')\mathrm{e}^{-\mathrm{i}\omega_k t'}\mathrm{e}^{-\mathrm{i}\omega_k a}\mathrm{d}t' \\ &= \mathrm{e}^{-\mathrm{i}\omega_k a} C_k^{\text{old}}. \quad \square \end{aligned}$$

We integrate over a full period, that's why shifting the limits of the interval by a does not make any difference.

The proof is trivial, the result of the shifting along the time axis not! The new Fourier coefficient results from the old coefficient C_k by multiplying it with the phase factor $\mathrm{e}^{-\mathrm{i}\omega_k a}$. As C_k generally is complex, shifting "shuffles" real and imaginary parts.

Without using complex notation we get:

$$\begin{aligned} f(t) &\leftrightarrow \{A_k; B_k; \omega_k\}, \\ f(t-a) &\leftrightarrow \{A_k \cos\omega_k a - B_k \sin\omega_k a; A_k \sin\omega_k a + B_k \cos\omega_k a; \omega_k\}. \end{aligned} \tag{1.30}$$

Two examples follow:

Example 1.5 (Quarter period shifted "triangular function"). "Triangular function" (with average = 0) (see Fig. 1.8):

$$f(t) = \begin{cases} \dfrac{1}{2} + \dfrac{2t}{T} \text{ for } -T/2 \le t \le 0 \\[2ex] \dfrac{1}{2} - \dfrac{2t}{T} \text{ for } 0 < t \le T/2 \end{cases}$$
$$\text{with } C_k = \begin{cases} \dfrac{1-\cos\pi k}{\pi^2 k^2} = \dfrac{2}{\pi^2 k^2} & \text{for } k \text{ odd} \\[2ex] 0 & \text{for } k \text{ even} \end{cases}. \tag{1.31}$$

Now let's shift this function to the right by $a = T/4$:

$$f_{\text{new}} = f_{\text{old}}(t - T/4).$$

So the new coefficients can be calculated as follows:

$$\begin{aligned} C_k^{\text{new}} &= C_k^{\text{old}} \mathrm{e}^{-\mathrm{i}\pi k/2} && (k \text{ odd}) \\ &= \frac{2}{\pi^2 k^2}\left(\cos\frac{\pi k}{2} - \mathrm{i}\sin\frac{\pi k}{2}\right) && (k \text{ odd}) \\ &= -\frac{2\mathrm{i}}{\pi^2 k^2}(-1)^{\frac{k-1}{2}} && (k \text{ odd}). \end{aligned} \tag{1.32}$$

It's easy to realise that $C_{-k}^{\text{new}} = -C_k^{\text{new}}$.
In other words: $A_k = 0$.
Using $\mathrm{i}B_k = C_{-k} - C_k$ we finally get:

$$B_k^{\text{new}} = \frac{4}{\pi^2 k^2}(-1)^{\frac{k-1}{2}} \qquad k \text{ odd}.$$

Using the above shifting we get an odd function (see Fig. 1.9b).

Example 1.6 (Half period shifted "triangular function"). Now we'll shift the same function to the right by $a = T/2$:

$$f_{\text{new}} = f_{\text{old}}(t - T/2).$$

The new coefficients then are:

$$
\begin{aligned}
C_k^{\text{new}} &= C_k^{\text{old}} \mathrm{e}^{-\mathrm{i}\pi k} && (k \text{ odd}) \\
&= \frac{2}{\pi^2 k^2} (\cos \pi k - \mathrm{i} \sin \pi k) && (k \text{ odd}) \\
&= -\frac{2}{\pi^2 k^2} && (k \text{ odd})
\end{aligned}
\tag{1.33}
$$

$(C_0 = 0$ stays).

So we've only changed the sign. That's okay, as the function now is upside-down (see Fig. 1.9c).

Warning: Shifting by $a = T/4$ will result in alternating signs for the coefficients (Fig. 1.9b). The series of Fourier coefficients, that are decreasing monotonically with k according to Fig. 1.9a, looks pretty "frazzled" after shifting the function by $a = T/4$, due to the alternating sign.

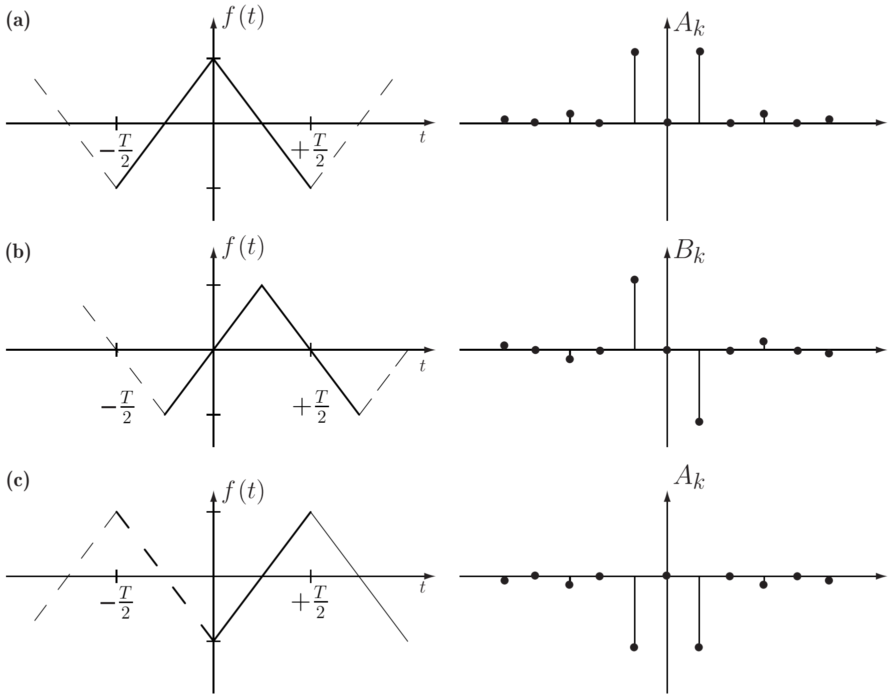

Fig. 1.9. (**a**) "Triangular function" (with average = 0); (**b**) right-shifted by $T/4$; (**c**) right-shifted by $T/2$

