6

Optimal Control of Evolution Systems in Banach Spaces

The next two chapters are on *optimal control*, which is among the most important motivations and fruitful applications of modern methods of variational analysis and generalized differentiation. It is not accidental that the very concepts of basic normals, subgradients, and coderivatives used in this book were introduced and applied by the author in connection with problems of optimal control. In fact, already the simplest and historically first problems of optimal control are *intrinsically nonsmooth*, even in the case of smooth functional data describing dynamics and constraints on feasible arcs. The crux of the matter is that a characteristic feature of optimal control problems, in contrast to the classical calculus of variations, is the presence of *pointwise* constraints on control functions, which may be (and often are) defined by *highly irregular* sets consisting, e.g., of finitely many points. In particular, this is the case of typical problems in automatic control that provided the primary motivation for developing optimal control theory.

The principal goal of the following developments is to derive necessary optimality conditions in a range of optimal control problems for evolution systems by using methods of variational analysis and generalized differentiation. This chapter concerns dynamical systems governed by *ordinary* differential equations and inclusions in Banach spaces; control problems for systems with *distributed parameters* governed by functional-differential and partial differential relations will be mostly considered in Chap. 7.

The main attention is paid in this chapter to optimal control/dynamic optimization problems of the Bolza and Mayer types governed by infinite-dimensional evolution inclusions and control systems with both *discrete-time* and *continuous-time* dynamics in the presence of endpoint constraints. Along with the variational principles in infinite dimensions and tools of generalized differentiation developed above, we employ special techniques of dynamic optimization and optimal control. The basic approach developed below is the *method of discrete approximations*, which allows us to approximate continuous-time control problems by those involving discrete dynamics. The relationship between continuous-time and discrete-time control systems is one

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of the *central topics* of this chapter. The results obtained in this direction shed new light upon both *qualitative* and *numerical* aspects of optimal control from the viewpoint of the theory and applications.

6.1 Optimal Control of Discrete-Time and Continuous-time Evolution Inclusions

This section concerns optimal control problems for dynamic/evolution systems governed by *differential inclusions* and their *finite-difference approximations* in appropriate (quite general) *Banach spaces*. The models under consideration capture more conventional problems of optimal control described by parameterized differential equations. Our primary method to study continuous-time control systems is to construct *well-posed discrete approximations* and to establish their *variational stability* with respect to the *value convergence* as well as a suitable *strong convergence* of their optimal solutions. Then we derive necessary optimality conditions for discrete-time optimal control problems governed by *finite-difference inclusions*. The latter problems can be reduced to non-dynamic optimization problems considered in the previous chapter in the presence of many geometric constraints. On the other hand, they have specific structural features exploited in what follows. In this way, applying generalized differential and SNC calculi from Chap. 3, we obtain necessary optimality conditions for discrete approximations in both fuzzy and exact forms under fairly general assumptions on the initial data. Passing to the limit with the use of coderivative characterizations of Lipschitzian stability from Chap. 4 allows us to derive necessary optimality conditions for *intermediate local minimizers* (that provide a local minimum lying between the classical weak and strong ones) in the *extended Euler-Lagrange* form for continuous-time systems under certain relaxation/convexification with respect to velocity variables. To avoid such a relaxation under appropriate assumptions, we develop an additional approximation procedure in the next section.

6.1.1 Differential Inclusions and Their Discrete Approximations

Let X be a Banach space (called the *state space* in what follows), and let $T := [a, b]$ be a *time interval* of the real line. Consider a set-valued mapping $F\colon X \times T \rightrightarrows X$ and define the *differential/evolution inclusion*

$$\dot{x}(t) \in F(x(t), t) \quad \text{a.e. } t \in [a, b] \tag{6.1}$$

generated by F, where $\dot{x}(t)$ stands for the time derivative of $x(t)$, and where a.e. (almost everywhere) means as usual that the relation holds up to the Lebesgue measure zero on $I\!R$. Let us give the precise definition of solutions to the differential inclusion (6.1), which is used in this chapter.

Definition 6.1 (solutions to differential inclusions). *By a* SOLUTION *to inclusion* (6.1) *we understand a mapping $x: T \to X$, which is Fréchet differentiable for a.e. $t \in T$ and satisfies* (6.1) *and the* NEWTON-LEIBNIZ FORMULA

$$x(t) = x(a) + \int_a^t \dot{x}(s)\,ds \quad \textit{for all} \;\; t \in T \;,$$

where the integral is taken in the BOCHNER SENSE.

It is well known that for $X = I\!R^n$, $x(t)$ is a.e. differentiable on T and satisfies the Newton-Leibniz formula if and only if it is *absolutely continuous* on T in the standard sense, i.e., for any $\varepsilon > 0$ there is δ such that

$$\sum_{j=1}^{l} \|x(t_{j+1}) - x(t_j)\| \le \varepsilon \;\; \text{whenever} \;\; \sum_{j=1}^{l} |t_{j+1} - t_j| \le \delta$$

for the disjoint intervals $(t_j, t_{j+1}] \subset T$. However, for infinite-dimensional spaces X even the Lipschitz continuity may not imply the a.e. differentiability. On the other hand, there is a *complete characterization* of Banach spaces X, where the absolute continuity of every $x: T \to X$ is *equivalent* to its a.e. differentiability and the fulfillment of the Newton-Leibniz formula. This is the class of spaces with the so-called *Radon-Nikodým property* (RNP).

Definition 6.2 (Radon-Nikodým property). *A Banach space X has the* RADON-NIKODÝM PROPERTY *if for every finite measure space (Ξ, Σ, μ) and for each μ-continuous vector measure $m: \Sigma \to X$ of bounded variation there is $g \in L^1(\mu; \Xi)$ such that*

$$m(E) = \int_E g\,d\mu \quad \textit{for} \;\; E \in \Sigma \;.$$

This fundamental property is well investigated in the general vector measure theorem and the geometric theory of Banach spaces; we refer the reader to the classical texts by Diestel and Uhl [334] and Bourgin [169] for the comprehensive study of the RNP and its applications. In particular, in [334, pp. 217–219] one can find the summary of equivalent formulations/characterizations of the RNP and the list of specific Banach spaces for which the RNP automatically holds. It is important to observe that the latter list contains every *reflexive* space and every *weakly compactly generated dual* space, hence all *separable duals*. On the other hand, the classical spaces c_0, c, l^∞, $L^1[0,1]$, and $L^\infty[0,1]$ *don't* have the RNP. Let us mention a nice relationship between the RNP and Asplund spaces used in what follows: *given a Banach space X, the dual space X^* has the RNP if and only if X is Asplund.*

Thus for Banach spaces with the RNP (and only for such spaces) the solution concept of Definition 6.1 agrees with the standard definition of

Carathéodory solutions dealing with absolutely continuous mappings. In general, Definition 6.1 postulates what we actually need for our purposes without appealing to Carathéodory solutions and the RNP. However, the RNP along with the Asplund property of X are essentially used for deriving major results in this chapter (but not all of them) from somewhat different prospectives not directly related to the adopted concept of solutions to differential inclusions.

It has been well recognized that differential inclusions, which are certainly of their own interest, provide a useful generalization of *control systems* governed by differential/evolution *equations* with control parameters:

$$\dot{x} = f(x, u, t), \quad u \in U(t) , \tag{6.2}$$

where the control sets $U(\cdot)$ may also depend on the state variable x via $F(x, t) = f(x, U(x, t), t)$. In some cases, especially when the sets $F(x, t)$ are convex, the differential inclusions (6.1) admit parametric representations of type (6.2), but in general they cannot be reduced to parametric control systems and should be studied for their own sake. Note also that the *ODE form* (6.2) in Banach spaces is strongly related to various control problems for evolution *partial differential equations* of parabolic and hyperbolic types, where solutions may be understood in some other appropriate senses; see, e.g., the books by Fattorini [432] and by Li and Yong [789] as well as the results and discussions presented in Remark 6.26 and Chap. 7 below.

Our principal method to study differential inclusions involves *finite-difference* replacements of the derivative

$$\dot{x}(t) \approx \frac{x(t+h) - x(t)}{h}, \quad h \to 0 ,$$

where the *uniform Euler scheme* is considered for simplicity. To formalize this process, we take any natural number $N \in I\!N$ and consider the *discrete grid/mesh* on T defined by

$$T_N := \big\{a, a + h_N, \ldots, b - h_N, b\big\}, \quad h_N := (b - a)/N ,$$

with the *stepsize of discretization* h_N and the *mesh points* $t_j := a + jh_N$ as $j = 0, \ldots, N$, where $t_0 = a$ and $t_N = b$. Then the differential inclusion (6.1) is replaced by a sequence of its *finite-difference/discrete approximations*

$$x_N(t_{j+1}) \in x_N(t_j) + h_N F(x_N(t_j), t_j), \quad j = 0, \ldots, N - 1 . \tag{6.3}$$

Given a discrete trajectory $x_N(t_j)$ satisfying (6.3), we consider its *piecewise linear extension* $x_N(t)$ to the continuous-time interval T, i.e., the *Euler broken lines*. We also define the *piecewise constant extension* to T of the corresponding *discrete velocity* by

$$v_N(t) := \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N}, \quad t \in [t_j, t_{j+1}), \quad j = 0, \ldots, N - 1 .$$

It follows from the very definition of the Bochner integral that

$$x_N(t) = x_N(a) + \int_a^t v_N(s)\,ds \quad \text{for} \quad t \in T \; .$$

Our first goal is to show that *every* solution to the differential inclusion (6.1) can be *strongly approximated*, under reasonable assumptions, by extended trajectories to the discrete inclusions (6.3). By strong approximation we understand the convergence in the norm topology of the classical Sobolev space $W^{1,2}\big([a,b];X\big)$ with the norm

$$\|x(\cdot)\|_{W^{1,2}} := \max_{t\in[a,b]} \|x(t)\| + \Big(\int_a^b \|\dot{x}(t)\|^2\,dt\Big)^{1/2} \; ,$$

where the norm on the right-hand side is taken in the space X. Note that the convergence in $W^{1,2}\big([a,b];X\big)$ implies the (uniform) convergence of the trajectories on $[a,b]$ and the *pointwise* (a.e. $t \in [a,b]$) convergence of (some subsequence of) their derivatives. The latter is crucial for our purposes, especially in the case of *nonconvex* values $F(x,t)$.

Let us formulate the *basic assumptions* for our study that apply not only to the next theorem but also to the subsequent results on differential inclusions via discrete approximations. Nevertheless, these assumptions can be relaxed in some settings; see the remarks and discussions below. Roughly speaking, we assume that the set-valued mapping $F\colon X \times [a,b] \rightrightarrows X$ is compact-valued, locally Lipschitzian in x, and Hausdorff continuous in t a.e. on $[a,b]$. More precisely, the following hypotheses are imposed along a given trajectory $\bar{x}(\cdot)$ to (6.1), which is arbitrary in the next theorem but then will be a reference optimal solution to the variational problem under consideration.

(H1) There are an open set $U \subset X$ and positive numbers m_F and ℓ_F such that $\bar{x}(t) \in U$ for all $t \in [a,b]$, the sets $F(x,t)$ are nonempty and compact for all $(x,t) \in U \times [a,b]$, and one has the inclusions

$$F(x,t) \subset m_F I\!B \quad \text{for all} \quad (x,t) \in U \times [a,b] \; , \tag{6.4}$$

$$F(x_1,t) \subset F(x_2,t) + \ell_F\|x_1 - x_2\| I\!B \quad \text{for all} \quad x_1,x_2 \in U, \quad t \in [a,b] \; . \tag{6.5}$$

(H2) $F(x,\cdot)$ is Hausdorff continuous for a.e. $t \in [a,b]$ uniformly in $x \in U$.

Note that inclusion (6.5) is equivalent to the uniform Lipschitz continuity

$$\text{haus}\big(F(x,t),F(u,t)\big) \le \ell_F\|x-u\|, \quad x,u \in U \; ,$$

of $F(\cdot,t)$ with respect to the *Pompieu-Hausdorff metric* haus$(\cdot,\cdot)$ on the space of nonempty and compact subsets of X; see Subsect. 1.2.2.

To handle efficiently the Hausdorff continuity of $F(x,\cdot)$ for a.e. $t \in [a,b]$, define the *averaged modulus of continuity* for F in $t \in [a,b]$ while $x \in U$ by

$$\tau(F;h) := \int_a^b \sigma(F;t,h)\,dt \;, \tag{6.6}$$

where $\sigma(F;t,h) := \sup\big\{\omega(F;x,t,h)\big|\; x \in U\big\}$ with

$$\omega(F;x,t,h) := \sup\Big\{\text{haus}\big(F(x,t_1),F(x,t_2)\big)\Big|\; t_1,t_2 \in \big[t-\tfrac{h}{2},t+\tfrac{h}{2}\big]\cap[a,b]\Big\}\;.$$

The following observation is easily implied by the definitions.

Proposition 6.3 (averaged modulus of continuity). *Property* (H2) *holds if and only if* $\tau(F;h) \to 0$ *as* $h \to 0$.

Note that for single-valued mapping $f\colon [a,b] \to X$ the property $\tau(f;h) \to 0$ as $h \to 0$ is *equivalent to the Riemann integrability* of f on $[a,b]$; see Sendov and Popov [1201]. The latter holds, as well known, if and only if f is continuous at almost all $t \in [a,b]$.

The following *strong approximation* theorem plays a crucial role in further results based on discrete approximations.

Theorem 6.4 (strong approximation by discrete trajectories). *Let* $\bar{x}(\cdot)$ *be a solution to the differential inclusion* (6.1) *under assumptions* (H1) *and* (H2), *where* X *is an arbitrary Banach space. Then there is a sequence of solutions* $\widehat{x}_N(t_j)$ *to the discrete inclusions* (6.3) *such that*

$$\widehat{x}_N(a) = \bar{x}(a) \;\; \textit{for all} \;\; N \in I\!N$$

and the extensions $\widehat{x}_N(t)$, $a \le t \le b$, *converge to* $\bar{x}(t)$ *strongly in the space* $W^{1,2}\big([a,b];X\big)$ *as* $N \to \infty$.

Proof. By Definition 6.1 involving the Bochner integral, the derivative mapping $\dot{\bar{x}}(\cdot)$ is *strongly measurable* on $[a,b]$, and hence we can find (rearranging the mesh points t_j if necessary) a sequence of *simple/step mappings* $w_N(\cdot)$ on T such that $w_N(t)$ are constant on $[t_j,t_{j+1})$ for every $j = 0,\ldots,N-1$ and $w_N(\cdot)$ converge to $\dot{\bar{x}}(\cdot)$ in the norm topology of $L^1\big([a,b];X\big)$ as $N \to \infty$. Combining this convergence with (6.1) and (6.4), we get

$$\int_a^b \|w_N(t)\|\,dt = \sum_{j=0}^{N-1} \|w_N(t_j)\|\,(t_{j+1}-t_j) \le (m_F+1)(b-a) \tag{6.7}$$

for all large N. In the estimates below we use the numerical sequence

$$\xi_N := \int_a^b \|\dot{\bar{x}}(t) - w_N(t)\|\,dt \to 0 \;\text{ as }\; N \to \infty\;.$$

Let us define the discrete functions $u_N(t_j)$ by

$$u_N(t_{j+1}) = u_N(t_j) + h_N w_N(t_j), \quad j = 0, \ldots, N-1, \quad u_N(t_0) := \bar{x}(a)$$

and observe that the functions

$$u_N(t) := \bar{x}(a) + \int_a^t w_N(s)\,ds, \quad a \le t \le b\ ,$$

are piecewise linear extensions of $u_N(t_j)$ to the interval $[a, b]$ and that

$$\|u_N(t) - \bar{x}(t)\| \le \int_a^t \|w_N(s) - \dot{\bar{x}}(s)\|\,ds \le \xi_N \ \text{ for } \ t \in [a, b]\ . \tag{6.8}$$

Therefore $u_N(t) \in U$ for all $t \in [a, b]$ whenever N is sufficiently large.

Taking the *distance function* $\operatorname{dist}(\cdot; \Omega)$ to a set in X, one can check that the Lipschitz condition (6.5) is equivalent to

$$\operatorname{dist}\big(w; F(x_1, t)\big) \le \operatorname{dist}\big(w; F(x_2, t)\big) + \ell_F\|x_1 - x_2\|$$

whenever $w \in X$, $x_1, x_2 \in U$, and $t \in [a, b]$; cf. the proof of Theorem 1.41. By the construction of $\tau(F; h)$ in (6.6) and the obvious relation

$$\operatorname{dist}\big(w; F(x, t_1)\big) \le \operatorname{dist}\big(w; F(x, t_2)\big) + \operatorname{haus}\big(F(x, t_1), F(x, t_2)\big)$$

one has the estimate

$$\begin{aligned}
\zeta_N :&= \sum_{j=0}^{N-1} h_N \operatorname{dist}\big(w_N(t_j); F(u_N(t_j), t_j)\big) \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \operatorname{dist}\big(w_N(t_j); F(u_N(t_j), t)\big)\,dt \\
&\le \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \operatorname{dist}\big(w_N(t_j); F(u_N(t), t)\big)\,dt + \tau(F; 2h_N)\ .
\end{aligned}$$

The Lipschitz property of F and the construction of $w_N(\cdot)$ imply

$$\begin{aligned}
\operatorname{dist}\big(w_N(t_j); F(u_N(t_j), t)\big) \le\ & \operatorname{dist}\big(w_N(t); F(u_N(t), t)\big) \\
&+ \ell_F(m_F + 1) w_N(t_j)(t - t_j)
\end{aligned}$$

whenever $t \in [t_j, t_{j+1})$, and then

$$\begin{aligned}
\operatorname{dist}\big(w_N(t); F(u_N(t), t)\big) &\le \operatorname{dist}\big(w_N(t); F(\bar{x}(t), t)\big) + \ell_F\|u_N(t) - \bar{x}(t)\| \\
&\le \|w_N(t) - \dot{\bar{x}}(t)\| + \ell_F \xi_N \ \text{ a.e. } \ t \in [a, b]\ .
\end{aligned}$$

Employing further (6.7) and (6.8), we arrive at the estimate

$$\zeta_N \le \gamma_N := \big(1 + \ell_F(b - a)\big)\xi_N + \ell_F h_N^2 (b - a)(m_F + 1)/2 + \tau(F; 2h_N)\ . \tag{6.9}$$

Observe that the functions $u_N(t_j)$ built above are *not* trajectories for the discrete inclusions (6.3), since one doesn't have $w_N(t_j) \in F(u_N(t_j), t_j)$. Now we use $w_N(t_j)$ to construct *actual trajectories* $\widehat{x}_N(t_j)$ for (6.3) that are close to $u_N(t_j)$ and enjoy the convergence property stated in the theorem.

Let us define $\widehat{x}_N(t_j)$ recurrently by the following *proximal algorithm*, which is realized due to the compactness assumption on the values of F:

$$\begin{cases} \widehat{x}_N(t_0)=\bar{x}(a), \quad \widehat{x}_N(t_{j+1})=\widehat{x}_N(t_j)+h_N v_N(t_j), \quad j=0,\ldots,N-1\,, \\ \text{where } \; v_N(t_j)\in F(\widehat{x}_N(t_j),t_j) \; \text{ with} \\ \|v_N(t_j)-w_N(t_j)\|=\operatorname{dist}\big(w_N(t_j);F(\widehat{x}_N(t_j),t_j)\big)\,. \end{cases} \tag{6.10}$$

First we prove that algorithm (6.10) keeps $\widehat{x}_N(t_j)$ inside the neighborhood U from (H1) whenever N is sufficiently large. Indeed, let us consider any number $N\in I\!N$ satisfying $\bar{x}(t)+\eta_N I\!B\subset U$ for all $t\in[a,b]$, where

$$\eta_N:=\gamma_N\exp\big(\ell_F(b-a)\big)+\xi_N$$

with ξ_N and γ_N defined above. We have $\eta_N\to 0$ as $N\to\infty$, since $\xi_N\to 0$ by the construction of ξ_N and since $\gamma_N\to 0$ due to assumption (H2) is equivalent to $\tau(F;h_N)\to 0$ by Proposition 6.3. Arguing by induction, we suppose that $\widehat{x}_N(t_i)\in U$ for all $i=0,\ldots,j$ and show that this also holds for $i=j+1$. Using (6.5), (6.9), and (6.10), one gets

$$\begin{aligned} \|\widehat{x}_N(t_{j+1})-u_N(t_{j+1})\| &\le \|\widehat{x}_N(t_j)-u_N(t_j)\|+h_N\|v_N(t_j)-w_N(t_j)\| \\ &\le \|\widehat{x}_N(t_j)-u_N(t_j)\|+h_N\operatorname{dist}\big(w_N(t_j);F(u_N(t_j),t_j)\big) \\ &\quad +\ell_F h_N\|\widehat{x}_N(t_j)-u_N(t_j)\|\le\ldots \\ &\le h_N\sum_{i=0}^{j}(1+\ell_F h_N)^{j-i}\operatorname{dist}\big(w_N(t_i);F(u_N(t_i),t_i)\big) \\ &\le h_N\exp\big[\ell_F(b-a)\big]\sum_{i=0}^{j}\operatorname{dist}\big(w_N(t_i);F(u_N(t_i),t_i)\big) \\ &\le \gamma_N\exp\big(\ell_F(b-a)\big)\,. \end{aligned}$$

Due to (6.8) the latter implies that

$$\|\widehat{x}_N(t_{j+1})-\bar{x}(t_{j+1})\|\le\gamma_N\exp\big(\ell_F(b-a)\big)+\xi_N=:\eta_N\,, \tag{6.11}$$

which proves that $\widehat{x}_N(t_j)\in U$ for all $j=0,\ldots,N$. Taking this into account, we have by the previous arguments that

$$\sum_{j=0}^{N}\|\widehat{x}_N(t_j)-u_N(t_j)\|\le(b-a)\exp\big(\ell_F(b-a)\big)\sum_{j=0}^{N-1}\operatorname{dist}\big(w_N(t_j);F(u_N(t_j),t_j)\big)\,.$$

Now let us estimate the quantity

$$\vartheta_N := \int_a^b \|\dot{\widehat{x}}_N(t) - w_N(t)\|\, dt \quad \text{as} \quad N \to \infty .$$

Using the last estimate above together with (6.9) and (6.11), we have

$$\begin{aligned}
\vartheta_N &= \sum_{j=0}^{N-1} h_N \|\dot{\widehat{x}}_N(t_j) - w_N(t_j)\| = \sum_{j=0}^{N-1} h_N \operatorname{dist}\big(w_N(t_j); F(\widehat{x}_N(t_j), t_j)\big) \\
&\le \sum_{j=0}^{N-1} h_N \operatorname{dist}\big(w_N(t_j); F(u_N(t_j), t_j)\big) + \ell_F \sum_{j=0}^{N-1} h_N \|\widehat{x}_N(t_j) - u_N(t_j)\| \\
&\le \gamma_N \big(1 + \ell_F(b-a) \exp\big(\ell_F(b-a)\big)\big) .
\end{aligned}$$

Thus one finally gets

$$\begin{aligned}
\int_a^b \|\dot{\widehat{x}}_N(t) - \dot{\bar{x}}(t)\|\, dt &\le \int_a^b \|\dot{\widehat{x}}_N(t) - w_N(t)\|\, dt + \int_a^b \|w_N(t) - \dot{\bar{x}}(t)\|\, dt \\
&\le \gamma_N \big(1 + \ell_F(b-a) \exp\big(\ell_F(b-a)\big)\big) + \xi_N := \alpha_N .
\end{aligned} \tag{6.12}$$

Since $\alpha_N \to 0$ as $N \to \infty$, this obviously implies the desired convergence $\widehat{x}_N(\cdot) \to \bar{x}(\cdot)$ in the norm of $W^{1,2}\big([a,b]; X\big)$ due to the Newton-Leibniz formula for $\widehat{x}_N(t)$ and $\bar{x}(t)$ and due to the boundedness assumption (6.4). $\triangle$

Remark 6.5 (numerical efficiency of discrete approximations). It follows from (6.12) by the Newton-Leibniz formula that

$$\|\widehat{x}_N(t) - \bar{x}(t)\| \le \int_a^b \|\dot{\widehat{x}}_N(s) - \dot{\bar{x}}(s)\|\, ds \le \alpha_N \quad \text{for all} \quad t \in [a,b] .$$

Thus the error estimate and numerical efficiency of the discrete approximation in Theorem 6.4 depend on the evaluation of the averaged modulus of continuity $\tau(F;h)$ from (6.6) and the approximating quantity ξ_N defined in the proof of Theorem 6.4. Denoting

$$v(F) := \sup_k \Big\{ \sum_{i=1}^{k-1} \sup_x \big[\operatorname{haus}\big(F(x,t_{i+1}), F(x,t_i)\big), x \in U\big], a \le t_1 \le \ldots \le t_k \le b \Big\} ,$$

it is not hard to check that

$$\tau(F;h) \le v(F)h = O(h)$$

whenever $F(x,\cdot)$ has a *bounded variation* $v(F) < \infty$ uniformly in $x \in U$; see Dontchev and Farkhi [354]. Furthermore, one has the estimate

$$\xi_N \le 2\tau(\dot{\bar{x}}; h_N)$$

by taking $w_N(t) = \dot{\bar{x}}_N(t) = \dot{\bar{x}}(t_j)$ for $t \in [t_j, t_j + h_N)$ if $\dot{\bar{x}}(\cdot)$ is *Riemann integrable* on $[a,b]$.

Remark 6.6 (discrete approximations of one-sided Lipschitzian differential inclusions). The Lipschitz continuity and compact-valuedness assumptions on F in Theorem 6.4 can be relaxed under additional requirements on the state space X in question. In particular, some counterparts of the $\mathcal{C}\big([a,b];X\big)$-approximation and $W^{1,2}\big([a,b];X\big)$-approximation results in the above theorem are obtained by Donchev and Mordukhovich [346] for the Hilbert pace setting with replacing the classical Lipschitz continuity in (H1) by the following *one-sided Lipschitzian property* of F in x: there is a constant $\ell \in I\!R$ (not necessarily positive) such that

$$\sigma\big(x_1 - x_2; F(x_1,t) - F(x_2,t)\big) \le \ell\|x_1 - x_2\|^2$$
$$\text{whenever } \ x_1, x_2 \in U, \quad \text{a.e. } \ t \in [a,b]\ ,$$

where $\sigma(x;Q) := \sup_{q\in Q}\langle x, q\rangle$ stands for the *support function* of $Q \subset X$. Moreover, the compact-valuedness assumption on the mapping $F(\cdot,t)$ may be replaced by imposing its *boundedness on bounded sets*: see the mentioned paper for more details and discussions.

6.1.2 Bolza Problem for Differential Inclusions and Relaxation Stability

In this subsection we start considering the following problem of dynamic optimization over solutions (in the sense of Definition 6.1) to differential inclusions in Banach spaces: minimize the *Bolza functional*

$$J[x] := \varphi\big(x(a), x(b)\big) + \int_a^b \vartheta\big(x(t), \dot{x}(t), t\big)\,dt \tag{6.13}$$

over trajectories $x\colon [a,b] \to X$ for the differential inclusion (6.1) such that $\vartheta\big(x(t), \dot{x}(t), t\big)$ is Bochner integrable on the fixed time interval $T := [a,b]$ subject to the *endpoint constraints*

$$\big(x(a), x(b)\big) \in \Omega \subset X^2\ . \tag{6.14}$$

This problem is labeled by (P) and called the (generalized) *Bolza problems for differential inclusions*. We use the term *arc* for any solution $x = x(\cdot)$ to (6.1) with $J[x] < \infty$ and the term *feasible arc* for arcs $x(\cdot)$ satisfying the endpoint constraints (6.14). Since the dynamic (6.1) and endpoint (6.14) constraints are given explicitly, we may assume that both functions φ and ϑ in the cost functional (6.13) take finite values.

The formulated problem (P) covers a broad range of various problems of dynamic optimization in finite-dimensional and infinite-dimensional spaces. In particular, it contains both standard and nonstandard models in optimal control for parameterized control systems (6.2) with possibly closed-loop control sets $U(x,t)$. Note also that problems with free time (non-fixed time intervals), with integral constraints on $(x,\dot{x})$, and with some other types of state constraints can be reduced to the form of (P).

Aiming to derive necessary conditions for optimal solutions to (P) that would apply not only to *global* but also to *local* minimizers, we first introduce appropriate concepts of local minima. Our basic notion is as follows.

Definition 6.7 (intermediate local minima). *A feasible arc $\bar{x}$ is an* INTERMEDIATE LOCAL MINIMIZER *(i.l.m.) of rank $p \in [1,\infty)$ for (P) if there are numbers $\varepsilon > 0$ and $\alpha \geq 0$ such that $J[\bar{x}] \leq J[x]$ for any feasible arcs to (P) satisfying*

$$\|x(t) - \bar{x}(t)\| < \varepsilon \;\; \textit{for all} \;\; t \in [a,b] \quad \textit{and} \tag{6.15}$$

$$\alpha \int_a^b \|\dot{x}(t) - \dot{\bar{x}}(t)\|^p \, dt < \varepsilon \, . \tag{6.16}$$

Relationships (6.15) and (6.16) actually mean that we consider a neighborhood of $\bar{x}$ in the Sobolev space $W^{1,p}\big([a,b];X\big)$. If there is only requirement (6.15) in Definition 6.7, i.e., $\alpha = 0$ in (6.16), that one gets the classical *strong* local minimum corresponding to a neighborhood of $\bar{x}$ in the norm topology of $\mathcal{C}\big([a,b];X\big)$. If instead of (6.16) one puts the more restrictive requirement

$$\|\dot{x}(t) - \dot{\bar{x}}(t)\| < \varepsilon \;\; \text{a.e.} \;\; t \in [a,b] \, ,$$

then we have the classical *weak* local minimum in the framework of Definition 6.7. This corresponds to considering a neighborhood of $\bar{x}$ in the topology of $W^{1,\infty}\big([a,b];X\big)$. Thus the introduced notion of i.l.m. takes, for any $p \in [1,\infty)$, an *intermediate* position between the classical concepts of strong $(\alpha = 0)$ and weak $(p = \infty)$ local minima. Clearly all the necessary conditions for i.l.m. automatically hold for strong (and hence for global) minimizers. Let us consider some examples that illustrate relationships between weak, intermediate, and strong local minimizers in variational problems.

The first example is standard showing that the notions of weak and strong minimizers are distinct in the simplest problems of the classical calculus of variations with endpoint constraints.

Example 6.8 (weak but not strong minimizers). *There is a problem of the classical calculus of variations for which a weak local minimizer is not a strong local minimizer.*

Proof. Consider the variational problem:

$$\text{minimize} \;\; J[x] := \int_0^\pi x^2(t)[1 - \dot{x}^2(t)] \, dt$$

over absolutely continuous functions $x\colon [0,\pi] \to I\!R$ satisfying the endpoint constraints $x(0) = x(\pi) = 0$. Let us first show that $\bar{x}(\cdot) \equiv 0$ is a *weak local minimizer*. Indeed, taking any $\varepsilon \in (0,1)$ and any feasible arc $x \neq \bar{x}$ satisfying

$$|x(t) - \bar{x}(t)| \le \varepsilon, \ t \in [0,\pi], \quad \text{and} \quad |\dot{x}(t) - \dot{\bar{x}}(t)| \le \varepsilon \ \text{ a.e. } \ t \in [0,\pi]\,,$$

one has $0 < 1-\varepsilon^2 \le 1-\dot{x}^2(t)$ for almost all $t \in [0,\pi]$. Thus $x^2(t)[1-\dot{x}^2(t)] > 0$ a.e. $t \in [0,\pi]$ with $J[x] > 0 = J[\bar{x}]$, i.e., $\bar{x}$ is a weak local minimizer. On the other hand, $\bar{x}$ is *not a strong local minimizer*, which can be justified as follows. Take feasible arcs $x_k(t) := (1/\sqrt{k})\sin(kt)$ for any $k \in I\!N$ and observe that

$$J[x_k] = \frac{\pi}{2}\Big(\frac{1}{k} - \frac{1}{4}\Big) < 0 \ \text{ for } \ k \ge 5$$

while $|x_k(t) - \bar{x}(t)| \le 1/\sqrt{k}$ for all $t \in [0,\pi]$ and $k \in I\!N$. Thus, given any $\varepsilon > 0$, we can always find a feasible arc x_k that belongs to the ε-neighborhood of $\bar{x}$ in $\mathcal{C}([0,\pi]; I\!R)$ with $J[x_k] < J[\bar{x}]$. $\triangle$

Next let us consider a less standard situation when a weak local minimizer may not be an intermediate local minimizer in the sense of Definition 6.7 for any rank $p \in [1,\infty)$. Again it happens in the one-dimensional framework of the classical calculus of variations.

Example 6.9 (weak but not intermediate minimizers). *There is a one-dimensional problem of the calculus of variations for which a weak local minimizer is not an intermediate local minimizer of any rank $p \ge 1$.*

Proof. Consider the variational problem:

$$\text{minimize } \ J[x] := \int_0^1 \big[\dot{x}^3(t) + 3\dot{x}^2(t)\big]\,dt$$

over absolutely continuous function $x\colon [0,1] \to I\!R$ satisfying the endpoint constraints $x(0) = x(1) = 0$. To show that $\bar{x}(\cdot) \equiv 0$ is a *weak local minimizer*, we observe that the integrand is non-negative whenever $\dot{x}(t) \ge -3$, and hence $J[x] > 0$ for every feasible arc x with

$$0 < |\dot{x}(t) - \dot{\bar{x}}(t)| \le \varepsilon < 3 \ \text{ a.e. } \ t \in [0,1]\,.$$

Given any $p \ge 1$, let us now prove that $\bar{x}$ is *not an intermediate local minimizer* of rank p. To proceed, we consider the family of feasible arcs

$$x_k(t) := \begin{cases} -k^{\frac{1}{2p}}t & \text{if } 0 \le t \le \frac{1}{k}\,, \\[2ex] \dfrac{-k^{\frac{1}{2p}}(1-t)}{k-1} & \text{if } \frac{1}{k} < t \le 1 \end{cases}$$

for natural numbers $k \geq 3^{4p}$. One can check that

$$J[x_k] = -\frac{k^{\frac{1}{p}}}{(k-1)^2}\Big[(k^{\frac{1}{2p}} - 3)(k-2) - 3\Big] < 0 \quad \text{and}$$

$$\int_0^1 |\dot{x}_k(t) - \dot{\bar{x}}(t)|^p = \frac{1}{\sqrt{k}^p}\Big(1 + \frac{1}{(k-1)^{p-1}}\Big)^p \leq \Big(\frac{2}{\sqrt{k}}\Big)^p .$$

Thus for any $\varepsilon > 0$ and any $p \geq 1$ we have

$$\int_0^1 |\dot{x}_k(t) - \dot{\bar{x}}(t)|^p \leq \varepsilon^p \;\text{ with }\; J[x_k] < 0 \;\text{ whenever }\; k \geq \max\big\{\varepsilon^{-2p}, 3^{4p}\big\} ,$$

which shows that $\bar{x}$ cannot be an intermediate minimizer of rank p.

Considering the *simplified version*

$$\text{minimize }\; J[x] := \int_0^1 \dot{x}^3(t)\,dt \;\text{ subject to }\; x(0) = 0, \quad x(1) = 1$$

of the above problem, observe that the arc $\bar{x}(t) = t$ is a weak local minimizer while not an intermediate local minimizer of any rank $p \geq 2$ (but not of $p \geq 1$). To show the latter, we take the functions $x_k(t) = \bar{x}(t) + y_k(t)$ with $y_k(0) = y_k(1) = 0$ and

$$\dot{y}_k(t) = \begin{cases} -\sqrt{k} & \text{if } 0 \leq t \leq \frac{1}{k} , \\ \sqrt{k}(k-1)^{-1} & \text{if } \frac{1}{k} < t \leq 1 \end{cases}$$

and check directly that

$$J[x_k] = -\sqrt{k} + O(1) \to -\infty \;\text{ while }\; \int_0^1 |\dot{x}_k(t) - \dot{\bar{x}}(t)|^p\,dt \to 0 \;\text{ as }\; k \to \infty$$

for each $p \in [2, \infty)$, which completes the discussion. △

The previous examples concerned problems of the calculus of variations with no differential inclusion/dynamic constraints. The next example deals with *autonomous, convex-valued, Lipschitzian* differential inclusions and demonstrates that the concepts of strong and intermediate local minimizers may be different in this case.

Example 6.10 (intermediate but not strong minimizers for bounded, convex-valued, and Lipschitzian differential inclusions). *There is an optimal control problem of minimizing a linear cost function over trajectories of an autonomous, uniformly bounded, and Lipschitzian differential inclusion with compact and convex values for which an intermediate local minimizer of any rank $p \in [1, \infty)$ is not a strong local minimizer.*

Proof. Let $x = (x_1, x_2, x_3, x_4) \in I\!R^4$, and let

$$\psi(x_1, x_2) := \begin{cases} x_2^2 \cos\left(\frac{\pi x_1}{x_2}\right) & \text{for } x_2 \neq 0 \,, \\ 0 & \text{for } x_2 = 0 \,. \end{cases}$$

It is easy to check that ψ is continuously differentiable on $I\!R^4$. Consider the following problem:

$$\text{minimize} \quad J[x] := -x_2(1)$$

over absolutely continuous trajectories for the differential inclusion

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} \in \left\{ \begin{bmatrix} 1 \\ 0 \\ v \\ |\psi(x_1, x_2) - x_2 x_3| \end{bmatrix} \,\middle|\, v \in [-4, 4] \right\} \quad \text{a.e. } \ t \in [0, 1]$$

with the endpoint constraints

$$x_1(0) = x_4(0) = x_4(1) = 0, \qquad x_1(1) = 1 \,.$$

Take a feasible arc $\bar{x}(t) = (t, 0, 0, 0)$ and show first that it is *not a strong local minimizer*. Indeed, for any $\varepsilon \in (0, 2\sqrt{2})$ the function

$$x(t) = \left(t, \ \frac{\varepsilon}{\sqrt{2}}, \ \frac{\varepsilon}{\sqrt{2}} \cos\left(\frac{\sqrt{2}\pi t}{\varepsilon}\right), \ 0\right)$$

is a feasible arc from the ε-neighborhood of $\bar{x}$ in the space $\mathcal{C}\big([0, 1]; I\!R^4\big)$ with the cost $J[x] = -\varepsilon/\sqrt{2} < 0 = J[\bar{x}]$.

Next let us show that $\bar{x}$ is an *intermediate local minimizer* of rank $p = 1$, and hence of any rank $p \in [1, \infty)$, for the problem under consideration. Choose any $\varepsilon \in (0, 1/2)$ and assume on the contrary that there is a feasible arc $x(\cdot)$ satisfying the relations (6.15) and (6.16) in Definition 6.7 and such that $J[x] < J[\bar{x}]$. Then

$$x_1(t) = t, \quad x_2(t) \equiv c, \quad \text{and} \quad |\psi(t, c) - c x_3(t)| \equiv 0$$

on $[0, 1]$ for some $c \in (0, 1/2)$. This gives

$$x_3(t) = \frac{\psi(t, c)}{c} = c \cos\left(\frac{\pi t}{c}\right), \quad \text{and hence} \quad \dot{x}_3 = -\pi \sin\left(\frac{\pi t}{c}\right) .$$

Therefore one has

$$\int_0^1 \|\dot{x}(t) - \dot{\bar{x}}(t)\| \, dt = \pi \int_0^1 \left| \sin\left(\frac{\pi t}{c}\right) \right| dt = \pi c \int_0^{c^{-1}} \big|\sin(\pi s)\big| \, ds$$

$$\geq \pi c \int_0^{[c^{-1}]} \big|\sin(\pi s)\big| \, ds = 2c \left[\frac{1}{c}\right] \geq \frac{2}{3}$$

due to $c \in (0, 1/2)$, where $[a]$ stands as usual for the greatest integer less than or equal to $a \in I\!R$. The latter clearly contradicts the choice of $\varepsilon < 1/2$, which proves that $\bar{x}$ is an intermediate local minimizer of rank $p = 1$. $\triangle$

In what follows, along with the original problem (P), we consider its *relaxed* counterpart that, roughly speaking, is obtained from (P) by the *convexification* procedure with respect to the velocity variable. Taking the integrand $\vartheta(x, v, t)$ in (6.13), we consider its restriction

$$\vartheta_F(x, v, t) := \vartheta(x, v, t) + \delta\big(v; F(x, t)\big)$$

to the sets $F(x, t)$ in (6.1) and denote by $\widehat{\vartheta}_F(x, v, t)$ the *biconjugate* (bipolar) function to $\vartheta_F(x, \cdot, t)$, i.e.,

$$\widehat{\vartheta}_F(x, v, t) = \big(\vartheta_F\big)^{**}_v(x, v, t) \;\text{ for all }\; (x, v, t) \in X \times X \times [a, b]\;.$$

It is well known that $\widehat{\vartheta}_F(x, v, t)$ is the *greatest proper, convex, l.s.c.* function with respect to v, which is *majorized* by ϑ_F. Moreover, $\vartheta_F = \widehat{\vartheta}_F$ *if and only if* ϑ_F is proper, convex, and l.s.c. with respect to v.

Given the original variational problem (P), we define the *relaxed problem* (R), or the *relaxation* of (P), as follows:

$$\text{minimize }\; \widehat{J}[x] := \varphi\big(x(a), x(b)\big) + \int_a^b \widehat{\vartheta}_F\big(x(t), \dot{x}(t), t\big)\, dt \tag{6.17}$$

over a.e. differentiable arcs $x\colon [a, b] \to X$ that are Bochner integrable on $[a, b]$ together with $\vartheta_F\big(x(t), \dot{x}(t), t\big)$, satisfy the Newton-Leibniz formula on $[a, b]$ and the endpoint constraints (6.14). Note that, in contrast to (6.13), the integrand in (6.17) is extended-real-valued. Furthermore, the natural requirement $\widehat{J}[x] < \infty$ yields that $x(\cdot)$ is a solution (in the sense of Definition 6.1) to the *convexified differential inclusion*

$$\dot{x}(t) \in \operatorname{clco} F\big(x(t), t\big) \;\text{ a.e. }\; t \in [a, b]\;. \tag{6.18}$$

Thus the relaxed problem (R) can be considered the explicit dynamic constraint given by the convexified differential inclusion (6.18). Any trajectory for (6.18) is called a *relaxed trajectory* for (6.1), in contrast to *original trajectories*/arcs for the latter inclusion.

There are deep relationships between relaxed and original trajectories for differential inclusion, which reflect *hidden convexity* inherent in continuous-time (nonatomic measure) dynamic systems defined by differential operators. We'll see various realizations of this phenomenon in the subsequent material of this chapter. In particular, *any relaxed trajectory* of compact-valued and Lipschitz in x differential inclusion in Banach spaces may be *uniformly approximated* (in the space $\mathcal{C}\big([a, b]; X\big)$ by original trajectories starting with the same initial state $x(a) = x_0$; see, e.g., Theorem 2.2.1 in Tolstonogov [1258] with the references therein. We need a version of this approximation/density property involving not only differential inclusions but also minimizing functionals. The following result, which holds when the underlying Banach space is *separable*, is proved by De Blasi, Pianigiani and Tolstonogov [308]. Results of this type go back to the classical theorems of Bogolyubov [121] and Young [1350] in the calculus of variations.

Theorem 6.11 (approximation property for relaxed trajectories). *Let $x(\cdot)$ be a relaxed trajectory for the differential inclusion (6.1), where X is separable, and where $F\colon X \times [a,b] \rightrightarrows X$ is compact-valued and uniformly bounded by a summable function, locally Lipschitzian in x, and measurable in t. Assume also that the integrand ϑ in (6.13) is continuous in (x,v), measurable in t, and uniformly bounded by a summable function near $x(\cdot)$. Then there is sequence of the original trajectories $x_k(\cdot)$ for (6.1) satisfying the relations*

$$x_k(a) = x(a), \quad x_k(\cdot) \to x(\cdot) \ \ \text{in} \ \ \mathcal{C}\big([a,b];X\big), \quad \text{and}$$

$$\liminf_{k\to\infty} \int_a^b \vartheta\big(x_k(t), \dot{x}_k(t), t\big)\, dt \le \int_a^b \widehat{\vartheta}_F\big(x(t), \dot{x}(t), t\big)\, dt \ .$$

Note that Theorem 6.11 *doesn't* assert that the approximating trajectories $x_k(\cdot)$ satisfy the endpoint constraints (6.14). Indeed, there are examples showing that the latter may not be possible. If they do, then problem (P) has the property of *relaxation stability*:

$$\inf(P) = \inf(R)\ , \tag{6.19}$$

where the infima of the cost functionals (6.13) and (6.17) are taken over all the feasible arcs in (P) and (R), respectively.

An obvious sufficient condition for the relaxation stability is the *convexity* of the sets $F(x,t)$ and of the integrand ϑ in v. However, the relaxation stability goes far beyond the standard convexity due to the hidden convexity property of continuous-time differential systems. In particular, Theorem 6.11 ensures the relaxation stability of nonconvex problems (P) with no constraints on $x(b)$. There are other efficient conditions for the relaxation stability of nonconvex problems discussed, e.g., in Ioffe and Tikhomirov [617], Mordukhovich [888, 915], and Tolstonogov [1258]. Let us mention the classical Bogolyubov theorem ensuring the relaxation stability in variational problems of minimizing (6.13) with endpoint constraint (6.14) but with *no dynamic constraints* (6.1); relaxation stability of control systems *linear in state variables* via the fundamental Lyapunov theorem on the range convexity of nonatomic vector measures that largely justifies the hidden convexity; the *calmness* condition by Clarke [246, 255] for differential inclusion problems with endpoint constraints of the inequality type; the *normality* condition by Warga [1315, 1321] involving parameterized control systems (6.2), etc.

An essential part of our study relates to *local minima* that are *stable with respect to relaxation*. The corresponding counterpart of Definition 6.7 is formulated as follows.

Definition 6.12 (relaxed intermediate local minima). *The arc $\bar{x}$ is a* RELAXED INTERMEDIATE LOCAL MINIMIZER *(r.i.l.m.) of rank $p \in [1, \infty)$ for the original problem (P) if $\bar{x}$ is a feasible solution to (P) and provides an intermediate local minimum of this rank to the relaxed problem (R) with the same cost $J[\bar{x}] = \widehat{J}[\bar{x}]$.*

The notions of *relaxed weak* and *relaxed strong local minima* are defined similarly, with the same relationships between them as discussed above. Of course, there is no difference between the corresponding relaxed and usual (non-relaxed) notions of local minima for problems (P) with convex sets $F(x,t)$ and integrands ϑ convex with respect to velocity. It is also clear that any relaxed intermediate (weak, strong) minimizer for (P) provides the corresponding non-relaxed local minimum to the original problem. The opposite requires a kind of *local* relaxation stability. Note that any necessary condition for r.i.l.m. holds for relaxed strong local minima, and hence for optimal solutions to (P) (global or absolute minimizers) under the relaxation stability (6.19) of this problem.

Our primary goal is to derive general necessary optimality conditions for r.i.l.m. in the Bolza problem (P) under consideration; some results will be later obtained without any relaxation as well. To proceed, we employ the *method of discrete approximations*, which relates variational/optimal control problems for continuous-time systems to their finite-difference counterparts. The first step in this direction is to build *well-posed* discrete approximations of a *given* r.i.l.m. $\bar{x}(\cdot)$ in problem (P) such that optimal solutions to discrete-time problems *strongly converge* to $\bar{x}(\cdot)$ in the space $W^{1,2}\big([a,b];X\big)$. This will be accomplished in the next subsection.

6.1.3 Well-Posed Discrete Approximations of the Bolza Problem

Considering differential inclusions and their finite-difference counterparts in Subsect. 6.1.1, we established there that *every* trajectory for a differential inclusion in a general Banach space can be *strongly approximated* by extended trajectories for finite-difference inclusions under the natural assumptions made. This result doesn't directly relate to optimization problems involving differential inclusions, but we are going to employ it now in the optimization framework. The primary objective of this subsection is as follows.

Given a trajectory $\bar{x}(\cdot)$ for the differential inclusion (6.1), which provides a *relaxed intermediate local minimum* (r.i.l.m.) to the optimization problem (P) defined above, construct a *well-posed* family of approximating optimization problems (P_N) for finite-difference inclusions (6.3) such that (extended) *optimal solutions* $\bar{x}_N(\cdot)$ to (P_N) *strongly converge* to $\bar{x}(\cdot)$ in the norm topology of $W^{1,2}\big([a,b];X\big)$.

Imposing the standing hypotheses (H1) and (H2) formulated in Subsect. 6.1.1, we observe that the boundedness assumption (6.4) implies that the notion of r.i.l.m. from Definition 6.12 *doesn't depend on rank* p from the interval $[1,\infty)$. This means that $\bar{x}(\cdot)$ is an r.i.l.m. of some rank $p\in[1,\infty)$, then it is also an r.i.l.m. of any other rank $p\geq 1$. In what follows we take $p=2$ and $\alpha=1$ in (6.16) for simplicity.

To proceed, one needs to impose proper assumptions on the other data ϑ, φ, and Ω of problem (P) in addition to those imposed on F. Dealing with the Bochner integral, we always identify measurability of mappings $f\colon[a,b]\to X$ with *strong measurability.* Recall that f is strongly measurable if it can be a.e. approximated by a sequence of step X-valued functions on measurable subsets of $[a,b]$. Given a neighborhood U of $\bar{x}(\cdot)$ and a constant m_F from (H1), we further assume that:

(H3) $\vartheta(\cdot,\cdot,t)$ is continuous on $U\times(m_F I\!\!B)$ uniformly in $t\in[a,b]$, while $\vartheta(x,v,\cdot)$ is measurable on $[a,b]$ and its norm is majorized by a summable function uniformly in $(x,v)\in U\times(m_F I\!\!B)$.

(H4) φ is continuous on $U\times U$; $\Omega\subset X\times X$ is locally closed around $\big(\bar{x}(a),\bar{x}(b)\big)$ and such that the set $\mathrm{proj}_1\Omega\cap\big(\bar{x}(a)+\varepsilon I\!\!B\big)$ is compact for some $\varepsilon>0$, where $\mathrm{proj}_1\Omega$ stands for the projection of Ω on the first space X in the product space $X\times X$.

Note that symmetrically we may assume the local compactness of the second projection of $\Omega\subset X\times X$; the first one is selected in (H4) just for definiteness.

Now taking the r.i.l.m. $\bar{x}(\cdot)$ under consideration, let us apply to this feasible arc Theorem 6.4 on the strong approximation by discrete trajectories. Thus we find a sequence of the extended discrete trajectories $\widehat{x}_N(\cdot)$ approximating $\bar{x}(\cdot)$ and compute the numbers η_N in (6.11). Having $\varepsilon>0$ from relations (6.15) and (6.16) for the intermediate minimizer $\bar{x}(\cdot)$ with $p=2$ and $\alpha=1$, we always suppose that $\bar{x}(t)+\varepsilon/2\in U$ for all $t\in[a,b]$. Let us construct the sequence of discrete approximation problems (P_N), $N\in I\!\!N$, as follows: minimize the discrete-time Bolza functional

$$\begin{aligned} J_N[x_N]:&=\varphi\big(x_N(t_0),x_N(t_N)\big)+\|x_N(t_0)-\bar{x}(a)\|^2\\ &+\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\vartheta\Big(x_N(t_j),\frac{x_N(t_{j+1})-x_N(t_j)}{h_N},t\Big)\,dt\\ &+\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\Big\|\frac{x_N(t_{j+1})-x_N(t_j)}{h_N}-\dot{\bar{x}}(t)\Big\|^2\,dt \end{aligned}\tag{6.20}$$

over discrete trajectories $x_N = x_N(\cdot) = \big(x_N(t_0), \ldots, x_N(t_N)\big)$ for the difference inclusions (6.3) subject to the constraints

$$\big(x_N(t_0), x_N(t_N)\big) \in \Omega + \eta_N I\!B \ , \tag{6.21}$$

$$\|x_N(t_j) - \bar{x}(t_j)\| \le \frac{\varepsilon}{2} \quad \text{for} \quad j = 1, \ldots, N, \quad \text{and} \tag{6.22}$$

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left\| \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N} - \dot{\bar{x}}(t) \right\|^2 dt \le \frac{\varepsilon}{2} \ . \tag{6.23}$$

As in Subsect. 6.1.1, we consider (without mentioning any more) piecewise linear extensions of $x_N(\cdot)$ to the whole interval $[a,b]$ with piecewise constant derivatives for which one has

$$\begin{cases} x_N(t) = x_N(a) + \displaystyle\int_a^t \dot{x}_N(s)\,ds \quad \text{for all } t \in [a,b] \quad \text{and} \\[2ex] \dot{x}_N(t) = \dot{x}_N(t_j) \in F(x_N(t_j), t_j), \quad t \in [t_j, t_{j+1}), \quad j = 0, \ldots, N-1 \ . \end{cases} \tag{6.24}$$

The next theorem establishes that the given local minimizer $\bar{x}(\cdot)$ to (P) can be approximated by *optimal solutions* to (P_N) *strongly* in $W^{1,2}\big([a,b];X\big)$, which implies the a.e. *pointwise* convergence of the derivatives essential in what follows. To justify such an approximation, we need to impose both the Asplund structure and the Radon-Nikodým property (RNP) on the space X in question, which ensure the validity of the classical Dunford theorem on the weak compactness in $L^1\big([a,b];X\big)$. It is worth noting that every *reflexive* space is Asplund and has the RNP simultaneously. Furthermore, the second dual space X^{**} enjoys the RNP (and hence so does $X \subset X^{**}$) if X^* is Asplund. Thus the class of Banach spaces X for which both X and X^* are Asplund satisfies the properties needed in the next theorem. As discussed in the beginning of Subsect. 3.2.5, there are *nonreflexive* (even separable) spaces that fall into this category.

Theorem 6.13 (strong convergence of discrete optimal solutions). *Let $\bar{x}(\cdot)$ be an r.i.l.m. for the Bolza problem (P) under assumptions* (H1)–(H4), *and let (P_N), $N \in I\!N$, be a sequence of discrete approximation problems built above. The following hold:*

(i) *Each (P_N) admits an optimal solution.*

(ii) *If in addition X is Asplund and has the RNP, then any sequence $\{\bar{x}_N(\cdot)\}$ of optimal solutions to (P_N) converges to $\bar{x}(\cdot)$ strongly in $W^{1,2}\big([a,b];X\big)$.*

Proof. To justify (i), we observe that the set of feasible trajectories to each problem (P_N) is nonempty for all large N, since the extended functions $\widehat{x}_N(\cdot)$

from Theorem 6.4 satisfy (6.3) and the constraints (6.21)–(6.23) by construction. This follows immediately from (6.11) in the case of (6.21) and (6.22). In the case of (6.23) we get from (6.4) and (6.12) that

$$\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\left\|\frac{\widehat{x}_N(t_{j+1})-\widehat{x}_N(t_j)}{h_N}-\dot{\bar{x}}(t)\right\|^2 dt=\int_a^b\|\dot{\widehat{x}}_N(t)-\dot{\bar{x}}(t)\|^2\,dt$$
$$\le 2m_F\alpha_N\le\frac{\varepsilon}{2}$$

for large N by the formula for α_N in (6.12). The existence of optimal solutions to (P_N) follows now from the classical Weierstrass theorem due to the compactness and continuity assumptions made in (H1), (H3), and (H4).

It remains to prove the convergence assertion (ii). Check first that

$$J_N[\widehat{x}_N]\to J[\bar{x}]\quad\text{as}\quad N\to\infty \tag{6.25}$$

along some sequence of $N\in I\!N$. Considering the expression (6.20) for $J_N[\widehat{x}_N]$, we deduce from Theorem 6.4 that the second terms therein vanishes, the forth term tends to zero due to (6.4) and (6.12), and the first term tends to $\varphi\big(\bar{x}(a),\bar{x}(b)\big)$ due to the continuity assumption on φ in (H4). It is thus sufficient to show that

$$\sigma_N:=\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\vartheta\left(\widehat{x}_N(t_j),\frac{\widehat{x}_N(t_{j+1})-\widehat{x}_N(t_j)}{h_N},t\right)dt\to\int_a^b\vartheta(\bar{x}(t),\dot{\bar{x}}(t),t)\,dt$$

as $N\to\infty$. Using the sign "$\sim$" for expressions that are equivalent as $N\to\infty$, we get the relationships

$$\sigma_N=\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\vartheta\big(\widehat{x}_N(t_j),\dot{\widehat{x}}_N(t),t\big)\,dt\sim\int_a^b\vartheta\big(\widehat{x}_N(t),\dot{\widehat{x}}_N(t),t\big)\,dt$$

$$\sim\int_a^b\vartheta\big(\bar{x}(t),\dot{\widehat{x}}_N(t),t\big)\,dt\sim\int_a^b\vartheta\big(\bar{x}(t),\dot{\bar{x}}(t),t\big)\,dt$$

by Theorem 6.4 ensuring the a.e. convergence $\dot{\widehat{x}}_N(t)\to\dot{\bar{x}}(t)$ along a subsequence of $N\to\infty$ and by the Lebesgue dominated convergence theorem for the Bochner integral that is valid under (H3).

Note that we have justified (6.25) for any intermediate (not relaxed) local minimizer $\bar{x}(\cdot)$ for the original problem (P) in an arbitrary Banach space X. Next let us show that (6.25) implies that

$$\lim_{N\to\infty}\left[\beta_N:=\|\bar{x}_N(a)-\bar{x}(a)\|^2+\int_a^b\|\dot{\bar{x}}_N(t)-\dot{\bar{x}}(t)\|^2\,dt\right]=0 \tag{6.26}$$

for every sequence of optimal solutions $\bar{x}_N(\cdot)$ to (P_N) provided that $\bar{x}(\cdot)$ is a *relaxed* intermediate local minimizer for the original problem, where the state space X is assumed to be Asplund and to satisfy the RNP.

Suppose that (6.26) is not true. Take a limiting point $\beta > 0$ of the sequence $\{\beta_N\}$ in (6.26) and let for simplicity that $\beta_N \to \beta$ for all $N \to \infty$. We are going to apply the Dunford theorem on the relative *weak compactness* in the space $L^1\big([a,b];X\big)$ (see, e.g., Diestel and Uhl [334, Theorem IV.1]) to the sequence $\{\dot{\bar{x}}_N(\cdot)\}$, $N \in I\!N$. Due to (6.24) and (H1) this sequence satisfies the assumptions of the Dunford theorem. Furthermore, both spaces X and X^* have the RNP, since the latter property for X^* is equivalent to the Asplund structure on X, as mentioned above. Hence we suppose without loss of generality that there is $v \in L^1\big([a,b];X\big)$ such that

$$\dot{\bar{x}}_N(\cdot) \to v(\cdot) \;\text{ weakly in }\; L^1\big([a,b];X\big) \;\text{ as }\; N \to \infty\,.$$

Since the Bochner integral is a linear continuous operator from $L^1\big([a,b];X\big)$ to X, it remains continuous if the spaces $L^1\big([a,b];X\big)$ and X are endowed with the weak topologies. Due to (6.21) and the assumptions on Ω in (H4), the set $\{\bar{x}_N(a)|\; N \in I\!N\}$ is relatively compact in X. Using (6.24) and the *compactness* property of solution sets for differential inclusions under the assumptions made in (H1) and (H2) (see, e.g., Tolstonogov [1258, Theorem 3.4.2]), we conclude that the sequence $\{\bar{x}_N(\cdot)\}$ contains a subsequence that converges to some $\widetilde{x}(\cdot)$ in the norm topology of the space $\mathcal{C}\big([a,b];X\big)$. Now passing to the limit in the Newton-Leibniz formula for $\bar{x}_N(\cdot)$ in (6.24) and taking into account the above convergences, one has

$$\widetilde{x}(t) = \widetilde{x}(a) + \int_a^t v(s)\,ds \;\text{ for all }\; t \in [a,b]\,,$$

which implies the absolute continuity and a.e. differentiability of $\widetilde{x}(\cdot)$ on $[a,b]$ with $v(t) = \dot{\widetilde{x}}(t)$ for a.e. $t \in [a,b]$. Observe that $\widetilde{x}(\cdot)$ is a solution to the convexified differential inclusion (6.18). Indeed, since a subsequence of $\{\dot{\bar{x}}_N(\cdot)\}$ converges to $\dot{\widetilde{x}}(\cdot)$ weakly in $L^1\big([a,b];X\big)$, some *convex combinations* of $\dot{\bar{x}}_N(\cdot)$ converge to $\dot{\widetilde{x}}(\cdot)$ in the norm topology of $L^1\big([a,b];X\big)$, and hence *pointwisely* for a.e. $t \in [a,b]$. Passing to the limit in the differential inclusions for $\bar{x}_N(\cdot)$ in (6.24), we conclude that $\widetilde{x}(\cdot)$ satisfies (6.18). By passing to the limit in (6.21) and (6.22), we also conclude that $\widetilde{x}(\cdot)$ satisfies the endpoint constraints in (6.14) as well as

$$\|\widetilde{x}(t) - \bar{x}(t)\| \le \varepsilon/2 \;\text{ for all }\; t \in [a,b]\,.$$

Furthermore, the integral functional

$$I[v] := \int_a^b \|v(t) - \dot{\bar{x}}(t)\|^2\,dt$$

is lower semicontinuous in the weak topology of $L^2\big([a,b];X\big)$ due to the convexity of the integrand in v. Since the weak convergence of $\dot{\bar{x}}_N(\cdot) \to \dot{\widetilde{x}}(\cdot)$ in $L^1\big([a,b];X\big)$ implies the one in $L^2\big([a,b];X\big)$ by the boundedness assumption (6.4), and since

$$\int_a^b \|\dot{\bar{x}}_N(t) - \dot{\bar{x}}(t)\|^2\,dt = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left\| \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} - \dot{\bar{x}}(t) \right\|^2 dt\;,$$

the above lower semicontinuity and relation (6.23) imply that

$$\int_a^b \|\dot{\widetilde{x}}(t) - \dot{\bar{x}}(t)\|^2\,dt \le \liminf_{N\to\infty} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left\| \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} - \dot{\bar{x}}(t) \right\|^2 dt \le \frac{\varepsilon}{2}\;.$$

Thus the arc $\widetilde{x}(\cdot)$ belongs to the ε-neighborhood of $\bar{x}(\cdot)$ in the space $W^{1,2}\big([a,b];X\big)$.

Let us finally show that the arc $\widetilde{x}(\cdot)$ gives a smaller value to cost functional (6.17) than $\bar{x}(\cdot)$. One always has

$$J_N[\bar{x}_N] \le J_N[\widehat{x}_N] \;\text{ for all large }\; N \in I\!N\;,$$

since each $\widehat{x}_N(\cdot)$ is feasible to (P_N). Now passing to the limit as $N\to\infty$ and taking into account the previous discussions as well as the construction of the convexified integrand $\widehat{\vartheta}_F$ in (6.17), we get from (6.25) that

$$\varphi\big(\widetilde{x}(a),\widetilde{x}(b)\big) + \int_a^b \widehat{\vartheta}_F(\widetilde{x}(t),\dot{\widetilde{x}}(t),t)\,dt + \beta \le J[\bar{x}]\;,$$

which yields by $\beta > 0$ that $\widehat{J}[\widetilde{x}] < J[\bar{x}] = \widehat{J}[\bar{x}]$. The latter is impossible, since $\bar{x}(\cdot)$ is a r.i.l.m. for (P). Thus (6.26) holds, which obviously implies the desired convergence $\bar{x}_N(\cdot) \to \bar{x}(\cdot)$ in the norm topology of the space $W^{1,2}\big([a,b];X\big)$ and completes the proof of the theorem. △

The arguments developed in the proof of Theorem 6.13 allow us to establish efficient conditions for the *value convergence* of discrete approximations, which means that the optimal/infimal values of the cost functionals in the discrete approximation problems converge to the one in the original problem (P). Moreover, using the approximation property for relaxed trajectories from Theorem 6.11, we obtain in fact a *necessary and sufficient* condition for the value convergence in terms of an intrinsic property of the original problems.

Observe that the cost functional (6.20) as well as the constraints (6.22) and (6.23) in the discrete approximation problems (P_N) explicitly contain the given local minimizer $\bar{x}(\cdot)$ to (P). Considering below the value convergence of discrete approximations, we are *not* going to involve *any local minimizer* in the construction of discrete approximations and/or even to assume the *existence of optimal solutions* to the original problem. To furnish this, we consider a sequence of new discrete approximation problems $(\widetilde{P}_N)$ built as follows: minimize

$$\widetilde{J}_N[x_N] := \varphi\big(x_N(t_0),x_N(t_N)\big) + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \vartheta\left(x_N(t_j), \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N}, t\right) dt$$

subject to the discrete inclusions (6.3) and the *perturbed* endpoint constraints (6.21), where the sequence η_N is not yet specified. Note that problems $(\widetilde{P}_N)$ are constructively built upon the initial data of the original continuous-time problem. In the next theorem the notation $\widetilde{J}_N^0 := \inf(\widetilde{P}_N)$, $\inf(P)$, and $\inf(R)$ stands for the optimal value of the cost functional in problems $(\widetilde{P}_N)$, (P), and (R), respectively. Observe that optimal solutions to the discrete-time problems $(\widetilde{P}_N)$ always *exist* due to the assumptions (H1)–(H4) made in Theorem 6.13 under proper perturbations η_N of the endpoint constraints; see its proof.

Theorem 6.14 (value convergence of discrete approximations). *Let $U \subset X$ be an open subset of a Banach space X such that $x_k(t) \in U$ as $t \in [a,b]$ and $k \in I\!N$ for a minimizing sequence of feasible solutions to (P). Assume that hypotheses* (H1)–(H4) *are fulfilled with this set U, where $\bar{x}(a) + \varepsilon I\!B$ is replaced by* $\operatorname{cl} U$ *in* (H4). *The following assertions hold:*

(i) *There is a sequence of the endpoint constraint perturbations $\eta_N \downarrow 0$ in* (6.21) *such that*

$$\inf(R) \le \liminf_{N\to\infty} \widetilde{J}_N^0 \le \limsup_{N\to\infty} \widetilde{J}_N^0 \le \inf(P)\ , \tag{6.27}$$

where the left-hand side inequality requires that X is Asplund and has the RNP. Therefore the relaxation stability (6.19) *of (P) is sufficient for the value convergence of discrete approximations*

$$\inf(\widetilde{P}_N) \to \inf(P) \quad \text{as} \quad N \to \infty$$

provided that X is Asplund and has the RNP.

(ii) *Conversely, the relaxation stability of (P) is also a necessary condition for the value convergence $\inf(\widetilde{P}_N) \to \inf(P)$ of the discrete approximations with arbitrary perturbations $\eta_N \downarrow 0$ of the endpoint constraints provided that X is reflexive and separable.*

Proof. Let us first prove that the right-hand side inequality in (6.27) holds in any Banach space X. Taking the minimizing sequence of feasible arcs $x_k(\cdot)$ to (P) specified in the theorem, we apply to each $x_k(\cdot)$ Theorem 6.4 on the strong approximation by discrete trajectories. Involving the diagonal process, we build the extended discrete trajectories $\widehat{x}_N(\cdot)$ for (6.3) such that

$$\eta_N := \big\|\big(\widehat{x}_N(a), \widehat{x}_N(b)\big) - \big(x_{k_N}(a), x_{k_N}(b)\big)\big\| \to 0 \quad \text{as} \quad N \to \infty$$

and consider the sequence of discrete approximation problems $(\widetilde{P}_N)$ with these constraint perturbations η_N in (6.21). Similarly to the proof of the first part of Theorem 6.13, we show that each $(\widetilde{P}_N)$ admits an optimal solution and, arguing by contradiction, one has the right-hand side inequality in (6.27). To justify the left-hand side inequality in (6.27), we follow the proof of the second part of Theorem 6.13 assuming that X is Asplund and enjoys the RNP. This automatically implies the value convergence of $\inf(\widetilde{P}_N) \to \inf(P)$ under the relaxation stability of (P).

To prove the converse assertion (ii) in the theorem, we first observe that the relaxed problem (R) admits an optimal solution under the assumptions made; see Tolstonogov [1258, Theorem A.1.3]. It follows from the arguments in the second part of Theorem 6.13 that actually justify, under the assumptions made, the compactness of feasible solutions to the relaxed problem and the lower semicontinuity of the minimizing functional (6.17) in the topology on the set of feasible solutions $x(\cdot)$ induced by the weak convergence of the derivatives $\dot{x}(\cdot) \in L^1\big([a,b];X\big)$ provided that X is Asplund and has the RNP. Assume now that X is reflexive and separable and, employing Theorem 6.11, approximate a certain relaxed optimal trajectory $\bar{x}(\cdot)$ by a sequence of the original trajectories $x_k(\cdot)$ converging to $\bar{x}(\cdot)$ as established in that theorem. In turn, each $x_k(\cdot)$ can be strongly approximated in $W^{1,2}\big([a,b];X\big)$ by discrete trajectories $\widehat{x}_{k_N}(\cdot)$ due to Theorem 6.4. Using the diagonal process, we get a sequence of the discrete trajectories $\widehat{x}_N(\cdot)$ approximating $\bar{x}(\cdot)$ and put

$$\eta_N := \|\big(\widehat{x}_N(a), \widehat{x}_N(b)\big) - \big(\bar{x}(a), \bar{x}(b)\big)\| \to \infty \quad \text{as} \quad N \to \infty\,.$$

Now assume that problem (P) is not stable with respect to relaxation, i.e., $\inf(R) < \inf(P)$, and show that

$$\liminf_{N\to\infty} \widetilde{J}_N^0 < \inf(P)$$

for a sequence of discrete approximation problems $(\widetilde{P}_N)$ with some perturbations η_N of the endpoint constraints (6.21). Indeed, having

$$\inf(R) = \varphi(\bar{x}(a), \bar{x}(b)\big) + \int_a^b \widehat{\vartheta}_F(\bar{x}(t), \dot{\bar{x}}(t), t)\,dt < \inf(P)$$

for the relaxed optimal trajectory $\bar{x}(\cdot)$, we build η_N as above and consider problems $(\widetilde{P}_N)$ with these perturbations of the endpoint constraints. Taking into account the approximation of $\bar{x}(\cdot)$ by $x_k(\cdot)$ due to Theorem 6.11, the strong approximation of $x_k(\cdot)$ by the discrete trajectories $\widehat{x}_N(\cdot)$ in Theorem 6.4, and the relations

$$\begin{aligned}
\widetilde{J}_N^0 &\le \varphi\big(\widehat{x}_N(t_0), \widehat{x}_N(t_N)\big) + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \vartheta\left(\widehat{x}_N(t_j), \frac{\widehat{x}_N(t_{j+1}) - \widehat{x}_N(t_j)}{h_N}, t\right) dt \\
&= \varphi\big(\widehat{x}_N(a), \widehat{x}_N(b)\big) + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \vartheta\left(\widehat{x}_N(t_j), \dot{\widehat{x}}_N(t), t\right) dt\,,
\end{aligned}$$

we get by the absence of the relaxation stability that

$$\liminf_{N\to\infty} \widehat{J}_N^0 \le \liminf_{N\to\infty} \Big[\varphi\big(\widehat{x}_N(a), \widehat{x}_N(b)\big) + \int_a^b \vartheta\left(\widehat{x}_N(t), \dot{\widehat{x}}_N(t), t\right) dt\Big]$$

$$\le \varphi\big(\bar{x}(a), \bar{x}(b)\big) + \int_a^b \widehat{\vartheta}_F\left(\bar{x}(t), \dot{\bar{x}}(t), t\right) dt < \inf(P) \ .$$

Therefore we don't have the value convergence of discrete approximations for problems $(\widetilde{P}_N)$ corresponding to the above perturbations of the endpoint constraints. This justifies (ii) and completes the proof of the theorem. △

Thus the relaxation stability of (P), which is an intrinsic and natural property of continuous-time dynamic optimization problems, is actually a *criterion* for the value convergence of discrete approximations under appropriate perturbations of the endpoint constraints in (6.21). It follows from the proof of Theorem 6.14 that one can express the corresponding perturbations η_N via the averaged modulus of continuity (6.6) by

$$\eta_N = \tau(\dot{\bar{x}}; h_N) \to \infty \quad \text{as} \quad N \to \infty$$

provided that (P) admits an optimal solution $\bar{x}(\cdot)$ with the Riemann integrable derivative $\dot{\bar{x}}(\cdot)$ on $[a, b]$. Moreover, $\eta_N = O(h_N)$ if $\dot{\bar{x}}(t)$ is of bounded variation on this interval; see Subsect. 6.1.1.

Remark 6.15 (simplified form of discrete approximations). Observe that if $\vartheta(x, v, \cdot)$ is *a.e. continuous* on $[a, b]$ uniformly in (x, v) in some neighborhood of the optimal solution $\bar{x}(\cdot)$, then the cost functional in (6.20) in problem (P_N) can be replaced in Theorem 6.13 by

$$\begin{aligned} J_N[x_N] :&= \varphi\big(x_N(t_0), x_N(t_N)\big) + \|x_N(t_0) - \bar{x}(a)\|^2 \\ &+ h_N \sum_{j=0}^{N-1} \vartheta\Big(x_N(t_j), \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N}, t_j\Big) \\ &+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \Big\|\frac{x_N(t_{j+1}) - x_N(t_j)}{h_N} - \dot{\bar{x}}(t)\Big\|^2 dt \ ; \end{aligned} \tag{6.28}$$

and similarly for the cost functional in problem $(\widetilde{P}_N)$ used in Theorem 6.14. Indeed, this is an easy consequence of the fact that $\tau(\vartheta; h_N) \to 0$ as $N \to \infty$ for the averaged modulus of continuity (6.6) when $\vartheta(x, v, \cdot)$ is a.e. continuous. Denote by $(\overline{P}_N)$ the discrete approximation problem that differs from (P_N) of that the cost functional (6.20) is replaced by the simplified one (6.28). In what follows we consider both problems (P_N) and $(\overline{P}_N)$ using them to derive necessary optimality conditions for the original problem. The results obtained in these ways are distinguished by the assumptions on the initial data that allow us to justify the desired necessary optimality conditions. Namely, while the use of the simplified problems $(\overline{P}_N)$ as $N \to \infty$ requires the a.e. continuity assumption on ϑ with respect of t (versus the measurability), it relaxes the requirements on the state space X needed in the case of (P_N); see below.

6.1.4 Necessary Optimality Conditions for Discrete-Time Inclusions

Theorem 6.13 on the strong convergence of discrete approximations makes a *bridge* between optimal solutions to the discrete-time problems (P_N), as well as their simplified versions $(\overline{P}_N)$ from Remark 6.15, and the given relaxed intermediate local minimizer $\bar{x}(\cdot)$ for the original continuous-time problem (P). Our further strategy is as follows: first to establish necessary optimality conditions in the sequences of discrete approximation problems (P_N) and $(\overline{P}_N)$ and then to obtain, by passing to the limit as $N \to \infty$, necessary conditions for the given local minimizer to the original optimal control problem (P) governed by differential inclusions.

This subsection is devoted to the derivation of necessary optimality conditions in general discrete-time Bolza problems and their special counterparts for the discrete approximations problems (P_N) and $(\overline{P}_N)$. We explore *two approaches* to these issues. The first one involves reducing general dynamic optimization problems for discrete-time inclusions to non-dynamic problems of *mathematical programming with operator constraints* and then employing necessary optimality conditions for such problems obtained in Subsect. 5.1.2. The second approach is based on the specific features of the discrete approximation problems (P_N) and $(\overline{P}_N)$ and the use of *fuzzy calculus* results from Chaps. 2–4. The results derived by using these two approaches are not reduced to each other, and they require different assumptions. It happens, however, that the *approximate* necessary optimality conditions obtained via the second approach are more suitable for deriving the corresponding results for the continuous-time problem (P) in the next subsection, while those obtained via the first one are definitely of independent interest.

Let us start with the *first approach* and consider the following (non-dynamic) problem of *mathematical programming* (MP) with operator, inequality, and geometric constraints to which we can reduce our discrete-time problems of dynamic optimization:

$$\begin{cases} \text{minimize } \ \varphi_0(z) \ \text{ subject to} \\ \varphi_j(z) \le 0, \quad j = 1, \ldots, s\,, \\ f(z) = 0\,, \\ z \in \Xi_j \subset Z, \quad j = 1, \ldots, l\,, \end{cases} \tag{6.29}$$

where φ_j are real-valued functions on Z, where $f: Z \to E$ is a mapping between Banach spaces, and where $\Xi_j \subset Z$. This is a problem with operator constraints of the type considered in the end of Subsect. 5.1.2 with the only difference that now we have *many geometric constraints* given by the sets Ξ_j. As we see below, the geometric constraints in (6.29) arise from the discretized differential inclusions (6.3), and the number l of them is increasing as $N \to \infty$. Note that problem (MP) is *intrinsically nonsmooth*, even in the case of the smooth data f and φ_j in (6.29) and in the generating dynamic problems. Indeed, the nonsmoothness comes from the geometric constraints in (6.29), which reflect the *dynamics* governed by differential and finite-difference inclusions in the original problem (P) and its discrete approximations.

To derive necessary optimality conditions in problem (MP), one may apply Corollary 5.18 that concerns the problem like (6.29) but with many geometric constraints. Denote

$$\Xi := \Xi_1 \cap \ldots \cap \Xi_l$$

and replace the geometric constraints in (6.29) by $z \in \Xi$. Employing now the result of Corollary 5.18, we need to present necessary optimality conditions for problem (MP) via its initial data. This can be done by using calculus rules for generalized normals and the SNC property of set intersections developed in Chap. 3.

Proposition 6.16 (necessary conditions for mathematical programming with many geometric constraints). *Let $\bar{z}$ be a local optimal solution to problem* (6.29), *where the spaces Z and E are Asplund and where the sets Ξ_j are locally closed around $\bar{z}$. Assume also that all φ_i are Lipschitz continuous around $\bar{z}$, that f is generalized Fredholm at $\bar{z}$, and that each Ξ_j is SNC at this point. Then there are real numbers $\{\mu_j \in I\!R \mid j = 0, \ldots, s\}$ as well as linear functionals $e^* \in E^*$ and $\{z_j^* \in Z^* \mid j = 1, \ldots, l\}$, not all zero, such that $\mu_j \geq 0$ for $j = 0, \ldots, s$ and*

$$\mu_j \varphi_j(\bar{z}) = 0 \quad \text{for} \quad j = 1, \ldots, s \, , \tag{6.30}$$

$$z_j^* \in N(\bar{z}; \Xi_j) \quad \text{for} \quad j = 1, \ldots, l \, , \tag{6.31}$$

$$-\sum_{j=1}^{l} z_j^* \in \partial\Big(\sum_{j=0}^{s} \mu_j \varphi_j\Big)(\bar{z}) + D_N^* f(\bar{z})(e^*) \, . \tag{6.32}$$

Proof. Apply Corollary 5.18 to problem (6.29) with the condensed geometric constraint $z \in \Xi$ given by the intersection of the sets Ξ_j. Then we find $\{\mu_j \geq 0 \mid j = 0, \ldots, s\}$ and $e^* \in E^*$, not all zero, such that μ_j satisfy the complementary slackness conditions in (6.30) and

$$0 \in \partial\Big(\sum_{j=0}^{s} \mu_j \varphi_j\Big)(\bar{z}) + D_N^* f(\bar{z})(e^*) + N(\bar{z}; \Xi) \tag{6.33}$$

provided that the intersection set Ξ is SNC at $\bar{z}$. The latter holds, by Corollary 3.81, if each Ξ_j is SNC at this point and the qualification condition

$$\Big[z_1^* + \ldots + z_l^* = 0, \quad z_j^* \in N(\bar{z}; \Xi_j)\Big] \Longrightarrow \Big[z_j^* = 0, \quad j = 1, \ldots, s\Big]$$

is fulfilled. Furthermore, the same qualification condition ensures, by Corollary 3.37, the intersection formula

$$N(\bar{z}; \Xi) \subset N(\bar{z}; \Xi_1) + \ldots + N(\bar{z}; \Xi_l)$$

when all but one of Ξ_j are SNC at $\bar{z}$. Substituting this into (6.33), we conclude that the fulfillment of the above qualification condition implies (6.32) with $(\mu_j, e^*) \neq 0$. At the same time, the violation of the qualification condition means that (6.32) holds with $(z_1^*, \ldots, z_l^*) \neq 0$ and all zero μ_j and e^*. This completes the proof of the proposition. △

Now let us consider the application of Proposition 6.16 to the following constrained *Bolza problem for discrete-time inclusions* labeled as (DP):

$$\begin{cases} \text{minimize } \ \varphi(x_0, x_N) + h \displaystyle\sum_{j=0}^{N-1} \vartheta_j\Big(x_j, \frac{x_{j+1} - x_j}{h}\Big) \ \text{ subject to} \\ x_{j+1} \in x_j + hF_j(x_j) \ \text{ for } \ j = 0, \ldots, N-1 \,, \\ (x_0, x_N) \in \Xi \subset X^2 \,, \end{cases}$$

where $F_j\colon X \rightrightarrows X$, where φ and ϑ_j are real-valued functions on X^2, and where $h > 0$ and $N \in I\!N$ are fixed. Observe that problem (DP) incorporates the basic structure of discrete approximation problems from the preceding subsection, for any fixed N, without taking into account the terms therein related to approximating the given intermediate local minimizer $\bar{x}(\cdot)$ for the original continuous-time problem (P).

Theorem 6.17 (necessary optimality conditions for discrete-time inclusions). *Let $\{\bar{x}_j |\ j = 0, \ldots, N\}$ be a local optimal solution to problem (DP). Assume that X is Asplund, that the sets Ξ and $\operatorname{gph} F_j$ are locally closed and SNC at $(\bar{x}_0, \bar{x}_N)$ and $\big(\bar{x}_j, (\bar{x}_{j+1} - \bar{x}_j)/h\big)$, respectively, and that the functions φ and ϑ_j are locally Lipschitzian around the corresponding points $\bar{x}_j$ for all $j = 0, \ldots, N-1$. Then there are $\lambda \geq 0$ and $\{p_j \in X^* |\ j = 0, \ldots, N\}$, not simultaneously zero, such that one has the extended Euler-Lagrange inclusion*

$$\Big(\frac{p_{j+1} - p_j}{h}, p_{j+1}\Big) \in \lambda \partial \vartheta_j\Big(\bar{x}_j, \frac{\bar{x}_{j+1} - \bar{x}_j}{h}\Big) + N\Big(\Big(\bar{x}_j, \frac{\bar{x}_{j+1} - \bar{x}_j}{h}\Big); \operatorname{gph} F_j\Big)$$

for all $j = 0, \ldots, N-1$ with the transversality inclusion

$$(p_0, -p_N) \in \lambda \partial \varphi(\bar{x}_0, \bar{x}_N) + N\big((\bar{x}_0, \bar{x}_N)); \Xi\big) \,.$$

Proof. It is easy to see that the discrete-time *dynamic optimization* problem (DP) can be equivalently written in the *non-dynamic* form of mathematical programming given by (6.29) with

$$z := (x_0, \ldots, x_N, v_0, \ldots, v_{N-1}) \in Z := X^{2N+1}, \;\; E := X^N, \;\; l := N+1 \;,$$

$$\varphi_0(z) := \varphi(x_0, x_N) + h \sum_{j=0}^{N-1} \vartheta_j(x_j, v_j), \quad \varphi_j(z) := 0 \;\text{ as }\; j \geq 1 \;,$$

$$f(z) = \big(f_0(z), \ldots, f_{N-1}(z)\big) \;\text{ with}$$

$$f_j(z) := x_{j+1} - x_j - hv_j, \quad j = 0, \ldots, N-1 \;,$$

$$\Xi_j := \big\{ z \in X^{2N+1} \big|\; v_j \in F_j(x_j) \big\} \;\text{ for }\; j = 0, \ldots, N-1 \;,$$

$$\Xi_N := \big\{ z \in X^{2N+1} \big|\; (x_0, x_N) \in \Xi \big\} \;.$$

Thus $\bar{z} := \big(\bar{x}_0, \ldots, \bar{x}_N, (\bar{x}_1 - \bar{x}_0)/h, \ldots, (\bar{x}_N - \bar{x}_{N-1})/h\big)$ is a local optimal solution to the (MP) problem (6.29) with the data defined above. The operator constraint mapping f is surely *generalized Fredholm* at $\bar{z}$; moreover, the sets Ξ_j, $j = 0, \ldots, N$, are obviously SNC at $\bar{z}$ under the assumptions imposed on F_j and Ξ. Since the cost function φ_0 is locally Lipschitzian around $\bar{z}$ and the product spaces Z and E are Asplund, we apply the necessary optimality conditions from Proposition 6.16 to the (MP) problem under consideration, which give us a number $\mu_0 \geq 0$ as well as linear functionals $z_j^* = (x_{0j}^*, \ldots, x_{Nj}^*, v_{0j}^*, \ldots, v_{(N-1)j}^*) \in (X^*)^{2N+1}$ for $j = 0, \ldots, N$ and $e^* = (e_0^*, \ldots, e_{N-1}^*) \in (X^*)^N$, not all zero, such that conditions (6.30)–(6.32) hold with the data defined above. It follows from the above structure of Ξ_j that (6.31) is equivalent to

$$\begin{cases} (x_{jj}^*, v_{jj}^*) \in N\Big(\Big(\bar{x}_j, \dfrac{\bar{x}_{j+1} - \bar{x}_j}{h}\Big); \operatorname{gph} F_j\Big) \quad \text{and} \\[2ex] x_{ij}^* = v_{ij}^* = 0 \;\text{ if }\; i \neq j \;\text{ for all }\; j = 0, \ldots, N-1 \;; \\[2ex] (x_{0N}^*, x_{NN}^*) \in N\big((\bar{x}_0, \bar{x}_N); \Xi\big) \;\text{ and }\; x_{iN}^* = v_{iN}^* = 0 \;\text{ otherwise} \;. \end{cases}$$

Denoting $\lambda := \mu_0$ and employing the sum rule for basic subgradients of locally Lipschitzian functions in Theorem 3.36, we get from (6.32) and the structures of φ_0 and f that there are

$$(x_0^*, x_N^*) \in \partial\varphi(\bar{x}_0, \bar{x}_N) \;\text{ and }\; (u_j^*, w_j^*) \in \partial\vartheta_j\Big(\bar{x}_j, \frac{\bar{x}_{j+1} - \bar{x}_j}{h}\Big)$$

for $j = 0, \ldots, N-1$ satisfying the relations

$$\begin{cases} -x_{00}^* - x_{0N}^* = \lambda\big(x_0^* + hu_0^*\big) - e_0^* \;, \\ -x_{jj}^* = \lambda h u_j^* + e_{j-1}^* - e_j^*, \quad j = 1, \ldots, N-1 \;, \\ -x_{NN}^* = \lambda x_N^* + e_{N-1}^* \;, \\ -v_{jj}^* = h\big(\lambda w_j^* - e_j^*\big), \quad j = 0, \ldots, N-1 \;. \end{cases}$$

Denoting finally

$$p_0 := -x_{0N}^* - \lambda x_0^* + e_0^* \;\text{ and }\; p_j := h e_{j-1}^*, \quad j = 1, \ldots, N \;,$$

we arrive at the desired Euler-Lagrange and transversality inclusions with $\lambda \geq 0$ and $\{p_j \in X^* |\; j = 0, \ldots, N\}$ not equal to zero simultaneously. This completes the proof of the theorem. △

Let us return to our *discrete approximation* problems (P_N) and $(\overline{P}_N)$. Fixed any $N \in I\!N$, observe that problem $(\overline{P}_N)$ defined in (6.3), (6.21)–(6.23), and (6.28) reduces to the form of mathematical programming (6.29) that is just slightly different from the one for (DP). Indeed, letting

$$z := (x_0, \ldots, x_N, v_0, \ldots, v_{N-1}) \in Z := X^{2N+1}, \; E := X^N, \; s := N+2, \; l := N \;,$$

we rewrite $(\overline{P}_N)$ as (6.29) with the following data:

$$\begin{aligned} \varphi_0(z) :&= \varphi(x_0, x_N) + \|x_0 - \bar{x}(a)\|^2 + h_N \sum_{j=0}^{N-1} \vartheta_j(x_j, v_j) \\ &+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \|v_j - \dot{\bar{x}}(t)\|^2 \, dt \;, \end{aligned} \tag{6.34}$$

$$\varphi_j(z) := \begin{cases} \|x_{j-1} - \bar{x}(t_{j-1})\| - \varepsilon/2 \;\text{ for }\; j = 1, \ldots, N+1 \;, \\ \displaystyle\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|v_i - \dot{\bar{x}}(t)\|^2 \, dt - \varepsilon/2 \;\text{ for }\; j = N+2 \;, \end{cases} \tag{6.35}$$

$$\begin{aligned} &f(z) = \big(f_0(z), \ldots, f_{N-1}(z)\big) \;\text{ with} \\ &f_j(z) := x_{j+1} - x_j - h_N v_j, \quad j = 0, \ldots, N-1 \;, \end{aligned} \tag{6.36}$$

$$\begin{aligned} &\Xi_j := \big\{ z \in X^{2N+1} \big|\; v_j \in F_j(x_j) \big\} \;\text{ for }\; j = 0, \ldots, N-1 \;, \\ &\Xi_N := \big\{ z \in X^{2N+1} \big|\; (x_0, x_N) \in \Omega_N \big\} \;, \end{aligned} \tag{6.37}$$

where $\vartheta_j(x,v) := \vartheta(x,v,t_j)$, $F_j(x) := F(x,t_j)$, and $\Omega_N := \Omega + \eta_N I\!\!B$. Notice that the only difference between the (MP) forms for (DP) and $(\overline{P}_N)$ is reflected by the terms in the cost functions and inequality constraints involving the given intermediate local minimizer $\bar{x}(\cdot)$ for the original continuous-time problem (P). These terms can be easily treated in deriving necessary optimality conditions similarly to the proof of Theorem 6.17. Moreover, the impact of these terms to necessary optimality conditions *disappears* in the limiting procedure as $N \to \infty$, i.e., they can be actually *ignored* from the viewpoint of necessary optimality conditions in the original problem (P); see below.