1.2.3 The Second Shifting Rule (Shifting within the Frequency Domain)

The First Shifting Rule showed us that shifting within the time domain leads to a multiplication by a phase factor in the frequency domain. Reversing this statement gives us the Second Shifting Rule:

$$\begin{aligned} f(t) &\leftrightarrow \{C_k;\omega_k\}, \\ f(t)\mathrm{e}^{\mathrm{i}\frac{2\pi at}{T}} &\leftrightarrow \{C_{k-a};\omega_k\}. \end{aligned} \tag{1.34}$$

In other words, a multiplication of the function $f(t)$ by the phase factor $\mathrm{e}^{\mathrm{i}2\pi at/T}$ results in frequency ω_k now being related to "shifted" coefficient C_{k-a} – instead of the former coefficient C_k. A comparison between (1.34) and (1.29) demonstrates the two-sided character of the two Shifting Rules. If a is an integer, there won't be any problem if you simply take the coefficient shifted by a. But what if a is not an integer?

Strangely enough nothing serious will happen. Simply shifting like we did before won't work any more, but who is to keep us from inserting $(k-a)$ into the expression for old C_k, whenever k occurs.

(If it's any help to you, do commit another venial sin and temporarily consider k to be a continuous variable.) So, in the case of non-integer a we didn't really "shift" C_k, but rather recalculated it using "shifted" k.

Caution: If you have simplified a k-dependency in the expressions for C_k, for example:

$$1-\cos\pi k = \begin{cases} 0 \text{ for k even} \\ 2 \text{ for k odd} \end{cases}$$

(as in (1.16)), you'll have trouble replacing the "vanished" k with $(k-a)$. In this case, there's only one way out: back to the expressions with *all* k-dependencies *without* simplification.

Before we present examples, two more ways of writing down the Second Shifting Rule are in order:

$$\begin{aligned} f(t) &\leftrightarrow \{A_k;B_k;\omega_k\}, \\ f(t)\mathrm{e}^{\frac{2\pi \mathrm{i}at}{T}} &\leftrightarrow \left\{\frac{1}{2}[A_{k+a}+A_{k-a}+\mathrm{i}(B_{k+a}-B_{k-a})];\right. \\ &\qquad \left.\frac{1}{2}[B_{k+a}+B_{k-a}+\mathrm{i}(A_{k-a}-A_{k+a})];\omega_k\right\}. \end{aligned} \tag{1.35}$$

Caution: This is true for $k\neq 0$.

Old A_0 then becomes $A_a/2+\mathrm{i}B_a/2$!

This is easily proved by solving (1.25) for A_k and B_k and inserting it in (1.34):

$$\begin{aligned} A_k &= C_k + C_{-k}, \\ -\mathrm{i}B_k &= C_k - C_{-k}, \end{aligned} \tag{1.36}$$

$$A_k^{\text{new}} = C_k + C_{-k} = \frac{A_{k-a} - \mathrm{i}B_{k-a}}{2} + \frac{A_{k+a} + \mathrm{i}B_{k+a}}{2},$$

$$-\mathrm{i}B_k^{\text{new}} = C_k - C_{-k} = \frac{A_{k-a} - \mathrm{i}B_{k-a}}{2} - \frac{A_{k+a} + \mathrm{i}B_{k+a}}{2},$$

which leads to (1.35). We get the special treatment for A_0 from:

$$A_0^{\text{new}} = C_0^{\text{new}} = \frac{A_{-a} - \mathrm{i}B_{-a}}{2} = \frac{A_{+a} + \mathrm{i}B_{+a}}{2}.$$

The formulas become a lot simpler in case $f(t)$ is real. Then we get:

$$f(t)\cos\frac{2\pi at}{T} \leftrightarrow \left\{\frac{A_{k+a} + A_{k-a}}{2}; \frac{B_{k+a} + B_{k-a}}{2}; \omega_k\right\}, \tag{1.37}$$

old A_0 becomes $A_a/2$ and also:

$$f(t)\sin\frac{2\pi at}{T} \leftrightarrow \left\{\frac{B_{k+a} - B_{k-a}}{2}; \frac{A_{k-a} - A_{k+a}}{2}; \omega_k\right\},$$

old A_0 becomes $B_a/2$.

Example 1.7 ("Constant").

$$f(t) = 1 \quad \text{for } -T/2 \leq t \leq +T/2\ .$$

$A_k = \delta_{k,0}$ (Kronecker symbol, see Sect. 4.1.2) or $A_0 = 1$, all other A_k, B_k vanish. Of course, we've always known that $f(t)$ is a cosine wave with frequency $\omega = 0$ and therefore, only requires the coefficient for $\omega = 0$.

Now, let's multiply function $f(t)$ by $\cos(2\pi t/T)$, i.e. $a = 1$. From (1.37) we can see:

$$\begin{array}{lll} & A_k^{\text{new}} = \delta_{k-1,0}, & \text{i.e.} \quad A_1 = 1 \text{ (all others are 0)}, \\ \text{or} & C_1 = 1/2, & \qquad C_{-1} = 1/2. \end{array}$$

So, we have shifted the coefficient by $a = 1$ (to the right and to the left, and gone halves, like "between brothers").