Similarly we observe that problem (P_N) defined in (6.3), (6.20)–(6.23) equivalently reduces to the (MP) form (6.29) with the cost function

$$\begin{aligned}\varphi_0(z) :&= \varphi(x_0,x_N) + \|x_0 - \bar{x}(a)\|^2\\ &\quad + \sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\Big[\vartheta(x_j,v_j,t) + \|v_j - \dot{\bar{x}}(t)\|^2\Big]\,dt\end{aligned} \tag{6.38}$$

and the same constraints (6.35)–(6.37). The difference between (6.34) and (6.38) consists of replacing

$$h_N\sum_{j=0}^{N-1}\vartheta_j(x_j,v_j) \quad \text{by} \quad \sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\vartheta(x_j,v_j,t)\,dt\ ,$$

where the latter allows us to deal with *summable* (in Bochner's sense) integrands $\vartheta(x,v,\cdot)$. In order to derive necessary optimality conditions for problems involving measurable/summable integrands, we need an auxiliary result (certainly important for its own sake) ensuring the *subdifferentiation under the integral sign*, which can be viewed as an "infinite sum" (continuous measure) extension of the subdifferential sum rule for finite sums of Lipschitzian functions obtained in Subsect. 3.2.1. However, the validity of the integral result requires more restrictions on the space in question: we assume its reflexivity and separability versus the Asplund structure in the finite sum rule used in Theorem 6.17. Although the following subdifferential formula holds in rather general measure spaces, we present it only for the case of real intervals, say $T = [0,1]$, needed in subsequent applications. Recall that the *integral of a set-valued mapping* is always understood as the collection of integrals of its summable selections.

Lemma 6.18 (basic subgradients of integral functional). *Let X be a reflexive and separable Banach space. Given $\bar{x} \in X$, assume that $\varphi\colon X\times[0,1] \to I\!\!R$ is measurable in t for each x near $\bar{x}$ and locally Lipschitzian around $\bar{x}$ with a summable modulus on $[0,1]$. Then one has*

$$\partial\Big(\int_0^1 \varphi(\cdot,t)\,dt\Big)(\bar{x}) \subset \mathrm{cl}\int_0^1 \partial\varphi(\bar{x},t)\,dt\ , \tag{6.39}$$

where the subdifferential on the right-hand side is taken with respect to x, *and where the closure "*cl*" is taken with respect to the norm topology in* X^*.

Proof. First we observe that the mapping $\partial\varphi(\bar{x}, \cdot)\colon [0, 1] \rightrightarrows X^*$ is closed-valued and *measurable* in the standard sense for set-valued mappings $F\colon T \rightrightarrows Y$, i.e., that the inverse image $F^{-1}(\Theta)$ is measurable for any open subset $\Theta \subset Y$; for closed-valued mappings such a measurability admits many other equivalent descriptions; see, e.g., Theorems 14.3 and 14.56 in Rockafellar and Wets [1165] that hold in infinite dimensions. Note also that, in the case of separable image spaces, this measurability is equivalent to *strong measurability* (i.e., the possibility of the a.e. pointwise approximation by a sequence of step mappings) that is specific for the Bochner integral under consideration. By the well-known theorems on *measurable selections* (see, e.g., the afore-mentioned book [1165] as well as the early book by Castaing and Valadier [229]) there are measurable singe-valued mappings $\xi\colon [0, 1] \to X^*$ such that $\xi(t) \in \partial\varphi(\bar{x}, t)$ for a.e. $t \in [0, 1]$. Moreover, since X^* is separable and $\partial\varphi(\bar{x}; \cdot)$ is *integrably bounded* by the summable Lipschitz modulus of $\varphi(\cdot, t)$ as easily follows from the assumptions made (see Corollary 1.81), every measurable selector ξ of $\partial\varphi(\bar{x}; \cdot)$ is Bochner integrable on $[0, 1]$. Hence the multivalued integral on the right-hand side of (6.39) is well-defined and nonempty.

It follows from Clarke [255, Theorem 2.7.2] that a counterpart of (6.39) holds with the replacement of the basic subdifferential by the Clarke generalized gradient of Lipschitz functions on both sides. Using now Theorem 3.57 and the reflexivity of X, we have

$$\partial\Big(\int_0^1 \varphi(\cdot, t)\, dt\Big)(\bar{x}) \subset \int_0^1 \operatorname{clco} \partial\varphi(\bar{x}, t)\, dt\ ,$$

since the weak closure agrees with the norm closure for convex sets in reflexive spaces by the Mazur theorem. On the other hand, it is known as an infinite-dimensional extension of the celebrated Lyapunov-Aumann theorem (see, e.g., Sect. 1.1 in Tolstonogov [1258]) that

$$\int_0^1 \operatorname{clco} F(t)\, dt = \operatorname{cl} \int_0^1 F(t)\, dt$$

for every compact-valued, strongly measurable, and integrable bounded mapping. This gives (6.39) and ends the proof of the lemma. △

Based on Theorem 6.17 and the subsequent discussions, we can similarly formulate and justify the extended Euler-Lagrange and transversality inclusions for optimal solutions to both discrete approximation problems (P_N) and $(\overline{P}_N)$. The differences between the above ones for problem (DP) in Theorem 6.17 and those for problem $(\overline{P}_N)$ are just in terms converging to zero as $N \to \infty$. The Euler-Lagrange inclusion for problem (P_N) is parallel to the one in $(\overline{P}_N)$ with replacing

$$\lambda_N \partial\vartheta\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}, t_j\Big)$$

by the norm-closure of

$$\frac{\lambda_N}{h_N}\int_{t_j}^{t_{j+1}} \partial\vartheta\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}, t\Big)\,dt$$

on the right-hand side, which comes from the integration formula of Lemma 6.18. The latter terms converges to $\lambda\partial\vartheta(\bar{x}(t), \dot{\bar{x}}(t), t)$ as $N \to \infty$ for a.e. $t \in [a, b]$; see the proof of Theorem 6.21 in the next subsection.

The results obtained by this approach employing the exact/limiting optimality conditions in the general mathematical programming problems from Theorem 6.16 require the *SNC assumptions* on the sets gph F_j and Ω_N in problems (P_N) and $(\overline{P}_N)$. These assumptions may be restrictive for the limiting procedure to derive necessary optimality conditions in the original continuous-time problem (P); so we'll try to avoid or essentially relax them in what follows. This can be done by starting with *approximate/fuzzy* necessary optimality conditions for problems of mathematical programming that strongly take into account specific features of the discrete-time problems (P_N) and $(\overline{P}_N)$. It happens that to realize this approach, we need to impose the *Lipschitz-like* property of the set-valued mappings F_j generating the graphical geometric constraints in problem (DP), and hence in (P_N) and $(\overline{P}_N)$, which is not assumed in Theorem 6.17. On the other hand, the Lipschitz continuity of the original mapping $F(\cdot, t)$ in (6.1) is among our standing assumptions (see (H1) in Subsect. 6.1.1), and thus we don't have any reservations to employ it in the context of necessary optimality conditions for discrete approximations.

The next two theorems give *approximate* necessary optimality conditions for local minimizers in *sequences* of discrete-time problem $(\overline{P}_N)$ and (P_N). Their proofs involve the use of some *fuzzy/neighborhood* calculus results from the prior chapters. In particular, we employ the semi-Lipschitzian sum rule for Fréchet subgradients from Theorem 2.33 and the fuzzy intersection rule for Fréchet normals from Lemma 3.1. These results provide representations of Fréchet subgradients and normals of sums and intersections at the reference points via those at points that are arbitrarily close to the reference ones. *Just for notational simplicity* we suppose in the formulation and proof of the following theorem that these *arbitrarily close points reduce to the reference points in question.* This agreement doesn't actually restrict the generality from the viewpoint of our main goal in this section to derive necessary optimality conditions in the continuous-time problem (P), which is finalized in the next subsection. Indeed, the possible difference between the mentioned points obviously disappears in the limiting procedure. The interested reader may readily proceed with all the details.

Let us start with approximate necessary optimality conditions for the *simplified* discrete approximation problems $(\overline{P}_N)$ as $N \to \infty$ described in Remark 6.15, which are efficient under the a.e. continuity assumption on the integrand $\vartheta(x, v, \cdot)$ in the original problem (P). In what follows $I\!B^*$ stands as usual for the dual closed unit ball *regardless* of the space in question, and subdifferential of ϑ is taken with respect to the first two variables.

Theorem 6.19 (approximate Euler-Lagrange conditions for simplified discrete-time problems). *Let $\bar{x}_N(\cdot) = \{\bar{x}_N(t_j)|\ j = 0, \ldots, N\}$ be local optimal solutions to problems $(\overline{P}_N)$ as $N \to \infty$. Assume that X is Asplund, that Ω_N is locally closed around $\big(\bar{x}_N(t_0), \bar{x}_N(t_N)\big)$, that F_j is closed-graph and Lipschitz-like around $\big(\bar{x}_N(t_j), [\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)]/h_N\big)$, and that the functions φ and $\vartheta(\cdot, \cdot, t_j)$ are locally Lipschitzian around $\bar{x}_N(\cdot)$ for every $j = 0, \ldots, N-1$. Consider the quantities*

$$\theta_{Nj} := 2\int_{t_j}^{t_{j+1}} \Big\|\frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} - \dot{\bar{x}}(t)\Big\|\, dt, \quad j = 0, \ldots, N-1\,.$$

Then there exists a number $\gamma > 0$ independent of N such that for some sequences of natural numbers $N \to \infty$ and positive numbers $\varepsilon_N \downarrow 0$ there are multipliers $\lambda_N \ge 0$ and adjoint trajectories $p_N(\cdot) = \{p_N(t_j) \in X^*|\ j = 0, \ldots, N\}$ satisfying the nontriviality condition

$$\lambda_N + \|p_N(t_N)\| \ge \gamma \quad \text{as} \quad N \to \infty\,, \tag{6.40}$$

the approximate Euler-Lagrange inclusion

$$\begin{aligned}
&\Big(\frac{p_N(t_{j+1}) - p_N(t_j)}{h_N}, p_N(t_{j+1}) - \lambda_N \frac{\theta_{Nj}}{h_N} b^*_{N_j}\Big) \\
&\in \lambda_N \widehat{\partial}\vartheta\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}, t_j\Big) \\
&+\widehat{N}\Big(\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}\Big); \operatorname{gph} F_j\Big) + \varepsilon_N I\!B^*
\end{aligned} \tag{6.41}$$

for $j = 0, \ldots, N-1$, and the approximate transversality inclusion

$$\begin{aligned}
&\Big(p_N(t_0) - 2\lambda_N b^*_N \|\bar{x}(a) - \bar{x}_N(t_0)\|, -p_N(t_N)\Big) \\
&\in \lambda_N \widehat{\partial}\varphi\big(\bar{x}_N(t_0), \bar{x}_N(t_N)\big) + \widehat{N}\big((\bar{x}_N(t_0), \bar{x}_N(t_N)); \Omega_N\big) + \varepsilon_N h_N I\!B^*
\end{aligned} \tag{6.42}$$

with some $b^*_N, b^*_{Nj} \in I\!B^*$.

Proof. Fixed $N \in I\!N$, consider problem $(\overline{P}_N)$ in the equivalent (MP) form (6.29) with the data defined in (6.34)–(6.37). Denote

$$\bar{z} := \big(\bar{x}_N(t_0), \ldots, \bar{x}_N(t_N), \bar{v}_N(t_0), \ldots, \bar{v}_N(t_{N-1})\big)$$

and take N so large that constraints (6.22) and (6.23) for $\bar{x}_N(\cdot)$ hold with the strict inequality. The latter can be clearly done by the strong convergence result of Theorem 6.13.

Suppose first that f in (6.36) is *metrically regular* at $\bar{z}$ relative to the intersection $\Xi := \Xi_0 \cap \ldots \cap \Xi_N$, where the sets Ξ_j are constructed in (6.37). Since φ_0 in (6.34) is locally Lipschitzian around $\bar{z}$ and by the choice of N, we employ Theorem 5.16 and find $\mu > 0$ such that $\bar{z}$ is a local optimal solution to the unconstrained problem:

$$\text{minimize } \; \varphi_0(z) + \mu \left(\|f(z)\| + \operatorname{dist}(z; \Xi) \right) .$$

Therefore, by the generalized Fermat rule, one has

$$0 \in \widehat{\partial}\Big(\varphi_0(\cdot) + \mu\, \|f(\cdot)\| + \mu \operatorname{dist}(\cdot; \Xi)\Big)(\bar{z}) .$$

Now using the fuzzy sum rule from Theorem 2.33 and remembering our notational agreement, we fix any $\varepsilon > 0$ and get

$$0 \in \widehat{\partial}\varphi_0(\bar{z}) + \mu\widehat{\partial}\|f(\cdot)\|(\bar{z}) + \mu\, \widehat{\partial}\operatorname{dist}(\bar{z}; \Xi) + (\varepsilon/3)I\!B^* .$$

By Proposition 1.95 on Fréchet subgradients of the distance function and by the elementary chain rule for the composition $\|f(z)\| = (\psi \circ f)(z)$ with $\psi(y) := \|y\|$ and the smooth mapping f from (6.36) one has

$$0 \in \widehat{\partial}\varphi_0(\bar{z}) + \sum_{j=0}^{N-1} \nabla f_j(\bar{z})^* e_j^* + \widehat{N}(\bar{z}; \Xi) + (\varepsilon/3)I\!B^*$$

with some $e_j^* \in X^*$. Observe that

$$\sum_{j=0}^{N-1} \nabla f_j(\bar{z})^* e_j^* = \big(-e_0^*, e_0^* - e_1^*, \ldots, e_{N-2}^* - e_{N-1}^*, e_{N-1}^*, -h_N e_0^*, \ldots, -h_N e_{N-1}^*\big)$$

by the structure of $f(z)$ in (6.36). Further, it follows from the fuzzy intersection rule in Lemma 3.1 and the discussion right after it that, taking into account the notational agreement, we get

$$\widehat{N}(\bar{z}; \Xi) \subset \widehat{N}(\bar{z}; \Xi_0) + \ldots + \widehat{N}(\bar{z}; \Xi_N) + (\varepsilon/3)I\!B^* .$$

To justify it, one needs to check the *fuzzy qualification condition* (3.9) for the sets involved. It obviously holds for the set intersections of Ξ_j, with $j = 0, \ldots, N-1$ by the structure of these sets in (6.37). To verify this condition at the last step, let us show that there is $\gamma > 0$ for which

$$\Big(\widehat{N}\Big(z; \bigcap_{j=0}^{N-1} \Xi_j\Big) + \gamma I\!B^*\Big) \bigcap \Big(-\widehat{N}\big(z_N; \Xi_N\big) + \gamma I\!B^*\Big) \bigcap I\!B^* \subset \frac{1}{2} I\!B^*$$

whenever $z \in \Xi_j \cap (\bar{z} + \gamma I\!B)$, $j = 0, \ldots, N-1$, and $z_N \in \Xi_N \cap (\bar{z} + \gamma I\!B)$. It follows directly from the set structures in (6.37) that for any $z_j^* \in \widehat{N}(z_j; \Xi_j)$ with $z_j^* =$

$(x_{0j}^*,\ldots,x_{Nj}^*,v_{0j}^*,\ldots,v_{N-1j}^*)$ and $z_j=(x_{0j},\ldots,x_{Nj},v_{0j},\ldots,v_{N-1j})$ close to $\bar{z}$ one has the relations

$$x_{ij}^*\in\widehat{D}^*F_j(x_{jj},v_{jj})(-v_{jj}^*),\quad x_{ij}^*=v_{ij}^*=0\ \text{ if }\ i\neq j,\quad j=0,\ldots,N-1\,;$$

$$(x_{0N}^*,x_{NN}^*)\in\widehat{N}\big((x_{0N},x_{NN});\Omega_N\big)\ \text{ with }\ x_{iN}^*=v_{iN}^*=0\ \text{ otherwise}\,.$$

Therefore, by Theorem 1.43 on Fréchet coderivatives of Lipschitzian mappings, we get the estimates

$$\|x_{jj}^*\|\le\ell\|v_{jj}^*\|\ \text{ for all }\ j=0,\ldots,N-1$$

provided that F_j are Lipschitz-like around (x_{jj},v_{jj}) with modulus ℓ. This easily implies the above fuzzy qualification condition at the last step by taking into account that it holds at all the previous steps with $\varepsilon_N:=\varepsilon/N$.

Next we proceed with estimating Fréchet subgradients of the cost function φ_0 in (6.34). It is well known from convex analysis that

$$\partial\|\cdot\|^2(x)\subset 2\|x\|I\!B^*\ \text{ for any }\ x\in X$$

in arbitrary Banach spaces. Using this and applying the fuzzy sum rule from Theorem 2.33 to the specific form of φ_0 in (6.34), we have

$$\widehat{\partial}\varphi_0(\bar{z})\subset\widehat{\partial}\varphi\big(\bar{x}_N(t_0),\bar{x}_N(t_N)\big)+2\|\bar{x}_N(t_0)-\bar{x}(a)\|I\!B^*$$
$$+h_N\sum_{j=0}^{N-1}\Big[\widehat{\partial}\vartheta_j\big(\bar{x}_N(t_j),\bar{v}_N(t_j)\big)+\big(0,2\theta_{Nj}I\!B^*\big)\Big]+(\varepsilon/3)I\!B^*$$

with taking into account our notational agreement and the construction of θ_{Nj}. Now combining the above relationships and estimates in generalized Fermat rule, one gets

$$\begin{cases}-x_{00}^*-x_{0N}^*-x_0^*-2b_N^*\big\|\bar{x}_N(t_0)-\bar{x}(a)\big\|-u_0^*+e_0^*\in\varepsilon I\!B^*\,,\\[2mm] -x_{jj}^*-h_Nu_j^*-e_{j-1}^*+e_j^*\in\varepsilon I\!B^*,\quad j=0,\ldots,N-1\,,\\[2mm] -x_{NN}^*-x_N^*-e_{N-1}^*\in\varepsilon I\!B^*\,,\\[2mm] -v_{jj}^*-h_Nw_j^*-\theta_{Nj}b_{Nj}^*+h_Ne_j^*\in\varepsilon I\!B^*,\quad j=0,\ldots,N-1\end{cases}$$

with some $b_{Nj}^*,b^*\in I\!B^*$,

$$(x_{ij}^*,v_{ij}^*)\in\widehat{N}\Big(\Big(\bar{x}_N(t_j),\frac{\bar{x}_N(t_{j+1})-\bar{x}_N(t_j)}{h_N}\Big);\operatorname{gph}F_j\Big),\quad\text{and}$$

$$(x_0^*,x_N^*)\in\widehat{\partial}\varphi\big(\bar{x}_N(t_0),\bar{x}_N(t_N)\big),\quad(u_j^*,w_j^*)\in\widehat{\partial}\vartheta_j\Big(\bar{x}_N(t_j),\frac{\bar{x}_N(t_{j+1})-\bar{x}_N(t_j)}{h_N}\Big)$$

for $j=0,\ldots,N-1$. Denoting

$$p_N(t_0):=-x_{0N}^*-\lambda_N x_0^*+e_0^* \quad\text{and}\quad p_N(t_j):=h_N e_{j-1}^*,\quad j=1,\ldots,N\ ,$$

we arrive at the approximate Euler-Lagrange and transversality inclusions (6.41) and (6.42) with $\lambda_N=1$ for *any* $N\in I\!N$ sufficiently large and *any* $\varepsilon=\varepsilon_N$. Note that the nontriviality condition (6.40) is obviously fulfilled with $\gamma_N=1$ in the metric regularity case under consideration.

It remains to consider the case when the mapping f from (6.36) is *not metrically regular* at $\bar z$ relative to the set intersection $\Xi:=\Xi_0\cap\ldots\cap\Xi_N$. In this case the extended mapping $f_\Xi(z):=f(z)+\Delta(z;\Xi)$ is not metrically regular around $\bar z$ in the sense of Definition 1.47(ii). We now apply the *neighborhood characterization* of metric regularity in Asplund spaces obtained in Theorem 4.5. It is not hard to observe that this criterion can be equivalently written as follows: a closed-graph mapping $F\colon X\rightrightarrows Y$ between Asplund spaces is metrically regular around $(\bar x,\bar y)\in\operatorname{gph}F$ *if and only if* there is a positive number ν such that

$$\ker\widehat D^*F(x,y)\subset I\!B^*\quad\text{whenever}\ \ x\in\bar x+\nu I\!B,\quad y\in F(x)\cap(\bar y+\nu I\!B)\ .$$

Applying this result to the mapping $f(z)+\Delta(z;\Xi)$ that is *not* metrically regular around $\bar z$, we have the following assertion as N is fixed: for any $\eta>0$ there are $z\in\bar z+\eta I\!B$ and $e^*\in\ker\widehat D^*f_\Xi(z)$ with $e^*=(e_0^*,\ldots,e_{N-1}^*)\in(X^*)^N$ satisfying $\|e^*\|>1$. Thus

$$0\in\widehat D^*f_\Xi(z)(e^*)\ \ \text{for some}\ \ \|e^*\|>1\ \ \text{and}\ \ z\in\bar z+\nu I\!B\ .$$

Fixed $\varepsilon>0$, we employ the coderivative sum rule from Theorem 1.62(i) and then the above intersection rule for Fréchet normals that give

$$0\in\sum_{j=0}^{N-1}\nabla f_j(z)^*e_j^*+\sum_{j=0}^{N}\widehat N(z_j;\Xi_j)+\varepsilon I\!B^*$$

with some $z_j\in\Xi_j\cap(z+\varepsilon I\!B)$. According to our notation agreement we may put $z_j=z=\bar z$ for simplicity. Thus there are $z_j^*\in\widehat N(\bar z;\Xi_j)$ satisfying

$$-\sum_{j=0}^{N}z_j\in\sum_{j=0}^{N-1}\nabla f_j(\bar z)^*e_j^*+\varepsilon I\!B^*\ .$$

Taking into account the structures of the mapping f in (6.36) and the sets Ξ_j in (6.37), we find as above dual elements

$$(x_{ij}^*,v_{ij}^*)\in\widehat N\Big(\Big(\bar x_N(t_j),\,\frac{\bar x_N(t_{j+1})-\bar x_N(t_j)}{h_N}\Big);\operatorname{gph}F_j\Big)$$

for $j=0,\ldots,N-1$ and

$$(x_{0N}^*, x_{NN}^*) \in \widehat{N}\big((\bar{x}_N(t_0), \bar{x}_N(t_N)); \Omega_N\big)$$

satisfying the relations

$$\begin{cases} -x_{00}^* - x_{0N}^* + e_0^* \in \varepsilon I\!B^* \;, \\ -x_{jj}^* - e_{j-1}^* + e_j^* \in \varepsilon I\!B^*, \quad j = 0, \ldots, N-1 \;, \\ -x_{NN}^* - x_N^* - e_{N-1}^* \in \varepsilon I\!B^* \;, \\ -v_{jj}^* + h_N e_j^* \in \varepsilon I\!B^*, \quad j = 0, \ldots, N-1 \;. \end{cases}$$

Define the adjoint discrete trajectory $p_N(t_j)$, $j = 0, \ldots, N$, by

$$p_N(t_0) := -x_{0N}^* + e_0^* \;\text{ and }\; p_N(t_j) := e_{j-1}^*, \quad j = 1, \ldots, N \;.$$

It follows from the above constructions that the pair $\big(\bar{x}_N(\cdot), p_N(\cdot)\big)$ satisfies the Euler-Lagrange inclusion (6.41) and the transversality inclusion (6.42) with $\lambda_N = 0$ and arbitrary $\varepsilon_N = \varepsilon > 0$. Moreover, the adjoint trajectory $p_N(\cdot)$ obeys the following nontriviality condition:

$$\|p_N(t_1)\| + \ldots + \|p_N(t_N)\| \ge 1 \;\text{ for all large }\; N \in I\!N \;.$$

Let us finally prove that, by the *Lipschitz-like* assumption on F_j, the nontriviality condition in this case can be equivalently written as $\|p_N(t_N)\| \ge 1$, which agrees with (6.40) as $\lambda_N = 0$. The approximate Euler-Lagrange inclusion (6.41) can be now rewritten in the form

$$\frac{p_N(t_{j+1}) - p_N(t_j)}{h_N} \in \widehat{D}^* F_j\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}\Big)\big(-p_N(t_{j+1}) + I\!B^*\big)$$

$$+\varepsilon I\!B^* \quad \text{for} \quad j = 0, \ldots, N-1 \;.$$

Then the Lipschitz-like property of F_j assumed in the theorem with modulus $\ell = \ell_F$ yields by Theorem 1.43 that

$$\|x_j^*\| \le \ell \|v_j^*\| \;\text{ whenever }\; x_j^* \in \widehat{D}^* F_j(x_j, v_j)(v_j^*)$$

and (x_j, v_j) around $\big(\bar{x}_N(t_j), [\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)]/h_N\big)$. Thus

$$\|p_N(t_{N-1})\| \le \|p_N(t_N)\|\big(1 + h_N \ell\big) + h_N \varepsilon \;.$$

Continuing this process, one has

$$\|p_N(t_j)\| \le \exp\big(\ell(b-a)\big)\|p_N(t_N)\| + \varepsilon(b-a) \;\text{ for all }\; j = 0, \ldots, N \;.$$

Suppose that the nontriviality condition (6.40) doesn't hold along with (6.41) and (6.42) in the case of $\lambda_N = 0$ under consideration. Take a sequence $\gamma_k \downarrow 0$ as $k \to \infty$ and choose numbers N_k and ε_k such that

$$N_k := \big[1/\gamma_k\big], \quad \varepsilon_k \le \gamma_k^2, \text{ and } \|p_N(t_N)\| \le \gamma_k^2, \quad k \in I\!N \ ,$$

where $[\cdot]$ stands for the greatest integer less than or equal to the given real number. By the adjoint trajectory estimate we have

$$\begin{aligned}\sum_{j=1}^{N_k} \|p_{N_k}(t_j)\| &\le 2N_k\gamma_k \exp\big(\ell(b-a)\big) + \varepsilon_k N_k(b-a) \\ &\le \gamma_k \exp\big(\ell(b-a)\big) + \gamma_k(b-a) \downarrow 0 \ \text{ as } \ k \in I\!N \ ,\end{aligned}$$

which contradicts the fact established above. This therefore completes the proof of the theorem. $\triangle$

Finally in this subsection, we obtain *approximate* necessary optimality conditions for the sequence of discrete-time problems (P_N) defined in (6.3), (6.20)–(6.23). The difference between these problems and the simplified problems $(\overline{P}_N)$ is that (P_N) deal with approximating *summable* integrands $\vartheta(x, v, \cdot)$ in the original problem (P), which is reflected by the integral term involving ϑ in the cost function (6.20). The latter term makes the analysis of problems (P_N) to be more complicated in comparison with the one for $(\overline{P}_N)$. To proceed, we need to use Lemma 6.18 on the subdifferentiation under the (Bochner) integral sign, which requires additional assumptions on the space X. The next theorem incorporates these developments in the framework of the extended Euler-Lagrange inclusion for (P_N). We keep our notational agreement discussed before the formulation of Theorem 6.19.

Theorem 6.20 (approximate Euler-Lagrange conditions for discrete problems involving summable integrands). *Let $\bar{x}_N(\cdot) = \{\bar{x}_N(t_j)|\ j = 0, \ldots, N\}$ be local optimal solutions to problems (P_N) as $N \to \infty$. Assume that X is reflexive and separable, that φ, F_j, Ω_N, and θ_{N_j} are the same as in Theorem 6.19, and that ϑ satisfies assumption (H3) of Subsect. 6.1.3 with the replacement of continuity by Lipschitz continuity. Then there exists a number $\gamma > 0$ independent of N such that for some sequences of natural numbers $N \to \infty$ and positive numbers $\varepsilon_N \downarrow 0$ there are multipliers $\lambda_N \ge 0$ and adjoint trajectories $p_N(\cdot) = \{p_N(t_j) \in X^*|\ j = 0, \ldots, N\}$ satisfying the nontriviality condition (6.40), the approximate transversality inclusion (6.42), and the Euler-Lagrange inclusion in the modified form*

$$\begin{aligned}&\Big(\frac{p_N(t_{j+1}) - p_N(t_j)}{h_N}, p_N(t_{j+1}) - \lambda_N \frac{\theta_{Nj}}{h_N} b^*_{Nj}\Big) \\ &\in \frac{\lambda_N}{h_N} \operatorname{cl} \int_{t_j}^{t_{j+1}} \partial\vartheta\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}, t\Big)\, dt \\ &+ \widehat{N}\Big(\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}\Big); \operatorname{gph} F_j\Big) + \varepsilon_N h_N I\!B^*\end{aligned} \tag{6.43}$$

for all $j = 0, \ldots, N-1$ with some $b^*_{Nj} \in I\!B^*$.

Proof. Each problem (P_N) can be equivalently written in the (MP) form (6.29) with the data defined in (6.35)–(6.38). Now we proceed similarly to the proof of Theorem 6.19 using additionally Lemma 6.18 to calculate subgradients of integral function. This becomes possible under the additional assumptions on X made in the theorem and gives the modified form (6.43) of the approximate Euler-Lagrange inclusion. $\triangle$

Taking into account the value convergence results of Theorem 6.14, we can treat the necessary optimality conditions obtained in this subsection for the discrete approximation problems under consideration as *suboptimality conditions* for the original problem (P). Moreover, the strong convergence results presented in Theorem 6.13 and Remark 6.15 allow us to view the above necessary optimality conditions for the discrete-time problems as suboptimality conditions concerning a *given* relaxed intermediate local minimizer for the original problem. Note that the assumptions made in Theorems 6.13 and 6.14 ensure the *existence* of optimal solutions to the discrete approximations, while it is *not* the case for the original continuous-time problem (P) in either finite-dimensional or infinite-dimensional setting. Necessary *optimality* conditions for relaxed local minimizers to problem (P) are considered next.

6.1.5 Euler-Lagrange Conditions for Relaxed Minimizers

The aim of this subsection is to derive necessary conditions for the underlying r.i.l.m. to the original Bolza problem (P) involving constrained differential inclusions by *passing to the limit* from the ones for discrete approximations obtained in the preceding subsection. This is based on the strong convergence result for discrete approximations given in Theorem 6.13, on the approximate necessary optimality conditions for the discrete problems (P_N) and $(\overline{P}_N)$ from Theorems 6.19 and 6.20, and on stability properties of the generalized differential constructions. The major ingredient involved in this limiting procedure is the possibility to establish an appropriate *convergence of adjoint trajectories*, which allows us to pass to the limit in the approximate Euler-Lagrange inclusions. This is done below by employing the *coderivative characterization of Lipschitzian stability* used also in the preceding subsection.

Let us first clarify the assumptions needed for the main results of this subsection. They involve of course those ensuring the strong convergence of discrete approximations and the fulfillment of the (approximate) necessary optimality conditions in discrete-time problems (P_N) and $(\overline{P}_N)$ used below. In fact, not too much has to be added for furnishing the limiting process to derive pointwise necessary optimality conditions in the original Bolza problem (P) via discrete approximations.

In what follows we keep assumptions (H1) and (H2) from Subsect. 6.1.1 on the mapping F in (6.1) and consider the Lipschitzian modification of assumptions (H3) and (H4) from Subsect. 6.1.3:

(H3′) $\vartheta(\cdot,\cdot,t)$ is Lipschitz continuous on $U \times (m_F I\!B)$ uniformly in $t \in [a,b]$, while $\vartheta(x,v,\cdot)$ is measurable on $[a,b]$ and its norm is majorized by a summable function uniformly in $(x,v) \in U \times (m_F I\!B)$.

(H4′) φ is Lipschitz continuous on $U \times U$; $\Omega \subset X \times X$ is locally closed around $(\bar{x}(a),\bar{x}(b))$ and such that the set $\mathrm{proj}_1\Omega \cap \big(\bar{x}(a)+\varepsilon I\!B\big)$ is compact for some $\varepsilon > 0$.

Note that (H3′) contains the *measurability* assumption on $\vartheta(x,v,\cdot)$, which corresponds to Theorem 6.20. The latter imposes more restrictive requirement on the state space X in comparison with Theorem 6.19, which however relates to the *a.e. continuity* of $\vartheta(x,v,\cdot)$ in the convergence result for problem $(\overline{P}_N)$; see Remark 6.15. Taking this into account, we consider also another modification of (H3) that is an alternative to the above assumption (H3′):

(H3′′) $\vartheta(x,v,\cdot)$ is a.e. continuous on $[a,b]$ and bounded on this interval uniformly in $(x,v) \in U \times (m_F I\!B)$, while $\vartheta(\cdot,\cdot,t)$ is Lipschitz continuous on

$$\Theta_\nu(t) := \big\{(x,v) \in U \times (m_F+\nu)I\!B \big|\ \exists \tau \in (t-\nu,t] \ \text{ with } \ v \in F(x,\tau)\big\}$$

uniformly in $t \in [a,b]$ for some $\nu > 0$.

Dealing with the a.e. continuous mappings $F(x,\cdot)$ and $\vartheta(x,v,\cdot)$ in the limiting procedures involving t, we use *extended* normal cone N_+ from Definition 5.69 to the *moving* sets $\mathrm{gph}\, F(\cdot)$ and the corresponding subdifferential of $\vartheta(x,v,t)$. Although these constructions may be different from the basic normal cone and subdifferential in the case of *non-autonomous* objects, they agree with the latter in general settings ensuring *normal semicontinuity*; see the results and discussions after Definition 5.69. Note that we *don't need* to replace the basic subdifferential of the integrand ϑ by the extended one assuming the *measurability* of ϑ in t as in (H3′). We also don't need to replace the basic normal cone to $\mathrm{gph}\, F$ in the next Subsect. 6.1.6 dealing with measurable set-valued mappings in differential inclusions.

Recall that, given $(\bar{x},\bar{v},\bar{t})$ with $\bar{v} \in F(\bar{x},\bar{t})$, the *extended normal cone* to the moving set $\mathrm{gph}\, F(t)$ at $(\bar{x},\bar{v}) \in \mathrm{gph}\, F(\bar{t})$ is, in the case of closed subsets in Asplund spaces,

$$N_+\big((\bar{x},\bar{v});\mathrm{gph}\, F(\bar{t})\big) := \mathop{\mathrm{Lim\,sup}}_{(x,v,t)\to(\bar{x},\bar{v},\bar{t})} \widehat{N}\big((x,v);\mathrm{gph}\, F(t)\big)\ .$$

Correspondingly, the *extended subdifferential* of $\vartheta(\cdot,\cdot,\bar{t})$ at $(\bar{x},\bar{v})$ is

$$\partial_+\vartheta(\bar{x},\bar{v},\bar{t}) := \mathop{\mathrm{Lim\,sup}}_{(x,v,t)\to(\bar{x},\bar{v},\bar{t})} \widehat{\partial}\vartheta(x,v,t)\ ,$$

where $\widehat{\partial}\vartheta(\cdot,\cdot,t)$ is taken with respect to (x,v) under fixed t. Note that $\partial_+\vartheta(\bar{x},\bar{v},\bar{t})$ can be equivalently described via the extended normal cone N_+ to the moving epigraphical set $\operatorname{epi}\vartheta(t)$. One can see that these extended objects reduce to the basic ones $N(\cdot;\operatorname{gph}F)$ and $\partial\vartheta$ when F and ϑ are independent of t, as well as in the more general settings discussed above.

Now we are ready to formulate and prove the *extended Euler-Lagrange conditions* for relaxed intermediate minimizers in the original Bolza problem (P). We consider separately the two cases: when the integrand ϑ is a.e. continuous in t, and when it is summable. Although the second case imposes less requirements on the integrand and gives a better form of the Euler-Lagrange inclusion, in the first case we are able to obtain necessary optimality conditions in more general Banach spaces. Let us start with the first one. The *strong PSNC* property used below is defined and discussed in Subsect. 3.1.1.

Theorem 6.21 (extended Euler-Lagrange conditions for relaxed local minimizers in Bolza problems with a.e. continuous integrands). *Let $\bar{x}(\cdot)$ be a relaxed intermediate local minimizer for the Bolza problem (P) under assumptions* (H1), (H2), (H4′), *and* (H3″). *Suppose also that both spaces X and X^* are Asplund and that the set Ω is strongly PSNC at $(\bar{x}(a),\bar{x}(b))$ with respect to the second component. Then there are $\lambda\ge 0$ and an absolutely continuous mapping $p\colon[a,b]\to X^*$, not both zero, satisfying the extended Euler-Lagrange inclusion*

$$\dot{p}(t)\in\operatorname{clco}\Big\{u\in X^*\Big|\ \big(u,p(t)\big)\in\lambda\partial_+\vartheta(\bar{x}(t),\dot{\bar{x}}(t),t)+N_+\big((\bar{x}(t),\dot{\bar{x}}(t));\operatorname{gph}F(t)\big)\Big\}\tag{6.44}$$

for a.e. $t\in[a,b]$ and the transversality inclusion

$$\big(p(a),-p(b)\big)\in\lambda\partial\varphi\big(\bar{x}(a),\bar{x}(b)\big)+N\big((\bar{x}(a),\bar{x}(b));\Omega\big)\ .\tag{6.45}$$

Proof. We derive these conditions by passing to the limit in the necessary optimality conditions for discrete-time problems $(\overline{P}_N)$ from Theorem 6.19 with taking into account the strong convergence of the simplified discrete approximations; see Theorem 6.13 and Remark 6.15. Recall that the Asplund property of X is *equivalent* to the Radon-Nikodým property of X^*; see Subsect. 6.1.1. Since X is a closed subspace of X^{**} and X^* is assumed to be Asplund, this yields that X has the Radon-Nikodým property. Thus all the assumptions of Theorem 6.13 are fulfilled, which allows us to employ the strong convergence of discrete approximations.

Note that the assumptions made clearly ensure the fulfillment of the ones in Theorem 6.19. Employing the necessary optimality conditions for $(\overline{P}_N)$ obtained therein, we find (sub)sequences of numbers $\lambda_N\ge 0$ and discrete adjoint trajectories $p_N(\cdot)=\{p_N(t_j)|\ j=0,\ldots,N\}$ satisfying inclusions (6.40)–(6.42) with some $\varepsilon_N\downarrow 0$ as $N\to\infty$. Observe that without loss of generality the nontriviality condition (6.40) can be equivalently written as

$$\lambda_N + \|p_N(t_N)\| = 1 \quad \text{for all} \quad N \in I\!N \; ,$$

because the number $\gamma > 0$ is independent of N. Also one can always suppose that $\lambda_N \to \lambda \geq 0$ as $N \to \infty$.

In what follows we use the notation $\bar{x}_N(t)$ and $p_N(t)$ for piecewise linear extensions of the corresponding discrete trajectories to $[a, b]$ with their piecewise constant derivatives $\dot{\bar{x}}_N(t)$ and $\dot{p}_N(t)$. Having θ_{Nj} defined in Theorem 6.19, we consider a sequence of functions $\theta_N\colon [a, b] \to I\!R$ given by

$$\theta_N(t) := \frac{\theta_{Nj}}{h_N} b^*_{Nj} \quad \text{for} \quad t \in [t_j, t_{j+1}), \quad j = 0, \ldots, N-1 \; .$$

Invoking Theorem 6.13, we get

$$\begin{aligned}\int_a^b \big\|\theta_N(t)\big\| \, dt \leq \sum_{j=0}^{N-1} \theta_{Nj} \leq 2 \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \Big\| \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} - \dot{\bar{x}}_N(t) \Big\| \, dt \\ = 2 \int_a^b \|\dot{\bar{x}}_N(t) - \dot{\bar{x}}(t)\| \, dt =: \nu_N \to 0 \quad \text{as} \quad N \to \infty \; .\end{aligned}$$

This allows us to suppose without loss of generality that

$$\dot{\bar{x}}_N(t) \to \dot{\bar{x}}(t) \quad \text{and} \quad \theta_N(t) \to 0 \quad \text{a.e.} \quad t \in [a, b] \quad \text{as} \quad N \to \infty \; .$$

Consider the approximate discrete Euler-Lagrange inclusions (6.41) along the designated sequence of $N \to \infty$, which is identified with the whole set of natural numbers $I\!N$. By (6.41) we find

$$(x^*_{Nj}, v^*_{Nj}) \in \widehat{\partial} \vartheta_j \Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} \Big), \qquad j = 0, \ldots, N-1 \; ,$$

and $e^*_{N_j}, \widetilde{e}^*_{Nj} \in I\!B^*$ such that the inclusions

$$\begin{aligned}&\Big(\frac{p_N(t_{j+1}) - p_N(t_j)}{h_N} - \lambda_N x^*_{Nj} \Big) + \varepsilon_N e^*_{N_j} \\ &\in \widehat{D}^* F_j \Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} \Big) \Big(\lambda_N v^*_{Nj} + \lambda_N \frac{\theta_{Nj}}{h_N} b^*_{Nj} - p_N(t_{j+1}) + \varepsilon_N \widetilde{e}^*_{N_j} \Big)\end{aligned}$$

hold for all $j = 0, \ldots, N-1$ and all $N \in I\!N$. It follows from the local Lipschitz continuity of ϑ assumed in (H3$'$) and from Proposition 1.85 that

$$\|(x^*_{Nj}, v^*_{Nj})\| \leq \ell_\vartheta \quad \text{for all} \quad j = 0, \ldots, N-1 \quad \text{and} \quad N \in I\!N \; ,$$

where ℓ_ϑ is a uniform Lipschitz modulus of $\vartheta(\cdot, \cdot, t)$ independent of $t \in [a, b]$. By the Lipschitz continuity of F in (H1) and the coderivative condition of Theorem 1.43 we get the estimates

$$\left\|\frac{p_N(t_{j+1})-p_N(t_j)}{h_N}-\lambda_N x^*_{Nj}+\varepsilon_N e^*_{N_j}\right\|$$

$$\le \ell_F\left\|\lambda_N v^*_{Nj}+\lambda_N\frac{\theta_{Nj}}{h_N}b^*_{Nj}-p_N(t_{j+1})+\varepsilon_N\widetilde{e}^*_{N_j}\right\|$$

for $j=0,\ldots,N-1$. Similarly to the proof of Theorem 6.19 with taking $\|p_N(t_N)\|\le 1$ into account, we derive from these estimates that $p_N(t)$ is uniformly bounded on $[a,b]$ and that

$$\|\dot p_N(t)\|\le\alpha+\beta\|\theta_N(t)\|\ \text{ a.e. }\ t\in[a,b]$$

with some positive numbers α and β independent of N. Since both spaces X and X^* have the RNP, it follows from the Dunford theorem on the weak compactness in $L^1\big([a,b];X^*\big)$ that a subsequence of $\{\dot p_N(\cdot)\}$ converges to some $v(\cdot)\in L^1\big([a,b];X^*\big)$ weakly in this space. Employing the weak continuity of the Bochner integral as a linear operator from $L^1\big([a,b];X^*\big)$ to X^* and the estimate $\|p_N(b)\|\le 1$, we conclude that there is an absolutely continuous mapping $p\colon[a,b]\to X^*$ satisfying

$$p(t):=p(b)+\int_t^b v(s)\,ds,\quad a\le t\le b\;,$$

where $p(b)$ is a limiting point of $\{p_N(b)\}$ in the weak* topology of X^*, and such that the values $p_N(t)$ converge to $p(t)$ weakly in X^* (and hence weak* in this space) for all $t\in[a,b]$. Furthermore, $\dot p_N(\cdot)\to\dot p(\cdot)=v(t)$ in the weak topology of $L^1\big([a,b];X^*\big)$. Then the classical Mazur theorem ensures that some sequence of *convex combinations* of $\{\dot p_N(\cdot)\}$ converges to $\dot p(\cdot)$ strongly in $L^1\big([a,b];X^*\big)$ as $N\to\infty$, and hence (passing to a subsequence with no relabeling) it converges to $\dot p(t)$ almost everywhere on $[a,b]$.