This example demonstrates that the frequency $\omega = 0$ is as good as any other function. No kidding! If you know, for example, the Fourier series of a function $f(t)$ and consequently the solution for integrals of the form:

$$\int_{-T/2}^{+T/2} f(t)\mathrm{e}^{-\mathrm{i}\omega_k t}\mathrm{d}t$$

then you already have, using the Second Shifting Rule, solved all integrals for $f(t)$, multiplied by $\sin(2\pi at/T)$ or $\cos(2\pi at/T)$. No wonder, you only had to combine phase factor $\mathrm{e}^{\mathrm{i}2\pi at/T}$ with phase factor $\mathrm{e}^{-\mathrm{i}\omega_k t}$!

Example 1.8 ("Triangular function" multiplied by cosine). The function:

$$f(t) = \begin{cases} 1 + \dfrac{2t}{T} & \text{for } -T/2 \le t \le 0 \\[2ex] 1 - \dfrac{2t}{T} & \text{for } 0 \le t \le T/2 \end{cases}$$

is to be multiplied by $\cos(\pi t/T)$, i.e. we shift the coefficients C_k by $a = 1/2$ (see Fig. 1.10). The new function still is even, and therefore we only have to look after A_k:

$$A_k^{\text{new}} = \frac{A_{k+a}^{\text{old}} + A_{k-a}^{\text{old}}}{2}.$$

We use (1.16) for the old A_k (and stop using the simplified version (1.17)!):

$$A_k^{\text{old}} = \frac{2(1-\cos \pi k)}{\pi^2 k^2}.$$

We then get:

$$\begin{aligned} A_k^{\text{new}} &= \frac{1}{2}\left[\frac{2(1-\cos\pi(k+1/2))}{\pi^2(k+1/2)^2} + \frac{2(1-\cos\pi(k-1/2))}{\pi^2(k-1/2)^2}\right] \\ &= \frac{1-\cos\pi k\cos(\pi/2)+\sin\pi k\sin(\pi/2)}{\pi^2(k+1/2)^2} \\ &\quad + \frac{1-\cos\pi k\cos(\pi/2)-\sin\pi k\sin(\pi/2)}{\pi^2(k-1/2)^2} \\ &= \frac{1}{\pi^2(k+1/2)^2} + \frac{1}{\pi^2(k-1/2)^2} \\ A_0^{\text{new}} &= \frac{A_{1/2}^{\text{old}}}{2} = \frac{2(1-\cos(\pi/2))}{2\pi^2\left(\frac{1}{2}\right)^2} = \frac{4}{\pi^2}\ . \end{aligned} \tag{1.38}$$

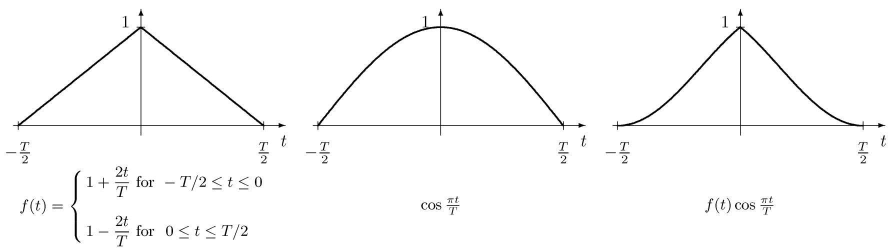

Fig. 1.10. "Triangular function" (*left*); $\left(\cos\frac{\pi t}{T}\right)$-function (*middle*); "Triangular function" with $\left(\cos\frac{\pi t}{T}\right)$-weighting (*right*)

The new coefficients then are:

$$
\begin{aligned}
A_0 &= \frac{4}{\pi^2}, \\
A_1 &= \frac{1}{\pi^2}\left(\frac{1}{\left(\frac{3}{2}\right)^2} + \frac{1}{\left(\frac{1}{2}\right)^2}\right) = \frac{4}{\pi^2}\left(\frac{1}{9} + \frac{1}{1}\right) = \frac{4}{\pi^2}\frac{10}{9}, \\
A_2 &= \frac{1}{\pi^2}\left(\frac{1}{\left(\frac{5}{2}\right)^2} + \frac{1}{\left(\frac{3}{2}\right)^2}\right) = \frac{4}{\pi^2}\left(\frac{1}{25} + \frac{1}{9}\right) = \frac{4}{\pi^2}\frac{34}{225}, \\
A_3 &= \frac{1}{\pi^2}\left(\frac{1}{\left(\frac{7}{2}\right)^2} + \frac{1}{\left(\frac{5}{2}\right)^2}\right) = \frac{4}{\pi^2}\left(\frac{1}{49} + \frac{1}{25}\right) = \frac{4}{\pi^2}\frac{74}{1225}, \quad \text{etc.}
\end{aligned}
\tag{1.39}
$$

A comparison of these coefficients with the ones without the $\left(\cos\frac{\pi t}{T}\right)$-weighting shows what we've done:

	without weighting	with $\left(\cos\frac{\pi t}{T}\right)$-weighting
A_0	$\frac{1}{2}$	$\frac{4}{\pi^2}$
A_1	$\frac{4}{\pi^2}$	$\frac{4}{\pi^2}\frac{10}{9}$
A_2	0	$\frac{4}{\pi^2}\frac{34}{225}$
A_3	$\frac{4}{\pi^2}\frac{1}{9}$	$\frac{4}{\pi^2}\frac{74}{1225}$.