Given any $N\in I\!\!N$, the approximate Euler-Lagrange inclusion (6.41) can be rewritten as

$$\dot p_N(t)\in\Big\{u\in X^*\Big|\ \big(u,\,p_N(t_{j+1})-\lambda_N\theta_N(t)\big)\in\lambda_N\widehat\partial\vartheta(\bar x_N(t_j),\dot{\bar x}_N(t),t_j)$$

$$+\widehat N\big((\bar x_N(t_j),\dot{\bar x}_N(t));\text{gph}\,F(t_j)\big)+\varepsilon_N I\!\!B^*\Big\}$$

for $t\in[t_j,t_{j+1})$ with $j=0,\ldots,N-1$. Now passing to the limit as $N\to\infty$ and using the *pointwise* convergence results established below, we arrive at the extended Euler-Lagrange inclusion (6.44).

To derive the transversality inclusion (6.45), we take the limit in the discrete ones (6.42) as $N\to\infty$. The only thing to clarify is the possibility to pass from Fréchet normals to $\Omega_N=\Omega+\eta_N I\!\!B$ to the basic normals to Ω. The latter can be easily done by using the sum rule from Theorem 3.7(i) and the fact that $\eta_N\downarrow 0$ as $N\to\infty$.

It remains to justify the nontriviality condition $\big(\lambda,p(\cdot)\big)\ne 0$. Assuming that $\lambda=0$, one may put $\lambda_N=0$ for all $N\in I\!\!N$ without loss of generality.

We need to show that $p(\cdot)$ is not identically equal to zero on $[a,b]$. Suppose the contrary, i.e., $p(t) = 0$ whenever $t \in [a,b]$. Then it follows from the above proof that $p_N(t) \xrightarrow{w^*} 0$ for all $t \in [a,b]$; in particular, $p_N(t_0) \xrightarrow{w^*} 0$ and $p_N(t_N) \xrightarrow{w^*} 0$ as $N \to \infty$. The discrete transversality inclusion (6.42) is written in this case as

$$\big(p_N(t_0), -p_N(t_N)\big) \in \widehat{N}\big((\bar{x}_N(t_0), \bar{x}_N(t_N)); \Omega + \eta_N I\!B\big) + \varepsilon_N I\!B^* \, . \tag{6.46}$$

Using again Theorem 3.7(i) for the Fréchet normals cone to the sum in (6.46) and then employing the strong PSNC property of Ω at $\big(\bar{x}(a), \bar{x}(b)\big)$ with respect to the second component, we get $\|p_N(t_N)\| \to 0$ as $N \to \infty$, which contradicts the nontriviality condition (6.42) in Theorem 6.19 and completes the proof of this theorem. △

The next theorem gives necessary optimality conditions in the extended Euler-Lagrange form for the original Bolza problem (P) derived by passing to the limit from the approximate necessary optimality in the discrete-time problems (P_N). In contrast to Theorem 6.21, this theorem applies to the *summable* integrands $\vartheta(x, v, \cdot)$ and gives a better form of the Euler-Lagrange inclusion. On the other hand, it imposes more restrictive assumptions on the state space X in question. In the formulations and proof of this theorem we keep the same notational agreement as for Theorem 6.21 discussed above.

Theorem 6.22 (extended Euler-Lagrange conditions for relaxed local minimizers in Bolza problems with summable integrands). *Let $\bar{x}(\cdot)$ be a relaxed intermediate local minimizer for the Bolza problem (P) under assumptions* (H1), (H2), (H3′), *and* (H4′). *Suppose also that the space X is reflexive and separable and that the set Ω is strongly PSNC at $\big(\bar{x}(a), \bar{x}(b)\big)$ with respect to the second component. Then there are a number $\lambda \geq 0$ and an absolutely continuous mapping $p\colon [a,b] \to X^*$, not both zero, satisfying the extended Euler-Lagrange inclusion*

$$\dot{p}(t) \in \operatorname{co}\Big\{u \in X^* \Big| \begin{array}{l} \big(u, p(t)\big) \in \lambda\partial\vartheta\big(\bar{x}(t), \dot{\bar{x}}(t), t\big) \\ \qquad + N_+\big((\bar{x}(t), \dot{\bar{x}}(t)); \operatorname{gph} F(t)\big)\end{array}\Big\} \tag{6.47}$$

for a.e. $t \in [a,b]$ and the transversality inclusion (6.45).

Proof. We follow the lines in the proof of Theorem 6.21 using the sequence of discrete approximation problems (P_N) instead of $(\overline{P}_N)$. The only difference is in the justification of the extended Euler-Lagrange inclusion (6.47) in comparison with (6.44) that are based on generally different discrete-time counterparts (6.43) and (6.41) under somewhat different assumptions.

To proceed, we suppose for notation convenience that the discrete Euler-Lagrange inclusions (6.43) hold as $N \to \infty$ without taking the closure of the set-valued integral therein; this doesn't restrict the generality as follows from the proof below. Then, by (6.43) and the definition of the Fréchet coderivative, there are dual elements

$$(x^*_{Nj}, v^*_{Nj}) \in \int_{t_j}^{t_{j+1}} \partial\vartheta\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}, t\Big)\, dt, \quad j = 0, \ldots, N-1\,,$$

as well as $e^*_{Nj}, \widetilde{e}^*_{Nj} \in I\!B^*$ satisfying the inclusions

$$\Big(\frac{p_N(t_{j+1}) - p_N(t_j)}{h_N} - \lambda_N x^*_{Nj}\Big) + \varepsilon_N e^*_{N_j}$$

$$\in \widehat{D}^* F_j\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N}\Big)\Big(\lambda_N v^*_{Nj} + \lambda_N \frac{\theta_{Nj}}{h_N} b^*_{Nj} - p_N(t_{j+1}) + \varepsilon_N \widetilde{e}^*_{N_j}\Big)$$

that are fulfilled for all $j = 0, \ldots, N-1$ along a sequence of $N \to \infty$; put below $N \in I\!N$ for simplicity. Following the proof of Theorem 6.21, we find an absolutely continuous mapping $p\colon [a,b] \to X^*$ such that $p_N(t) \to p(t)$ weakly in X^* for all $t \in [a,b]$ and a sequence of convex combinations of $\dot{p}_N(t)$ converges to $\dot{p}(t)$ almost everywhere on $[a,b]$ as $N \to \infty$. Then rewrite the above discrete-time inclusions in the form

$$\dot{p}_N(t) \in \Big\{ u \in X^* \Big|\ \big(u, p_N(t_{j+1}) - \lambda_N \theta_N(t)\big) \in \frac{\lambda_N}{h_N}(x^*_{Nj}, v^*_{Nj})$$

$$+ \widehat{N}\big((\bar{x}_N(t_j), \dot{\bar{x}}_N(t)); \operatorname{gph} F(t_j)\big) + \varepsilon_N I\!B^* \Big\}$$

for $t \in [t_j, t_{j+1})$ with $j = 0, \ldots, N-1$. By the construction of (x^*_{Nj}, v^*_{Nj}) there are summable mappings $u^*_{Nj}\colon [t_j, t_{j+1}] \to X^*$ and $w^*_{Nj}\colon [t_j, t_{j+1}] \to X^*$ satisfying the relations

$$\big(u^*_{Nj}(t), w^*_{Nj}(t)\big) \in \partial\vartheta\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_j) - \bar{x}_N(t_{j+1})}{h_N}, t\Big) \ \text{ a.e. } \ t \in [t_j, t_{j+1}]\,,$$

$$\frac{(x^*_{Nj}, v^*_{Nj})}{h_N} = \frac{1}{h_N}\int_{t_j}^{t_{j+1}} \big(u^*_{Nj}(t), w^*_{Nj}(t)\big)\, dt \ \text{ for } \ j = 0, \ldots, N-1\,.$$

Define the sequences of mappings $u^*_N\colon [a,b] \to X^*$ and $w^*_N\colon [a,b] \to X^*$ on the whole interval $[a,b]$ by

$$\big(u^*_N(t), w^*_N(t)\big) := \big(u^*_{Nj}(t), w^*_{Nj}(t)\big) \ \text{ for } \ t \in [t_j, t_{j+1}), \quad j = 0, \ldots, N-1\,.$$

Since $u^*_N(\cdot)$ and $w^*_N(\cdot)$ are integrable bounded on $[a,b]$, there are subsequences of them that converge, by the Dunford theorem, to some $u^*(\cdot)$ and $w^*(\cdot)$ in the weak topology of $L^1\big([a,b]; X^*\big)$. Invoking again the Mazur weak closure theorem and using the strong convergence of $\bar{x}_N(\cdot) \to \bar{x}(\cdot)$ from Theorem 6.13, one has the relations

$$\big(u^*(t), w^*(t)\big) \in \operatorname{clco} \partial\vartheta\big(\bar{x}(t), \dot{\bar{x}}(t), t\big) = \operatorname{co} \partial\vartheta\big(\bar{x}(t), \dot{\bar{x}}(t), t\big) \ \text{ a.e. } t \in [a, b] \ ,$$

where the *closure operation can be omitted* due to the reflexivity of X and the compactness of $\operatorname{co} \partial\vartheta(\bar{x}(t), \dot{\bar{x}}(t), t)$ in the weak topology of X^*, and hence this set is closed in the strong topology of X. Employing now the infinite-dimensional counterpart of the Lyapunov-Aumann theorem mentioned in the proof of Lemma 6.18, the well-known property

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f(s)\, ds = f(t) \ \text{ a.e. } \ t \in [a, b]$$

of the Bochner integral, and also the weak closedness of the basic subdifferential for locally Lipschitzian functions on reflexive spaces (cf. Theorem 3.59), we conclude that there are subgradients $\big(x^*(t), v^*(t)\big)$ of $\vartheta(\cdot, \cdot, t)$ such that

$$\frac{\lambda_N}{h_N}(x^*_{Nj}, v^*_{Nj}) \xrightarrow{w^*} \big(x^*(t), v^*(t)\big) \in \partial\vartheta\big(\bar{x}(t), \dot{\bar{x}}(t), t\big) \ \text{ a.e. } \ t \in [a, b] \ .$$

Passing finally to the limit in the above inclusions for $\dot{p}_N(\cdot)$ as $N \to \infty$, we arrive at the desired extended Euler-Lagrange inclusion (6.47), where the closure operation can be dropped in the reflexive case under consideration due to the uniform boundedness of $p_N(\cdot)$ and $\dot{p}_N(\cdot)$; see the discussion above. Note that it is sufficient to use the basic subdifferential in the integrand $\vartheta(\cdot, \cdot, t)$ in (6.47), but not the extended one as in (6.44), in the case under consideration. Thus we complete the proof of the theorem. △

The nontriviality condition in both Theorems 6.21 and 6.22 ensures that the pair $\big(\lambda, p(\cdot)\big)$ satisfying the Euler-Lagrange and transversality inclusions is not zero. The next result presents additional assumptions under which we have the *enhanced* nontriviality conditions: $\big(\lambda, p(b)\big) \neq 0$.

Corollary 6.23 (extended Euler-Lagrange conditions with enhanced nontriviality). *Let $\bar{x}(\cdot)$ be an r.i.l.m. for the Bolza problem (P). In addition to the assumptions in Theorems 6.21 and 6.22, respectively, suppose that*

(a) *either $\Omega = \Omega_a \times \Omega_b$, where Ω_b is SNC at $\bar{x}(b)$;*

(b) *or Ω is strongly PSNC at $\big(\bar{x}(a), \bar{x}(b)\big)$ relative to the second component, $F(\cdot, t)$ is strongly coderivatively normal at $\big(\bar{x}(t), \dot{\bar{x}}(t)\big)$, and $\operatorname{gph} F(t)$ is normally semicontinuous at this point for a.e. $t \in [a, b]$.*

Then one has the extended Euler-Lagrange and transversality inclusions (6.44) *and* (6.45) $\big($*respectively,* (6.47) *and* (6.45)$\big)$ *with the replacement of*

$$N_+\big((\bar{x}(t), \dot{\bar{x}}(t)); \operatorname{gph} F(t)\big) \quad \text{by} \quad N\big((\bar{x}(t), \dot{\bar{x}}(t)); \operatorname{gph} F(t)\big)$$

in case (b) *and with the enhanced nontriviality condition $\lambda + \|p(b)\| = 1$.*

Proof. Following the (same) proof of the nontriviality condition in Theorems 6.21 and 6.22, one has the transversality inclusion (6.46) for the adjoint trajectories $p_N(\cdot)$ in the discrete approximations with $\lambda_N = 0$. Assuming (a), we arrive at

$$-p_N(t_N) \in \widehat{N}\big(\bar{x}_N(t_N); \Omega_b + \eta I\!B\big) + \varepsilon_N I\!B^* \quad \text{as} \quad N \to \infty ,$$

which implies, by Theorem 3.7(i) and the SNC property of Ω_b at $\bar{x}(b)$, that $\|p_N(t_N)\| \to 0$ whenever $p_N(t_N) \stackrel{w^*}{\to} 0$ as $N \to \infty$. This clearly contradicts the nontriviality condition for the discrete-time problems $(\overline{P}_N)$ and (P_N) from Theorems 6.19 and 6.20, respectively.

It remains to justify the nontriviality condition $\lambda + \|p(b)\| \neq 0$ in case (b). It follows from the fact that, under the assumptions made in (b), $p(t) = 0$ for all $t \in [a, b]$ whenever $p(\cdot)$ satisfies the extended Euler-Lagrange inclusion (6.44) with $\lambda = 0$ and $p(b) = 0$. Indeed, invoking the normal semicontinuity of $\operatorname{gph} F(t)$ in this case, we write (6.44) as

$$\dot{p}(t) \in \operatorname{clco}\Big\{u \in X^* \Big|\ \big(u, p(t)\big) \in N\big((\bar{x}(t), \dot{\bar{x}}(t)); \operatorname{gph} F(t)\big)\Big\} \ \text{ a.e. } \ t \in [a, b]$$

that is equivalent, by the strong coderivative normality assumption in (b), to

$$\dot{p}(t) \in \operatorname{clco} D^*_M F\big(\bar{x}(t), \dot{x}(t)\big)\big(-p(t)\big) \ \text{ a.e. } \ t \in [a, b] .$$

The latter clearly implies, due to the mixed coderivative condition for the Lipschitz continuity from Theorem 1.44, that

$$p(t) \equiv 0 \ \text{ on } \ [a, b] \ \text{ when } \ p(b) = 0 ,$$

which completes the proof of the corollary. △

If X is *finite-dimensional*, any set is SNC and any mapping $F\colon X \rightrightarrows X$ is strongly coderivatively normal at every point. Thus we automatically have the extended Euler-Lagrange conditions in Theorem 6.22 and Corollary 6.23. Another setting that *doesn't require any SNC/PSNC assumptions* on the constraint set Ω is the case of endpoint constraints given by a *finite number of equalities and inequalities* with locally Lipschitzian functions considered next.

Corollary 6.24 (extended Euler-Lagrange conditions for problems with functional endpoint constraints). *Let the endpoint constraint set Ω in problem (P) be given by*

$$\Omega := \Big\{(x_a, x_b) \in X^2 \Big|\ \varphi_i(x_a, x_b) \le 0, \quad i = 1, \ldots, m ,$$

$$\varphi_i(x_a, x_b) = 0, \quad i = m+1, \ldots, m+r\Big\} ,$$

where each φ_i is locally Lipschitzian around $\big(\bar{x}(a), \bar{x}(b)\big)$ together with the cost function $\varphi_0 := \varphi$. Suppose that all the assumptions of Corollary 6.23 *hold except those related to the SNC/PSNC properties of Ω. Then there are nonnegative multipliers* $(\lambda_0, \ldots, \lambda_{m+r}) \neq 0$ *with*

$$\lambda_i \varphi_i\big(\bar{x}(a), \bar{x}(b)\big) = 0, \quad i = 1, \ldots, m\ ,$$

and an absolutely continuous adjoint arc $p\colon [a,b] \to X^*$ *satisfying the extended Euler-Lagrange inclusions mentioned therein as well as the following transversality condition:*

$$\begin{aligned}\big(p(a), -p(b)\big) \in &\sum_{i=0}^{m} \lambda_i \partial\varphi_i\big(\bar{x}(a), \bar{x}(b)\big)\\ &+ \sum_{i=m+1}^{m+r} \lambda_i \Big[\partial\varphi_i\big(\bar{x}(a), \bar{x}(b)\big) \bigcup \partial\big(-\varphi_i\big)\big(\bar{x}(a), \bar{x}(b)\big)\Big]\ .\end{aligned}$$

If, in particular, all φ_i *are strictly differentiable at* $\big(\bar{x}(a), \bar{x}(b)\big)$, *then there are* $(\lambda_0, \ldots, \lambda_{m+r}) \neq 0$ *satisfying the above complementary slackness condition and the standard sign condition*

$$\lambda_i \geq 0 \ \ \textit{for} \ \ i = 0, \ldots, m$$

and such that the transversality condition

$$\big(p(a), -p(b)\big) = \sum_{i=0}^{m+r} \lambda_i \nabla\varphi_i\big(\bar{x}(a), \bar{x}(b)\big)$$

supplements the corresponding Euler-Lagrange inclusion of Corollary 6.23.

Proof. Suppose first that the locally Lipschitzian functions $\varphi_1, \ldots, \varphi_{m+r}$ satisfy the nonsmooth counterpart of the Mangasarian-Fromovitz constraint qualification formulated in Theorem 3.86. Then the constraint set Ω defined in this corollary is SNC at $\big(\bar{x}(a), \bar{x}(b)\big)$. Furthermore, it follows from the calculus rule of Theorem 3.8 specified for $F := (\varphi_1, \ldots, \varphi_{m+r})$ and

$$\begin{aligned}\Theta := \Big\{(\alpha_1, \ldots, \alpha_{m+r}) \in I\!R^{m+r} \Big|\ &\alpha_i \leq 0, \quad i = 1, \ldots, m\ ,\\ &\alpha_i = 0, \quad i = m+1, \ldots, m+r\Big\}\end{aligned}$$

therein that the same constraint qualification ensures the inclusion

$$\begin{aligned}N(\bar{z}; \Omega) \subset \Big\{ &\sum_{i=1}^{m} \lambda_i \partial\varphi_i(\bar{z}) + \sum_{i=m+1}^{m+r} \lambda_i \Big[\partial\varphi_i(\bar{z}) \bigcup \partial\big(-\varphi_i\big)(\bar{z})\Big] \Big|\\ &\lambda_i \geq 0, \quad i = 1, \ldots, m+r; \quad \lambda_i\varphi_i(\bar{z}) = 0, \quad i = 1, \ldots, m\Big\}\end{aligned}$$

for basic normals to the constraint set Ω at the point $\bar{z} := \big(\bar{x}(a), \bar{x}(b)\big)$. Then the transversality inclusion formulated at this corollary follows from (6.45) with $\lambda_0 = \lambda$, where the nontriviality condition $\big(\lambda, p(b)\big) \neq 0$ is equivalent to $(\lambda_0, \ldots, \lambda_{m+r}) \neq 0$. Assuming finally that the qualification conditions of Theorem 3.86 don't hold, we immediately arrive at the desired transversality inclusion with $(\lambda_1, \ldots, \lambda_{m+r}) \neq 0$ and complete the proof. △

Note that the enhanced nontriviality condition $\big(\lambda_0, p(b)\big) \neq 0$, inspired by the one in Corollary 6.23, may *not* hold in the framework of Corollary 6.24 if the constraint set Ω is not SNC (or strongly PSNC); in particular, when the Mangasarian-Fromovitz type constraint qualification of Theorem 3.86 is not fulfilled. It may happen, for instance, for a *two-point boundary problem* with $x(a) = x_0$ and $x(b) = x_1$ involving smooth parabolic systems of optimal control; see the well-known examples in Fattorini [432] and Li and Yong [789]. On the other hand, the SNC requirement is met in case (a) of Corollary 6.23 when $x(a) = x_0$ and $x(b) \in x_1 + r I\!B$ with $r > 0$, since the latter ball is always SNC (it is actually epi-Lipschitzian by Proposition 1.25).

Observe also that, using the *smooth variational description* of Fréchet subgradients similarly to the proof of Theorem 5.19 for nondifferentiable programming and employing the results of Corollary 6.24 in the case of smooth endpoint functions, we can derive counterparts of Theorems 6.21 and 6.22 with *upper subdifferential* transversality conditions; see Remark 6.30 for the exact formulation and more details.

To conclude this section, let us discuss some particular issues mostly related to the above Euler-Lagrange conditions for differential inclusions with infinite-dimensional state spaces.

Remark 6.25 (discussion on the Euler-Lagrange conditions).

(i) It follows from the proof of Theorems 6.21 and 6.22 that the strong PSNC assumption imposed on Ω to ensure the nontriviality condition may be replaced by the following *alternative* assumption on F written as: there is $t \in [a, b]$ such that for any sequences $t_k \to t$, $x_k \to \bar{x}(t)$, $v_k \in F(x_k, t_k)$, and $(x_k^*, v_k^*) \in \widehat{N}\big((x_k, v_k); \operatorname{gph} F(t_k)\big)$ one has

$$(x_k^*, v_k^*) \xrightarrow{w^*} (0, 0) \Longrightarrow \|v_k^*\| \to 0 \quad \text{as} \quad k \to \infty \, .$$

This property is closely related to the *strong PSNC* property of F at $(\bar{x}(t), t)$ with respect to the *image component*; cf. also its SNC analog for moving sets in Definition 5.71.

(ii) Recall that the SNC property of *convex sets* with nonempty relative interiors is *equivalent* by Theorem 1.21 to the *finite codimension* property of their closed affine hulls. The *strong PSNC* property may be essentially weaker than the SNC one; see, e.g., Theorem 1.75.

(iii) If the velocity sets $F(x, t)$ and the integrand $\vartheta(x, \cdot, t)$ are *convex* around the given local minimizer, then the Euler-Lagrange inclusion of Theorem 6.21 easily implies the *Weierstrass-Pontryagin maximum condition*

$$\langle p(t), \dot{\bar{x}}(t) \rangle - \lambda\vartheta\big(\bar{x}(t), \dot{\bar{x}}(t), t\big) = \max_{v \in F(\bar{x}(t),t)} \Big\{ \langle p(t), v \rangle - \lambda\vartheta\big(\bar{x}(t), v, t\big) \Big\}$$

for a.e. $t \in [a, b]$. It can be directly derived from the extremal property of the coderivative of convex-valued mappings in Theorem 1.34. The latter is the underlying condition of the results unified under the label "(Pontryagin) maximum principle" in optimal control. It will be shown in the next subsection that the maximum condition supplements, at least in the case of reflexive and separable state spaces under some additional assumptions, the extended Euler-Lagrange inclusion with *no convexity* requirements. To this end we note that the SNC (actually strong PSNC) properties required in Theorems 6.21 and 6.22 may be viewed as *nonconvex counterparts* of *finite codimension* requirements in the theory of necessary optimality conditions for controlled *evolution equations* of type (6.2) and their *PDE specifications* known in the case of *smooth* velocity mappings f and *convex* constraint/target sets Ω; cf. the afore-mentioned books by Fattorini [432] and Li and Yong [789] with the references and discussions therein.

Remark 6.26 (optimal control of semilinear unbounded differential inclusions). Many important models involving *semilinear partial differential equations* can be appropriately described by $\mathcal{C}_0$ *semigroups*; we again refer to the books by Fattorini [432] and Li and Yong [789] as well as to the subsequent material of Sects. 7.2–7.4 in this book. In this way an analog of the optimal control problem (P) from this section can be considered with the replacement of the differential inclusion (6.1) by the evolution model

$$\dot{x}(t) \in Ax(t) + F\big(x(t), t\big) ,$$

where A is an *unbounded* infinitesimal generator of a *compact* $\mathcal{C}_0$ semigroup on X, and where *continuous* solutions $x(\cdot)$ to this inclusion are understood in the *mild* sense. The latter means that there is a Bochner integrable mapping $v(\cdot) \in L^1\big([a, b]; X\big)$ such that

$$v(t) \in F\big(x(t), t\big) \quad \text{a.e. } \ t \in [a, b] \quad \text{and}$$

$$x(t) = e^{A(t-a)}x(a) + \int_a^t e^{A(t-s)}v(s)\,ds, \qquad t \in [a, b] .$$

Developing the above approach in the case of the *Mayer cost functional*

$$\text{minimize } \ \varphi\big(x(a), x(b)\big) \ \text{ with } \ \big(x(a), x(b)\big) \in \Omega \subset X^2 ,$$

we derive necessary optimality conditions under the additional *convexity* assumption of the velocity sets $F(x, t)$ around the optimal solution. Then the *extended Euler-Lagrange inclusion* in the case of reflexive and separable state spaces X and autonomous systems (for simplicity) is formulated as follows:

$$\begin{cases} p(t) \in e^{A^*(b-t)}p(b) \\ \qquad + \displaystyle\int_b^t \Big\{ e^{A^*(s-t)} D_N^* F\big(\bar{x}(s), v\big)\big(-p(s)\big) \Big| \; v \in M\big(\bar{x}(s), p(s)\big) \Big\}\, ds \end{cases}$$

for all $t \in [a, b]$, where $p\colon [a, b] \to X^*$ is a *continuous* mapping satisfying the *transversality* and *nontriviality* conditions

$$\big(p(a), -p(b)\big) \in \lambda \partial\varphi\big(\bar{x}(a), \bar{x}(b)\big) + N\big((\bar{x}(a), \bar{x}(b)); \Omega\big), \quad \lambda + \|p(b)\| \neq 0$$

with $\lambda \geq 0$, where the *argmaximum sets* $M(x, p)$ are defined by

$$M(x, p) := \big\{ v \in F(x) \big|\; \langle p, v\rangle = \mathcal{H}(x, p) \big\}$$

with

$$\mathcal{H}(x, p) := \max\big\{ \langle p, v\rangle \big|\; v \in F(x) \big\}\ .$$

Moreover, the extended Euler-Lagrange inclusion implies in this case the *Weierstrass-Pontryagin maximum condition*

$$\langle p(t), \bar{v}(t)\rangle = \mathcal{H}\big(\bar{x}(t), p(t)\big) \ \text{ a.e. } \ t \in [a, b]$$

with a measurable mapping $\bar{v}(t) \in F\big(\bar{x}(t)\big)$ satisfying

$$p(t) \in e^{A^*(b-t)}p(b) + \int_b^t \Big\{ e^{A^*(s-t)} D_N^* F\big(\bar{x}(s), \bar{v}(s)\big)\big(-p(s)\big) \Big\}\, ds, \quad t \in [a, b]\ ;$$

see Mordukhovich and D. Wang [970, 971] for proofs and more discussions on these and related results.

6.2 Necessary Optimality Conditions for Differential Inclusions without Relaxation

This section is mainly devoted to deriving necessary optimality conditions for nonconvex differential inclusions *without any relaxation* based on approximating the original constrained problem by a family of nonsmooth Bolza problems with no differential inclusions and no endpoint constraints. The extended Euler-Lagrange conditions for the latter class of *unconstrained* Bolza problems and the assumptions made allow essential specifications in comparison with the general results established in the preceding section. By passing to the limit, we obtain necessary optimality conditions of the Euler-Lagrange type for arbitrary (i.e., *non-relaxed*) intermediate minimizers for the original control problems with *reflexive and separable* state spaces. Moreover, they are supplemented by the *Weierstrass-Pontryagin maximum condition* valid in the general nonconvex setting. If the state space X is *finite-dimensional* and the velocity sets $F(x,t)$ are *convex*, the above Euler-Lagrange and maximum conditions are *equivalent* to the *extended Hamiltonian inclusion* expressed via a *partial convexification* of the basic subdifferential of the Hamiltonian function associated with $F(x,t)$. We also discuss various generalizations of the results obtained and present some illustrative examples.

6.2.1 Euler-Lagrange and Maximum Conditions for Intermediate Local Minimizers

The realization of the approach mentioned above requires some additional assumptions on the initial data in comparison with Theorem 6.22, while the a.e. continuity assumption on the velocity mapping $F(x,\cdot)$ can be replaced by its measurability; see below. Furthermore, it is more convenient in this section to consider the following *Mayer form* (P_M) of problem (P) studied in the preceding section, with a fixed left endpoint of feasible arcs:

$$\text{minimize } \varphi\big(x(b)\big) \text{ subject to } x(b) \in \Omega \subset X$$

over absolutely continuous trajectories of the differential inclusion

$$\dot{x}(t) \in F(x(t),t) \text{ a.e. } t \in [a,b], \quad x(a) = x_0 \,. \tag{6.48}$$

The general case of nonzero integrands f in the Bolza problem can be reduced to the Mayer one by standard state augmentation techniques. Note also that, since the state space X is assumed to be reflexive and separable in what follows, this notion of absolutely continuous solutions to (6.48) agrees with the one given in Definition 6.1.

We first formulate the assumptions on the set-valued mapping F in (6.48) that are *weaker* than those imposed in Theorem 6.22. Keeping assumption (H1) from Subsect. 6.1.1 on the compactness and Lipschitz continuity of F in x with possibly *summable* functions $m_F(\cdot)$ and $\ell_F(\cdot)$ on $[a,b]$ (although it may also be loosen in some directions by various standard reductions as, e.g., in [255, 261, 598, 1289]), we replace the a.e. continuity assumption (H2) by the *measurability* assumption on F in the time variable $t \in [a,b]$. Note that all the reasonable notions of measurability are *equivalent* for set-valued mappings with closed values in *separable* spaces (cf. the discussion in the proof of Lemma 6.18), which is the case in this section.

(H2′) $F(x,\cdot)$ is measurable on the interval $[a,b]$ uniformly in x on the open set $U \subset X$ taken from (H1).

We also weaken the continuity and Lipschitz continuity assumptions on the cost function $\varphi = \varphi(x)$ from (H4) and (H4′) observing that this leads to the modified (more general) transversality condition for the Mayer problem under consideration. Namely, we replace the latter assumptions by the following one:

(H4″) φ is l.s.c. around $\bar{x}(b)$ relative to Ω, which is suppose to be locally closed around this point.

On the other hand, the following theorem imposes the *additional* coderivative normality and SNC assumptions on F in comparison with Theorem 6.22 and Corollary 6.23. Observe that the coderivative form of the extended Euler-Lagrange inclusion given below is *equivalent* to the one from Corollary 6.23 for $\vartheta = 0$ *without* imposing the normal semicontinuity assumptions on $\operatorname{gph} F(t)$. In the rest of this subsection we study intermediate local minimizers of *rank one* from Definition 6.7. Recall that $\varphi_\Omega(\cdot) = \varphi(\cdot) + \delta(\cdot;\Omega)$ as usual.

Theorem 6.27 (Euler-Lagrange and Weierstrass-Pontryagin conditions for nonconvex differential inclusions). *Let $\bar{x}(\cdot)$ be an intermediate local minimizer for the Mayer problem (P_M) under assumptions* (H1), H2′), *and* (H4″)*. Suppose in addition that:*

(a) *the Banach space X is reflexive, separable, and admits an equivalent Kadec norm;*

(b) *the function φ_Ω is SNEC at $\bar{x}(b)$, and its epigraph is weakly closed;*

(c) *the mapping $F(\cdot,t)\colon X \rightrightarrows X$ is SNC at $\big(\bar{x}(t),\dot{\bar{x}}(t)\big)$, strongly coderivatively normal around this point, and its graph is weakly closed for a.e. $t \in [a,b]$.*

Then there exist a number $\lambda \ge 0$ and an absolutely continuous adjoint arc $p\colon [a,b] \to X^*$, not both zero, satisfying the Euler-Lagrange inclusion

$$\dot{p}(t) \in \operatorname{co} D_x^* F\big(\bar{x}(t),\dot{\bar{x}}(t),t\big)\big(-p(t)\big) \quad \text{a.e. } t \in [a,b]\,, \tag{6.49}$$

the Weierstrass-Pontryagin maximum condition

$$\big\langle p(t),\dot{\bar{x}}(t)\big\rangle = \max_{v \in F(\bar{x}(t),t)} \big\langle p(t),v\big\rangle \quad \text{a.e. } t \in [a,b]\,, \tag{6.50}$$

and the transversality inclusion

$$\big(-p(b),-\lambda\big) \in N\big((\bar{x}(b),\bar{\beta});\operatorname{epi}\varphi_\Omega\big)\,. \tag{6.51}$$

Moreover, (6.51) *always implies*

$$-p(b) \in \partial\big[\lambda\varphi + \delta(\cdot;\Omega)\big]\big(\bar{x}(b)\big) \tag{6.52}$$

being equivalent to the latter condition if φ is Lipschitz continuous around $\bar{x}(b)$ relative to Ω.

Proof. Consider the parametric functional

$$\theta_\beta(x) := \operatorname{dist}\big((x(b),\beta);\operatorname{epi}\varphi_\Omega\big) \text{ as } \beta \in I\!R$$

over feasible arcs/trajectories to the original differential inclusion (6.1) with no other constraints. In what follows we fix the open set $U \subset X$ from assumption (H1) regarding $\bar{x}(\cdot)$. For every $\beta \in I\!R$ one obviously has

$$\theta_\beta(\bar{x}) \le |\beta - \bar{\beta}|$$

whenever β is sufficiently close to $\bar{\beta} = \varphi(\bar{x}(b))$. Since $\bar{x}(\cdot)$ is an intermediate local minimizer for (P_M) and by the structure of $\theta_\beta(x)$, we get

$$\theta_\beta(x) > 0 \;\text{ for any }\; \beta < \bar{\beta}$$

whenever a trajectory $x(t)$ for (6.48) belongs to some $W^{1,1}$-neighborhood of the local minimizer under consideration and such that

$$x(t) \in U \;\text{ for all }\; t \in [a,b] \; .$$

Form now the space $\mathcal{X}$ of all the trajectories $x(\cdot)$ for (6.48) satisfying the only constraint $x(t) \in \operatorname{cl} U$ as $t \in (a,b]$ with the metric

$$d(x,y) := \int_a^b \|\dot{x}(t) - \dot{y}(t)\|\, dt \; .$$

It is easy to see, from Definition 6.1 of solutions to the original differential inclusion and standard properties of the Bochner integral, that the metric space $\mathcal{X}$ is complete and that the function $\theta_\beta(\cdot)$ is (Lipschitz) continuous on $\mathcal{X}$ for any $\beta \in I\!R$. It follows from the above constructions that for every $\varepsilon > 0$ there is $\beta_\varepsilon < \bar{\beta}$ such that $\beta_\varepsilon \to \bar{\beta}$ as $\varepsilon \downarrow 0$ and

$$0 \le \theta_\varepsilon(\bar{x}) < \varepsilon \le \inf_{x \in \mathcal{X}} \theta_\varepsilon(x) + \varepsilon \quad \text{with} \;\; \theta_\varepsilon := \theta_{\beta_\varepsilon} \; .$$

Applying the *Ekeland variational principle* from Theorem 2.26(i), we find an arc $x_\varepsilon(\cdot) \in \mathcal{X}$ satisfying

$$d(x_\varepsilon, \bar{x}) \le \sqrt{\varepsilon} \;\text{ and }\; \theta_\varepsilon(x) + \sqrt{\varepsilon} d(x, x_\varepsilon) \ge \theta_\varepsilon(x_\varepsilon)$$

for all $x \in \mathcal{X}$. Note that the distance estimate above yields that $x_\varepsilon(t) \in U$ as $t \in (a,b]$ and that $x_\varepsilon(\cdot)$ belongs to the fixed $W^{1,1}$-neighborhood of the intermediate local minimizer $\bar{x}(\cdot)$ for small $\varepsilon > 0$. Hence $\theta_\varepsilon(x_\varepsilon) > 0$.

Next, given any $\alpha, \varepsilon > 0$ and the summable Lipschitz constant $\ell_F(\cdot)$ from (6.5), we define the Bolza-type functional

$$J_\varepsilon^\alpha[x] := \theta_\varepsilon(x) + \sqrt{\varepsilon} d(x, x_\varepsilon) + \alpha \int_a^b \sqrt{1 + \ell_F^2(t)}\, \operatorname{dist}\big((x(t), \dot{x}(t)); \operatorname{gph} F(t)\big)\, dt$$

on the sets of all absolutely continuous mappings $x\colon [a,b] \to X$, not necessarily trajectories for (6.48), satisfying $x(t) \in U$ as $t \in (a,b]$. To proceed, we need the following auxiliary result.

Claim. *There is a number $\alpha \ge 1$ such that for every $\varepsilon \in (0, 1/\alpha)$ the absolutely continuous mapping $x_\varepsilon : [a, b] \to X$ built above provides an intermediate local minimum for the Bolza functional J_ε^α subject to*

$$x(a) = x_0 \quad and \quad x(t) \in U \quad for \quad t \in (a, b] \; .$$

To prove this claim, we first observe that there are positive numbers ν, γ such that for every arc $y(\cdot)$ satisfying $y(a) = x_0$, $y(t) \in U$ as $t \in (a, b]$, and

$$\int_a^b \operatorname{dist}\big(\dot{y}(t); F(y(t), t)\big)\, dt < \nu$$

there exists a trajectory $x(\cdot)$ for (6.28) with

$$d(x, y) \le \gamma \int_a^b \sqrt{1 + \ell_F^2(t)}\, \operatorname{dist}\big((y(t), \dot{y}(t)); \operatorname{gph} F(t)\big)\, dt \; . \tag{6.53}$$

Indeed, this follows directly from Filippov's theorem on quasitrajectories of differential inclusions (see, e.g., Theorem 1 on p. 120 in Aubin and Cellina [50] whose proof holds true for infinite-dimensional inclusions under the assumptions made in (H1) and (H2$'$)) and from the estimate

$$\operatorname{dist}\big(v, F(u, t)\big) \le \sqrt{1 + \ell_F^2(t)}\, \operatorname{dist}\big((u, v); \operatorname{gph} F(t)\big)$$

that is obviously valid under (H1). Suppose now that the above claim doesn't hold. Then for each $k \in I\!N$ there are $\varepsilon_k \in (0, 1/k)$ and an arc $y_i(\cdot) \in \mathcal{X}$ satisfying $y_k(t) \in U$ as $t \in (a, b]$,

$$\max_{t \in [a,b]} \|y_k(t) - x_{\varepsilon_k}(t)\| + \int_a^b \|\dot{y}_k(t) - \dot{x}_{\varepsilon_k}(t)\|\, dt < \frac{1}{k} \; ,$$

and $J_{\varepsilon_k}^k[x_{\varepsilon_k}] > J_{\varepsilon_k}^k[y_k]$. Hence $y_k(\cdot) \to \bar{x}(\cdot)$ in the norm topology of $W^{1,1}\big([a, b]; X\big)$ and, moreover,

$$J_{\varepsilon_k}^k[x_{\varepsilon_k}] = \theta_{\varepsilon_k}(x_{\varepsilon_k}) \downarrow 0 \quad \text{as} \quad k \to \infty \; .$$

Therefore, given any $\nu > 0$, we get

$$\int_a^b \operatorname{dist}\big(\dot{y}_k(t); F(y_k(t), t)\big)\, dt < J_{\varepsilon_k}^k[x_{\varepsilon_k}] < \nu$$

for large k. This implies, by (6.53), that there are a number $\gamma > 0$ independent of k and trajectories $x_k(\cdot)$ for (6.28) as $k \to \infty$ such that

$$d(x_k, y_k) \le \gamma \int_a^b \sqrt{1 + \ell_F^2(t)}\, \operatorname{dist}\big((y_k(t), \dot{y}_k(t)); \operatorname{gph} F(t)\big)\, dt \; . \tag{6.54}$$

Since the right-hand side of (6.54) converges to zero and since $y_k(\cdot) \to \bar{x}(\cdot)$ strongly in $W^{1,1}\big([a,b];X\big)$, we get the strong $W^{1,1}$-convergence $x_k(\cdot) \to \bar{x}(\cdot)$ as $k \to \infty$, which ensures that all the trajectories $x_k(\cdot) \in \mathcal{X}$ belong to the fixed $W^{1,1}\big([a,b];X\big)$-neighborhood of the intermediate local minimizer $\bar{x}(\cdot)$ for large $k \in I\!\!N$. This gives

$$J^k_{\varepsilon_k}[x_k] \ge J^k_{\varepsilon_k}[x_{\varepsilon_k}] > J^k_{\varepsilon_k}[y_k] = \theta_{\varepsilon_k}(y_k) + \sqrt{\varepsilon_k} d(y_k, x_{\varepsilon_k})$$

$$+k\int_a^b \operatorname{dist}\big(\dot{y}_k(t); F(y_k(t),t)\big)\,dt =: k\xi_k \ .$$

Now taking into account (6.54) and the construction of θ_ε, we arrive at

$$k\xi_k < \sqrt{\varepsilon}_k\big(d(x_k, x_{\varepsilon_k}) - d(y_k, x_{\varepsilon_k})\big) + \theta_{\varepsilon_k}(x_k) - \theta_{\varepsilon_k}(y_k) \le 3\gamma\xi_k$$

for large k. This is a contradiction, which ends the proof of the claim.

Note that, since U is open in X, the constraint $x(t) \in U$ as $t \in (a,b]$ can be *ignored* from the viewpoint of necessary optimality conditions. Thus we may treat $x_\varepsilon(\cdot)$ is an intermediate local minimizer for the *unconstrained Bolza problem* with finite-valued and *Lipschitzian data*:

$$\text{minimize} \;\; \varphi_\varepsilon\big(x(b)\big) + \int_a^b \vartheta_\varepsilon(x(t), \dot{x}(t), t)\,dt \tag{6.55}$$

over absolutely continuous arcs $x\colon [a,b] \to X$ satisfying $x(a) = x_0$ and lying in a $W^{1,1}$-neighborhood of $\bar{x}(\cdot)$, where the endpoint cost function is given by

$$\varphi_\varepsilon(x) := \operatorname{dist}\big((x, \beta_\varepsilon); \operatorname{epi}\varphi_\Omega\big) \ , \tag{6.56}$$

and where the integrand is

$$\vartheta_\varepsilon(x,v,t) := \alpha\sqrt{1+\ell_F^2(t)}\operatorname{dist}\big((x,v);\operatorname{gph}F(t)\big) + \sqrt{\varepsilon}\|v - \dot{x}_\varepsilon(t)\| \ . \tag{6.57}$$

Note that *any intermediate local minimizer* for the unconstrained problem (6.55) provides a *relaxed* intermediate local minimum to this problem. It can be observed from the relaxation result in Theorem 6.11 and its "intermediate" modification given by Ioffe and Rockafellar in Theorem 4 of [616], which is valid in infinite dimensions under the assumptions made. Note also that assumptions (H1), (H2$'$), and (H3$''$) ensure that problem (6.55) with the data defined in (6.56) and (6.57) satisfies all the assumptions of Theorem 6.22 except for the compactness of the velocity sets in (P), which in fact is *not needed* in the unconstrained and $W^{1,1}$-bounded framework of (6.55); cf. the proof of Theorem 6.22 and the preceding results it is based on.