(1.40)

We can see the following:

i. The average A_0 got somewhat smaller, as the rising and falling flanks were weighted with the cosine, which, except for $t = 0$, is less than 1.
ii. We raised coefficient A_1 a bit, but lowered all following odd coefficients a bit, too. This is evident straight away, if we convert:

$$\frac{1}{(2k+1)^2} + \frac{1}{(2k-1)^2} < \frac{1}{k^2} \quad \text{to} \quad 8k^4 - 10k^2 + 1 > 0.$$

This is not valid for $k = 1$, yet all bigger k.
iii. Now we've been landed with even coefficients, that were 0 before.

We now have twice as many terms in the series as before, though they go down at an increased rate when k increases. The multiplication by $\cos(\pi t/T)$ caused the kink at $t = 0$ to turn into a much more pointed "spike". This should actually make for a worsening of convergence or a slower rate of decrease of the coefficients. We have, however, rounded the kink at the interval-boundary $\pm T/2$, which naturally helps, but we couldn't reasonably have predicted what exactly was going to happen.

1.2.4 Scaling Theorem

Sometimes we happen to want to scale the time axis. In this case, there is no need to re-calculate the Fourier coefficients. From:

$$\begin{aligned} & f(t) \leftrightarrow \{C_k; \omega_k\} \\ \text{we get:} \qquad & f(at) \leftrightarrow \left\{C_k; \frac{\omega_k}{a}\right\}. \end{aligned} \tag{1.41}$$

Here, a must be real! For $a > 1$ the time axis will be stretched and, hence, the frequency axis will be compressed. For $a < 1$ the opposite is true. The proof for (1.41) is easy and follows from (1.27):

$$C_k^{\text{new}} = \frac{a}{T} \int_{-T/2a}^{+T/2a} f(at) \mathrm{e}^{-\mathrm{i}\omega_k t} \,\mathrm{d}t = \frac{a}{T} \int_{-T/2}^{+T/2} f(t') \mathrm{e}^{-\mathrm{i}\omega_k t'/a} \frac{1}{a} \,\mathrm{d}t'$$

$$\text{with } t' = at$$

$$= C_k^{\text{old}} \text{ with } \omega_k^{\text{new}} = \frac{\omega_k^{\text{old}}}{a}.$$

Please note that we also have to stretch or compress the interval limits because of the requirement of periodicity. Here, we have tacitly assumed $a > 0$. For $a < 0$, we would only reverse the time axis and, hence, also the frequency axis. For the special case $a = -1$ we have:

$$\begin{aligned} f(t) &\leftrightarrow \{C_k, \omega_k\}, \\ f(-t) &\leftrightarrow \{C_k; -\omega_k\}. \end{aligned} \tag{1.42}$$

1.3 Partial Sums, Bessel's Inequality, Parseval's Equation

For practical work, infinite Fourier series have to get terminated at some stage, regardless. Therefore, we only use a partial sum, say until we reach $k_{\max} = N$. This Nth partial sum then is:

$$S_N = \sum_{k=0}^{N} (A_k \cos \omega_k t + B_k \sin \omega_k t). \tag{1.43}$$

Terminating the series results in the following squared error:

$$\delta_N^2 = \frac{1}{T} \int_T [f(t) - S_N(t)]^2 \mathrm{d}t. \tag{1.44}$$

The "T" below the integral symbol means integration over a full period. This definition will become plausible in a second if we look at the discrete version:

$$\delta_N^2 = \frac{1}{N}\sum_{i=1}^{N}(f_i - s_i)^2.$$

Please note that we divide by the length of the interval, to compensate for integrating over the interval T. Now we know that the following is correct for the infinite series:

$$\lim_{N\to\infty} S_N = \sum_{k=0}^{\infty}(A_k \cos\omega_k t + B_k \sin\omega_k t) \tag{1.45}$$

provided the A_k and B_k happen to be the Fourier coefficients. Does this also have to be true for the Nth partial sum? Isn't there a chance the mean squared error would get smaller, if we used other coefficients instead of Fourier coefficients? That's not the case! To prove it, we'll now insert (1.43) and (1.44) in (1.45), leave out $\lim_{N\to\infty}$ and get:

$$\begin{aligned}
\delta_N^2 &= \frac{1}{T}\left\{\int_T f^2(t)\mathrm{d}t - 2\int_T f(t)S_N(t)\mathrm{d}t + \int_T S_N^2(t)\mathrm{d}t\right\} \\
&= \frac{1}{T}\left\{\int_T f^2(t)\mathrm{d}t\right. \\
&\quad - 2\int_T \sum_{k=0}^{\infty}(A_k \cos\omega_k t + B_k \sin\omega_k t)\sum_{k=0}^{N}(A_k \cos\omega_k t + B_k \sin\omega_k t)\mathrm{d}t \\
&\quad \left. + \int_T \sum_{k=0}^{N}(A_k \cos\omega_k t + B_k \sin\omega_k t)\sum_{k=0}^{N}(A'_k \cos\omega'_k t + B'_k \sin\omega'_k t)\mathrm{d}t\right\} \\
&= \frac{1}{T}\left\{\int_T f^2(t)\mathrm{d}t - 2TA_0^2 - 2\frac{T}{2}\sum_{k=1}^{N}(A_k^2 + B_k^2) + TA_0^2\right. \\
&\quad \left. + \frac{T}{2}\sum_{k=1}^{N}(A_k^2 + B_k^2)\right\} \\
&= \frac{1}{T}\int_T f^2(t)\mathrm{d}t - A_0^2 - \frac{1}{2}\sum_{k=1}^{N}(A_k^2 + B_k^2).
\end{aligned} \tag{1.46}$$

Here, we made use of the somewhat cumbersome orthogonality properties of (1.10), (1.11) and (1.12). As the A_k^2 and B_k^2 always are positive, the mean squared error will drop *monotonically* while N increases.

Example 1.9 (Approximating the "triangular function"). The "Triangular function":

$$f(t) = \begin{cases} 1 + \dfrac{2t}{T} & \text{for } \; -T/2 \leq t \leq 0 \\ \\ 1 - \dfrac{2t}{T} & \text{for } \; 0 \leq t \leq T/2 \end{cases} \tag{1.47}$$

has the mean squared "signal":

$$\frac{1}{T} \int\limits_{-T/2}^{+T/2} f^2(t) \mathrm{d}t = \frac{2}{T} \int\limits_{0}^{+T/2} f^2(t) \mathrm{d}t = \frac{2}{T} \int\limits_{0}^{+T/2} \left(1 - 2\frac{t}{T}\right)^2 \mathrm{d}t = \frac{1}{3}. \tag{1.48}$$

The most coarse, meaning 0th, approximation is:

$$\begin{aligned} S_0 &= 1/2, \text{ i.e.} \\ \delta_0^2 &= 1/3 - 1/4 = 1/12 = 0.0833\ldots \end{aligned}$$

The next approximation results in:

$$\begin{aligned} S_1 &= 1/2 + \tfrac{4}{\pi^2} \cos \omega t, \quad \text{i.e.} \\ \delta_1^2 &= 1/3 - 1/4 - 1/2 \left(\tfrac{4}{\pi^2}\right)^2 = 0.0012\ldots \end{aligned}$$

For δ_3^2 we get $0.0001915\ldots$, the approximation of the partial sum to the "triangle" quickly gets better and better.

As δ_N^2 is always positive, we finally arrive from (1.46) at Bessel's inequality:

$$\frac{1}{T} \int\limits_{T} f^2(t) \mathrm{d}t \geq A_0^2 + \frac{1}{2} \sum_{k=1}^{N} (A_k^2 + B_k^2). \tag{1.49}$$

For the border-line case of $N \to \infty$ we get Parseval's equation:

$$\frac{1}{T} \int\limits_{T} f^2(t) \mathrm{d}t = A_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (A_k^2 + B_k^2). \tag{1.50}$$

Parseval's equation may be interpreted as follows: $1/T \int f^2(t) \mathrm{d}t$ is the mean squared "signal" within the time domain, or – more colloquially – the "information content". Fourier series don't lose this information content: it's in the squared Fourier coefficients.

The rule of thumb, therefore, is:

"The information content isn't lost"
or
"Nothing goes missing in this house."

Here, we simply have to mention an analogy with the energy density of the electromagnetic field: $w = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)$ with $\epsilon_0 = \mu_0 = 1$, as often is customary in theoretical physics. The comparison has got some weak sides, as $\boldsymbol{E}$ and $\boldsymbol{B}$ have nothing to do with even and odd components.

Parseval's equation is very useful: you can use it to easily sum up infinite series. I think you'd always have been curious how we arrive at formulas such as, for example,

$$\sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{96}. \tag{1.51}$$

Our "triangular function" (1.47) is behind it! Insert (1.48) and (1.17) in (1.50), and you'll get:

$$\begin{aligned} \frac{1}{3} &= \frac{1}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \left(\frac{4}{\pi^2 k^2} \right)^2 \\ \text{or} \quad \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \frac{1}{k^4} &= \frac{2}{12} \frac{\pi^4}{16} = \frac{\pi^4}{96}. \end{aligned} \tag{1.52}$$

1.4 Gibbs' Phenomenon

So far we've only been using smooth functions as examples for $f(t)$, or – like the much-used "triangular function" – functions with "a kink", that's a discontinuity in the first derivative. This pointed kink made sure that we basically needed an infinite number of terms in the Fourier series. Now, what will happen if there is a step, a discontinuity, in the function itself? This certainly won't make the problem with the infinite number of elements any smaller. Is there any way to approximate such a step by using the Nth partial sum, and will the mean squared error for $N \to \infty$ approach 0? The answer is clearly **"Yes and No"**. Yes, because it apparently works, and no, because Gibbs' phenomenon happens at the steps, an overshoot or undershoot, that doesn't disappear for $N \to \infty$.

In order to understand this, we'll have to dig a bit wider.

1.4.1 Dirichlet's Integral Kernel

The following expression is called Dirichlet's integral kernel:

$$\begin{aligned} D_N(x) &= \frac{\sin\left(N + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} \\ &= \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos Nx. \end{aligned} \tag{1.53}$$

The second equal sign can be proved as follows:

$$\begin{aligned}\left(2\sin\tfrac{x}{2}\right) D_N(x) &= 2\sin\tfrac{x}{2}\times\left(\tfrac{1}{2}+\cos x+\cos 2x+\cdots+\cos Nx\right)\\ &= \sin\tfrac{x}{2}+2\cos x\sin\tfrac{x}{2}+2\cos 2x\sin\tfrac{x}{2}+\cdots\\ &\quad + 2\cos Nx\sin\tfrac{x}{2}\\ &= \sin\left(N+\tfrac{1}{2}\right)x.\end{aligned} \tag{1.54}$$

Here we have used the identity:

$$\begin{aligned}&2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)\\ &\qquad\text{with } \alpha = x/2 \text{ and } \beta = nx, \qquad n = 1,2,\ldots,N.\end{aligned}$$

By insertion, we see that all pairs of terms cancel out each other, except for the last one.

Figure 1.11 shows a few examples for $D_N(x)$. Please note that $D_N(x)$ is periodic in 2π. This is immediately evident from the cosine notation. With $x = 0$ we get $D_N(0) = N+1/2$, between 0 and 2π $D_N(x)$ oscillates around 0.