We now apply the necessary optimality conditions from Theorem 6.22 to problem (6.55) for any fixed $\varepsilon > 0$. Using the extended Euler-Lagrange inclusion (6.47) with the integrand ϑ_ε in (6.57) and then employing the sum rule from Theorem 2.33(c), find an absolutely continuous adjoint arc $p_\varepsilon\colon [a,b] \to X^*$ satisfying

$$\dot p_\varepsilon(t) \in \operatorname{co}\Big\{u \in X^* \Big|\ \big(u, p_\varepsilon(t)\big) \in \mu(t)\partial\operatorname{dist}\big((x_\varepsilon(t), \dot x_\varepsilon(t)); \operatorname{gph} F(t)\big) + \sqrt{\varepsilon}\big(0, I\!B^*\big)\Big\}$$

for a.e. $t \in [a,b]$ with $\mu(t) := \alpha\sqrt{1+\ell_F^2(t)}$. Fixed $t \in [a,b]$, consider the two cases regarding $\big(x_\varepsilon(t), \dot x_\varepsilon(t)\big)$:

$$\textbf{(i)}\ \ \dot x_\varepsilon(t) \in F(x_\varepsilon(t), t) \quad \text{and} \quad \textbf{(ii)}\ \ \dot x_\varepsilon(t) \notin F(x_\varepsilon(t), t)\ .$$

In case (i) we use Theorem 1.97 on basic subgradients of the distance function at set points, which gives the *approximate adjoint inclusion*

$$\dot p_\varepsilon(t) \in \operatorname{co}\Big\{u \in X^* \Big|\ \big(u, p_\varepsilon(t)\big) \in N\big((x_\varepsilon(t), \dot x_\varepsilon(t)); \operatorname{gph} F(t)\big) + \sqrt{\varepsilon}\big(0, I\!B^*\big)\Big\}\ .$$

Considering case (ii) and employing the first projection formula from Theorem 1.105 for basic subgradients of the distance function at out-of-set points under the Kadec norm structure of X assumed in (a) (see Corollary 1.106 of that theorem), we have the inclusion

$$\partial\operatorname{dist}\big((x_\varepsilon(t), \dot x_\varepsilon(t)); \operatorname{gph} F(t)\big) \subset \bigcup_{(x,v)\in\Pi\left((x_\varepsilon(t),\dot x_\varepsilon(t));\operatorname{gph}F(t)\right)} N\big((x,v); \operatorname{gph} F(t)\big)\ .$$

Taking now into account the pointwise convergence $\big(x_\varepsilon(t), \dot x_\varepsilon(t)\big) \to \big(\bar x(t), \dot{\bar x}(t)\big)$ as $\varepsilon \downarrow 0$, one has

$$\partial\operatorname{dist}\big((x_\varepsilon(t), \dot x_\varepsilon(t)); \operatorname{gph} F(t)\big) \subset N\big((\widetilde x_\varepsilon, \widetilde v_\varepsilon); \operatorname{gph} F(t)\big)$$

for some $(\widetilde x_\varepsilon, \widetilde v_\varepsilon) \in \operatorname{gph} F(t)$ converging to $\big(\bar x(t), \dot{\bar x}(t)\big)$ as $\varepsilon \downarrow 0$. Thus in case (ii) we get the *approximate adjoint inclusion*

$$\dot p_\varepsilon(t) \in \operatorname{co}\Big\{u \in X^* \Big|\ \big(u, p_\varepsilon(t)\big) \in N\big((\widetilde x_\varepsilon, \widetilde v_\varepsilon); \operatorname{gph} F(t)\big) + \sqrt{\varepsilon}\big(0, I\!B^*\big)\Big\}\ .$$

To derive the extended Euler-Lagrange inclusion (6.49) in problem (P_M), one needs to pass to the limit as $\varepsilon \downarrow 0$ in the approximate adjoint inclusions for $p_\varepsilon(\cdot)$ in both cases (i) and (ii). Since the two approximate adjoint inclusions are similar, we may consider only the first one for definiteness. Observe that

$$\operatorname*{Lim\,sup}_{\varepsilon\downarrow 0} N\big((x_\varepsilon(t), \dot x_\varepsilon(t)); \operatorname{gph} F(t)\big) = N\big((\bar x(t), \dot{\bar x}(t)); \operatorname{gph} F(t)\big)$$

by the pointwise convergence of $\big(x_\varepsilon(t), \dot x_\varepsilon(t)\big) \to \big(\bar x(t), \dot{\bar x}(t)\big)$ and the robustness property of the basic normal cone from Theorem 3.60 held due to the SNC assumption on F. Note also that the approximate adjoint inclusion for $p_\varepsilon(\cdot)$ can be equivalently rewritten via the *normal* coderivative of F and hence, by the strong coderivative normality assumption of the theorem, in terms of the *mixed* coderivative $D^*_M F$. Proceeding similarly to the proof of Theorem 6.21 with the use of the mixed coderivative condition for the Lipschitzian continuity from Theorem 1.44 as well as the classical Dunford and Mazur theorems as above, we surely arrive at (6.49).

Consider next the transversality inclusion for $p_\varepsilon(b)$ in problem (6.55) with the cost function φ_ε in (6.56). Employing the transversality condition (6.45) from Theorem 6.22 in this setting, we have just the first terms in (6.45), where $\lambda = 1$ and $\varphi(x_a, x_b) = \varphi_\varepsilon(x_b)$. The crucial condition

$$\operatorname{dist}\big((x_\varepsilon(b), \beta_\varepsilon); \operatorname{epi}\varphi_\Omega\big) > 0$$

ensures that $(x_\varepsilon(b), \beta_\varepsilon) \notin \operatorname{epi}\varphi_\Omega$ for all $\varepsilon > 0$ sufficiently small. Employing again Theorem 1.105/Corollary 1.106, one has

$$\big(-p_\varepsilon(b), -\lambda_\varepsilon\big) \in \bigcup_{(x,b)\in\Pi((x_\varepsilon,\beta_\varepsilon);\,\operatorname{epi}\varphi_\Omega)} N\big((x,\beta); \operatorname{epi}\varphi_\Omega\big)$$

with some $\lambda_\varepsilon \geq 0$. Moreover, we can put $\lambda_\varepsilon + \|p_\varepsilon(b)\| = 1$ due to the SNEC property of φ_Ω at $\bar{x}(b)$ and hence *around* this point; see Remark 1.27(ii). Passing to the limit as $\varepsilon \downarrow 0$ and taking into account the robustness result of Theorem 3.60, we arrive at the desired transversality inclusion (6.51) with $\lambda \geq 0$ by putting $\varepsilon \downarrow 0$. The nontriviality condition $\lambda + \|p(b)\| = 1$ follows from the one for $\big(\lambda_\varepsilon, p_\varepsilon(b)\big)$ due to the SNEC property of φ_Ω that surely holds if Ω is SNC at $\bar{x}(b)$ and φ is Lipschitz continuous around this point. The latter is an easy consequence of Theorem 3.90, which ensures even the stronger SNC property of φ at $\bar{x}(b)$. The equivalence between the transversality inclusions (6.51) and (6.52) whenever φ is locally Lipschitzian around $\bar{x}(b)$ relative to Ω follows from Lemma 5.23. Note that inclusion (6.52) further implies

$$-p(b) \in \lambda\partial\varphi\big(\bar{x}(b)\big) + N(\bar{x}(b); \Omega)$$

for Lipschitz continuous cost functions.

The above proof justifies the extended Euler-Lagrange and transversality conditions in the theorem for arbitrary intermediate local minimizers to problem (P_M) with *no relaxation.* In this general nonconvex setting the extended Euler-Lagrange inclusion (6.49) doesn't automatically imply the maximum condition (6.50). To establish the latter condition supplementing (6.49) and (6.51), we follow the proof of Theorem 7.4.1 in Vinter [1289] given for a Mayer problem of the type (P_M) involving nonconvex differential inclusions in finite-dimensional spaces. The proof of the latter theorem is based on reducing the constrained Mayer problem for nonconvex differential inclusions to an unconstrained Bolza (finite Lagrangian) problem, which in turn is reduced to a problem of optimal control with *smooth dynamics* admitting a direct way to derive the maximum principle; cf. also Sect. 6.3. One can check that the tools of infinite-dimensional variational analysis developed above and the assumptions made allow us to extend the given proof to the case of reflexive and separable spaces under consideration. In this way we establish the maximum condition (6.50) in addition to the other necessary optimality conditions of the theorem and complete the proof. $\triangle$

Remark 6.28 (necessary conditions for nonconvex differential inclusions under weakened assumptions). Some assumptions of Theorem 6.27, particularly those on the Kadec norm and on the weakly closed graph and epigraph in (a)–(c), can be relaxed under a certain modification of the proof. This concerns the application of necessary optimality conditions from Theorem 6.22 to the unconstrained Bolza problem (6.55). The latter conditions are expressed in terms of the basic/limiting constructions and then require the usage of the projection result from Corollary 1.106 to efficiently estimate basic subgradients of the distance function at out-of-set points under the mentioned assumptions. To avoid these extra requirements, one may apply first a *fuzzy* discrete approximation version of Theorem 6.27 to the unconstrained problem (6.55), involving Fréchet normals and subgradients as in the proof of Theorem 6.21, and then pass to the limit as $N \to \infty$ and $\varepsilon \downarrow 0$. In this way, the realization of which is more involved, we replace the usage of the distance function result of Corollary 1.106 via basic subgradients by its Fréchet subgradient counterpart from Theorem 1.103 that holds under milder assumptions.

Observe that the SNC and strong coderivative normality properties of F are automatic when X is *finite-dimensional*, which also implies the SNEC property of the extended endpoint function φ_Ω assumed in Theorem 6.27. Furthermore, the latter property is *not needed* (actually it holds *automatically* under qualification conditions of the Mangasarian-Fromovitz type) in the general infinite-dimensional case of the theorem if the cost function is locally Lipschitzian and the endpoint constraint set given via a *finite number* of equalities and inequalities defined by locally Lipschitzian functions.

Corollary 6.29 (transversality conditions for differential inclusions with equality and inequality constraints). *Let $\bar{x}(\cdot)$ be an intermediate local minimizer for the Mayer problem (P_M) with the endpoint constraint set*

$$\Omega := \left\{x \in X \,\middle|\, \varphi_i(x) \le 0,\ i = 1, \ldots, m;\ \ \varphi_i(x) = 0,\ i = m+1, \ldots, m+r\right\},$$

where each φ_i is locally Lipschitzian around $\bar{x}(b)$ together with the cost function $\varphi_0 := \varphi$. Suppose that all the assumptions of Theorem 6.27 hold except the SNEC property of the extended endpoint function φ_Ω. Then there are nonnegative multipliers $(\lambda_0, \ldots, \lambda_{m+r}) \ne 0$ and an absolutely continuous adjoint arc $p\colon [a,b] \to X^*$ satisfying the Euler-Lagrange and maximum conditions (6.49) and (6.50) together with the complementary slackness condition

$$\lambda_i \varphi_i\big(\bar{x}(b)\big) = 0 \ \ \textit{for} \ \ i = 1, \ldots, m$$

and the transversality inclusion

$$-p(b) \in \sum_{i=0}^{m} \lambda_i \partial\varphi_i\big(\bar{x}(b)\big) + \sum_{i=m+1}^{m+r} \lambda_i \Big[\partial\varphi_i\big(\bar{x}(b)\big) \bigcup \partial\big(-\varphi_i\big)\big(\bar{x}(b)\big)\Big] .$$

If furthermore all φ_i, $i = 0, \ldots, m + r$, are strictly differentiable at $\bar{x}(b)$, then there are multipliers $(\lambda_0, \ldots, \lambda_{m+r}) \neq 0$ with $\lambda_i \geq 0$ as $i = 0, \ldots, m$ and an adjoint arc $p\colon [a, b] \to X^*$ satisfying

$$-p(b) = \sum_{i=0}^{m+r} \lambda_i \nabla\varphi_i\big(\bar{x}(b)\big)$$

together with the above Euler-Lagrange, Weierstrass-Pontryagin, and complementary slackness conditions.

Proof. It follows from (6.52) with $\lambda := \lambda_0$ that

$$-p(b) \in \lambda_0 \partial\varphi_0\big(\bar{x}(b)\big) + N\big(\bar{x}(b); \Omega\big) .$$

Moreover, φ_Ω is SNEC at $\bar{x}(b)$ provided that Ω is SNC at this point; see Corollary 3.89. Then we proceed similarly to the proof of Corollary 6.24 and complete the proof of this corollary. △

6.2.2 Discussion and Examples

In this subsection we consider certain generalizations and variants of the above results, discuss some interrelations and examples. First note that the comprehensive generalized differential and SNC calculi developed in Chap. 3 allow us to derive various consequences and extensions of Theorem 6.27 in the case of *operator* endpoint constraints given by

$$x(b) \in F^{-1}(\Theta) \cap \Omega$$

with $F\colon X \rightrightarrows Y$ and $\Theta \subset Y$; cf. Sect. 5.1 for problems of mathematical programming. Let us discuss in more details some other important issues related to obtained necessary optimality conditions for differential inclusions.

Remark 6.30 (upper subdifferential transversality conditions). Suppose in addition to the assumptions of Theorem 6.21 that the space X admits a $\mathcal{C}^1$ Lipschitzian bump function; this is automatic under the reflexivity assumption on X in Theorems 6.22 and 6.27. Then employing the results of Sects. 6.1 and 6.2 together with the *smooth variational description* of Fréchet subgradients in Theorem 1.88(ii), we derive necessary optimality conditions for problems (P) and (P_M), as well as for their discrete-time counterparts, with transversality relations expressed via *upper subgradients* of functions that describe the objective and inequality constraints. This can be done by reducing them to the case of *smooth* functions describing the objective and inequality constraints; cf. the proof of Theorem 5.19 for nondifferentiable programming. Considering, in particular, the Mayer problem of minimizing $\varphi_0(x(b))$ over absolutely continuous trajectories $x\colon [a,b] \to X$ for the differential inclusion (6.48) subject to the endpoint constraints

$$\varphi_i\big(x(b)\big) \le 0, \quad i = 1,\ldots,m\ ,$$

under the assumptions made on F and X in Theorem 6.27 and *no* assumptions on φ_i, we have the following necessary optimality conditions for an *intermediate* local minimizer $\bar{x}(\cdot)$: given *every* set of Fréchet *upper subgradients* $x_i^* \in \widehat{\partial}^+\varphi_i\big(\bar{x}(b)\big)$, $i = 0,\ldots,m$, there are multipliers

$$(\lambda_0,\ldots,\lambda_m) \ne 0 \ \text{ with } \ \lambda_i \ge 0 \ \text{ for all } \ i = 0,\ldots,m$$

and an absolutely continuous mapping $p\colon [a,b] \to X^*$ satisfying the Euler-Lagrange and maximum conditions (6.49) and (6.50) together with

$$\lambda_i\varphi_i\big(\bar{x}(b)\big) = 0 \ \text{ for } \ i = 1,\ldots,m \ \text{ and } \qquad p(b) + \sum_{i=0}^m \lambda_i x_i^* = 0\ .$$

To justify these conditions via the above arguments, it remains to check the SNEC property of the extended endpoint function φ_Ω in Theorem 6.27 with

$$\Omega := \big\{x \in X \big|\ \varphi_i(x) \le 0, \ \ i = 1,\ldots,m\big\}$$

and the *smooth* data φ, φ_i. It follows from Corollary 3.87 ensuring the SNC property of the classical constraint set in nonlinear programming; cf. the proof of Corollaries 6.24 and 6.29.

Remark 6.31 (necessary optimality conditions for multiobjective control problems). The methods and results developed above can be extended to *multiobjective optimization* problems governed by differential inclusions. Given a mapping $f\colon X \to Z$ and a subset $\Theta \subset Z$ of a Banach space with $0 \in \Theta$, consider a multiobjective counterpart of the above Mayer problem (P_M), where the *generalized order* (f,Θ)-*optimality* of a trajectory $\bar{x}(\cdot)$ for (6.48) subject to $x(b) \in \Omega$ is understood in the sense that there is a sequence $\{z_k\} \subset Z$ with $z_k \to 0$ as $k \to \infty$ such that

$$f\big(x(b)\big) - f\big(\bar{x}(b)\big) \notin \Theta - z_k, \quad k \in I\!N\ ,$$

for any feasible trajectory $x(\cdot)$ from a $W^{1,1}\big([a,b];X\big)$-neighborhood of $\bar{x}(\cdot)$; cf. Definition 5.53 and the related discussions in Subsect. 5.3.1. Let

$$\mathcal{E}(f,\Omega,\Theta) = \big\{(x,z) \in X \times Z \big|\ f(x) - z \in \Theta, \ \ x \in \Omega\big\}$$

be the "generalized epigraph" of the restrictive mapping $f_\Omega = f + \Delta(\cdot;\Omega)$ with respect to the ordering set Θ. Taking a sequence $z_k \to 0$ from the above definition of the (f,Θ)-optimality for $\bar{x}(\cdot)$, we define the functions

$$\theta_k(x) := \operatorname{dist}\big((x, f(\bar{x}) - z_k); \mathcal{E}(f,\Omega,\Theta)\big), \quad k \in I\!N .$$

and proceed similarly to the proof of Theorem 6.27 with the replacement of $\theta_\beta(x)$ therein by the sequence of $\theta_k(x)$. In this way we arrive at necessary optimality conditions in the multiobjective control problem under consideration that are different from the ones in Theorem 6.27 only in transversality relations. Namely, suppose in addition to the assumptions on X and F in Theorem 6.27 that the space Z is WCG and Asplund and that the generalized epigraphical set $\mathcal{E}(f,\Omega,\Theta)$ is locally closed around $(\bar{x},\bar{z})$ and SNC at this point with $\bar{z} := f(\bar{x})$. Then there are an adjoint arc $p\colon [a,b] \to X^*$ and an adjoint vector $z^* \in N(0;\Theta)$, not both zero, satisfying the extended Euler-Lagrange inclusion (6.49), the Weierstrass-Pontryagin maximum condition (6.50), and the *transversality inclusion*

$$\big(-p(b), -z^*\big) \in N\big((\bar{x}(b),\bar{z}); \mathcal{E}(f,\Omega,\Theta)\big) .$$

The latter inclusion is equivalent, by Lemma 5.23, to

$$-p(b) \in \partial\langle z^*, f_\Omega\rangle(\bar{x}), \quad z^* \in N(0;\Theta)$$

if the mapping f is Lipschitz continuous around $\bar{x}$ relative to Ω and strongly coderivatively normal at this point, and if the sets Ω and Θ are locally closed around the points $\bar{x}$ and 0, respectively. Note that multiobjective optimal control problems of the above type but with respect to *closed preference relations* can be treated similarly; cf. Subsect. 5.3.4. In this way we can also derive necessary optimality conditions for multiobjective (as well as of the Mayer and Bolza types) optimal control problems governed by differential inclusions with *equilibrium constraints*, which are dynamic counterparts of MPEC and EPEC problems studied in Sect. 5.2 and Subsect. 5.3.5.

Remark 6.32 (Hamiltonian inclusions). When $X = I\!R^n$, an additional optimality condition can be obtained for *relaxed* intermediate local minimizers to problem (P_M) (as well as to (P) and the counterparts of these problems discussed in the preceding remarks), which is expressed via basic subgradients to the *Hamiltonian* function defined by

$$\mathcal{H}(x,p,t) := \sup\{\langle p, v\rangle \,\big|\, v \in F(x,t)\} .$$

It follows from Rockafellar's dualization theorem ([1162, Theorem 3.3]) that

$$\operatorname{co}\Big\{u \in I\!R^n \,\Big|\, (u,p) \in N\big((\bar{x},\bar{v}); \operatorname{gph} F\big)\Big\} = \operatorname{co}\Big\{u \in I\!R^n \,\Big|\, (-u,\bar{v}) \in \partial\mathcal{H}(\bar{x},p)\Big\}$$

if F is *convex-valued* and satisfies some requirements around $(\bar{x},\bar{v})$ that are automatic under the assumptions made on F in (H1); dependence on t is not important and is thus suppressed. The proof of the latter *dualization relationship* is essentially finite-dimensional; cf. also the proofs in Ioffe [604, Theorem 4] and in Vinter [1289, Theorem 7.6.5]. Since the Hamiltonian of the convexified inclusion (6.18) is obviously agrees with the original one $\mathcal{H}(x,p,t)$, we deduce from the above duality relation that the Euler-Lagrange inclusion (6.49) in Theorem 6.27 implies the *extended Hamiltonian inclusion*

$$\dot p(t)\in\operatorname{co}\Big\{u\in I\!\!R^n\Big|\ \big(-u,\dot{\bar x}(t)\big)\in\partial\mathcal{H}\big(\bar x(t),p(t),t\big)\Big\}\ \text{ a.e }\ t\in[a,b] \tag{6.58}$$

as a *necessary optimality condition* for *relaxed minimizers* in the case of finite-dimensional state spaces. Moreover, the Euler-Lagrange inclusion (6.49) and the Hamiltonian inclusion (6.58) are *equivalent* for problems (P_M) with the *convex* velocity sets $F(x,t)$. Note that (6.58) is a refined Hamiltonian inclusion involving a *partial convexification* of the basic subdifferential $\partial\mathcal{H}(\bar x(t),p(t),t)$, which clearly supersedes the *fully convexified* one

$$\big(-\dot p(t),\dot{\bar x}(t)\big)\in\operatorname{co}\partial\mathcal{H}(\bar x(t),p(t),t)\ \text{ a.e. }\ t\in[a,b] \tag{6.59}$$

involving Clarke's generalized gradient $\partial_C\mathcal{H}(\bar x(t),p(t),t)=\operatorname{co}\partial\mathcal{H}(\bar x(t),p(t),t)$ of the Hamiltonian with respect to (x,p). It is worth observing that both Hamiltonian inclusions (6.58) and (6.59) are *invariant* with respect to the convexification of $F(x,t)$, which is *not* the case for the extended Euler-Lagrange inclusion (6.49).

Remark 6.33 (local controllability). The approach developed in the preceding subsection for necessary optimality conditions allows us to study also related issues concerning the so-called *local controllability* of nonconvex differential inclusions in the case of *finite-dimensional* spaces. Given $x_0\in X$, we denote by $\mathcal{R}(x_0)$ the *reachable set* for the differential inclusion (6.48), which is the set of all $z\in X$ such that $x(b)=z$ for some arc $x\colon[a,b]\to X$ admissible to (6.48). The meaning of local controllability is to derive efficient conditions for *boundary trajectories* of the differential inclusion (6.48), in a certain generalized sense. To be more precise, we consider a mapping $g\colon X\to X$ locally Lipschitzian mapping around $\bar x(b)$ and a trajectory $\bar x\colon[a,b]\to X$ for (6.48) such that $g\big(\bar x(b)\big)\in\operatorname{bd}\mathcal{R}(x_0)$. Then assuming that $X=I\!\!R^n$ in addition to (H1) and (H2$'$), we find a vector $x^*\in I\!\!R^n$ with $\|x^*\|=1$ and an adjoint arc $p(\cdot)$ satisfying the extended Euler-Lagrange inclusion (6.49) with the *boundary/transversality condition*

$$-p(b)\in\partial\langle x^*,g\rangle\big(\bar x(b)\big) \tag{6.60}$$

and the Weierstrass-Pontryagin maximum condition (6.50). Moreover, if the reachable set $\mathcal{R}(x_0)$ is *locally closed* around $\bar x(b)$, then the extended Hamiltonian inclusion (6.58) is also satisfied.

To justify the Euler-Lagrange and maximum conditions (6.49) and (6.50) with the new transversality condition (6.60), we follow the proof of Theorem 6.27 and, given any $\varepsilon>0$, find a vector $c_\varepsilon\in I\!\!R^n$ and a trajectory $x_\varepsilon(\cdot)$ for (6.48) such that $\|g\big(x_\varepsilon(b)\big)-c_\varepsilon\|>0$,

$$c_\varepsilon \to g\big(\bar{x}(b)\big),\quad x_\varepsilon(\cdot) \to \bar{x}(\cdot)\ \text{ strongly in }\ W^{1,1}\big([a,b];I\!\!R^n\big)\ \text{ as }\ \varepsilon \downarrow 0\,,$$

and $x_\varepsilon(\cdot)$ is an unconditional *strong* local minimizer for problem (6.55) with the same integrand (6.57) and the endpoint function

$$\varphi_\varepsilon(z) := \|g(z) - c_\varepsilon\|\,.$$

Then we proceed as in the proof of Theorem 6.27 with the only difference that now we need to compute the basic subdifferential of the new function $\varphi_\varepsilon(\cdot)$ at the point $\bar{x}_\varepsilon(b)$ with $\|g\big(x_\varepsilon(b)\big) - c_\varepsilon\| > 0$. Using the subdifferential chain rule of Corollary 3.43 and then passing to the limit as $\varepsilon \downarrow 0$ while taking into account the *compactness of the unit sphere* in $I\!\!R^n$, we arrive at the transversality condition (6.60) that supplements (6.49) and (6.50). To justify the extended Hamiltonian inclusion (6.58), we observe that the assumptions made ensure the closedness of the reachable set $\widetilde{\mathcal{R}}(x_0)$ generated by the *convexified* differential inclusion

$$\dot{x}(t) \in \operatorname{co} F(x(t),t)\ \text{ a.e. }\ t \in [a,b],\quad x(a) = x_0$$

and the density of $\mathcal{R}(x_0)$ in $\widetilde{\mathcal{R}}(x_0)$; cf. Theorem 6.11. Thus the local closedness assumption on $\mathcal{R}(x_0)$ yields that $\bar{x}(b)$ is a boundary point of $\widetilde{\mathcal{R}}(x_0)$, and so (6.58) follows from the discussion in Remark 6.32.

Note that the finite dimensionality of the state space X is needed in the above proof for local controllability to guarantee the compactness of the dual unit sphere in the weak* topology of X^*, which never holds in infinite dimensions due to the fundamental Josefson-Nissenzweig theorem. Such a difference with the infinite-dimensional setting of Theorem 6.27 is due to the fact that in the proof of the latter theorem we actually applied the *exact extremal principle* to the local extremal system of sets $\mathcal{R}(x_0) \times \{\varphi(\bar{x}(b))\}$ and $\operatorname{epi}\varphi_\Omega$ (in the notation of Theorem 6.27) with the SNC assumption imposed on the second set in the extremal system. In the setting of local controllability we deal with the local extremal system of sets $\mathcal{R}(x_0)$ and $\{\bar{x}(b)\}$, where the second singleton set is *never SNC* in infinite dimensions. Observe however that we didn't explore in the proof of Theorem 6.27, as well as in the framework of local controllability, the possibility of imposing a *SNC requirement on the reachable set* $\mathcal{R}(x_0)$, which may lead to *alternative* assumptions ensuring the fulfillment of necessary optimality and local controllability conditions in infinite dimensions; cf. the result and discussion in Remark 6.25(i).

To conclude this section, we present some examples illustrating the results obtained and the relationships between them. First let us show that the *partial convexification* can *not* be avoided in both extended Euler-Lagrange and Hamiltonian inclusion (6.49) and (6.58).

Example 6.34 (partial convexification is essential in Euler-Lagrange and Hamiltonian optimality conditions). *There is a two-dimensional*

Mayer problem of minimizing a linear function over absolutely continuous trajectories of a convex-valued differential inclusion with no endpoint constraints such that analogs of the Euler-Lagrange inclusion (6.49) *and the Hamiltonian inclusion* (6.58) *with no* (*partial*) *convexification* "co" *therein don't hold as necessary optimality conditions.*

Proof. Consider the following Mayer problem for a convex-valued differential inclusion with $x = (x_1, x_2) \in I\!\!R^2$:

$$\begin{cases} \text{minimize} \;\; J[x] := x_2(1) \;\; \text{subject to} \\ \dot{x}_1 \in [-\nu, \nu], \quad x_1(0) = 0\;, \\ \dot{x}_2 = |x_1|, \quad x_2(0) = 0\;, \\ \;\text{for a.e.} \;\; t \in [0,1] \;\; \text{with some} \;\; \nu > 0\;. \end{cases}$$

It is easy to see that $\bar{x}(t) \equiv 0$ is the only optimal solution to this problem, and that an analog of the Euler-Lagrange inclusion (6.49) for the adjoint arc $(p(t), -1) \in I\!\!R^2$ without "co" therein gives, along this $\bar{x}(\cdot)$, the relation

$$\dot{p}(t) \in \big\{ -1, 1 \big\} \;\; \text{a.e.}\; t \in [0,1]$$

with the transversality condition $p(1) = 0$. Furthermore, the maximum condition, implied by the Euler-Lagrange inclusion in this case due to Theorem 1.34, takes the form

$$\langle p(t), \dot{\bar{x}}(t) \rangle = \max_{v \in [-\nu, \nu]} \langle p(t), v \rangle \;\; \text{a.e.} \;\; t \in [0,1]\;,$$

which yields that $p(t) \equiv 0$; a contradiction. Since $\mathcal{H}(p, x) = \nu \operatorname{sign} p - |x_1|$, the Hamiltonian inclusion

$$\big(-\dot{p}(t), \dot{\bar{x}}(t) \big) \in \partial \mathcal{H}\big(\bar{x}(t), p(t) \big) \;\; \text{a.e.} \;\; t \in [0,1]\;,$$

which is (6.58) with no "co" therein, leads to the same relations as above and hence doesn't hold as a necessary optimality condition. △

The next two examples illustrate relationships between the extended Euler-Lagrange inclusion (6.49) and the extended Hamiltonian inclusion (6.58) with the (fully) convexified Hamiltonian inclusion (6.59).

Example 6.35 (extended Euler-Lagrange inclusion is strictly better than convexified Hamiltonian inclusion). *There is a compact-valued and convex-valued multifunction* $F\colon I\!\!R^2 \rightrightarrows I\!\!R^2$, *which is Lipschitz continuous on* $I\!\!R^2$ *and such that*

$$(-w, v) \in \operatorname{co} \partial \mathcal{H}(x, p) \;\; \textit{but} \;\; w \notin \operatorname{co} \big\{ u \in I\!\!R^2 \big|\, u \in D^*F(x, v)(-p) \big\}$$

for some points x, v, w, p *in the plane.*

Proof. Define $F\colon I\!R^2 \rightrightarrows I\!R^2$ by

$$F(x_1,x_2) := \big\{(\tau, \tau\,|x_1| + \nu) \in I\!R^2 \big|\ \tau \in [-1,1],\ \nu \in [0,\mu]\big\} \ \text{ with some } \ \mu > 0\ ,$$

where the sets $F(x)$ are parallelograms in the plane for all $x = (x_1, x_2) \in I\!R^2$. The corresponding Hamiltonian is

$$\mathcal{H}(x_1,x_2,p_1,p_2) = \big|p_1 + p_2|x_1|\big| + \max\big\{p_2, 0\big\}\ .$$

Considering the points $x = (0,0)$, $v = (0,0)$, and $p = (0,-1)$, we see that the corresponding set $F(x)$ is the rectangle $[-1,1] \times [0,\mu]$, and that p is an outward normal vector to this set at the boundary point v. The crucial feature of this example is that the hyperplane $x_2 = 0$ supporting the set $F(x)$ at v intersects this set in *more than one point*. In other words, the maximum of $\langle p, v\rangle$ over $v \in F(x)$ is attained at *infinitely many points*. The basic subdifferential of $\mathcal{H}$ at the point $(0,0,0,-1)$ and its convexification (Clarke's generalized gradient) are actually calculated in Example 2.49; thus

$$\operatorname{co}\partial\mathcal{H}(0,0,0,-1) = [-1,1] \times \{0\} \times [-1,1] \times \{0\} \subset I\!R^4\ .$$

Taking $w = (-1,0)$, one has $(-w, v) \in \operatorname{co}\partial\mathcal{H}(0,0,0,-1)$. Let us show that

$$(w,p) = (-1,0,0,-1) \notin \operatorname{clco} N\big((x,v);\operatorname{gph} F\big)\ ,$$

which definitely justifies the claim of this example.

To proceed, we note that, up to a permutation of the coordinates, the graph of F can be represented as

$$\operatorname{gph} F = E \times I\!R \ \text{ with } \ E := \big\{(x_1, \tau, |x_1|\tau + \nu) \in I\!R^3 \big|\ \tau \in [-1,1],\ \nu \in [0,\mu]\big\}\ ,$$

where the set E obviously coincides around the point $(0,0,0)$ with the *epigraph* of the Lipschitzian function $\varphi\colon I\!R^2 \to I\!R$ defined by $\varphi(y,\tau) := \tau|y|$. It is easy to see that

$$\operatorname{co}\partial\varphi(0,0) = \partial\varphi(0,0) = \big\{(0,0)\big\}\ .$$

One therefore calculates

$$N\big((0,0,\varphi(0,0));\operatorname{epi}\varphi\big) = \bigcup_{\lambda \ge 0} \lambda\big[\partial\varphi(0,0) \times \{-1\}\big] = \big\{(0,0)\big\} \times (-\infty, 0]\ ,$$

and hence we deduce that

$$\operatorname{clco} N\big((0,0,0,0);\operatorname{gph} F\big) = \big\{(0,0,0)\big\} \times (-\infty,0]\ .$$

In particular, the latter cone doesn't contain the point $(w,p) = (-1,0,0,-1)$, even though $(-w,v) \in \operatorname{co}\partial\mathcal{H}(x,p)$. $\triangle$

The last example shows that the extended/refined Hamiltonian condition (6.58) *strictly supersedes* the fully convexified one (6.59) in both settings of convex-valued and nonconvex-valued differential inclusions.

Example 6.36 (partially convexified Hamiltonian condition strictly improves its fully convexified counterpart). *There is a set-valued mapping $F\colon I\!\!R^n \rightrightarrows I\!\!R^n$ in the form $F(x) = g(x)S$, where $S \subset I\!\!R^n$ is a compact set and where $g(x)$, for each x, is a linear isomorphism of $I\!\!R^n$ depending continuously on x, such that for some $(\bar{x}, \bar{v}, \bar{p})$ one has*

$$\mathrm{co}\,\big\{u \in I\!\!R^n\big|\ (u, \bar{v}) \in \partial\mathcal{H}(\bar{x}, \bar{p})\big\} \neq \big\{u \in I\!\!R^n\big|\ (u, \bar{v}) \in \mathrm{co}\,\partial\mathcal{H}(\bar{x}, \bar{p})\big\}\ .$$

Proof. If F is given in the above form, then its Hamiltonian is calculated by

$$\mathcal{H}(x, p) = \sup\big\{\langle p, v\rangle\big|\ v \in g(x)S\big\} = \sup\{\langle p, g(x)s\rangle\big|\ s \in S\big\} =: \delta^*\big(g^*(x)p; S\big),$$

where $\delta^*(\cdot; S)$ stands for the standard support function of the set S. Since S is bounded, its support function is continuous. Denote

$$\psi_s(x, p) := \langle s, g^*(x)p\rangle = \langle g(x)s, p\rangle$$

and suppose that $g(\cdot)$ is Lipschitz continuous. Employing the scalarization formula and taking into account the structure of ψ, we have

$$\partial\mathcal{H}(\bar{x}, \bar{p}) = \bigcup_{s \in \partial\delta^*(0; S)} \partial\psi_s(\bar{x}, \bar{p})$$

at any given point $(\bar{x}, \bar{p})$. The linearity of ψ in p yields that

$$\partial\psi_s(\bar{x}, \bar{p}) = \big(\partial_x\psi_s(\bar{x}, \bar{p}), g(\bar{x})s\big)\ .$$

Therefore the inclusion $(u, 0) \in \partial\psi_s(\bar{x}, \bar{p})$ implies that $s = 0$ and thus $u = 0$.

Based on the above discussion, we need to find a set S, a Lipschitz continuous family of linear isomorphisms $g(x)$ of $I\!\!R^n$, and a point $(\bar{x}, \bar{p}) \in I\!\!R^n \times I\!\!R^n$ such that $0 \in S$ and $\mathrm{co}\,\partial\mathcal{H}(\bar{x}, \bar{p})$ contains a pair $(u, 0)$ with $u \neq 0$. In particular, it can be done as follows for $n = 2$. Let

$$S := \big\{(y_1, y_2) \in I\!\!R^2\big|\ |y_1| \le 1,\ y_2 = 0\big\}, \quad g^*(x) := \begin{pmatrix} 1 & |x_1| \\ 1 & 1 \end{pmatrix}\ ,$$

$\bar{x} := (0, 0)$, and $\bar{p} := (0, 1)$. Then

$$\delta^*\big((w_1, w_2); S\big) = w_1 \ \text{ and }\ \mathcal{H}(x, p) = \big|p_1 + p_2|x_1|\big|\ .$$

One can directly calculate (cf. Example 2.49) that the set $\mathrm{co}\,\partial\mathcal{H}(\bar{x}, \bar{p})$ is the convex hull of the following four points: $(1, 0, 1, 0)$, $(-1, 0 - 1, 0)$, $(1, 0, -1, 0)$, and $(-1, 0, 1, 0)$. Thus

$$\big\{u \in I\!\!R\big|\ (u, 0) \in \mathrm{co}\,\partial\mathcal{H}(\bar{x}, \bar{p})\big\} = [-1, 1]\ ,$$

which justifies the claim of this example. $\triangle$

6.3 Maximum Principle for Continuous-Time Systems with Smooth Dynamics

In this section we study optimal control problems governed by ordinary differential equations in infinite-dimensional spaces that explicitly involve constrained control inputs $u(\cdot)$ as follows:

$$\dot{x} = f(x,u,t), \quad u(t) \in U \ \text{ a.e. } \ t \in [a,b]\,, \tag{6.61}$$

where $f\colon X \times U \times [a,b] \to X$ with a Banach state space X and a metric control space U. Although control systems of this type can be reduced to differential inclusions $\dot{x} \in F(x,t)$ with $F(x,t) := f(x,U,t)$, the explicit control input in (6.61) with the control region U independent of x (it may depend on t) allows us to develop efficient methods of studying such dynamic systems that take into account their specific features.

Throughout the section we assume that system (6.61) is of *smooth dynamics*, which means that the velocity mapping f is *continuously differentiable* ($\mathcal{C}^1$) with respect to the state variable x around an optimal solution to be considered. Despite this assumption, the control system (6.61) and optimization problems over its feasible controls and trajectories intrinsically involve *nonsmoothness* due to the control geometric constraints $u(t) \in U$ a.e. $t \in [a,b]$ defined by control sets U of a general nature. For instance, it is the case of the simplest/classical optimal control problems with $U = \{0,1\}$.

In this section the main attention is paid to the Mayer-type control problem for systems (6.61) of smooth dynamics subject to *finitely many* endpoint constraints given by equalities and inequalities with functions merely *Fréchet differentiable* (possibly not strictly) *at* points of minima. Our goal is to derive necessary optimality conditions in the form of the *Pontryagin maximum principle* (PMP) for such problems in *general Banach spaces*, with *no* additional assumptions on the reflexivity and separability of X as well as on the sequential normal compactness and strong coderivative normality of $F(x,t) = f(x,U,t)$ imposed in Theorem 6.27 of the preceding section. The technique used for this purpose is different from those employed in Sects. 6.1 and 6.2; it goes back to the classical approach in optimal control theory involving *needle variations* of optimal controls. We also derive enhanced results of the maximum principle type with *upper subdifferential* transversality conditions in the case of nondifferentiable cost and inequality constraint functions. Such conditions are obtained without imposing *any smoothness* assumptions on the state space in question needed for the corresponding necessary optimality conditions derived above in both mathematical programming and dynamic optimization settings; cf. Theorem 5.19 and Remark 6.30. Thus the results of this section, which essentially exploit the specific structure of smooth control systems (6.61) and the imposed endpoint constraints, are generally independent of those obtained in Sects. 6.1 and 6.2.

This section is organized as follows. Subsect. 6.3.1 contains the formulation of the main assumptions and results as well as the derivation of the maximum principle with upper subdifferential transversality conditions from the one with Fréchet differentiable endpoint functions. We also discuss possible extensions of the maximum principle to control problems with intermediate state constraints as well as to some classes of time-delay systems. Subsection 6.3.2 is devoted to the proof of the PMP for free-endpoint control problems in Banach spaces, which is substantially simpler than that for problems with endpoint constraints. Subsection 6.3.3 deals with optimal control problems involving endpoint constraints of the inequality type. Finally, in Subsect. 6.3.4 we derive, with the use of the Brouwer fixed-point theorem, transversality conditions in the case of equality constraints given by continuous functions that are just differentiable at optimal endpoints.

6.3.1 Formulation and Discussion of Main Results

It is more simple and convenient (and in fact does't much restrict the generality) to formulate and then to prove the main results of this section for the case of control systems (6.61) with a *fixed left endpoint* $x(a) = x_0$; we discuss various extensions of the main results in the end of this subsection.

Denote by $\mathcal{A}$ the collection of *admissible* control-trajectory pairs $\{u(\cdot), x(\cdot)\}$ generated by *measurable* controls $u(\cdot)$ satisfying the *pointwise* constraints $u(t) \in U$ for a.e. $t \in [a, b]$ and the corresponding solutions $x(\cdot)$ to (6.61) with $x(a) = x_0$ defined by

$$x(t) = x_0 + \int_a^t f(x(s), u(s), s)\, ds \quad \text{for all} \quad t \in [a, b] \,, \tag{6.62}$$

where the integral is understood in the Bochner sense; cf. Definition 6.1. As is well known, any solution to (6.62) is absolutely continuous on $[a, b]$. Moreover, it is a.e. differentiable on $[a, b]$ and satisfies the differential equation (6.61) for a.e. $t \in [a, b]$ provided that X has the Radon-Nikodým property (see Subsect. 6.1.1), which is *not* assumed here. What we need in this section is the integral representation (6.62), which is taken as the definition of admissible solutions/arcs to the differential equation (6.61) in Banach spaces.

Given real-valued functions φ_i, $i = 0, \ldots, m + r$, on the state space X, we now formulate the optimal control problem studied below:

$$\text{minimize} \quad J[u, x] = \varphi_0\big(x(b)\big) \quad \text{over} \quad (u, x) \in \mathcal{A} \tag{6.63}$$

subject to the endpoint constraints

$$\varphi_i\big(x(b)\big) \le 0 \quad \text{for} \quad i = 1, \ldots, m \,, \tag{6.64}$$

$$\varphi_i\big(x(b)\big) = 0 \quad \text{for} \quad i = m + 1, \ldots, m + r \,. \tag{6.65}$$

Admissible solutions $(u, x) \in \mathcal{A}$ satisfying the endpoint constraints (6.64) and (6.65) are called *feasible solutions* to problem (6.63)–(6.65). So we don't distinguish between admissible and feasible solutions for problems with *free endpoints*, i.e., with no endpoint constraints (6.64) and (6.65). We always assume that the set of feasible solutions to (6.63)–(6.65) is *not empty*.

A feasible solution $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ is *optimal* to (6.63)–(6.65) if

$$J[\bar{u}, \bar{x}] \leq J[u, x] \text{ for all } (u, x) \in \mathcal{A}$$

satisfying the endpoint constraints (6.64) and (6.65). Our goal is to derive necessary conditions of the PMP type for a given optimal solution $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ to the problem under consideration. Although we present necessary conditions for (global) optimal solutions, one can observe from the proofs provided below that the results obtained hold true for *local minimizers* $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ in the sense that $J[\bar{u}, \bar{x}] \leq J[x, u]$ whenever (u, x) is feasible to (6.63)–(6.65) and $\|x(t) - \bar{x}(t)\| < \varepsilon$ for all $t \in [a, b]$ with some $\varepsilon > 0$. This corresponds to *strong* local minimizers in Subsect. 6.1.2 for $F(x, t) = f(x, U, t)$.