In the border-line case of $N \to \infty$ everything averages to 0, except for $x = 0$ (modulo 2π), that's where $D_N(x)$ grows beyond measure. Here we've found a notation for the δ-function (see Chap. 2)! Please excuse the two venial sins I've committed here: first, the δ-function is a distribution (and not a function!), and second, $\lim_{N\to\infty} D_N(x)$ is a whole "comb" of δ-functions 2π apart.

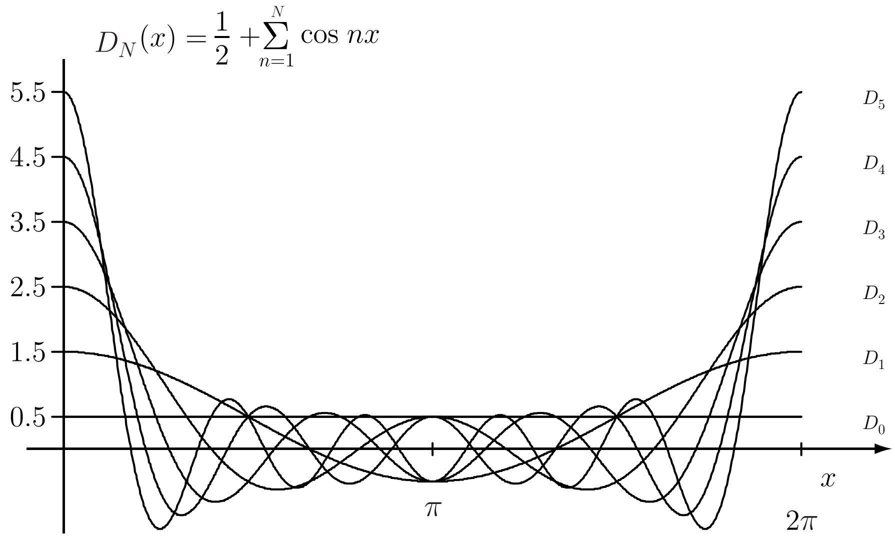

Fig. 1.11. $D_N(x) = 1/2+\cos x+\cos 2x+\cdots+\cos Nx$

1.4.2 Integral Notation of Partial Sums

We need a way to sneak up on the discontinuity, from the left and the right. That's why we insert the defining equations for the Fourier coefficients, (1.13)–(1.15), in (1.43):

$$\begin{aligned}
S_N(t) &= \frac{1}{T}\int_{-T/2}^{+T/2} f(x)\mathrm{d}x \qquad \begin{cases}(k=0)\text{-term taken out}\\ \text{of the sum}\end{cases}\\
&\quad+\sum_{k=1}^{N}\frac{2}{T}\int_{-T/2}^{+T/2}\Bigg(f(x)\cos\frac{2\pi kx}{T}\cos\frac{2\pi kt}{T}\\
&\quad+ f(x)\sin\frac{2\pi kx}{T}\sin\frac{2\pi kt}{T}\Bigg)\mathrm{d}x \qquad (1.55)\\
&= \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\left(\frac{1}{2}+\sum_{k=1}^{N}\cos\frac{2\pi k(x-t)}{T}\right)\mathrm{d}x\\
&= \frac{2}{T}\int_{-T/2}^{+T/2} f(x)D_N\left(\tfrac{2\pi(x-t)}{T}\right)\mathrm{d}x.
\end{aligned}$$

Using the abbreviation $x - t = u$ we get:

$$S_N(t) = \frac{2}{T}\int_{-T/2-t}^{+T/2-t} f(u+t)D_N(\tfrac{2\pi u}{T})\mathrm{d}u. \qquad (1.56)$$

As both f and D are periodic in T, we may shift the integration boundaries by t with impunity, without changing the integral. Now we split the integration interval from $-T/2$ to $+T/2$:

$$\begin{aligned}
S_N(t) &= \frac{2}{T}\left\{\int_{-T/2}^{0} f(u+t)D_N(\tfrac{2\pi u}{T})\mathrm{d}u + \int_{0}^{+T/2} f(u+t)D_N(\tfrac{2\pi u}{T})\mathrm{d}u\right\} \qquad (1.57)\\
&= \frac{2}{T}\int_{0}^{+T/2}[f(t-u)+f(t+u)]D_N(\tfrac{2\pi u}{T})\mathrm{d}u.
\end{aligned}$$

Here, we made good use of the fact that D_N is an even function (sum over cosine terms!).

Riemann's localisation theorem – which we won't prove here in the scientific sense, but which can be understood straight away using (1.57) – states that the convergence behaviour of $S_N(t)$ for $N \to \infty$ only depends on the immediate proximity to t of the function:

$$\lim_{N\to\infty} S_N(t) = S(t) = \frac{f(t^+) + f(t^-)}{2}. \tag{1.58}$$

Here t^+ and t^- mean the approach to t, from above and below, respectivly. Contrary to a continuous function with a non-differentiability ("kink"), where $\lim_{N\to\infty} S_N(t) = f(t)$, (1.58) means, that in the case of a discontinuity ("step") at t, the partial sum converges to a value that's "half-way" there.

That seems to make sense.

1.4.3 Gibbs' Overshoot

Now we'll have a closer look at the unit step (see Fig. 1.12):

$$f(t) = \begin{cases} -1/2 \text{ for } -T/2 \le t < 0 \\ +1/2 \text{ for } 0 \le t \le T/2 \end{cases} \text{ with periodic continuation.} \tag{1.59}$$

At this stage we're only interested in the case where $t > 0$, and $t \le T/4$. The integrand in (1.57) prior to Dirichlet's integral kernel is:

$$f(t-u) + f(t+u) = \begin{cases} 1 \text{ for } 0 \le u < t \\ 0 \text{ for } t \le u < T/2 - t \\ -1 \text{ for } (T/2) - t \le u < T/2 \end{cases}. \tag{1.60}$$

Inserting in (1.57) results in:

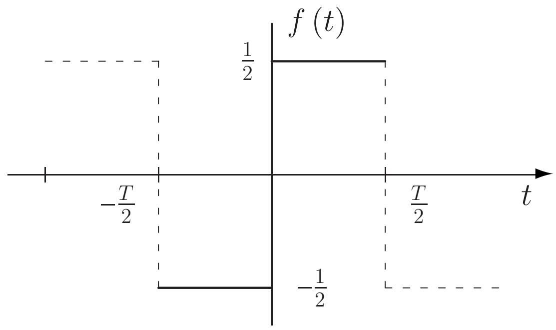

Fig. 1.12. Unit step

$$
\begin{aligned}
S_N(t) &= \frac{2}{T}\left\{\int\limits_0^t D_N(\tfrac{2\pi u}{T})\mathrm{d}u \;-\; \int\limits_{(T/2)-t}^{T/2} D_N(\tfrac{2\pi u}{T})\mathrm{d}u\right\} \\
&= \left\{\frac{1}{\pi}\int\limits_0^{2\pi t/T} D_N(x)\mathrm{d}x \;-\; \int\limits_{-2\pi t/T}^{0} D_N(x-\pi)\mathrm{d}x\right\} \\
&\quad\;\; (\text{with } x = \tfrac{2\pi u}{T}) \qquad\quad (\text{with } x = \tfrac{2\pi u}{T} - \pi).
\end{aligned}
\tag{1.61}
$$

Now we will insert the expression of Dirichlet's kernel as sum of cosine terms and integrate them:

$$
\begin{aligned}
S_N(t) &= \frac{1}{\pi}\left\{\frac{\pi t}{T} + \frac{\sin\frac{2\pi t}{T}}{1} + \frac{\sin 2\frac{2\pi t}{T}}{2} + \cdots + \frac{\sin N\frac{2\pi t}{T}}{N}\right. \\
&\quad \left. - \left(\frac{\pi t}{T} - \frac{\sin\frac{2\pi t}{T}}{1} + \frac{\sin 2\frac{2\pi t}{T}}{2} - \cdots + (-1)^N\frac{\sin N\frac{2\pi t}{T}}{N}\right)\right\} \\
&= \frac{2}{\pi}\sum_{\substack{k=1\\ \text{odd}}}^{N}\frac{1}{k}\sin\frac{2\pi k t}{T}.
\end{aligned}
\tag{1.62}
$$

This function is the expression of the partial sums of the unit step. In Fig. 1.13 we show some approximations.

Figure 1.14 shows the 49th partial sum. As we can see, we're already getting pretty close to the unit step, but there are overshoots and undershoots near the discontinuity. Electro-technical engineers know this phenomenon

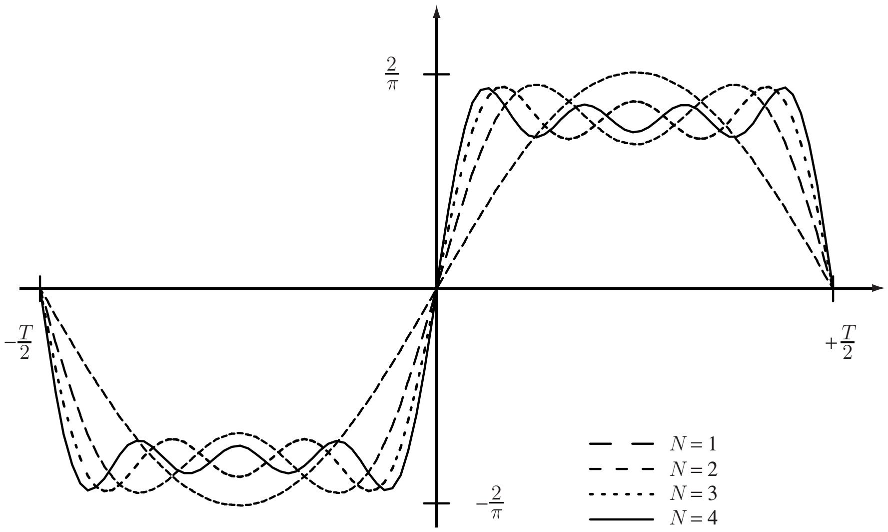

Fig. 1.13. Partial sum expression of unit step

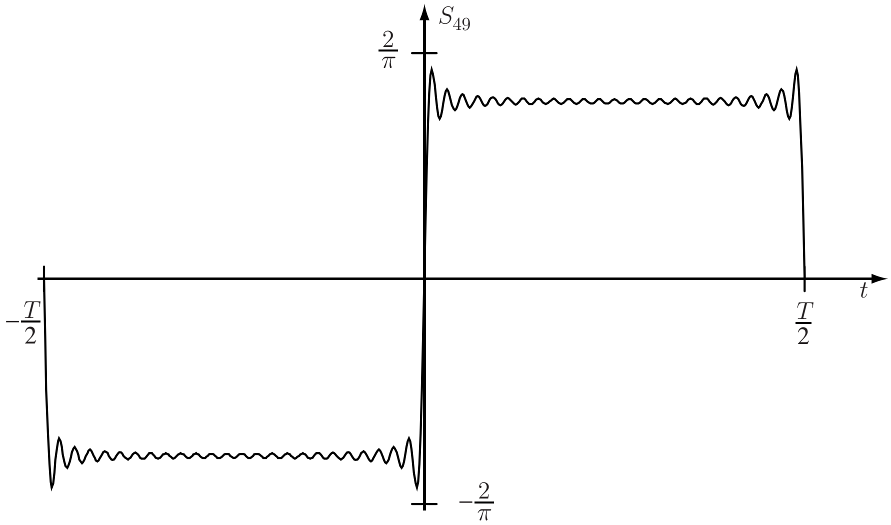

Fig. 1.14. Partial sum expression of unit step for $N = 49$

when using filters with very steep flanks: the signal "rings". We could be led to believe that the amplitude of these overshoots and undershoots will get smaller and smaller, provided only we make N big enough. We haven't got a chance! Comparing Fig. 1.13 with Fig. 1.14 should have made us think twice. We'll have a closer look at that, using the following approximation: N is to be very big and t (or x in (1.61), respectively) very small, i.e. close to 0.