Given an optimal solution $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ to (6.63)–(6.65), we impose the following *standing assumptions* throughout the whole section:

—the state space X is Banach;

—the control set U is a *Souslin subset* (i.e., a continuous image of a Borel subset) in a separable Banach space;

—there is an open set $O \subset X$ containing $\bar{x}(t)$ such that f is Fréchet differentiable in x with both $f(x, u, t)$ and $\nabla_x f(x, u, t)$ continuous in (x, u), measurable in t, and norm-bounded by a summable function for all $x \in O$, $u \in U$, and a.e. $t \in [a, b]$;

—the functions φ_i are continuous around $\bar{x}(b)$ and Fréchet differentiable at this point for $i = m+1, \ldots, m+r$.

Note that the control set U may depend on t in a general *measurable* way, which allows one to use standard *measurable selection* results; see, e.g., the books [54, 229, 1165] with the references therein.

Appropriate assumptions on the functions φ_i, $i = 0, \ldots, m$, describing the objective and inequality constraints will be presented in the main theorems stated below. Note that the basic assumptions on them require their Fréchet *differentiability at* $\bar{x}(b)$ (*not* even their *continuity around* this point), while upper subdifferential conditions hold for a broader class of nondifferentiable functions on *arbitrary* Banach spaces.

To formulate the relations of the maximum principle, let us define the *Hamilton-Pontryagin* function for system (6.61) by

$$H(x, p, u, t) := \langle p, f(x, u, t) \rangle \text{ with } p \in X^* .$$

Observe that the *Hamiltonian* defined in Sect. 6.2 for $F(x, t) = f(x, U, t)$ corresponds to the *maximization* of the function $H(x, p, u, t)$ with respect to u over the whole the control region:

$$\mathcal{H}(x, p, t) = \max\left\{H(x, p, u, t)\Big|\ u \in U\right\} .$$

Note also that H is *smooth* with respect to the state and adjoint variables (x, p), which of course is not the case for $\mathcal{H}$.

Theorem 6.37 (maximum principle for smooth control systems). *Let $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ be an optimal solution to problem (6.63)–(6.65) under the standing assumptions made. Suppose also that the functions φ_i, $i = 0, \ldots, m$, are Fréchet differentiable at the optimal endpoint $\bar{x}(b)$. Then there are multipliers $(\lambda_0, \ldots, \lambda_{m+r}) \neq 0$ satisfying*

$$\lambda_i \geq 0 \ \ \text{for} \ \ i = 0, \ldots, m ,$$

$$\lambda_i \varphi_i\big(\bar{x}(b)\big) = 0 \ \ \text{for} \ \ i = 1, \ldots, m ,$$

and such that the following maximum condition holds:

$$H\big(\bar{x}(t), p(t), \bar{u}(t), t\big) = \max_{u \in U} H\big(\bar{x}(t), p(t), u, t\big) \ \ \text{a.e.} \ \ t \in [a, b] , \tag{6.66}$$

where an absolutely continuous mapping $p\colon [a, b] \to X^*$ is a trajectory for the adjoint system

$$\dot{p} = -\nabla_x H(\bar{x}, p, \bar{u}, t) \ \ \text{a.e.} \ \ t \in [a, b] \tag{6.67}$$

with the transversality condition

$$p(b) = -\sum_{i=0}^{m+r} \lambda_i \nabla \varphi_i\big(\bar{x}(b)\big) . \tag{6.68}$$

Note that a solution (adjoint arc) to system (6.67) is understood in the *integral/mild* sense similarly to (6.61), i.e.,

$$p(t) = p(b) + \int_t^b \nabla_x H\big(\bar{x}(s), p(s), \bar{u}(t), s\big)\, ds, \quad t \in [a, b] ,$$

with $\nabla_x H(\bar{x}, p, \bar{u}, t) = \langle p, \nabla_x f(\bar{x}, \bar{u}, t)\rangle$. Observe also that the transversality condition (6.68) agrees with the one in Corollary 6.29. However, now the endpoint functions is *not* assumed to be *strictly* differentiable at $\bar{x}(b)$.

The proof of Theorem 6.37 will be given in Subsects. 6.3.2–6.3.4. Meantime let us formulate and prove an *upper subdifferential* counterpart of this theorem, which gives on one hand an extension of the transversality condition (6.68) to the case of nondifferentiable functions φ_i, $i = 0, \ldots, m$, while on the other hand follows from Theorem 6.37 and the *smooth variational description* of Fréchet subgradients.

Theorem 6.38 (maximum principle with transversality conditions via Fréchet upper subgradients). *Let $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ be an optimal solution to the control problem (6.63)–(6.65) under the standing assumptions made. Then for every collection of Fréchet upper subgradients $x_i^* \in \widehat{\partial}^+\varphi_i\big(\bar{x}(b)\big)$, $i = 0, \ldots, m$, there are multipliers $(\lambda_0, \ldots, \lambda_{m+r}) \neq 0$ satisfying the sign and complementary slackness conditions of Theorem 6.37 and such that the maximum condition (6.66) holds with the corresponding trajectory $p(\cdot)$ of the adjoint system (6.67) satisfying the transversality condition*

$$p(b) + \sum_{i=0}^{m+r} \lambda_i x_i^* = 0 \,. \tag{6.69}$$

Proof. Take an arbitrary set of Fréchet upper subgradients $x_i^* \in \widehat{\partial}^+\varphi_i\big(\bar{x}(b)\big)$, $i = 0, \ldots, m$, and employ the smooth variational description of $-x_i^*$ from assertion (i) of Theorem 1.88 held in any Banach space. In this way we find functions $s_i \colon X \to I\!R$ for $i = 0, \ldots, m$ satisfying the relations

$$s_i\big(\bar{x}(b)\big) = \varphi_i\big(\bar{x}(b)\big), \quad s_i(x) \geq \varphi_i(x) \ \text{ around } \bar{x}(b) \,,$$

and such that each $s_i(\cdot)$ is Fréchet differentiable at $\bar{x}(b)$ with $\nabla s_i\big(\bar{x}(b)\big) = x_i^*$, $i = 0, \ldots, m$. From the construction of these functions we easily deduce that the process $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ is an optimal solution to the following control problem:

$$\text{minimize } \ \widetilde{J}[u, x] = s_0\big(x(b)\big) \ \text{ over } \ (u, x) \in \mathcal{A}$$

subject to the inequality and equality endpoint constraints

$$s_i\big(x(b)\big) \leq 0 \ \text{ for } \ i = 1, \ldots, m$$

and (6.65), where $\mathcal{A}$ is the collection of admissible control-trajectory pairs defined in the beginning of this subsection. The initial data of the latter optimal control problem satisfy all the assumptions of Theorem 6.37. Thus applying the above maximum principle to this problem and taking into account that $\nabla s_i\big(\bar{x}(b)\big) = x_i^*$ for $i = 0, \ldots, m$, we complete the proof of the theorem. △

One can observe the difference between the formulations and proofs of Theorem 6.38, in the part related to upper subdifferential transversality conditions, and of Theorem 5.19 on upper subdifferential optimality conditions in mathematical programming. Both results reduce to their *smooth* (in difference senses) counterparts based on smooth variational descriptions of Fréchet subgradients. In the case of Theorem 5.19 we need to require the continuous differentiability (more precisely, *strict differentiability*) of the cost and constraint functions to be able to apply the corresponding necessary conditions in smooth nonlinear programming. In this way an additional assumption on the geometry of Banach spaces comes into play to ensure the $\mathcal{C}^1$ description of

Fréchet subgradients by Theorem 1.88(ii). On the other hand, Theorem 6.38 relies, by a milder smooth variational description from Theorem 1.88(i), on the preceding Theorem 6.37 that requires only the *Fréchet differentiability* of the endpoint functions *at* the optimal point. Note that Theorems 6.37 and 6.38 concerning optimal control problems obviously imply, by putting $f = 0$ in (6.61), the corresponding *improvements* of the results in Subsect. 5.1.3 for *mathematical programming* problems with equality and inequality constraints.

Remark 6.39 (control problems with constraints at both endpoints and at intermediate points of trajectories). One can see from the proof of Theorem 6.37 given in Subsects. 6.3.2–6.3.4 that a minor modification of this proof allows us to derive similar necessary optimality conditions (including those of the upper subdifferential type) for optimal control problems with endpoint constraints of form (6.64) and (6.65) at both $t = a$ and $t = b$ and with the cost function φ_0 depending on both $x(a)$ and $x(b)$ under the same assumptions on the initial data. In this case the transversality condition (6.68) on the absolutely continuous adjoint arc $p\colon [a, b] \to X^*$ is replaced by

$$\big(p(a), -p(b)\big) = \sum_{i=0}^{m+r} \lambda_i \nabla\varphi_i\big(\bar{x}(a), \bar{x}(b)\big) .$$

Furthermore, we may similarly derive necessary optimality conditions for control problems involving *intermediate state constraints*, i.e., with constraints on trajectories given at intermediate points $\tau_i \in [a, b]$ of the time interval. For example, consider the modified problem (6.63)–(6.65) with

$$\varphi_i = \varphi_i\big(x(a), x(\tau), x(b)\big), \quad i = 0, \ldots, m + r ,$$

where $\tau \in (a, b)$ is an intermediate moment of the time interval. Then the difference between the necessary optimality conditions of Theorem 6.37 and the ones for the modified state-constrained problem is that we now have a *discontinuous* adjoint arc $p(\cdot)$ with the *jump condition* at the intermediate point $t = \tau$ incorporated into the transversality conditions as follows:

$$\big(p(a), p(\tau + 0) - p(\tau - 0), -p(b)\big) = \sum_{i=0}^{m+r} \lambda_i \nabla\varphi_i\big(\bar{x}(a), \bar{x}(\tau), \bar{x}(b)\big) .$$

We can similarly modify the upper subdifferential conditions of Theorem 6.38 in the case of control problems with intermediate state constraints.

Remark 6.40 (maximum principle in time-delay control systems). The results of Theorems 6.37 and 6.38 can be extended to various systems with time delays in state and control variables. For example, let us consider the standard system with a constant time delay $\theta > 0$ in the state variable:

$$\begin{cases} \dot{x}(t) = f\big(x(t), x(t-\theta), u(t), t\big) \ \text{ a.e. } \ t \in [a,b] \,, \\[2ex] x(t) = c(t), \quad t \in [a-\theta, a] \,, \\[2ex] u(t) \in U \ \text{ a.e. } \ t \in [a,b] \end{cases}$$

over measurable controls and absolutely continuous trajectories with a Banach state space X and the initial "tail" mapping $c\colon [a-\theta,a] \to X$ that is necessary to start the time-delay process. Denote by $\mathcal{A}$ the collection of admissible pairs $\{u(\cdot), x(\cdot)\}$ satisfying the above delay system and define the corresponding Hamilton-Pontryagin function

$$H(x,y,p,u,t) := \big\langle p, f(x,y,u,t)\big\rangle, \quad p \in X^* \,,$$

where y stands for the delay variable $x(t-\theta)$. Considering now problem (6.63)–(6.65) with $\mathcal{A}$ signifying the collection of admissible pairs for the delay system, we get counterparts of Theorems 6.37 and Theorem 6.38 with the *adjoint system* given by

$$-\dot{p}(t) = \begin{cases} \nabla_x H\big(x(t), x(t-\theta), p(t), u(t), t\big) \\[2ex] +\nabla_y H\big(x(t+\theta), x(t), p(t+\theta), u(t+\theta), t\big) \ \text{ a.e. } \ t \in [a, b-\theta] \,; \\[2ex] \nabla_x H\big(x(t), x(t-\theta), p(t), u(t), t\big) \ \text{ a.e. } \ t \in [b-\theta, b] \,. \end{cases}$$

These results can be actually proved by reducing the time-delay control system in X to the one with *no* delay in the state space X^N, for some natural number N sufficiently large. Furthermore, the methods developed in the proofs of Theorems 6.37 and 6.38 allow us to derive similar results for control problems with more *general delays* depending on both time and state variables, as well as with time-distributed delays.

Remark 6.41 (functional-differential control systems of neutral type). The dynamics of such control systems is described by differential equations with time delays not only in state variables but in *velocity* variables as well. A typical model is given by

$$\dot{x}(t) = f\big(x(t), x(t-\theta), \dot{x}(t-\theta), u(t), t\big), \quad u(t) \in U, \quad \text{a.e. } \ t \in [a,b]$$

with proper initial conditions on $[a-\theta, a]$. Systems of this type are *fundamentally different* from the standard ODE control systems and time-delay systems considered in the preceding remark. They are substantially more difficult for variational analysis and exhibit a number of phenomena that are not inherent in the control systems considered above; the reader may find more discussions in Commentary to Chap. 7, where we consider such systems and their extensions in more details. Now observe that, although necessary optimality conditions in the form of Theorems 6.37 and 6.38 can be derived by similar methods in the case of *convex velocity sets* $f(x, y, z, U, t)$ with a Banach state space, a proper analog of the Pontryagin maximum principle *doesn't generally hold* for neutral control systems even with no endpoint constraints in finite dimensions. It happens, in particular, for the optimal control

$$\bar{u}(t) = 0 \;\text{ as }\; t \in [0, 1) \;\text{ and }\; \bar{u}(t) = 1 \;\text{ as }\; t \in [1, 2]$$

to the following two-dimensional control problem:

$$\begin{cases} \text{minimize } \; J[u, x] = x_2(2) \;\text{ subject to} \\ \dot{x}_1(t) = u(t), \quad \dot{x}_2(t) = \dot{x}_1^2(t-1) - u^2(t), \quad t \in [0, 2] , \\ x_1(t) = x_2(t) = 0, \quad t \in [-1, 0]; \quad |u(t)| \le 1, \quad t \in [0, 2] . \end{cases}$$

The reader can find complete calculations for this example in the book by Gabasov and Kirillova [485, Sect. 3.6]; see also Example 6.70 in Subsect. 6.4.6 below for similar calculations in a finite-difference analog of this control problem.

6.3.2 Maximum Principle for Free-Endpoint Problems

In this subsection we study problem (6.63), where $\mathcal{A}$ is the collection of admissible pairs $\{u(\cdot), x(\cdot)\}$ for the control system (6.61) with the fixed left endpoint $x(a) = x_0$; see the beginning of the preceding subsection for the exact formulation. This problem is labeled as a *free-endpoint problem of optimal control* despite the left endpoint is always fixed; we have in mind the absence of the constraints (6.64) and (6.65) on the right endpoint of admissible trajectories. As follows from the proofs below, the *free-endpoint* problem (6.63) is *significantly different* from the constrained problem (6.63)–(6.65); moreover, the problems with *inequality* and *equality* endpoint constraints are essentially different from each other as well. The principal difference between the unconstrained and constrained problems is that in case of (6.63) all admissible trajectories are feasible, and one doesn't need to care about satisfying the endpoint constraints while varying admissible controls $u(\cdot) \in U$. Note that the control constraints of the above (arbitrary) geometric type are *always present* in the problems under consideration, they distinguish optimal control problems from the classical calculus of variations and signify *intrinsic nonsmoothness* inherent in optimal control.

This subsection is devoted to the proof of the maximum principle from Theorem 6.37 for problem (6.63) under the assumptions made in the theorem on the given data (U, X, f, φ_0). Note that the transversality condition (6.68) reduces in this case to

$$p(b) = -\nabla\varphi_0\big(\bar{x}(b)\big) , \tag{6.70}$$

i.e., with $\lambda_0 = 1$ and $\lambda_i = 0$, $i = 1, \ldots, m+r$, in (6.68). Indeed, if $\lambda_0 = 0$ and $p(b) = 0$ in (6.68), then $p(t) \equiv 0$ for all $t \in [a, b]$ due to the linearity of the adjoint system (6.67) with respect to p, which would contradict the nontriviality condition $\big(p(\cdot), \lambda_0\big) \neq 0$ in Theorem 6.37.

The proof of Theorem 6.37 for the free-endpoint problem (6.63) is *purely analytic*, in the sense that it doesn't invoke any geometric facts and arguments in the vein of the convex separation theorem and the like. This is significantly different from the proofs of Theorem 6.37 in the case of inequality and equality endpoint constraints given in Subsect. 5.3.3 and 5.3.4. The basic ingredients in the proof of Theorem 6.37 for problem (6.63) are the *increment formula* for the cost functional in (6.63) and the use of the so-called *needle* variations (sometimes called "McShane variations") of the optimal control.

Let us start with the increment formula. Given two admissible controls $\bar{u}(t), u(t) \in U$ (observe that $\bar{u}(\cdot)$ may not be optimal before resuming it in the sequel) and the corresponding solutions $\bar{x}(\cdot), x(\cdot)$ in (6.62), we denote

$$\Delta\bar{u}(t) := u(t) - \bar{u}(t), \quad \Delta\bar{x}(t) := x(t) - \bar{x}(t), \quad \Delta J[\bar{u}] := \varphi_0\big(x(b)\big) - \varphi_0\big(\bar{x}(b)\big) \ .$$

Our intention is to obtain a convenient representation of the cost functional increment $\Delta J[\bar{u}]$ in terms of the Hamilton-Pontryagin function evaluated along the admissible pair $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ and the corresponding trajectory $p(\cdot)$ of the adjoint system (6.67) with the boundary condition (6.70). Recall that we use the same standard symbol $o(\cdot)$ for *all* expressions of this category.

Lemma 6.42 (increment formula for the cost functional). *Let*

$$\Delta_u H\big(\bar{x}(t), p(t), \bar{u}(t), t\big) := H\big(\bar{x}(t), p(t), u(t), t\big) - H\big(\bar{x}(t), p(t), \bar{u}(t), t\big)$$

in the notation above. Then one has

$$\Delta J[\bar{u}] = -\int_a^b \Delta_u H\big(\bar{x}(t), p(t), \bar{u}(t), t\big)\, dt + \eta \ ,$$

where the remainder η is given by $\eta = \eta_1 + \eta_2 + \eta_3$ with

$$\eta_1 := o\big(\|\Delta\bar{x}(b)\|\big), \quad \eta_2 := -\int_a^b o\big(\|\Delta\bar{x}(t)\|\big)\, dt, \quad \text{and}$$

$$\eta_3 := -\int_a^b \Big\langle \frac{\partial \Delta_u H\big(\bar{x}(t), p(t), \bar{u}(t), t\big)}{\partial x}, \Delta\bar{x}(t) \Big\rangle\, dt \ .$$

Proof. Since φ_0 is assumed to be Fréchet differentiable at $\bar{x}(b)$, we have the representation

$$\Delta J[\bar{u}] = \varphi_0\big(x(b)\big) - \varphi_0\big(\bar{x}(b)\big) = \big\langle \nabla\varphi_0\big(\bar{x}(b)\big), \Delta\bar{x}(b)\big\rangle + o\big(\|\Delta\bar{x}(b)\|\big) \ .$$

Taking into account that solutions to the state and adjoint equations satisfy (by definition) the Newton-Leibniz formula and using *integration by parts* held for the Bochner integral, one gets the identity

$$\big\langle p(b), \Delta\bar{x}(b)\big\rangle = \int_a^b \big\langle \dot{p}(t), \Delta\bar{x}(t)\big\rangle\,dt + \int_a^b \big\langle p(t), \Delta\dot{\bar{x}}(t)\big\rangle\,dt\;,$$

where $p\colon [a,b] \to X^*$ is an arbitrary absolutely continuous mapping from the solution class. Imposing the boundary condition (6.70) on $p(b)$, we arrive at

$$\Delta J[\bar{u}] = -\int_a^b \big\langle \dot{p}(t), \Delta\bar{x}(t)\big\rangle\,dt - \int_a^b \big\langle p(t), \Delta\dot{\bar{x}}(t)\big\rangle\,dt + o\big(\|\Delta\bar{x}(b)\|\big)\;.$$

Let us transform the second integral above. Using the equation

$$\Delta\dot{\bar{x}}(t) = f\big(\bar{x}(t)+\Delta\bar{x}(t), \bar{u}(t)+\Delta\bar{u}(t), t\big) - f\big(\bar{x}(t), \bar{u}(t), t\big)\;,$$

the definition of the Hamilton-Pontryagin function $H(x,p,u,t)$, and the smoothness of f in x, we have

$$\begin{aligned}
&\int_a^b \big\langle p(t), \Delta\dot{\bar{x}}(t)\big\rangle\,dt \\
&= \int_a^b \Big[H\big(\bar{x}(t)+\Delta\bar{x}(t), p(t), \bar{u}(t)+\Delta\bar{u}(t), t\big) - H\big(\bar{x}(t), p(t), \bar{u}(t), t\big)\Big]\,dt \\
&= \int_a^b \Big[H\big(\bar{x}(t), p(t), \bar{u}(t)+\Delta\bar{u}(t), t\big) - H\big(\bar{x}(t), p(t), \bar{u}(t), t\big)\Big]\,dt \\
&+ \int_a^b \Big\langle \frac{\partial \Delta_u H(\bar{x}(t), p(t), \bar{u}(t)+\Delta\bar{u}(t), t)}{\partial x}, \Delta\bar{x}(t)\Big\rangle\,dt + \int_a^b o\big(\|\Delta\bar{x}(t)\|\big)\,dt\;.
\end{aligned}$$

Remembering finally that $p(\cdot)$ is a solution to the adjoint system (6.67) generated by $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$, we complete the proof of the lemma. △

In the above increment formula both controls $\bar{u}(\cdot)$ and $u(\cdot)$ are arbitrary measurable mappings satisfying the pointwise control constraints. Now we build $u(\cdot)$ as a special perturbation of the reference control $\bar{u}(\cdot)$ that is called a *needle variation*, or sometimes a *single needle variation*, of this control. Namely, fix arbitrary numbers $\tau \in [a,b)$ and $\varepsilon > 0$ with $\tau + \varepsilon < b$, take an arbitrary point $v \in U$, and construct an admissible control $u(t)$, $t \in [a,b]$, in the following form

$$u(t) := \begin{cases} v, & t \in [\tau, \tau+\varepsilon)\;, \\ \bar{u}(t), & t \notin [\tau, \tau+\varepsilon)\;. \end{cases} \tag{6.71}$$

The obtained perturbed control differs from the reference one only on the small time interval $[\tau, \tau+\varepsilon)$, where its value is *arbitrary* in the control set U; the name "needle variation" comes from this. For the corresponding trajectory increment $\Delta\bar{x}(t)$, depending on the parameters (τ, ε, v), one clearly has

$$\Delta\bar{x}(t) = 0 \ \text{ for all } \ t \in [a, \tau] \ .$$

Let us estimate $\Delta\bar{x}(t)$ for $t \in (\tau, b]$, which is given in the next lemma. In what follows we denote by ℓ the uniform Lipschitz constant for $f(\cdot, v, t)$ whose existence is guaranteed by the standing assumptions. For simplicity we suppose that ℓ is independent of t although the assumptions made allow it to be summable on $[a, b]$ with no change of the result.

Lemma 6.43 (increment of trajectories under needle variations). *Let $\Delta\bar{x}(\cdot)$ be the increment of $\bar{x}(\cdot)$ corresponding to the needle variation (6.71) of $\bar{u}(\cdot)$ with parameters (τ, ε, v). Then there is a constant $K > 0$ independent of (τ, ε) (it may depend on v) such that*

$$\|\Delta\bar{x}(t)\| \le K\varepsilon \ \textit{ for all } \ t \in [a, b] \ .$$

Proof. Since $\Delta\bar{x}(\tau) = 0$, one has by (6.62) that

$$\Delta\bar{x}(t) = \int_\tau^t \Big[f\big(\bar{x}(s) + \Delta\bar{x}(s), v, s\big) - f\big(\bar{x}(s), \bar{u}(s), s\big) \Big] ds, \quad \tau \le t \le \tau + \varepsilon \ .$$

Taking into account the uniform Lipschitz continuity of f in x with the constant ℓ and denoting $\Delta_v f\big(\bar{x}(s), \bar{u}(s), s\big) := f\big(\bar{x}(s), v, s\big) - f\big(\bar{x}(s), \bar{u}(s), s\big)$, we have

$$\begin{aligned} \|\Delta\bar{x}(t)\| &\le \int_\tau^t \big\| f\big(\bar{x}(s) + \Delta\bar{x}(s), v, s\big) - f\big(\bar{x}(s), \bar{u}(s), s\big) \big\| ds \\ &\le \int_\tau^t \big\| \Delta_v f\big(\bar{x}(s), \bar{u}(s), s\big) \big\| \ ds + \ell \int_\tau^t \|\Delta\bar{x}(s)\| ds \ . \end{aligned}$$

Using the notation

$$\alpha(t) := \int_\tau^t \big\| \Delta_v f\big(\bar{x}(s), \bar{u}(s), s\big) \big\| ds \ \text{ and } \ \beta(t) := \|\Delta\bar{x}(t)\| \ ,$$

the above estimate can be rewritten as

$$\beta(t) \le \alpha(t) + \ell \int_\tau^t \beta(s) \, ds, \quad \tau \le t \le \tau + \varepsilon \ ,$$

which yields by the classical Gronwall lemma that

$$\|\Delta\bar{x}(t)\| \le \Big(\int_\tau^t \big\| \Delta_v f\big(\bar{x}(s), \bar{u}(s), s\big) \big\| ds \Big) \exp\big(\ell(t - \tau)\big) \le K\varepsilon$$

for $t \in [\tau, \tau + \varepsilon]$, where $K = K(v)$ is independent of ε and τ.

It remains to estimate $\Delta\bar{x}(t)$ on the last interval $[\tau + \varepsilon, b]$, where it satisfies the equation

$$\Delta\dot{\bar{x}}(t) = f\big(\bar{x}(t)+\Delta\bar{x}(t),\bar{u}(t),t\big) - f\big(\bar{x}(t),\bar{u}(t),t\big) \;\text{ with }\; \|\Delta\bar{x}(\tau+\varepsilon)\| \le K\varepsilon$$

the solution of which is understood in the integral sense (6.62). Since

$$\|\Delta\bar{x}(t)\| \le \|\Delta\bar{x}(\tau+\varepsilon)\| + \int_{\tau+\varepsilon}^{t} \big\|f\big(\bar{x}(s)+\Delta\bar{x}(s),\bar{u}(s),s\big) - f\big(\bar{x}(s),\bar{u}(s),s\big)\big\|\,ds$$

$$\le K\varepsilon + \ell\int_{\tau+\varepsilon}^{t} \|\Delta\bar{x}(s)\|\,ds, \quad \tau+\varepsilon\le t\le b\,,$$

we again apply the Gronwall lemma and arrive, by increasing K if necessary, at the desired estimate of $\|\Delta\bar{x}(t)\|$ on the whole interval $[a,b]$. △

Now we are ready to justify the maximum principle of Theorem 6.37 for the free-endpoint control problem under consideration.

Proof of Theorem 6.37 for the free-endpoint problem. Let $\{\bar{u}(\cdot),\bar{x}(\cdot)\}$ be an optimal solution to problem (6.63), and let $p(\cdot)$ be the corresponding solution to the adjoint system (6.67) with the boundary/transversality condition (6.70). We are going to show that the maximum condition (6.66) holds for a.e. $t\in[a,b]$. Assume on the contrary that there is a set $T\subset[a,b]$ of a positive measure such that

$$H\big(\bar{x}(t),p(t),\bar{u}(t),t\big) < \sup_{u\in U} H\big(\bar{x}(t),p(t),u,t\big) \;\text{ for } t\in T\,.$$

Then using standard results on *measurable selections* under the assumptions made, we find a measurable mapping $v\colon T\to U$ satisfying

$$\Delta_v H(t) := H\big(\bar{x}(t),p(t),v(t),t\big) - H\big(\bar{x}(t),p(t),\bar{u}(t),t\big) > 0, \quad t\in T\,.$$

Let $T_0\subset[a,b]$ be a set of *Lebesgue regular* points (or points of approximate continuity) for the function $\Delta_v H(t)$ on the interval $[a,b]$, which is of *full measure* on $[a,b]$ due to the classical Denjoy theorem. Given $\tau\in T_0$ and $\varepsilon>0$, consider a *needle variation* of the optimal control built by

$$u(t) := \begin{cases} v(t), & t\in T_\varepsilon := [\tau,\tau+\varepsilon)\cap T_0\,,\\[1ex] \bar{u}(t), & t\in[a,b]\setminus T_\varepsilon\,, \end{cases}$$

and apply to $\bar{u}(\cdot)$ and $u(\cdot)$ the *increment formula* for the cost functional from Lemma 6.42. By this formula we have the relation

$$\Delta J[\bar{u}] = -\int_{\tau}^{\tau+\varepsilon} \Delta_v H(t)\,dt + \eta_1+\eta_2+\eta_3$$

with the above positive increment of the Hamilton-Pontryagin function $\Delta_v H(t)$ and the remainders η_i, $i=1,2,3$, defined in Lemma 6.42 along the trajectory increment $\Delta\bar{x}(\cdot)$ corresponding to the needle variation $u(\cdot)$ under consideration. It follows from the proof of Lemma 6.43, with an easy modification to take into account the variable perturbation $v(\cdot)$ on T_ε instead of the constant one in (6.71), that $\|\Delta\bar{x}(t)\| = O(\varepsilon)$ for $t \in [a,b]$. Hence

$$\eta_1 = o\big(\|\Delta\bar{x}(b)\|\big) = o(\varepsilon), \quad \eta_2 = -\int_a^b o\big(\|\Delta\bar{x}(t)\|\big)\,dt = o(\varepsilon), \quad \text{and}$$

$$\begin{aligned}\eta_3 &\le \int_\tau^{\tau+\varepsilon} \Big|\Big\langle \frac{\partial\Delta_v H\big(\bar{x}(t),p(t),\bar{u}(t),t\big)}{\partial x}, \Delta\bar{x}(t)\Big\rangle\Big|\,dt \\ &\le K\varepsilon\int_\tau^{\tau+\varepsilon}\Big\|\frac{\partial\Delta_v H\big(\bar{x}(t),p(t),\bar{u}(t),t\big)}{\partial x}\Big\|\,dt = o(\varepsilon)\;.\end{aligned}$$

The choice of $\tau \in T_0$ as a Lebesgue regular point of the function $\Delta_v H(t)$ and the construction of the Bochner integral yield

$$\int_\tau^{\tau+\varepsilon}\Delta_v H(t)\,dt = \varepsilon\Big[H\big(\bar{x}(\tau),p(\tau),v(\tau),\tau\big) - H\big(\bar{x}(\tau),p(\tau),\bar{u}(\tau),\tau\big)\Big] + o(\varepsilon)\;.$$

Thus we get the representation

$$\Delta J[\bar{u}] = -\varepsilon\Big[H\big(\bar{x}(\tau),p(\tau),v(\tau),\tau\big) - H\big(\bar{x}(\tau),p(\tau),\bar{u}(\tau),\tau\big)\Big] + o(\varepsilon)\;,$$

which implies that $\Delta J[\bar{u}] < 0$ along the above needle variation of the optimal control $\bar{u}(\cdot)$ for all $\varepsilon > 0$ sufficiently small. This clearly contradicts the optimality of $\bar{u}(\cdot)$ in problem (6.63) and completes the proof of Theorem 6.37 for the free-endpoint optimal control problem. △

6.3.3 Transversality Conditions for Problems with Inequality Constraints

One can see from the preceding subsection that the analytic proof of the maximum principle given there for the free-endpoint optimal control problem doesn't hold in the case of endpoint constraints of types (6.64) and/or (6.65). Indeed, in that proof we didn't care about the *feasibility* with respect to these constraints of trajectories corresponding to needle control variations. Dealing with endpoint constraint problems requires a more sophisticated technique that involves the *geometry* of the reachable set for system (6.61) and its interaction with the cost functional and endpoint constraints. The crux of the matter is to show that there is a *convex* set generated by feasible endpoint variations of the given optimal trajectory that doesn't intersect some convex set "forbidden" by optimality, which allows us to employ the *convex separation*. This can be achieved by using *multineedle* variations of the optimal control in question. The latter is realized by the *continuity of time* in $[a,b]$ and actually reflects the *hidden convexity* of continuous-time control problems.

In this subsection we consider optimal control problems that involve only endpoint constraints of the *inequality type* (6.64). Control problems with the equality constraints (6.65) are somewhat different (more complicated); they will be studied in the next subsection. Our main goal is to derive the transversality condition (6.68) in the relations of the maximum principle from Theorem 6.37 in the case of inequality constraints given by differentiable functions. As discussed in Subsect. 6.3.1, transversality conditions in more general control problems and under less restrictive assumptions can be either reduced to the one in (6.68) or derived similarly.

Let us emphasize that, although we study optimal control problems with a *Banach* state space X, they involve only *finitely many* endpoint constraints on system trajectories. The method we develop allows us to take an advantage of this setting (which is somehow related to the *finite codimension* property of the constraint set; cf. Corollaries 6.29, 6.24 and Remark 6.25) and to deal with *finite-dimensional images* of endpoint variations under the derivative operators for the cost and constraint functions, employing thus the convex separation theorem in finite dimensions.

In the rest of this subsection we consider the optimal control problem (6.63) with the inequality endpoint constraints (6.64) and fix an optimal solution $\{\bar{u}(\cdot),\bar{x}(\cdot)\}$ to this problem. Assume without loss of generality that $\varphi_i\big(\bar{x}(b)\big)=0$ for all $i=1,\ldots,m$. It is easy to see from the proof (as usually with inequality constraints) that $\lambda_i=0$ if $\varphi_i\big(\bar{x}(b)\big)<0$ for some $i\in\{1,\ldots,m\}$, i.e., the corresponding function φ_i can be excluded from consideration. In this setting the complementary slackness conditions of Theorem 6.37 hold automatically, and we need to establish relations (6.66)–(6.68) with $r=0$ and $0\neq(\lambda_0,\ldots,\lambda_m)\in I\!\!R^m_+$.

Along with (single) needle variations introduced in the preceding subsection we now invoke "multineedle variations" built as follows. Fix a natural number $M\geq 1$ and M points $\tau_j\in[a,b]$ of the original time interval with $a\leq\tau_1<\tau_2\leq\ldots<\tau_M<b$. Consider also arbitrary numbers $N_j\in I\!\!N$ for $j=1,\ldots,M$ and $\alpha_{ij}\in[0,1]$ for $i=1,\ldots,N_j$ satisfying the relations

$$\tau_j+\varepsilon_0\sum_{i=1}^{N_j}\alpha_{ij}<\tau_{j+1},\quad j=1,\ldots,M-1,\quad\text{and}\quad\tau_M+\varepsilon_0\sum_{i=1}^{N_M}\alpha_{iM}<b$$

with some $\varepsilon_0>0$. We are going to construct a perturbation $u(\cdot)$ of the reference control $\bar{u}(\cdot)$ that is different from $\bar{u}(\cdot)$ on $N_1+\ldots+N_M$ time intervals of a *small total length*, while the difference between $u(\cdot)$ and $\bar{u}(\cdot)$ on these intervals is up to *any element* from the feasible control region U. To proceed, let us take arbitrary $v_{ij}\in U$ and $\varepsilon\in(0,\varepsilon_0]$ and define a *multineedle variation* $u(\cdot)$ of the reference control $\bar{u}(\cdot)$ by

$$u(t) := \begin{cases} v_{ij}, & t \in \Big[\tau_j + \sum_{\nu=0}^{i-1} \alpha_{\nu j}\varepsilon, \tau_j + \sum_{\nu=1}^{i} \alpha_{\nu j}\varepsilon\Big), \quad \alpha_{0j} := 0, \; i = 1, \ldots, N_j \;, \\ \bar{u}(t), & t \notin \Big[\tau_j, \tau_j + \sum_{i=1}^{N_j} \alpha_{ij}\varepsilon\Big), \quad j = 1, \ldots, M \;. \end{cases} \tag{6.72}$$

Note that, although there are M basic points τ_j, the multineedle variation (6.72) involves $N_1 + \ldots + N_M$ points of needle-type perturbations; this is different from a single needle variation (6.71) even in the case of $M = 1$. Actually the multineedle variation (6.72) is a *collection* of $N_1 + \ldots + N_M$ *single* needle variations of type (6.71) with the given parameters $(\tau_j, v_{ij}, \alpha_{ij}, \varepsilon)$.

Let $\Delta\bar{x}_{\tau_j, v_{ij}, \alpha_{ij}, \varepsilon}(b)$ be the *endpoint increment* of the trajectory $\bar{x}(\cdot)$ corresponding to the *single needle variation* of type (6.71) with the parameters $(\tau_j, v_{ij}, \alpha_{ij}, \varepsilon)$. Dealing with the differential equation (6.61) of smooth dynamics and its *linearization* in x along the process $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ as in the proof of Lemma 6.43, we can check the relationship

$$\Delta\bar{x}_{\tau_j, v_{ij}, \alpha_{ij}, \varepsilon}(b) = \big[\alpha_{ij}\Lambda\bar{x}_{\tau_j, v_{ij}, 1}(b)\big]\varepsilon + o(\varepsilon) \tag{6.73}$$

between $\Delta\bar{x}_{\tau_j, v_{ij}, \alpha_{ij}, \varepsilon}(b)$ and the corresponding *linearized endpoint increment* $\Lambda\bar{x}_{\tau_j, v_{ij}, \alpha_{ij}}(b)$ computed by

$$\Lambda\bar{x}_{\tau_j, v_{ij}, \alpha_{ij}}(b) = \alpha_{ij} R(b, \tau_j)\Delta_{v_{ij}} f\big(\bar{x}(\tau_j), \bar{u}(\tau_j), \tau_j\big) =: \alpha_{ij}\Lambda\bar{x}_{\tau_j, v_{ij}, 1}$$

via the *resolvent* (Green function) $R(t, \tau)$ of the linearized homogeneous equation for (6.61) with respect to x along $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ given as

$$\dot{x} = \nabla_x f\big(\bar{x}(t), \bar{u}(t), t\big)x \;.$$

Furthermore, the *endpoint increment* $\Delta\bar{x}(b)$ generated by the *multineedle variation* (6.72) is represented by

$$\Delta\bar{x}(b) = \Big[\sum_{j=1}^{M}\sum_{i=1}^{N_j} \alpha_{ij}\Lambda_{\tau_j, v_{ij}, 1}\bar{x}(b)\Big]\varepsilon + o(\varepsilon) \;.$$

Now we form the following finite-dimensional *linearized image set* generated by inner products involving derivatives of the cost and constraint functions and the linearized endpoint increments corresponding to *all the multineedle variations* (6.72) of the reference optimal control $\bar{u}(\cdot)$:

$$\begin{aligned} S := \Big\{(y_0, \ldots, y_m) \in I\!R^{m+1}\Big|\; & y_0 = \sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle \nabla\varphi_0\big(\bar{x}(b)\big), \Lambda_{\tau_j, v_{ij}, \alpha_{ij}}\bar{x}(b)\Big\rangle, \ldots \;, \\ & y_m = \sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle \nabla\varphi_m\big(\bar{x}(b)\big), \Lambda_{\tau_j, v_{ij}, \alpha_{ij}}\bar{x}(b)\Big\rangle\Big\} \end{aligned} \tag{6.74}$$

with arbitrary $\tau_j \in [a,b)$, $v_{ij} \in U$, $\alpha_{ij} \in [0,1]$, $i = 1,\ldots,N_j$, $N_j \in I\!N$, $j = 1,\ldots,M$, and $M \in I\!N$.

There are *two crucial facts* regarding the set S in (6.74). First of all, it happens to be *convex*, which is mainly due to the possibility of using arbitrary $\alpha_{ij} \in [0,1]$ in multineedle variations (6.72). The latter is based on the *time continuity* of $[a,b]$ and, as mentioned above, reflects the *hidden convexity* of continuous-time control systems. The second fact is due to the *optimality* of $\bar{u}(\cdot)$ in the constrained control problem (6.63), (6.64): it ensures that the linearized image set (6.74) *doesn't intersect* the convex set of *forbidden points* (from the viewpoint of optimality and inequality constraints in the problem under consideration) given by

$$I\!R^{m+1}_{<} := \left\{(y_0,\ldots,y_m) \in I\!R^{m+1} \,\middle|\, y_i < 0 \;\text{ for all }\; i = 0,\ldots,m\right\}.$$

Both of these facts are proved in the following lemma.

Lemma 6.44 (hidden convexity and primal optimality condition in control problems with inequality constraints). *Let $\{\bar{u}(\cdot),\bar{x}(\cdot)\}$ be an optimal solution to the inequality constrained problem* (6.63) *and* (6.64)*, where all the functions φ_i are supposed to be Fréchet differentiable at $\bar{x}(b)$ in addition to the standing assumptions of Subsect.* 6.3.1*. Then the linearized image set S in* (6.74) *is convex and doesn't intersect the set of forbidden points $I\!R^{m+1}_{<}$.*

Proof. Let us fix a collection of parameters (τ_i, v_{ij}, N_j, M) and show that the set (6.74), still denoted by S, is *convex* while the numbers α_{ij} are arbitrarily taken from $[0,1]$. This clearly implies the convexity of the "full" set S. Indeed, taking two different collections of (τ_i, v_{ij}, N_j, M), we may always unify them, which again gives an admissible multineedle variation (6.72). It is therefore sufficient to justify the convexity of S only in the case when parameters α_{ij} take values on the interval $[0,1]$.

To proceed, we fix (τ_i, v_{ij}, N_j, M) and take two collections $\{\alpha^{(1)}_{ij}\}$ and $\{\alpha^{(2)}_{ij}\}$ such that the corresponding points $y^{(1)}$ and $y^{(2)}$ in (6.74) belong to the linearized image set S. Then considering the point $\lambda y^{(1)} + (1-\lambda)y^{(2)}$ for any $\lambda \in [0,1]$ and taking into account the linear dependence of $\Lambda\bar{x}_{\tau_j,v_{ij},\alpha_{ij}}(b)$ on α_{ij}, we conclude that $\lambda y^{(1)} + (1-\lambda)y^{(2)}$ is an element of S corresponding to $\{\lambda\alpha^{(1)}_{ij} + (1-\lambda)\alpha^{(2)}_{ij}\}$, which justifies the convexity of S.

It remains to show that $S \cap I\!R^{m_1}_{<} = \emptyset$, where S stands for the "full" image set in (6.74) corresponding to all the admissible multineedle variations (6.72). Assuming the contrary, we find a multineedle variation (6.72) with some admissible parameters $(\tau_i, v_{ij}, \alpha_{ij}, N_j, M)$ such that

$$\sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle\nabla\varphi_0\big(\bar{x}(b)\big),\,\varLambda_{\tau_j,v_{ij},\alpha_{ij}}\bar{x}(b)\Big\rangle<0,\ldots\;,$$

$$\sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle\nabla\varphi_m\big(\bar{x}(b)\big),\,\varLambda_{\tau_j,v_{ij},\alpha_{ij}}\bar{x}(b)\Big\rangle<0\;.$$

Then using the Fréchet differentiability of the functions $\varphi_0,\ldots,\varphi_m$ at $\bar{x}(b)$ and the above relationship between the endpoint increment $\varDelta\bar{x}(b)$ generated by (6.72) and the linearized ones $\varLambda_{\tau_j,v_{ij},\alpha_{ij}}$ corresponding to each collection $(\tau_j,v_{ij},\alpha_{ij},N_j,M)$, we get

$$\varphi_k\big(x(b)\big)-\varphi_k\big(\bar{x}(b)\big)=\big\langle\nabla\varphi_k\big(\bar{x}(b)\big),\,\varDelta\bar{x}(b)\big\rangle+o(\varepsilon)$$

$$=\Big[\sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle\nabla\varphi_k\big(\bar{x}(b)\big),\,\varLambda_{\tau_j,v_{ij},\alpha_{ij}}\bar{x}(b)\Big\rangle\Big]\varepsilon+o(\varepsilon)<0$$

for all $k=0,\ldots,m$ and all $\varepsilon>0$ sufficiently small. The latter means that there is a multineedle control variation (6.72) such that the corresponding trajectory $x(\cdot)$ satisfies all the inequality constraints (6.64), being therefore *feasible* for the problem under consideration, and gives a smaller value to the cost functional in (6.63) in comparison with $\bar{x}(\cdot)$. This contradicts the optimality of the process $\{\bar{u}(\cdot),\bar{x}(\cdot)\}$ in problem (6.63), (6.64) and thus completes the proof of the lemma. △

The obtained relation $S\cap I\!R^{m+1}_{<}=\emptyset$ can be viewed as a *primal* necessary optimality condition, which is of course not efficient, since it depends on control variations and is not expressed in terms of the initial data of the problem under consideration. To proceed further, we pass to its *dual* form employing the *convex separation* theorem and then invoking the Hamilton-Pontryagin function by the constructions of the increment method in Lemma 6.42; see the arguments below.