Then we may neglect $1/2$ with respect to N in the numerator of Dirichlet's kernel and simply use $x/2$ in the denominator, instead of $\sin(x/2)$:

$$D_N(x) \to \frac{\sin Nx}{x}. \tag{1.63}$$

Therefore, the partial sum for large N and close to $t = 0$ becomes:

$$S_N(t) \to \frac{1}{\pi} \int_0^{2\pi Nt/T} \frac{\sin z}{z} \mathrm{d}z \qquad \text{with } z = Nx. \tag{1.64}$$

That is the sine integral. We'll get the extremes at $\mathrm{d}S_N(t)/\mathrm{d}t \stackrel{!}{=} 0$. Differentiating with respect to the upper integral boundary gives:

$$\frac{1}{\pi} \frac{2\pi N}{T} \frac{\sin z}{z} \stackrel{!}{=} 0 \tag{1.65}$$

or $z = l\pi$ with $l = 1, 2, 3, \ldots$ The first extreme on $t_1 = T/(2N)$ is a maximum, the second extreme at $t_2 = T/N$ is a minimum (as can easily be seen). The extremes get closer and closer to each other for $N \to \infty$. How big is $S_N(t_1)$? Insertion in (1.64) gives us the value of the "overshoot":

$$S_N(t_1) \to \frac{1}{\pi} \int\limits_0^{\pi} \frac{\sin z}{z} \mathrm{d}z = \frac{1}{2} + 0.0895. \tag{1.66}$$

Using the same method we get the value of the "undershoot":

$$S_N(t_2) \to \frac{1}{\pi} \int\limits_0^{2\pi} \frac{\sin z}{z} \mathrm{d}z = \frac{1}{2} - 0.048. \tag{1.67}$$

I bet you've noticed that, in the approximation of N big and t small, the value of the overshoot or undershoot doesn't depend on N at all any more. Therefore, it doesn't make sense to make N as big as possible, the overshoots and undershoots will settle at values of $+0.0895$ and -0.048 and stay there. We could still show that the extremes decrease monotonically until $t = T/4$; thereafter, they'll be mirrored and increase (cf. Fig. 1.14). Now what about our mean squared error for $N \to \infty$? The answer is simple: the mean squared error approaches 0 for $N \to \infty$, though the overshoots and undershoots stay. That's the trick: as the extremes get closer and closer to each other, the area covered by the overshoots and the undershoots with the function $f(t) = 1/2$ $(t > 0)$ approaches 0 all the same. Integration will only come up with areas of measure 0 (I'm sure I've committed at least a venial sin by putting it this way). The moral of the story: a kink in the function (non-differentiability) lands us with an infinite Fourier series, and a step (discontinuity) gives us Gibbs' "ringing" to boot. In a nutshell: avoid steps wherever it's possible!

Playground

1.1. Very Speedy
A broadcasting station transmits on 100 MHz. Calculate the angular frequency ω and the period T for one complete oscillation. How far travels an electromagnetic pulse (or a light pulse!) in this time? Use the vacuum velocity of light $c \approx 3 \times 10^8$ m/s.

1.2. Totally Odd
Given is the function $f(t) = \cos(\pi t/2)$ for $0 < t \leq 1$ with periodic continuation. Plot this function. Is this function even, odd, or mixed? If it is mixed, decompose it into even and odd components and plot them.

1.3. Absolutely True
Calculate the complex Fourier coefficients C_k for $f(t) = \sin \pi t$ for $0 \leq t \leq 1$ with periodic continuation. Plot $f(t)$ with periodic continuation. Write down the first four terms in the series expansion.

1.4. Rather Complex
Calculate the complex Fourier coefficients C_k for $f(t) = 2\sin(3\pi t/2)\cos(\pi t/2)$ for $0 \leq t \leq 1$ with periodic continuation. Plot $f(t)$.

1.5. Shiftily
Shift the function $f(t) = 2\sin(3\pi t/2)\cos(\pi t/2) = \sin \pi t + \sin 2\pi t$ for $0 \leq t \leq 1$ with periodic continuation by $a = -1/2$ to the left and calculate the complex Fourier coefficient C_k. Plot the shifted $f(t)$ and its decomposition into first and second parts and discuss the result.

1.6. Cubed
Calculate the complex Fourier coefficients C_k for $f(t) = \cos^3 2\pi t$ for $0 \leq t \leq 1$ with periodic continuation. Plot this function. Now use (1.5) and the Second Shifting Rule to check your result.

1.7. Tackling Infinity
Derive the result for the infinite series $\sum_{k=1}^{\infty} 1/k^4$ using Parseval's theorem. *Hint*: Instead of the triangular function try a parabola!

1.8. Smoothly
Given is the function $f(t) = [1-(2t)^2]^2$ for $-1/2 \leq t \leq 1/2$ with periodic continuation. Use (1.63) and argue how the Fourier coefficients C_k must depend on k. Check it by calculating the C_k directly.

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