Proof of Theorem 6.37 for problems with inequality constraints. Applying the classical *separation theorem* to the convex sets S and $I\!R^{m+1}_{<}$ from Lemma 6.44, we find a *nonzero* vector $(\lambda_0,\ldots,\lambda_m)\in I\!R^{m+1}$ such that

$$\sum_{i=0}^{m}\lambda_i y_i\ge\sum_{i=0}^{m}\lambda_i z_i\quad\text{for all}\quad(y_0,\ldots,y_m)\in S\quad\text{and}\quad(z_0,\ldots,z_m)\in I\!R^{m+1}_{<}\;.$$

This easily implies that $\lambda_i\ge 0$ for all $i=0,\ldots,m$ and that

$$\sum_{i=0}^{m}\lambda_i y_i\ge 0\quad\text{whenever}\quad(y_0,\ldots,y_m)\in S\;.\tag{6.75}$$

Note that the vector $(\lambda_0,\ldots,\lambda_m)$ *doesn't depend* on a specific multineedle variation (6.72); it separates the set of all such variations from $0\in I\!R^{m+1}$. In particular, employing (6.75) just for vectors $(y_0, \ldots, y_m)$ generated by *single* needle variations (6.71) with parameters (τ, v, ε) and taking into account the relationship (6.73) between the full and linearized increments of the optimal trajectory along (single) needle variations, one has

$$\sum_{i=0}^{m} \lambda_i \Big\langle \nabla\varphi_i\big(\bar{x}(b)\big), \varDelta_{\tau,v,\varepsilon}\bar{x}(b)\Big\rangle + o(\varepsilon) \geq 0$$

for all $\tau \in [a, b)$, $v \in U$, and $\varepsilon > 0$ sufficiently small. Putting now

$$p(b) := -\sum_{i=0}^{m} \lambda_i \nabla\varphi_i\big(\bar{x}(b)\big)$$

and proceeding as in the proof of Lemma 6.42 and Theorem 6.37 for the free-endpoint control problem in Subsect. 6.3.2 with the replacement of the boundary condition (6.70) by the latter one, we end the proof of Theorem 6.37 for problems with inequality endpoint constraints. $\triangle$

6.3.4 Transversality Conditions for Problems with Equality Constraints

To complete the proof of Theorem 6.37, it remains to justify it for the case of *equality* endpoint constraints in the problem under consideration. Without loss of generality we focus here on the optimal control problem given by (6.63) and (6.65), i.e., with no inequality constraints considered in the preceding subsection. For convenience, suppose that the equality constraints are given by the first m functions φ_i as

$$\varphi_i\big(x(b)\big) = 0, \quad i = 1, \ldots, m \,. \tag{6.76}$$

Having this in mind, form again the *linearized image set* S in (6.74) generated now by the images of multineedle variations under the gradient mappings for the cost and *equality* constraint functions. The set of *forbidden points* in the equality constrained problem is given by

$$S^{<} := \Big\{(y_0, \ldots, y_m) \in I\!\!R^{m+1} \Big|\; y_0 < 0, y_1 = 0, \ldots, y_m = 0\Big\} \,.$$

Our goal is to investigate all the possible relationships between the image set S and the above set of forbidden points that are allowed by the optimality of $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$. The most difficult case is considered in the next lemma, which establishes that the origin cannot be an *interior point* of the $I\!\!R^m$-projection of S. The proof given below involves the *Brouwer fixed-point theorem*. Note that, although this fundamental topological result is heavily finite-dimensional, it allows us to deal with the optimal control problems described by evolution equations in *infinite dimensions*. The crux of the matter is, as mentioned, that the control problem has *finitely many* endpoint constraints, which ensures the *finite codimension* property of the constraint set.

Lemma 6.45 (endpoint variations under equality constraints). *Let $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ be an optimal solution to the control problem (6.63), (6.76) under the standing assumptions on X, U, and f. Assume also that the functions $\varphi_0, \ldots, \varphi_m$ are Fréchet differentiable at $\bar{x}(b)$ and that $\varphi_1, \ldots, \varphi_m$ are in addition continuous around this point. Then one has*

$$0 \notin \operatorname{int}\big(\operatorname{proj}_{I\!R^m} S\big) ,$$

where the linearized image set S is generated in (6.74) by the endpoint equality constraints (6.76).

Proof. Assume the contrary and denote by B_η a closed ball in $I\!R^m$ of radius $\eta > 0$ centered at the origin. Let $\mathcal{T}$ be a regular "tetrahedron" with the vertices $q^{(s)}$, $s = 1, \ldots, m+1$, inscribed into $\mathcal{T}$. If η is sufficiently small, then for each $s = 1, \ldots, m+1$ there are numbers $\{\alpha_{ij}^{(s)}\}$ in the multineedle variation (6.72) and $\nu < 0$ such that

$$\begin{cases} \displaystyle\sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle \nabla\varphi_0\big(\bar{x}(b)\big), \varLambda_{\tau_j, v_{ij}, \alpha_{ij}^{(s)}}\bar{x}(b)\Big\rangle < \nu < 0 \ \text{ and} \\[2ex] \displaystyle q_k^{(s)} = \sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle \nabla\varphi_k\big(\bar{x}(b)\big), \varLambda_{\tau_j, v_{ij}, \alpha_{ij}^{(s)}}\bar{x}(b)\Big\rangle \end{cases}$$

for all $k = 1, \ldots, m$, where $q_k^{(s)}$ stands for the kth component of the vertex $q^{(s)}$. Each point $q = q(\beta) \in \mathcal{T}$ can be represented as a *convex combination* of the tetrahedron vertices by

$$q(\gamma) = \sum_{s=1}^{m+1}\gamma_s q^{(s)} \ \text{ with } \ \gamma = (\gamma_1, \ldots, \gamma_{m+1}) \in P ,$$

where P connotes the m-dimensional simplex. Let $u_{\gamma,\varepsilon}(\cdot)$ be a multineedle variation (6.72) with the parameters $(\tau_j, v_{ij}, \alpha_{ij}(\gamma), \varepsilon)$, where

$$\alpha_{ij}(\gamma) := \sum_{s=1}^{m+1}\gamma_s\alpha_{ij}^{(s)}, \quad \gamma = (\gamma_1, \ldots, \gamma_m) \in P .$$

Consider now an ε-parametric family of mappings $g(\cdot, \varepsilon)\colon P \to I\!R^m$ defined by

$$g(\gamma, \varepsilon) := \Big(\frac{\varphi_1\big(x_{\gamma,\varepsilon}(b)\big) - \varphi_1\big(\bar{x}(b)\big)}{\varepsilon}, \ldots, \frac{\varphi_m\big(x_{\gamma,\varepsilon}(b)\big) - \varphi_m\big(\bar{x}(b)\big)}{\varepsilon}\Big) ,$$

where $x_{\gamma,\varepsilon}(\cdot)$ signifies a trajectory for (6.61) corresponding to the multineedle control variation $u_{\gamma,\varepsilon}(\cdot)$. Putting also

$$g(\gamma,0):=\Big(\sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle\nabla\varphi_1\big(\bar{x}(b)\big),\alpha_{ij}(\gamma)\Lambda_{\tau_j,v_{ij},1}\bar{x}(b)\Big\rangle,\ldots\ ,$$

$$\sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle\nabla\varphi_m\big(\bar{x}(b)\big),\alpha_{ij}(\gamma)\Lambda_{\tau_j,v_{ij},1}\bar{x}(b)\Big\rangle\Big)\ ,$$

we conclude that the mapping $g(\cdot,\cdot)$ is *continuous* on $P\times[0,\varepsilon_0]$ with ε_0 sufficiently small. This is due to the standing assumptions on the Fréchet differentiability of $\varphi_1,\ldots,\varphi_m$ *at* $\bar{x}(b)$ and the continuity of these functions *around* this point. It follows from the above constructions that

$$g(\gamma,0)=\sum_{s=1}^{m+1}\gamma_s q^{(s)}\ \text{ and }\ G(P,0)=\mathcal{T}\ ;$$

thus the set $g(P,0)$ contains the origin as an interior point. Let us show that there is $\widehat{\varepsilon}>0$ such that

$$0\in\operatorname{int}g(P,\varepsilon)\ \text{ for all }\ \varepsilon<\widehat{\varepsilon}\ .$$

To proceed, we observe that the mapping $g(\cdot,0)$ is one-to-one and continuous from P into $\mathcal{T}$. Hence its *inverse mapping* is single-valued and continuous; let us denote it by $p(y)$ and put

$$h(y,\varepsilon):=g\big(p(y),\varepsilon\big)\ \text{ for all }\ y\in\mathcal{T}\ \text{ and }\ \varepsilon\in[0,\varepsilon_0]\ .$$

Take $\eta>0$ so small that the ball B_η of radius η centered at the origin belongs to the tetrahedron $\mathcal{T}$. Then the continuity of the mapping $h(\cdot,\cdot)$ yields the existence of $\widehat{\varepsilon}>0$ such that

$$\|h(y,0)-h(y,\varepsilon)\|<\eta\ \text{ whenever }\ \varepsilon<\widehat{\varepsilon}\ .$$

Thus, given any $\varepsilon\in(0,\widehat{\varepsilon})$, the continuous mapping $h(y,0)-h(y,\varepsilon)$ *transforms the ball* B_η *into itself*. Employing the *Brouwer fixed-point theorem*, we find a point $y^\varepsilon\in B_\eta$ satisfying

$$h(y^\varepsilon,0)-h(y^\varepsilon,\varepsilon)=y^\varepsilon\ \text{ for all }\ \varepsilon\in(0,\widehat{\varepsilon})\ .$$

This implies by $h(y,0)\equiv y$ that

$$h(y^\varepsilon,\varepsilon)=g\big(p(y^\varepsilon),\varepsilon\big)=g(\gamma^\varepsilon,\varepsilon)\ \text{ for some }\ \gamma^\varepsilon\in P\ \text{ with }\ g(\gamma^\varepsilon,0)=y^\varepsilon\ .$$

Taking into account the construction of $g(\cdot,\cdot)$, we conclude that the trajectories $x_{\gamma^\varepsilon,\varepsilon}(\cdot)$ generated by the multineedle variations $u_{\gamma^\varepsilon,\varepsilon}(\cdot)$ under consideration *satisfy the equality constraints* (6.76) for all $\varepsilon\in(0,\widehat{\varepsilon})$. Moreover, for the variations along the cost functional one has

$$\sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle\nabla\varphi_0\big(\bar{x}(b)\big),\,\varLambda_{\tau_j,v_{ij},\alpha_{ij}(\gamma^\varepsilon)}\bar{x}(b)\Big\rangle$$

$$=\sum_{s=1}^{m+1}\gamma_s^\varepsilon\Big(\sum_{j=1}^{M}\sum_{i=1}^{N_j}\Big\langle\nabla\varphi_0\big(\bar{x}(b)\big),\,\varLambda_{\tau_j,v_{ij},\alpha_{ij}^{(s)}}\bar{x}(b)\Big\rangle\Big)$$

$$<\sum_{s=1}^{m+1}\gamma_s^\varepsilon\nu<\nu\quad\text{whenever}\quad\varepsilon\in(0,\widehat{\varepsilon})\;.$$

The latter implies, similarly to the case of inequality constraints, that

$$\varphi_0\big(x_{\gamma^\varepsilon,\varepsilon}(b)\big)<\varphi_0\big(\bar{x}(b)\big)$$

along some *feasible solutions* to the equality constrained problem (6.63), (6.65). This contradicts the *optimality* of the process $\{\bar{u}(\cdot),\bar{x}(\cdot)\}$ in this problem and completes the proof of the lemma. △

Based on Lemma 6.45 and the arguments developed in Subsects. 6.3.2 and 6.3.3, we finally justify Theorem 6.37 in the remaining case of equality constraints and thus complete the whole proof of this theorem.

Proof of Theorem 6.37 for problems with equality constraints. Taking into account Lemma 6.45, there are the following two possible relationships between the linearized image set S in (6.72) corresponding the equality constraints (6.76) and the set of forbidden points $S^<$:

(a) $S\cap S^<=\emptyset$;

(b) $S\cap S^<\neq\emptyset$ and $0\in\mathrm{bd}\big(\mathrm{proj}_{I\!R^m}S\big)$.

Consider first case (a). Since both sets S and $S^<$ are convex, we employ the classical *separation theorem* for convex sets and find a *nonzero* vector $(\lambda_0,\ldots,\lambda_m)\in I\!R^{m+1}$ such that

$$\sum_{i=0}^{m}\lambda_i y_i\ge\sum_{i=0}^{m}\lambda_i z_i\quad\text{for all}\quad(y_0,\ldots,y_m)\in S\quad\text{and}\quad(z_0,\ldots,z_m)\in S^<\;.$$

It easily implies, by the structure of the forbidden set $S^<$, that $\lambda_0\ge 0$ and that the relation (6.75) holds. To complete the proof of the theorem in this case, we now proceed exactly as in the case of inequality constraints at the very end of Subsect. 6.3.3.

It remains to examine case (b). Denote $\varOmega:=\mathrm{proj}_{I\!R^m}S$ and observe that this set is closed and convex in $I\!R^m$. Since $0\in\mathrm{bd}\,\varOmega$, we apply the supporting hyperplane theorem for convex sets and find a nonzero m-vector $(\lambda_1,\ldots,\lambda_m)$ supporting $\varOmega$ at the origin. Then we again arrive at the basic relation (6.75) with the nontrivial $(m+1)$-vector $(0, \lambda_1, \ldots, \lambda_m)$ and complete the proof of the theorem similarly to the case of inequality constraints. $\triangle$

Note that the *continuity* assumption on the *equality constraint* functions φ_i *around* $\bar{x}(b)$, an addition to their Fréchet differentiability at this point, is essential for the validity of Theorem 6.37 even in the case of finite-dimensional state space X with the trivial dynamics $f = 0$; see Example 5.12.

6.4 Approximate Maximum Principle in Optimal Control

This section is devoted to optimal control problems for a *parametric family* of dynamical systems governed by *discrete approximations* of control systems with continuous time. Discrete/finite-difference approximations play a prominent role in both qualitative and numerical aspects of optimal control. While considered as a *process* with a decreasing step of discretization, they occupy an *intermediate position* between continuous-time control systems and discrete-time control systems with fixed steps. Recall that discrete approximations of general control problems for differential inclusions have been studied in Sect. 6.1, but the attitude there was different from that in this section. Our previous direction was *from discrete to continuous*: to establish necessary optimality conditions for discrete-time systems with *fixed* discretization steps and then to use well-posed discrete approximations as a *vehicle* in deriving optimality conditions for continuous-time control systems. The results obtained in this way in Sect. 6.1 provide necessary conditions of a *maximum principle* type only under some *convexity/relaxation* assumptions imposed *a priori* on the system dynamics.

Now we are going to explore the other direction in the relationship between discrete-time and continuous-time control systems: *from continuous to discrete*. Having in mind that the *Pontryagin maximum principle* (PMP) and its extensions to nonsmooth problems and differential inclusions hold *without any convexity/relaxation* assumptions on the continuous-time dynamics, it is challenging to clarify the possibility to establish necessary optimality conditions of the *maximum principle type* for discrete approximations. The results obtained in this direction are *rather surprising*; see below.

6.4.1 Exact and Approximate Maximum Principles for Discrete-Time Control Systems

As seen in Sects. 6.2 and 6.3, the relations of the maximum principle involving the Weierstrass-Pontryagin maximum condition hold for continuous-time control systems with *no* a priori *convexity* assumptions. This happens due to specific features of the continuous-time dynamics that generates some *hidden convexity* property inherent in such control systems. Probably the most striking and deep manifestation of the hidden convexity for continuous-time systems is given by the fundamental Lyapunov theorem on the range convexity of *nonatomic/continuous* vector measures, which is equivalent to the Aumann convexity theorem for set-valued integrals; see, e.g., the discussion in the proof of Lemma 6.18 and the references therein. In the proof of the maximum principle for control systems with smooth dynamics given in Sect. 6.3 we didn't invoke these results while exploiting *directly the time continuity* in the construction of needle (and multineedle) variations generating the *automatic convexity* of the linearized image set as in Lemma 6.44. One cannot expect such properties for *discrete-time* systems described by the general discrete inclusions of the type

$$x(t+1) \in F(x(t), t), \quad t = 0, \ldots, K-1 ,$$

or by their parameterized control representations

$$x(t+1) = f\big(x(t), u(t), t\big), \quad u(t) \in U, \quad t = 0, \ldots, K-1 ,$$

where $K \in I\!N$ signifies the number of steps (final discrete time) for the discrete dynamic process. However, the *discrete maximum principle* holds if the sets of "discrete velocities" $F(x,t)$, or their counterparts $f(x,U,t)$ for the parameterized control systems, are *assumed to be convex*. In this case the maximum condition is actually a *direct consequence* of the *Euler-Lagrange inclusion* as discussed above. Indeed, it follows from the extremal property of the coderivative to convex-valued mappings from Theorem 1.34 due to a special representation of the normal cone to *convex* sets.

As well known, the discrete maximum principle *may not hold*, even for simple control systems with smooth dynamics, if the above velocity sets are *not convex*. We now present an example of the failure of the discrete maximum principle (as a natural analog of the Pontryagin maximum principle for discrete-time control systems) for a *family* of simple free-endpoint problems with smooth dynamics. In this example the Hamilton-Pontryagin function achieves its *global minimum* (instead of maximum) along *any* feasible control. As always in this chapter, a "free-endpoint" problem means that there are no constraints on the right endpoint of the system trajectories, while the left endpoint may be fixed.

Example 6.46 (failure of the discrete maximum principle). *There is a family of optimal control problems of minimizing a linear function over two-dimensional discrete-time control systems with smooth dynamics and no endpoint constraints such that any feasible control for these problems doesn't satisfy the discrete maximum principle.*

Proof. Consider the following family of optimal control problems with a two-dimensional state vector $x = (x_1, x_2) \in I\!R^2$:

$$
\begin{cases}
\text{minimize } \ J[u,x] = \varphi\big(x(K)\big) := x_2(3) \ \text{ subject to} \\[2ex]
x_1(t+1) = \vartheta\big(u(t),t\big), \quad x_1(0) = 0\,, \\[2ex]
x_2(t+1) = \gamma\big(x_1(t)\big)^2 + \eta x_2(t) - \big(\gamma/\eta\big)\big(\vartheta(u(t),t)\big)^2, \quad x_2(0) = 0\,, \\[2ex]
u(t) \in U, \quad t = 0,1,2\,,
\end{cases}
$$

where the scalar function $\vartheta(\cdot,\cdot)$, the numbers γ, η, and the control set U are arbitrary. Then (a natural discrete counterpart of) the Hamilton-Pontryagin function for this system is

$$
\begin{aligned}
H\big(x(t), p(t+1), u, t\big) :&= \big\langle p(t+1), f\big(x(t),u,t\big)\big\rangle \\[1ex]
&= p_1(t+1)\vartheta(u,t) + \gamma p_2(t+1)\big(x_1(t)\big)^2 \\[1ex]
&\quad + \eta p_2(t+1)x_2(t) - (\gamma/\eta)p_2(t+1)\big(\vartheta(u,t)\big)^2\,,
\end{aligned}
$$

where the adjoint trajectory $p(\cdot)$ satisfies the corresponding discrete analog of the system (6.67) given by

$$
p(t) = \nabla_x H\big(x(t), p(t+1), u(t), t\big), \quad t \in \big\{0,\ldots,K-1\big\} = \big\{0,1,2\big\}\,,
$$

with the boundary/transversality condition

$$
p(K) = -\nabla\varphi\big(x(K)\big) = (0,-1) \ \text{ at } \ K = 3\,.
$$

For the problem under consideration one has

$$
\begin{aligned}
&p_2(3) = -1, \quad p_2(2) = -\eta, \quad p_2(1) = -\eta^2\,, \\[1ex]
&p_1(3) = 0, \quad p_1(2) = -\gamma x_1(2) = -2\gamma\vartheta\big(u(1),1\big)\,, \\[1ex]
&p_1(1) = -2\gamma\eta x_1(1) = -2\gamma\eta\vartheta\big(u(0),0\big)\,.
\end{aligned}
$$

Then considering only the terms depending on u in the Hamilton-Pontryagin function, we get

$$
\begin{aligned}
&H(u,0) = -\gamma\eta\Big[2\vartheta\big(u(0),0\big)\vartheta(u,0) - \big(\vartheta(u,0)\big)^2\Big]\,, \\[1ex]
&H(u,1) = -\gamma\Big[2\vartheta\big(u(1),1\big)\vartheta(u,1) - \big(\vartheta(u,1)\big)^2\Big]\,.
\end{aligned}
$$

This shows that, given an arbitrary $\vartheta(\cdot,\cdot)$ and U, the functions $H(u,0)$ and $H(u,1)$ attain their *global minimum* at any $u(0)$ and $u(1)$ whenever $\gamma > 0$ and $\gamma\eta > 0$, respectively. Thus the above relationships of the discrete maximum principle are *not necessary for optimality* in the family of optimal control problems under consideration. △

It is worth mentioning that the Hamilton-Pontryagin function in the above example *does attain its global maximum* over $u \in U$ for optimal controls when

$t = K - 1 = 2$. This can be shown by using the increment formula applied to *concave* cost functionals along needle variations of optimal controls; cf. the arguments below in Subsect. 6.4.2. Moreover, the discrete maximum principle *holds true* in the family of problems from Example 6.46 for *all* t, i.e., it provides necessary optimality conditions along optimal controls at every time moment, *if and only if*

$$\gamma \leq 0 \text{ and } \eta \geq 0 .$$

This follows from the above consideration and the results of Sect. 17 in Mordukhovich's book [901], where some *individual conditions* for the validity of the discrete maximum principle are given. Thus the simultaneous fulfillment of the conditions $\gamma \leq 0$ and $\eta \geq 0$ *fully describes* the relationships between the initial data of the problems from Example 6.46, which ensure the fulfillment of the discrete maximum principle. Note that overall the results in this direction obtained in the afore-mentioned book [901] strongly take into account *interconnections* between the *initial* data of *nonconvex* discrete-time control systems; see more discussions and examples therein.

The main attention in this section is paid not to optimal control problems governed by dynamical systems with *fixed* discrete time but to *finite-difference/discrete approximations* of continuous-time problems studied in the preceding section. This means that instead of the continuous-time control system (6.61) we consider a *sequence* of its finite-difference analogs given by

$$\begin{cases} x_N(t+h_N) = x_N(t) + h_N f\big(x_N(t), u_N(t), t\big), \quad x_N(a) = x_0 \in X \ , \\ u_N(t) \in U, \quad t \in T_N := \big\{a, a+h_N, \ldots, b-h_N\big\} \ , \end{cases} \tag{6.77}$$

with $N \in I\!N$ and $h_N := (b-a)/N$. Recall that discrete approximations of differential/evolution inclusions have been studied in Sect. 6.1 being used there as a vehicle to derive necessary optimality conditions for continuous-time control problems. Now our goal is quite opposite: to look at optimal control problems for discrete approximations from the viewpoint of their continuous-time counterparts. **The key question is**:

Would it be possible to obtain a certain natural analog of the Pontryagin maximum principle for optimal control problems governed by nonconvex finite-difference systems of type (6.77) *as* $N \to \infty$?

If the answer is *no*, then such a potential *instability* of the PMP may pose *serious challenges* to its implementation in any numerical algorithm involving finite-difference approximations of time derivatives.

To begin with, for each $N \in I\!N$ we consider the problem of minimizing a smooth endpoint function $\varphi_0\big(x(b)\big)$ over discrete-time process $\{u_N(\cdot), x_N(\cdot)\}$ satisfying (6.77). The *exact* PMP analog for each of these problems, the *discrete maximum principle*, is written as follows: given an optimal process $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$, there is an adjoint arc $p_N(\cdot)$, $t \in T_N \cup \{b\}$, satisfying

$$p_N(t) = p_N(t+h_N) + h_N \nabla_x H\big(\bar{x}_N(t), p_N(t+h_N), \bar{u}_N(t), t\big) \tag{6.78}$$

as $t \in T_N$ with the transversality condition

$$p_N(b) = -\nabla\varphi_0\big(\bar{x}_N(b)\big) \tag{6.79}$$

and such that the *exact maximum condition*

$$H\big(\bar{x}_N(t), p_N(t+h_N), \bar{u}_N(t), t\big) = \max_{u \in U} H\big(\bar{x}_N(t), p_N(t+h_N), u, t\big), \quad t \in T_N \ .$$

is valid whenever $N \in I\!N$, with the usual Hamilton-Pontryagin function

$$H(x, p, u, t) := \big\langle p, f(x, u, t)\big\rangle \ .$$

It follows from Example 6.46 (via standard rescaling) and the discussion above that this (*exact*) *discrete maximum principle* may be generally *violated* even for simple classes of optimal control problems governed by discrete approximation systems of type (6.77) whenever $N \in I\!N$. This may signify a possible instability of the PMP under discrete approximations. Note, however, that to require the fulfillment of such an *exact* counterpart of the PMP for discrete approximation systems is *too much* to ensure the PMP stability under discretization of continuous-time control systems.

What we really need for this purpose is the validity, along *every sequence* of optimal solutions $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ to the discrete approximation problems while $N \in I\!N$ becomes sufficiently large, of the *approximate maximum condition*

$$H\big(\bar{x}_N(t), p_N(t+h_N), \bar{u}_N(t), t\big) = \max_{u \in U} H\big(\bar{x}_N(t), p_N(t+h_N), u, t\big) + \varepsilon(t, h_N)$$

for all $t \in T_N$ with some $\varepsilon_N(t, h_N) \to 0$ as $N \to \infty$ uniformly in $t \in T_N$, where $p_N(\cdot)$ are the corresponding adjoint trajectories satisfying (6.78) and (6.79). In this case we say that the *approximate maximum principle* (AMP) holds for the discrete approximation problems under consideration. Such an approximate analog of the PMP ensures the *discretization stability* of the latter and thus justifies the possibility to employ the PMP in computer calculations and simulations of nonconvex continuous-time control systems. Furthermore, giving necessary optimality conditions for sequences of discrete approximation problems, the AMP plays *essentially the same role* as the (exact) discrete maximum principle in solving discrete-time control problems with sufficiently *small steps*; see particularly Example 6.68. However, in the case of large stepsizes h the approximate maximum condition, still being necessary for optimality, may be *far removed* from the exact maximum.

It is proved in Subsect. 6.4.3 that the *AMP holds*, with $\varepsilon(h_N, t) = O(h_N)$ in arbitrary Banach state spaces X, for *smooth free-endpoint* problems of optimal control, i.e., for problems of minimizing smooth (continuously differentiable) cost functions over discrete approximation systems (6.77) with smooth dynamics and no endpoint constraints. The proof is purely analytic based on using (single) needle control variations and a discrete counterpart of the increment formula from Subsect. 6.3.2.

The *crucial difference* between the PMP for continuous-time systems and the AMP for discrete approximations is that the latter result *doesn't have* an expected (lower) *subdifferential analog* for optimal control problems involving the simplest *nonsmooth* (even convex) cost functions! The corresponding counterexample is presented in Subsect. 6.4.3, together with those showing the violation of the AMP for optimal control problems with Fréchet *differentiable* (but *not continuously* differentiable) cost functions as well as for control problems with *nonsmooth dynamics*.

Thus the AMP happens to be very *sensitive to nonsmoothness*. On the other hand, in Subsect. 6.4.3 we derive an *upper subdifferential* version of the AMP, parallel to that in Subsect. 6.3.1 for continuous-time systems, which holds however for a more restrictive class of cost functions in comparison with the one for continuous-time systems. This class of *uniformly upper subdifferentiable* functions is introduced and studied in Subsect. 6.4.2.

The case of optimal control problems for discrete approximation systems (6.77) *with endpoint constraints* is much more involved. Considering control systems with *smooth inequality constraints* of the type

$$\varphi_i\big(x_N(b)\big) \le 0, \quad i = 1, \ldots, m \ ,$$

we formulate in Subsect. 6.4.4 the AMP with *perturbed complementary slackness conditions* under some *properness* assumption on the sequence of optimal controls, which can be treated as a discrete counterpart of piecewise continuity. The latter assumption happens to be essential for the validity of the AMP for nonconvex constrained systems as demonstrated by an example. The proof of the AMP given in Subsect. 6.4.5 reveals an *approximate counterpart* of the *hidden convexity* property for finite-difference control problems under consideration; see below for more details and discussions. We also derive the *upper subdifferential* form of the AMP for inequality constrained problems with uniformly upper subdifferentiable endpoint functions φ_i, $i = 0, \ldots, m$.

A proper setup for discrete approximations of continuous-time control problems with endpoint constraints of the *equality type*

$$\varphi_i\big(x(b)\big) = 0, \quad i = m+1, \ldots, m+r \ ,$$

involves the *constraint perturbations*

$$\big|\varphi_i\big(x_N(b)\big)\big| \le \xi_{iN}, \quad i = m+1, \ldots, m+r \ ,$$

with $\xi_{iN} \downarrow 0$ as $N \to \infty$. It is proved in Subsect. 6.4.5 that the *AMP holds* for discrete approximation problems with perturbed equality constraints described by smooth functions provided that the following *consistency condition*

$$\limsup_{N\to\infty}\frac{h_N}{\xi_{iN}}=0 \quad\text{for all}\quad i=m+1,\ldots,m+r\;. \tag{6.80}$$

is imposed. This means that the equality constraint perturbations ξ_{iN} should tend to zero *slower* than the discretization stepsize h_N, which particularly requires that $\xi_{iN}\neq 0$. We give an example showing the consistency condition (6.80) is *essential* for the fulfillment of the AMP, which may be violated even when $\xi_{iN}=O(h_N)$.

The results obtained admit an extension to discrete approximations of systems with *time delays* in state variables, which relates to the case of *incommensurability* between the length $b-a$ of the time interval and the approximation stepsize h_N; see Subsect. 6.4.6. On the other hand, we present an example showing the *AMP doesn't hold* for discrete approximations of *neutral systems*, even in the case of smooth free-endpoint control problems.

Before deriving the mentioned results on the AMP, let us describe and study the class of *uniformly upper subdifferentiable* functions on Banach spaces for which the *upper subdifferential form* of the AMP will be developed. This class particularly includes every continuously differentiable function as well as every concave continuous function that are of special interest for applications.

6.4.2 Uniformly Upper Subdifferentiable Functions

The main object of this subsection is the class of functions defined as follows.

Definition 6.47 (uniform upper subdifferentiability). *A real-valued function defined on a Banach space X is* UNIFORMLY UPPER SUBDIFFERENTIABLE *around a point $\bar{x}$ if for every x from some neighborhood V of $\bar{x}$ there exists a nonempty set $\mathcal{D}^+\varphi(x)\subset X^*$ described by: for any given $\varepsilon>0$ there is $\nu>0$ such that $x^*\in\mathcal{D}^+\varphi(x)$ if and only if*

$$\varphi(v)-\varphi(x)-\langle x^*,v-x\rangle\le\varepsilon\|v-x\| \tag{6.81}$$

whenever $v\in V$ with $\|v-x\|\le\nu$ and $x^*\in\mathcal{D}^+\varphi(x)$.

It is easy to see that this class contains every *smooth* (i.e., $\mathcal{C}^1$ around $\bar{x}$)) function with $\mathcal{D}^+\varphi(x)=\big\{\nabla\varphi(x)\big\}$ and also every *concave* continuous function with $\mathcal{D}^+\varphi(x)=\partial^+\varphi(x)$ as x is around $\bar{x}$ in *any Banach space*. Furthermore, one can derive from the definition that the above class is closed with respect to taking the *minimum* over compact sets. Note that even if φ is Lipschitz continuous around $\bar{x}$ and Fréchet differentiable *at* $\bar{x}$, it may *not* be uniformly upper subdifferentiable around this point. A simple example is provided by the standard function $\varphi\colon I\!\!R\to I\!\!R$ defined by $\varphi(x):=x^2\sin(1/x)$ for $x\neq 0$ and $\varphi(0):=0$ with $\bar{x}=0$.

Before formulating the main result of this subsection, we consider an arbitrary function $\varphi\colon X\to\overline{I\!\!R}$ finite at $\bar{x}$ and describe relationships between the *Fréchet upper subdifferential* of φ at $\bar{x}$ defined in (1.52) by

$$\widehat{\partial}^+\varphi(\bar{x}) := \Big\{x^* \in X^* \Big| \limsup_{x\to\bar{x}} \frac{\varphi(x)-\varphi(\bar{x})-\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le 0\Big\}$$

and the two modifications of the so-called *Dini* (or Dini-Hadamard) *upper directional derivative* of φ at $\bar{x}$ defined by

$$d^+\varphi(\bar{x};z) := \limsup_{\substack{y\to z\\ t\downarrow 0}} \frac{\varphi(\bar{x}+ty)-\varphi(\bar{x})}{t}$$

for the standard (strong) version and by

$$d^+_w\varphi(\bar{x};z) := \limsup_{\substack{y\xrightarrow{w} z\\ t\downarrow 0}} \frac{\varphi(\bar{x}+ty)-\varphi(\bar{x})}{t}$$

for its *weak* counterpart, where $y \xrightarrow{w} z$ signifies the weak convergence in X. The next proposition used below is definitely interesting for its own sake; it reveals the *duality* between the subgradient and directional derivative constructions under consideration that generally holds in *reflexive* spaces for the *weak* directional derivative and in *finite dimensions* for the *strong* one. We formulate it for the case of upper constructions needed in this section; it readily implies the lower counterpart.

Proposition 6.48 (relationships between Fréchet subgradients and Dini directional derivatives). *One always has*

$$\widehat{\partial}^+\varphi(\bar{x}) \subset \big\{x^* \in X^* \big| \langle x^*, z\rangle \ge d^+_w\varphi(\bar{x};z) \;\; \text{for all} \;\; z \in X\big\}$$

$$\subset \big\{x^* \in X^* \big| \langle x^*, z\rangle \ge d^+\varphi(\bar{x};z) \;\; \text{for all} \;\; z \in X\big\}\,,$$

where the equality holds in the first inclusion when X is reflexive, while it holds in the second one when $\dim X < \infty$. *Moreover,*

$$d^+\varphi(\bar{x};z) = \limsup_{t\downarrow 0} \frac{\varphi(\bar{x}+tz)-\varphi(\bar{x})}{t} \tag{6.82}$$

if φ is locally Lipschitzian around $\bar{x}$.

Proof. To prove the final inclusion in the proposition, it is sufficient to observe that for every $x^* \in \widehat{\partial}^+\varphi(\bar{x})$ and $z \in X$ one has

$$d^+\varphi(\bar{x};z) - \langle x^*, z\rangle = \limsup_{\substack{y\to z\\ t\downarrow 0}} \frac{\varphi(\bar{x}+ty)-\varphi(\bar{x})-t\langle x^*, y\rangle}{t} \le 0\,;$$

the other is similar. Let us prove that the first inclusion holds as equality if X is reflexive. To proceed, we pick $x^* \notin \widehat{\partial}^+\varphi(\bar{x})$ and take any $\gamma > 0$. Then there is a sequence $x_k \to \bar{x}$ such that

$$\varphi(x_k) - \varphi(\bar{x}) - \langle x^*, x_k - \bar{x} \rangle - \gamma \|x_k - \bar{x}\| > 0 \text{ for all } k \in I\!N .$$

Since X is reflexive, we suppose without loss of generality that the sequence $(x_k - \bar{x})/\|x_k - \bar{x}\|$ weakly converges to some $z \in X$. Then

$$d^+\varphi(\bar{x}; z) \geq \limsup_{k \to \infty} \frac{\varphi(x_k) - \varphi(\bar{x})}{\|x_k - \bar{x}\|} \geq \langle x^*, z \rangle + \gamma ,$$

which ensures the required equality, since γ was chosen arbitrarily.

It remains to justify representation (6.82) if φ is locally Lipschitzian around $\bar{x}$ with some modulus $\ell > 0$. Then we get

$$|\varphi(\bar{x} + ty) - \varphi(\bar{x} + tz)| \leq t\ell \|y - z\| \text{ for any } y, z \in X$$

when $t > 0$ is sufficiently small. Thus one has

$$\begin{aligned} d^+\varphi(\bar{x}; z) &= \limsup_{\substack{y \to z \\ t \downarrow 0}} \Big[\frac{\varphi(\bar{x} + tz) - \varphi(\bar{x})}{t} + \frac{\varphi(\bar{x} + ty) - \varphi(\bar{x} + tz)}{t} \Big] \\ &= \limsup_{t \downarrow 0} \frac{\varphi(\bar{x} + tz) - \varphi(\bar{x})}{t} \text{ whenever } z \in X , \end{aligned}$$

which justifies (6.82) and completes the proof of the proposition. △

Now we are ready to establish important properties of uniformly upper subdifferentiable functions that are employed in what follows being certainly of independent interest. It shows, in particular, that such functions enjoy the *upper regularity* property formulated right after Definition 1.91.

Theorem 6.49 (properties of uniformly upper subdifferentiable functions). *Let X be reflexive, and let φ be continuous at $\bar{x}$ and uniformly upper subdifferentiable around this point with the subgradient sets $\mathcal{D}^+\varphi(x)$ from Definition 6.47. Then there is a neighborhood of $\bar{x}$ in which φ is Lipschitz continuous and one can choose*

$$\mathcal{D}^+\varphi(x) = \widehat{\partial}^+\varphi(x) = \partial^+\varphi(x) .$$

Proof. The subgradient sets $\mathcal{D}^+\varphi(x)$ are obviously convex. Moreover, it is easy to check that each of these sets is norm-closed in X^* and hence also weakly closed due to its convexity and the assumed reflexivity of X. Let us show that $\mathcal{D}^+\varphi(x)$ is *uniformly bounded* in X^* around $\bar{x}$. Assume the contrary and select some sequences $x_k \to \bar{x}$ and $x_k^* \in \mathcal{D}^+\varphi(x_k)$ with $\|x_k^*\| \to \infty$ as $k \to \infty$. Then employing the Hahn-Banach theorem and taking into account the reflexivity of X, we find $u_k \in X$ satisfying the relations

$$\langle x_k^*, u_k \rangle = \|x_k^*\|^{1/2} \text{ and } \|u_k\| = \|x_k^*\|^{-1/2} \text{ for all } k \in I\!N .$$

Setting now $v_k := x_k - u_k$, one has from (6.81) that

$$\varphi(v_k) - \varphi(x_k) \le -\langle x_k^*, u_k \rangle + \varepsilon \|u_k\|$$

with $\|u_k\| \to 0$ and $\langle x_k^*, u_k \rangle \to \infty$ by the construction above. This yields that $\varphi(v_k) - \varphi(x_k) \to -\infty$ while $x_k, v_k \to \bar{x}$ as $k \to \infty$, which contradicts the required continuity of φ at $\bar{x}$ and thus justifies the uniform boundedness of $\mathcal{D}^+\varphi(x)$ around this point.

Next we show that φ is *locally Lipschitzian* around $\bar{x}$. It can be done similarly to the proof of Theorem 3.52 based on the mean value inequality from Theorem 3.49 that holds for $\mathcal{D}^+\varphi(\cdot)$. However, we may easier proceed directly invoking the uniform boundedness of the sets $\mathcal{D}^+\varphi(x)$ around $\bar{x}$ and property (6.81). Indeed, assume the contrary and find sequences $x_k \to \bar{x}$ and $v_k \to \bar{x}$ satisfying

$$|\varphi(v_k) - \varphi(x_k)| > k\|v_k - x_k\| \quad \text{as} \quad k \to \infty \, .$$

Suppose for definiteness that $\varphi(v_k) - \varphi(x_k) > k\|v_k - x_k\|$; the other case is symmetric. Now using the uniform upper subdifferentiability of φ, we find a sequence of $x_k^* \in \mathcal{D}^+\varphi(x_k)$ satisfying

$$\begin{aligned} k\|v_k - x_k\| &< \varphi(v_k) - \varphi(x_k) \le \langle x_k^*, v_k - x_k \rangle + \varepsilon \|v_k - x_k\| \\ &\le \big(\|x_k^*\| + \varepsilon\big)\|v_k - x_k\| \end{aligned}$$

for any given $\varepsilon > 0$ when k is sufficiently large. This yields that $\|x_k^*\| \to \infty$ as $k \to \infty$, which contradicts the uniform boundedness of the sets $\mathcal{D}^+\varphi(x)$ around $\bar{x}$ and thus justifies the local Lipschitzian property of φ.

It follows from the definition of Fréchet upper subgradients in (1.52) and the construction of $\mathcal{D}^+\varphi(x)$ in (6.81) that one always has $\mathcal{D}^+\varphi(x) \subset \widehat{\partial}^+\varphi(x)$. Let us show in fact that $\mathcal{D}^+\varphi(x) = \widehat{\partial}^+\varphi(x)$ around $\bar{x}$. First observe that the set-valued mapping $\mathcal{D}^+\varphi\colon V \rightrightarrows X^*$ is *closed-graph* in the norm$\times$weak topology of $X \times X^*$ on any closed subset of V. Using this fact and the local Lipschitz continuity of φ around $\bar{x}$, we derive from (6.81) that φ is *directionally differentiable* in the classical sense

$$\varphi'(x; z) := \lim_{t \downarrow 0} \frac{\varphi(x + tz) - \varphi(x)}{t}, \quad z \in X \, ,$$

whenever x is sufficiently close to $\bar{x}$; moreover, we have the representation

$$\varphi'(x; z) = \min \big\{ \langle x^*, z \rangle \big|\, x^* \in \mathcal{D}^+\varphi(x) \big\} \, , \tag{6.83}$$

where the minimum is attained due to the weak closedness of $\mathcal{D}^+\varphi(x)$ in X^*. Since $\mathcal{D}^+\varphi(x)$ is also convex, one gets from (6.83) and the results of Proposition 6.48 that $\widehat{\partial}^+\varphi(x) \subset \mathcal{D}^+\varphi(x)$. Indeed, assuming the opposite and then separating $x^* \notin \mathcal{D}^+\varphi(x)$ from the convex and norm-closed set $\mathcal{D}^+\varphi(x) \subset X^*$, we arrive at a contradiction with (6.82) and (6.83). Finally, the equality $\mathcal{D}^+\varphi(x) = \partial^+\varphi(x)$ and the upper regularity of φ around $\bar{x}$ follows from the mention closed-graph property of $\mathcal{D}^+\varphi(\cdot)$ by the upper subdifferential version of Theorem 2.34 on the limiting representation of basic subgradients. This completes the proof of the theorem. △

As mentioned above, properties of uniformly upper subdifferentiable functions allow us to derive the AMP in optimal control problems for discrete approximations with *upper subdifferential* transversality conditions; see the following subsections. This requires more from the functions and spaces under consideration in comparison with the assumptions needed to justify upper subdifferential transversality conditions in the PMP for continuous-time systems as well as upper subdifferential optimality conditions in problems of mathematical programming; cf. Sects. 5.1, 5.2, and 6.3. These significantly more restrictive requirements needed for the AMP are due to the *parametric* nature of finite-difference systems treated as a *process* as $N \to \infty$. We'll see in the next subsection that, even in the case of *differentiable* cost functions in free-endpoint control problems with finite-dimensional state spaces, the *continuity of the derivatives* is essential for the validity of the AMP in sequences of discrete approximations.

6.4.3 Approximate Maximum Principle for Free-Endpoint Control Systems

This subsection is devoted to optimal control problems for sequences of finite-difference systems (6.77) with *no endpoint constraints* on the right-hand end of trajectories. As in the case of continuous-time systems, free-endpoint problems for discrete approximations are essentially different from their constrained counterparts. The main *positive result* of this subsection is the *approximate maximum principle* for free-endpoint problems in Banach spaces with *upper subdifferential* transversality conditions valid for uniformly upper subdifferentiable cost functions. In particular, this justifies the AMP for control problems with continuously differentiable cost functions, where the boundary/transversality condition for the adjoint system (6.78) is written in the classical form (6.79). On the other hand, we present an example showing that the AMP *doesn't hold* when the cost function is *differentiable at* the point of interest but *not* $\mathcal{C}^1$ *around* it. Other examples show that the AMP is very *sensitive to nonsmoothness*: it doesn't hold for control problems with nonsmooth dynamics and—which is even more striking—for nice systems with *convex* nonsmooth cost functions.

Consider the *sequence* of optimal control problems (P_N^0) for discrete-time systems studied in this subsection:

$$\text{minimize} \quad J_N[u_N, x_N] := \varphi_0\big(x_N(b)\big) \tag{6.84}$$

over control-trajectory pairs $\{u_N(\cdot), x_N(\cdot)\}$ satisfying the control system (6.77) as $N \to \infty$. Given a sequence of optimal solutions $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ to problems (P_N^0), we impose the following *standing assumptions*:

—the control space U is metric, the state space X is Banach;

—there is an open set O containing $\bar{x}_N(t)$ for all $t \in T_N \cup \{b\}$ such that f is Fréchet differentiable in x with both $f(x,u,t)$ and its state derivative $\nabla_x f(x,u,t)$ continuous in (x,u,t) and uniformly norm-bounded whenever $x \in O$, $u \in U$, and $t \in T_N \cup \{b\}$ as $N \to \infty$;

—the sequence $\{\bar{x}_N(b)\}$ belongs to a compact subset of X.

The latter assumption is not restrictive at all in finite dimensions: it follows from standard conditions ensuring the uniform boundedness of admissible trajectories for continuous-time control systems. In infinite dimensions it can be derived from the conditions imposed in (H1) of Subsect. 6.1.1; cf. the proof of Theorem 6.13 and the references therein.

Here is the main *positive* result of this subsection.

Theorem 6.50 (AMP for free-endpoint control problems with upper subdifferential transversality conditions). *Let the pairs $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ be optimal to problems (P_N^0) under the standing assumptions made. Suppose in addition that the cost function φ_0 is uniformly upper subdifferentiable around the limiting point(s) of the sequence $\{\bar{x}_N(b)\}$ with the corresponding subgradient sets $\mathcal{D}^+(x)$. Then for every sequence of upper subgradients $x_N^* \in \mathcal{D}^+\varphi_0\big(\bar{x}(b)\big)$ there is $\varepsilon(t,h_N) \to 0$ as $N \to \infty$ uniformly in $t \in T_N$ such that one has the approximate maximum condition*

$$H\big(\bar{x}_N(t), p_N(t+h_N), \bar{u}_N(t), t\big) = \max_{u \in U} H\big(\bar{x}_N(t), p_N(t+h_N), u, t\big) \\ +\varepsilon(t,h_N), \quad t \in T_N\ , \tag{6.85}$$

where each $p_N(\cdot)$ is the corresponding trajectory for the adjoint system (6.78) with the boundary/transversality condition

$$p_N(b) = -x_N^* \ \ \textit{for all} \ \ N \in \mathbb{N}\ . \tag{6.86}$$

Furthermore, this result holds with any $x_N^* \in \widehat{\partial}^+\varphi\big(\bar{x}_N(b)\big)$ in (6.86) if in addition X is reflexive and φ_0 is continuous at the optimal points.

Proof. Considering a sequence of optimal solutions $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ to (P_N^0), we suppose that the trajectories $\bar{x}_N(t)$ belong to the uniform neighborhoods fixed in the assumptions made for all $N \in \mathbb{N}$. It follows from Definition 6.47 of the uniform upper subdifferentiability for φ_0 that $\mathcal{D}^+\varphi_0\big(\bar{x}_N(b)\big) \neq \emptyset$ and that inequality (6.81) holds for any $x_N^* \in \mathcal{D}^+\varphi_0\big(\bar{x}_N(b)\big)$ as $N \to \infty$. Now taking an arbitrary sequence of $x_N^* \in \mathcal{D}^+\varphi_0\big(\bar{x}_N(b)\big)$, we get

$$\varphi_0(x) - \varphi_0\big(\bar{x}_N(b)\big) \le \langle x_N^*, x - \bar{x}_N(b)\rangle + o\big(\|x - \bar{x}_N(b)\|\big) \tag{6.87}$$

$$\text{with} \quad \frac{o\big(\|x - \bar{x}_N(b)\|\big)}{\|x - \bar{x}_N(b)\|} \to 0 \quad \text{as} \quad x \to x_N(b) \quad \textit{uniformly} \quad \text{in} \quad N \ .$$

Letting $p_N(b) := -x_N^*$ as in (6.86), we derive from (6.87) that

$$J[u_N, x_N] - J[\bar{u}_N, \bar{x}_N] \le -\big\langle p_N(b), \Delta x_N(b)\big\rangle + o\big(\|\Delta x_N(b)\|\big) \ ,$$

with $\Delta x_N(t) := x_N(t) - \bar{x}_N(t)$, for all admissible processes in (P_N^0) whenever $x_N(b)$ is sufficiently close to $\bar{x}_N(b)$. Taking into account the identity

$$\begin{aligned}\big\langle p_N(b), \Delta x_N(b)\big\rangle &= \sum_{t \in T_N} \Big\langle p_N(t + h_N) - p_N(t), \Delta x_N(t)\Big\rangle \\ &\quad + \sum_{t \in T_N} \Big\langle p_N(t + h_N), \Delta x_N(t + h_N) - \Delta x_N(t)\Big\rangle\end{aligned}$$

and using the smoothness of f in x, we get from the above inequality that

$$\begin{aligned}&0 \le J[u_N, x_N) - J[\bar{u}_N, \bar{x}_N] \le -\sum_{t \in T_N} \Big\langle p_N(t + h_N) - p_N(t), \Delta x_N(t)\Big\rangle \\ &-h_N \sum_{t \in T_N} \Big\langle p_N(t + h_N), \nabla_x f\big(\bar{x}_N(t), \bar{u}_N(t), t\big)\Delta x_N(t)\Big\rangle \\ &-h_N \sum_{t \in T_N} \Delta_u H\big(\bar{x}_N(t), p_N(t + h_N), \bar{u}_N(t), t\big) \\ &-h_N \sum_{t \in T_N} \eta_N(t) + o\big(\|\Delta x_N(b)\|\big) \ ,\end{aligned} \tag{6.88}$$

where the remainder $\eta_N(t)$ is computed by

$$\begin{aligned}\eta_N(t) = \Big\langle &\nabla_x H\big(\bar{x}_N(t), p_N(t + h_N), u_N(t), t\big) \\ &-\nabla_x H\big(\bar{x}_N(t), p_N(t + h_N), \bar{u}_N(t), t\big), \Delta x_N(t)\Big\rangle + o\big(\|\Delta x_N(t)\|\big)\end{aligned}$$

with the quantity $o(\|\Delta x_N(t)\|)$ being uniform in N due to the assumptions on $\nabla_x f$, and where the increment $\Delta_u H$ is defined similarly to the one in Subsect. 6.3.2 for continuous-time systems.

Now we consider (single) *needle variations* of the optimal controls $\bar{u}_N(\cdot)$ in the following form:

$$u_N(t) = \begin{cases} v & \text{if } t = \tau \ , \\ \bar{u}_N(t) & \text{if } t \in T_N \setminus \{\tau\} \ , \end{cases}$$

where $v \in U$ and $\tau = \tau(N) \in T_N$ as $N \in I\!N$. All these controls are obviously feasible for the discrete approximation problems under consideration, which are not subject to endpoint constraints. The trajectory increments corresponding to the needle variations satisfy the relations

$$\Delta x_N(t) = 0 \ \text{ for } \ t = a, \ldots, \tau; \quad \|\Delta x_N(t)\| = O(h_N) \ \text{ for } \ t = \tau + h_N, \ldots, b\,.$$

Taking this into account and substituting the needle variations $u_N(\cdot)$ into the increment inequality (6.88), one gets

$$0 \le J[u_N, x_N] - J[\bar{u}_N, \bar{x}_N] \le -h_N \Delta_u H\big(\bar{x}_N(\tau), p_N(\tau + h_N), \bar{u}_N(\tau), \tau\big) + o(h_N)\,.$$

Arguing by contradiction, we directly derive from the latter inequality the approximative maximum condition (6.85).

To complete the proof of the theorem, it remains to apply Theorem 6.49 on uniform upper subdifferentiability to the cost function φ_0. This ensures that x_N^* may be taken from the whole Fréchet upper subgradient sets $\widehat{\partial}^+\varphi_0(\big(\bar{x}(b)\big)$ in the transversality conditions (6.86) as $N \to \infty$ provided that X is reflexive and that φ_0 is assumed to be continuous a priori. △

Remark 6.51 (discrete approximations versus continuous-time systems.) Observe that the proof of Theorem 6.50 is similar to the one for continuous-time systems with free endpoints; cf. the proofs of Theorem 6.37 in Subsect. 6.3.2 and of its upper subdifferential version (Theorem 6.38) in Subsect. 6.3.1. The given proofs in both continuous-time and discrete-time settings are based on using the *increment formulas* for cost functionals and (*single*) *needle variations* of optimal controls. In a sense, the proof for discrete approximations problems is a simplified version of that given for systems with continuous time (which is definitely not the case when endpoint constraints are involved; see the next subsection). On the other hand, there are two *significant differences* between the results and proofs for continuous-time systems and those for discrete approximations.

Firstly, in the case of continuous-time systems there is a possibility of using a *small parameter* ε as the length of needle variations, which ensures the smallness of trajectory increments $\Delta x(t) = O(\varepsilon)$ and happens to be *crucial* for establishing the *exact* maximum principle in continuous-time optimal control. In systems of discrete approximations the smallness of trajectory increments is provided by the *decreasing stepsize* h_N, which is a parameter of the problem but not of variations. This leads to the *approximate* maximum condition with the error as small as the step of discretization. Of course, such a device is not possible when $h_N \not\to 0$.

The *second* difference concerns the *parametric nature* of discrete approximation problems in contrast to problems with continuous time. This requires the more restrictive *uniformity* assumptions imposed on cost functions in comparison with the case of continuous-time systems.

The following two consequences of Theorem 6.50 and its proof deal with important classes of cost functions that are automatically uniformly upper subdifferentiable and admit *more precise* versions of the AMP. Note that these results don't require the reflexivity assumption on the state space X as in the second part of Theorem 6.50; they are valid in *arbitrary Banach spaces*.

Corollary 6.52 (AMP for free-endpoint control problems with smooth cost functions). *Let the pairs $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ be optimal to problems (P_N^0) under the standing assumptions made. Suppose in addition that the cost function φ_0 is continuously differentiable around the limiting point(s) of $\{\bar{x}_N(b)\}$. Then the approximate maximum principle of Theorem 6.50 holds with the transversality condition (6.79) for the corresponding adjoint trajectory $p_N(\cdot)$ whenever $N \in I\!N$. Moreover, we can take $\varepsilon(t, h_N) = O(h_N)$ in the maximum condition (6.85) if both $\nabla_x f(\cdot, u, t)$ and $\nabla\varphi_0(\cdot)$ are locally Lipschitzian around $\bar{x}_N(\cdot)$ uniformly in $u \in U$, $t \in T_N$, and $N \to \infty$.*

Proof. As mentioned above, in any Banach space X we have $\mathcal{D}^+\varphi(x) = \{\nabla\varphi(x)\}$ in a neighborhood of $\bar{x}$ if φ is $\mathcal{C}^1$ around this point. It can be easily shown that (6.87) holds as equality for smooth functions φ_0; moreover, one has $|o(\eta)| \le \ell\eta^2$ therein if $\nabla\varphi_0$ is locally Lipschitzian. Note further that the Lipschitzian assumption imposed on $\nabla_x f(\cdot, u, t)$ in the corollary implies that

$$o\big(\|\Delta x_N(t)\|\big) = O\big(\|\Delta x_N(t)\|^2\big)$$

uniformly in N for the "o" term in the remainder $\eta_N(\cdot)$ in the proof of the theorem. This yields that $\varepsilon(t, h_N) = O(h_N)$ in the approximate maximum condition (6.85) under the assumptions made. △

Corollary 6.53 (AMP for free-endpoint control problems with concave cost functions). *Let the pairs $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ be optimal to problems (P_N^0) under the standing assumptions made. Suppose in addition that the cost function φ_0 is concave on some open set containing all $\bar{x}_N(b)$. Then the approximate maximum principle of Theorem 6.50 holds along every sequence of subgradients $x_N^* \in \partial^+\varphi_0\big(\bar{x}_N(b)\big)$. Moreover, one can take $\varepsilon(t, h_N) = O(h_N)$ in (6.85) if $\nabla_x f(\cdot, u, t)$ is locally Lipschitzian around $\bar{x}_N(\cdot)$ uniformly in $u \in U$, $t \in T_N$, and $N \to \infty$.*

Proof. Recall that $\mathcal{D}^+\varphi(x) = \partial^+\varphi(x)$ for concave continuous functions in arbitrary Banach spaces. Furthermore, $o\big(\|x - \bar{x}_N(b)\|\big) \equiv 0$ in the inequality (6.87) under the concavity assumption of the corollary. Combining this with the estimate of $\eta_N(\cdot)$ in the proof of Corollary 6.52, we conclude that $\varepsilon(t, h_N) = O(h_N)$ in (6.85) under the assumptions made. △

Now we proceed with *counterexamples*, i.e., examples showing that the AMP may be violated if some of the assumptions in Theorem 6.50 are not satisfied. All the examples below are given for finite-dimensional control systems with nonconvex velocity sets. Our first example demonstrates that the

AMP doesn't hold in the *expected lower subdifferential form* (as the maximum principle for continuous-time control systems) even in the simplest nonsmooth case of minimizing convex functions over systems with linear dynamics.

Example 6.54 (AMP may not hold for linear control systems with nonsmooth and convex minimizing functions). *There is a one-dimensional control problem of minimizing a nonsmooth and convex cost function over a linear system with no endpoint constraints for which the AMP is violated.*

Proof. Consider the following sequence of one-dimensional optimal control problem (P_N^0), $N \in I\!N$, for discrete-time systems:

$$\begin{cases} \text{minimize } \ \varphi\big(x_N(1)\big) := |x_N(1) - \vartheta| \\ \\ \text{subject to} \\ \\ x_N(t + h_N) = x_N(t) + h_N u_N(t), \quad x_N(0) = 0\,, \\ \\ u_N(t) \in U := \big\{0, 1\big\}, \quad t \in T_N := \big\{0, h_N, \ldots, 1 - h_N\big\}\,, \end{cases} \tag{6.89}$$

where ϑ is a positive *irrational* number less than 1 whose choice will be specified below. The dynamics in (6.89) is a discretization of the simplest ODE control system $\dot{x} = u$. Observe that, since ϑ is irrational and h_N is rational, we have $\bar{x}_N(1) \neq \vartheta$ for the endpoint of an optimal trajectory to (6.89) as $N \in I\!N$, while obviously $\bar{x}(1) = \vartheta$ for optimal solutions to the continuous-time counterpart. It is also clear that for all sufficiently small stepsizes h_N an optimal control to (6.89) is neither $u_N(t) \equiv 0$ nor $u_N(t) \equiv 1$, but it has at least one point of *control switch*.

Suppose that for some subsequence $N_k \to \infty$ one has $\bar{x}_{N_k}(1) > \vartheta$; put $\{N_k\} = I\!N$ without loss of generality. Let us show that in this case the approximate maximum condition *doesn't hold* at points $t \in T_N$ for which $\bar{u}_N(t) = 1$. Indeed, we have

$$H\big(\bar{x}_N(t), p_N(t + h_N), u\big) = p_N(t + h_N)u \quad \text{and} \quad p_N(t) \equiv -1$$

for the Hamilton-Pontryagin function and the adjoint trajectory for this problems, since $\bar{x}_N(1) > \vartheta$ along the optimal solution to (6.89). Thus

$$\max_{u \in U} H\big(\bar{x}_N(t), p_N(t + h_N), u\big) = 0, \quad t \in T_N\,,$$

$$\text{while } \ H\big(\bar{x}_N(s), p_N(s + h_N), \bar{u}_N(s)\big) = -1$$

at the points $s \in T_N$ of control switch, where $\bar{u}_N(s) = 1$ regardless of h_N.

Let us specify the choice of ϑ in (6.89) ensuring that $\bar{x}_N(1) > \vartheta$ along some subsequence of natural numbers. We claim that $\bar{x}_N(1) > \vartheta$ if $\vartheta \in (0, 1)$ is an irrational number whose decimal representation contains infinitely many digits from the set $\{5, 6, 7, 8, 9\}$; e.g., $\vartheta = 0.676676667\ldots$. Indeed, put $h_N := 10^{-N}$, which is a subsequence of $h_N = N^{-1}$ as required in (6.89). It is easy to see that in this case the set of all reachable points at $t = 1$ is the set of rational numbers between 0 and 1 with exactly N digits in the fractional part of their decimal representations. In particular, for $N = 3$ this set is $\{0, 0.001, 0.002, \ldots, 0.999, 1\}$. Therefore, by the construction of ϑ, the closest point to ϑ from the reachable set is greater than ϑ, and this point must be the endpoint of the optimal trajectory $\bar{x}_N(1)$. △

The next example, complemented to Example 6.54, shows that the AMP fails even for problems with *differentiable* but *not continuously differentiable* cost functions.

Example 6.55 (AMP may not hold for linear systems with differentiable but not $\mathcal{C}^1$ cost functions). *There is a one-dimensional control problem of minimizing a Fréchet differentiable but not continuously differentiable cost function over a linear system with no endpoint constraints for which the AMP is violated.*

Proof. Consider the same control system as in (6.89) and construct a minimizing function $\varphi(x)$ that satisfies the requirements listed above. Let $\psi(x)$ be a $\mathcal{C}^1$ function with the properties:

$$\psi(x) \ge 0, \quad \psi(x) = \psi(-x), \quad \psi(x) \equiv 0 \text{ if } |x| > 2\,,$$

$$|\nabla\psi(x)| \le 1 \text{ for all } x, \quad \text{and } \nabla\psi(-1) = \vartheta > 0\,.$$

Define the cost function $\varphi(x)$ by

$$\varphi(x) := \Big(x - \frac{1}{9}\Big)^2 + \sum_{n=1}^{\infty} 10^{-2n-3}\psi\Big(10^{2n+3}\big(x - \sum_{k=1}^{n} 10^{-k}\big) - 1\Big)\,,$$

which is continuously differentiable around every point but $x = \frac{1}{9}$, where it is differentiable and attains its absolute minimum. As in Example 6.54, we put $h_N := 10^{-N}$, and then the point $x = \frac{1}{9}$ *cannot be reached* by discretization. It is not hard to check that the endpoint of the optimal trajectory $\bar{x}_N(\cdot)$ for each N is computed by

$$\bar{x}_N(1) = \sum_{k=1}^{N} 10^{-k} \quad \text{with} \quad \nabla\varphi\big(\bar{x}_N(1)\big) = \vartheta + \varepsilon_N\,,$$

where $\varepsilon_N \downarrow 0$ as $N \to \infty$. Proceeding as in Example 6.54 with the same sequence of optimal controls, we have

$$H\big(\bar{x}_N(t), p_N(t + h_N), u\big) \equiv -\vartheta u + O(\varepsilon_N)\,,$$

and the approximate maximum condition (6.85) doesn't hold at the points of control switch, where $\bar{u}_N(t) = 1$. △

The last example in this subsection concerns systems with *nonsmooth dynamics*. We actually consider a finite-difference analog of minimizing an integral functional subject to a one-dimensional control system, which is equivalent to a two-dimensional optimal control problem of the Mayer type. The discrete "integrand" in this problem is nonsmooth with respect to the state variable x; it happens to be continuously differentiable with respect to x *along* the optimal process $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ under consideration but *not uniformly* in N. Thus the example below demonstrates that the *uniform smoothness* assumption on f over an open set containing all the optimal trajectories $\bar{x}_N(\cdot)$ is essential for the validity of the AMP.

Example 6.56 (violation of AMP for control problems with nonsmooth dynamics). *The AMP doesn't hold in discrete approximations of a minimization problem for an integral functional over a one-dimensional linear control system with no endpoint constraints such that the integrand is linear with respect to the control variable while convex and nonsmooth with respect to the state one. Moreover, the integrand in this problem happens to be* $\mathcal{C}^1$ *with respect to the state variable along the sequence of optimal solutions to the discrete approximations* (P_N^0) *for all* $N \in \mathbb{N}$ *but not uniformly in* N.

Proof. First we consider the following continuous-time optimal control problem of the Bolza type:

$$\begin{cases} \text{minimize} \ \ J[u,x] := \displaystyle\int_0^b \big(u(t) + |x(t) - t^2/2|\big)\,dt \\[2ex] \text{subject to} \\[2ex] \dot{x} = tu, \quad x(0) = 0\,, \\[2ex] u(t) \in U := \big\{1, c\big\}, \quad 0 \le t \le b\,, \end{cases}$$

where the terminal time b and the number $c > 1$ will be specified below. It is obvious that the optimal control to this problem is $\bar{u}(t) \equiv 1$ and the corresponding optimal trajectory is $\bar{x}(t) = t^2/2$.

By discretization we get the sequence of finite-difference control problems:

$$\begin{cases} \text{minimize} \ \ J[u_N, x_N] := h_N \displaystyle\sum_{t \in T_N} \big(u_N(t) + |x_N(t) - t^2/2|\big) \\[2ex] \text{subject to} \\[2ex] x_N(t + h_N) = x_N(t) + h_N t u_N(t), \quad x_N(0) = 0\,, \\[2ex] u_N(t) \in U = \big\{1, c\big\}, \quad t \in T_N := \big\{0, \ldots, (N-1)h_N\big\}\,. \end{cases} \tag{6.90}$$

We first show that $\bar{u}_N(t) \equiv 1$ remains to be the (unique) optimal control to (6.90) if the stepsize h_N is sufficiently small and the numbers (b, c) are chosen appropriately. It is easy to check that the corresponding trajectory $\bar{x}(\cdot)$ is computed by

$$\bar{x}_N(t) = \frac{t^2}{2} - \frac{th_N}{2} \quad \text{for all} \quad N \in I\!N .$$

Then the value $\bar{J}_N$ of the cost functional at $\bar{u}_N(\cdot)$ equals

$$\bar{J}_N = b + h_N^2 \sum_{t \in T_N} \frac{t}{2} = b + \frac{b^2 h_N}{4} + o(h_N) .$$

If we replace $u_N(t) = 1$ by $u_N(t) = c$ at some point $t \in T_N$, then the increment of the summation $h_N \sum_{t \in T_N} u_N(t)$ equals $(c-1)h_N$. Hence the corresponding value of the cost functional is

$$\begin{aligned} J[u_N, x_N] &= h_N \sum_{t \in T_N} u_N(t) + h_N \sum_{t \in T_N} |x_N(t) - t^2/2| \\ &> h_N \sum_{t \in T_N} u_N(t) \geq b + (c-1)h_N \end{aligned}$$

for any feasible control $u_N(t)$ to (6.90) different from $\bar{u}_N(t) \equiv 1$. Comparing the latter with $\bar{J}_N$, we conclude that the control $\bar{u}_N(t) \equiv 1$ is indeed *optimal* to (6.90) if $b^2/4 < c - 1$ and N is sufficiently large.

We finally show that for $b > 2$ and $c > b^2/4 + 1$ (e.g., for $b = 3$ and $c = 4$) the sequence of optimal controls $\bar{u}_N(t) \equiv 1$ *doesn't* satisfy the approximate maximum condition (6.85) at all points $t \in T_N$ sufficiently close to $t = b/2$. Compute the Hamilton-Pontryagin function as a function of $t \in T_N$ and of $u \in U$ at the optimal trajectory $\bar{x}_N(t)$ corresponding to the optimal control under consideration with the adjoint trajectory $p_N(t)$ for (6.78). Reducing (6.90) to the standard Mayer form and taking into account that $\bar{x}_N(t) < t^2/2$ for all $t \in T_N$ due to above formula for $\bar{x}_N(t)$, we get

$$\begin{aligned} H\big(\bar{x}_N(t), p_N(t+h_N), u, t\big) &= tp_N(t+h_N)u - u - |\bar{x}_N(t) - t^2/2| \\ &= \big(tp_N(t+h_N) - 1\big)u + \big(\bar{x}_N(t) - t^2/2\big) , \end{aligned}$$

where $p_N(t)$ satisfies the equation

$$p_N(t) = p_N(t+h_N) + h_N, \quad p_N(b) = 0 ,$$

whose solution is $p_N(t) = b - t$. Therefore one has

$$\begin{aligned} H\big(\bar{x}_N(t), p_N(t+h_N), u, t\big) &= \big(t(b - t + h_N) - 1\big)u + O(h_N) \\ &= \big(-t^2 + bt - 1\big)u + O(h_N) . \end{aligned}$$

The multiplier $-t^2 + bt - 1$ is positive in the neighborhood of $t = b/2$ if its discriminant $b^2 - 4$ is positive. Thus $u = c$, but not $u = 1$, provides the maximum to the Hamilton-Pontryagin function around $t = b/2$ if h_N is sufficiently small, which justifies the claim of this example. $\triangle$

Finally in this subsection, we give a modification of Theorem 6.50 in the general case of possible *incommensurability* of the time interval $b - a$ and the stepsize h_N; note that $b - a = Nh_N$ as $N \in I\!N$ in Theorem 6.50. This is particularly important for the extension of the AMP to finite-difference approximations of time-delay systems in Subsect. 6.4.5. For simplicity we use the notation

$$f(x_N, u_N, t) := f\big(x_N(t), u_N(t), t\big) .$$

Given the time interval $[a, b]$, define the grid T_N on $[a, b]$ by

$$T_N := \big\{a, a + h_N, \ldots, b - \widetilde{h}_N - h_N\big\}$$

$$\text{with } \ h_N := \frac{b-a}{N} \quad \text{and} \quad \widetilde{h}_N := b - a - h_N\Big[\frac{b-a}{h_N}\Big] ,$$

where $[z]$ stands for the greatest integer less than or equal to the real number z. The modified discrete approximation problems $(\widetilde{P}_N^0)$ are written as

$$\begin{cases} \text{minimize } J[u_N, x_N] := \varphi_0\big(x_N(b)\big) \ \text{ subject to} \\[2ex] x_N(t + h_N) = x_N(t) + h_N f(x_N, u_N, t), \quad t \in T_N, \quad x_N(a) = x_0 \in X , \\[2ex] x_N(b) = x_N(b - \widetilde{h}_N) + \widetilde{h}_N f(x_N, u_N, b - \widetilde{h}_N) , \\[2ex] u_N(t) \in U, \quad t \in T_N . \end{cases}$$

Theorem 6.57 (AMP for problems with incommensurability). *Let the pairs $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ be optimal to problems $(\widetilde{P}_N^0)$. In addition to the standing assumptions, suppose that φ_0 is uniformly upper subdifferentiable around the limiting point(s) of the sequence $\{\bar{x}_N(b)\}$, $N \in I\!N$. Then for every sequence of upper subgradients $x_N^* \in \widehat{\partial}^+\varphi_0\big(\bar{x}_N(b)\big)$ there is $\varepsilon(t, h_N) \to 0$ as $N \to \infty$ uniformly in $t \in T_N$ such that the approximate maximum condition*

$$H(\bar{x}_N, p_N, \bar{u}_N, t) = \max_{u \in U} H(\bar{x}_N, p_N, u, t) + \varepsilon(t, h_N)$$

holds for all $t \in \widetilde{T}_N := T_N \cup \{b - \widetilde{h}_N\}$, where the Hamilton-Pontryagin function is defined by

$$H(\bar{x}_N, p_N, u, t) := \begin{cases} \big\langle p_N(t + h_N), f(\bar{x}_N, u, t)\big\rangle & \text{if } t \in T_N , \\[2ex] \big\langle p_N(t), f(\bar{x}_N, u, t - \widetilde{h}_N)\big\rangle & \text{if } t = b - \widetilde{h}_N , \end{cases}$$

and where each $p_N(\cdot)$ *satisfies the adjoint system*

$$\begin{cases} p_N(t) = p_N(t+h_N) + h_N \nabla_x f(\bar{x}_N, \bar{u}_N, t)^* \, p_N(t+h_N), \quad t \in T_N \ , \\ p_N(b - \widetilde{h}_N) = p_N(b) + \widetilde{h}_N \nabla_x f(b - \widetilde{h}_N, \bar{x}_N, \bar{u}_N, t)^* \, p_N(b) \end{cases}$$

with the transversality condition $p_N(b) = -x_N^*$. *Furthermore, specifications similar to the second part of Theorem 6.50 as well as Corollaries 6.52 and 6.53 are also fulfilled.*

Proof. It is similar to the proof of Theorem 6.50 and its corollaries with the following modification of the increment formula for the minimizing functional:

$$\begin{aligned} 0 \le J[u_N, x_N] - J[\bar{u}_N, \bar{x}_N] &\le -\big\langle p_N(b), \Delta x_N(b) \big\rangle + o\big(\|\Delta x_N(b)\|\big) \\ &= -\sum_{t \in T_N} \big\langle p_N(t+h_N) - p_N(t), \Delta x_N(t) \big\rangle \\ &-\big\langle p_N(b) - p_N(b - \widetilde{h}_N), \Delta x_N(b - \widetilde{h}_N) \big\rangle \\ &-h_N \sum_{t \in T_N} \big\langle p_N(t+h_N), \nabla f_x(\bar{x}_N, \bar{u}_N, t)\, \Delta x_N(t) \big\rangle \\ &-\widetilde{h}_N \big\langle p_N(b), \nabla_x f(\bar{x}_N, \bar{u}_N, b - \widetilde{h}_N)\, \Delta x_N(b - \widetilde{h}_N) \big\rangle \\ &-h_N \sum_{t \in \widetilde{T}_N} \Delta_u H(\bar{x}_N, p_N, \bar{u}_N) + h_N \sum_{t \in \widetilde{T}_N} \eta_N(t) + o\big(\|\Delta x_N(b)\|\big) \ , \end{aligned}$$

where $\Delta_u H$ and $\eta_N(t)$ are defined similarly to the non-delay problems. Substituting the adjoint trajectory into this formula and using needle variations of the optimal control, we arrive at the conclusions of the theorem. $\triangle$

6.4.4 Approximate Maximum Principle under Endpoint Constraints: Positive and Negative Statements

This subsection concerns discrete approximations of optimal control problems with endpoint constraints. Our primary goal here is to formulate the approximate maximum principle for discrete approximation problems under appropriate assumptions and to clarify whether these assumptions are essential for its validity; the proof of the AMP is given in the next subsection.

Constructing discrete approximations, it is natural to *perturb* endpoint constraints and to consider the following *sequence* of optimal control problems (P_N) for discrete-time systems:

$$\begin{cases} \text{minimize } \; J[u_N, x_N] := \varphi_0\big(x_N(b)\big) \; \text{ subject to} \\ x_N(t + h_N) = x_N(t) + h_N f\big(x_N(t), u_N(t), t\big), \quad x_N(a) = x_0 \in X \;, \\ u_N(t) \in U, \quad t \in T_N := \big\{a, a + h_N, \ldots, b - h_N\big\} \;, \\ \varphi_i\big(x_N(b)\big) \le \gamma_{iN}, \quad i = 1, \ldots, m \;, \\ \big|\varphi_i\big(x_N(b)\big)\big| \le \xi_{iN}, \quad i = m+1, \ldots, m+r \;, \\ h_N := \dfrac{b-a}{N}, \quad N = 1, 2, \ldots \;, \end{cases}$$

where $\gamma_{iN} \to 0$ and $\xi_{iN} \downarrow 0$ as $N \to \infty$ for all i. The main result of this subsection shows that, under standard smoothness assumptions on the initial data, the AMP holds for *proper* sequences of optimal controls to problems (P_N) with *arbitrary* perturbations of *inequality* constraints (in particular, one can put $\gamma_{iN} = 0$) while with *consistent* perturbations of *equality* constraints matched the step of discretization. Then we demonstrate that the mentioned properness and consistency requirements are *essential* for the validity of the AMP, and we also derive an appropriate *upper subdifferential* analog of the AMP for problems with nonsmooth cost and inequality constraint functions.

Throughout this subsection we keep the *standing assumptions* on the initial data listed in Subsect. 6.4.3 supposing in addition that the state space X is *finite-dimensional*, which is needed in the proofs below. Along with the conventional notation for the matrix product, we use the agreement

$$\prod_{j}^{k=i} A_k := \begin{cases} A_i A_{i-1} \cdots A_j & \text{if } i \ge j \;, \\ I & \text{if } i = j-1 \;, \\ 0 & \text{if } i < j-1 \;, \end{cases}$$

where i, j are any integers and where I stands as usual for the identity matrix.

As in the case of continuous-time systems, the proof of the AMP for problems (P_N) with endpoint constraints is essentially different and more involved in comparison with free-endpoint problems. Recalling the proof of Theorem 6.37 for continuous-time systems with inequality endpoint constraints in Subsect. 6.3.3, we observe that a crucial part of this proof is Lemma 6.44, which verifies that the linearized image set S in (6.74) is convex and doesn't intersect the set of forbidden points. These facts are definitely due to the time continuity reflecting the *hidden convexity* of continuous-time control systems. Note that the mentioned image set S in (6.74) is generated by *multineedle* variations of the optimal control the very construction of which in (6.82) is essentially based on the time continuity.

In what follows we establish a certain *finite-difference analog* of the hidden convexity property for control systems in (P_N) involving *convex hulls* of some linearized image sets S_N generated by *single needle* variations of optimal controls. We show that *small shifts* (up to $o(h_N)$) of these convex hulls don't intersect the set of forbidden points as $N \to \infty$. This basically leads, via the *convex separation* theorem, to the approximate maximum principle for problems (P_N) under endpoint constraints of the inequality type, with appropriately *perturbed complementary slackness* conditions.

Such a device (as well as any finite-difference counterparts of the construction in Subsect. 6.3.4) doesn't apply to problems (P_N) with *arbitrarily* perturbed *equality* constraints (in particular, when $\xi_N = 0$) for which the AMP is *generally violated*. Nevertheless, the complementary slackness conditions mentioned above allow us to derive a natural version of the AMP for problems (P_N) with *appropriately perturbed* equality constraints by reducing them to the case of inequalities.

Before formulating the main result of this subsection, we introduce an important notion specific for *sequences* of finite-difference control problems.

Definition 6.58 (control properness in discrete approximations). *Let $d(\cdot,\cdot)$ stand for the distance in the control space U is problems (P_N). We say that a sequence of discrete-time controls $\{u_N(\cdot)\}$ in (P_N) is* PROPER *if for every increasing subsequence $\{N\}$ of natural numbers and every sequence of mesh points $\tau_{\theta(N)} \in T_N$ satisfying*

$$\tau_{\theta(N)} = a + \theta(N)h_N \quad \textit{as} \quad \theta(N) = 0,\ldots,N-1 \quad \textit{and} \quad \tau_{\theta(N)} \to t \in [a,b]$$

one of the following properties holds:

$$\textit{either} \quad d\big(u_N(\tau_{\theta(N)}), u_N(\tau_{\theta(N)+q})\big) \to 0 \quad \textit{or} \quad d\big(u_N(\tau_{\theta(N)}), u_N(\tau_{\theta(N)-q})\big) \to 0$$

as $N \to \infty$ with any natural constant q.

The notion of properness for sequences of feasible controls in discrete approximation problems is a *finite-difference counterpart* of the piecewise continuity for continuous-time systems. It turns out that the situation when sequences of optimal controls are not proper in discrete approximations of constrained systems with nonconvex velocities is not unusual, and this leads to the violation of the AMP for standard problems with inequality constraints. Note that the properness assumption is *not needed* for the validity of the AMP in free-endpoint problems; see Theorem 6.50.

Now we are ready to formulate the AMP for constrained control problems (P_N) with endpoint constraints described by smooth functions.

Theorem 6.59 (AMP for control problems with smooth endpoint constraints). *Let the pairs $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ be optimal to (P_N) for all $N \in \mathbb{N}$ under the standing assumptions made. Suppose in addition that all the functions φ_i, $i = 0,\ldots,m+r$, are continuously differentiable around the limiting point(s) of $\{\bar{x}_N(b)\}$ and that:*

(a) *the sequence of optimal controls* $\{\bar{u}_N(\cdot)\}$ *is proper;*

(b) *the consistency condition* (6.80) *holds for the perturbations* ξ_{iN} *of all the equality constraints.*

Then there are numbers $\{\lambda_{iN} \big| \, i = 0, \ldots, m + r\}$ *satisfying*

$$\lambda_{iN}\big(\varphi_i(\bar{x}_N(b)) - \gamma_{iN}\big) = O(h_N), \quad i = 1, \ldots, m \ , \tag{6.91}$$

$$\lambda_{iN} \geq 0, \ \ i = 0, \ldots, m, \qquad \sum_{i=0}^{m+r} \lambda_{iN}^2 = 1 \ , \tag{6.92}$$

and such that the approximate maximum condition (6.85) *is fulfilled with* $\varepsilon_N(t, h_N) \to 0$ *uniformly in* $t \in T_N$ *as* $N \to \infty$*, where each* $p_N(t)$*,* $t \in T_N \cup \{b\}$*, is the corresponding trajectory of the adjoint system* (6.78) *with the endpoint transversality condition*

$$p_N(b) = -\sum_{i=0}^{m+r} \lambda_{iN} \nabla\varphi_i\big(\bar{x}_N(b)\big) \ . \tag{6.93}$$

We postpone the proof of this major theorem till the next subsection and now present two *counterexamples* showing the *properness* and *consistency* conditions are *essential* for the validity of the AMP under the other assumptions held. Our first example concerns the properness condition from Definition 6.58.

Example 6.60 (AMP may not hold in smooth control problems with no properness condition). *There is a two-dimensional linear control problem with an inequality constraint such that optimal controls in the sequence of its discrete approximations are not proper and don't satisfy the approximate maximum principle.*

Proof. Consider a linear continuous-time optimal control problem (P) with a two-dimensional state $x = (x_1, x_2) \in I\!R^2$ in the following form:

$$\begin{cases} \text{minimize } \varphi\big(x(1)\big) := -x_1(1) \ \text{ subject to} \\[2ex] \dot{x}_1 = u, \quad \dot{x}_2 = x_1 - ct, \quad x_1(0) = x_2(0) = 0 \ , \\[2ex] u(t) \in U := \{0, 1\}, \quad 0 \leq t \leq 1 \ , \\[2ex] x_2(1) \leq -\dfrac{c-1}{2} \ , \end{cases}$$

where $c > 1$ is a given constant. Observe that the only "unpleasant" feature of this problem is that the control set $U = \{0, 1\}$ is *nonconvex*, and hence the feasible velocity sets $f(x, U, t)$ are nonconvex as well. It is clear that $\bar{u}(t) \equiv 1$

is the unique optimal solution to problem (P) and that the corresponding optimal trajectory is $\bar{x}_1(t) = t$, $\bar{x}_2(t) = -\dfrac{c-1}{2}t^2$. Moreover, the inequality constraint is *active*, since $\bar{x}_2(1) = -\dfrac{c-1}{2}$.

Let us now discretize this problem with the stepsize $h_N := \frac{1}{2N}$, $N \in I\!N$. For the notation convenience we omit the index N in what follows. Thus the discrete approximation problems (P_N) corresponding to the above problem (P) are written as:

$$\begin{cases} \text{minimize } \varphi\big(x(1)\big) = -x_1(1) \ \text{ subject to} \\ x_1(t+h) = x_1(t) + hu(t), \quad x_1(0) = 0\ , \\ x_2(t+h) = x_2(t) + h\big(x_1(t) - ct\big), \quad x_2(0) = 0\ , \\ u(t) \in \big\{0, 1\big\}, \quad t \in \big\{0, h, \ldots, 1-h\big\}\ , \\ x_2(1) \le -\dfrac{c-1}{2} + h^2\ , \end{cases}$$

i.e., we put $\gamma_N := h_N^2$ in the constraint perturbation for (P_N).

To proceed, we compute the trajectories in (P_N) corresponding to $u(t) \equiv 1$. It is easy to see that $x_1(t) = t$ for this $u(\cdot)$. To compute $x_2(t)$, observe that

$$\big[x(t+h) = x(t) + ht,\ x(0) = 0\big] \Longrightarrow x(t) = \frac{t^2}{2} - \frac{th}{2}\ .$$

Indeed, one has by the direct calculation that

$$x(t) = h\sum_{\tau=0}^{t-h} = \big[\text{put } \tau = kh\big] = h^2 \sum_{k=0}^{\frac{t}{h}-1} k = h^2 \frac{\frac{t}{h}\big(\frac{t}{h} - 1\big)}{2} = \frac{t^2}{2} - \frac{th}{2}\ .$$

Therefore for $x_2(t)$ corresponding to $u(t) \equiv 1$ in (P_N) we have

$$x_2(t) = h\sum_{\tau=0}^{t-h}\big(\tau - c\tau\big) = -\frac{c-1}{2}t^2 + \frac{c-1}{2}ht\ .$$

By this calculation we see that, for h sufficiently small, $x_2(1)$ no longer satisfies the endpoint constraint, and thus $u(t) \equiv 1$ is not a feasible control to problem (P_N) for all h close to zero. This implies that an optimal control to (P_N) for small h, which obviously exists, must have at least one *switching point* s such that $\bar{u}(s) = 0$, and hence the maximum value of the corresponding endpoint $x_1(1)$ will be less than or equal to $1 - h$. Put

$$u(t) := \begin{cases} 1 & \text{if } t \ne s\ , \\ 0 & \text{if } t = s \end{cases}$$

and justify the formula

$$x_2(t) = \begin{cases} -\dfrac{c-1}{2}t^2 + \dfrac{c-1}{2}ht, & t \le s\ , \\ -\dfrac{c-1}{2}t^2 + \dfrac{c-1}{2}ht - h(t-s) + h^2, & t \ge s+h\ , \end{cases}$$

for the corresponding trajectories in (P_N) depending on h and s. We only need to justify the second part of this formula. To compute $x_2(t)$ for $t \ge s+h$, substitute $x_1(t) = t-h$ into the discrete system in (P_N). It is easy to see that the increment $\Delta x_2(t)$ compared to the case when $u(t) \equiv 1$ is

$$h \sum_{\tau=s+h}^{t-h} (-h) = -h(t-h-s) = -h(t-s) + h^2\ ,$$

which justifies the above formula for $x_2(t)$.

Now we specify the parameters of the above control putting $c = 2$ and $s = 0.5$ for all N, i.e., considering the discrete-time function

$$\bar{u}(t) := \begin{cases} 1 & \text{if } t \ne 0.5\ , \\ 0 & \text{if } t = 0.5\ . \end{cases}$$

Note that the point $t = 0.5$ belongs to the grid T_N for all N due to $h_N := \frac{1}{2N}$. Observe further that the sequence of these controls *doesn't satisfy the properness property* in Definition 6.58. It follows from the above formula for $x_2(t)$ that the corresponding trajectories obey the endpoint constraint in (P_N) whenever $N \in I\!N$, since $\bar{x}_2(1) = -\frac{1}{2} + h^2$. Moreover, it is clear from the given calculations that the control $\bar{u}(t)$ is optimal to problem (P_N) for any N.

Let us show that this sequence of optimal controls $\bar{u}(\cdot)$ doesn't satisfy the approximate maximum condition (6.85) at the point of switch. Indeed, the adjoint system (6.78) for the problems (P_N) under consideration is

$$p(t) = p(t+h) + h\nabla_x f(\bar{x}_1, \bar{x}_2, \bar{u}, t)^* p(t+h)\ ,$$

where the Jacobian matrix $\nabla_x f$ and its adjoint/transposed one are equal to

$$\nabla_x f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \nabla_x f^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ .$$

Thus we have the adjoint trajectories

$$p_1(t) = p_1(t+h) + hp_2(t+h) \quad \text{and} \quad p_2(t) \equiv \text{const},$$

where the pair (p_1, p_2) satisfies the transversality condition (6.93) with the corresponding sign and nontriviality conditions (6.92) written as

$$p_1(1) = \lambda_0, \; p_2(1) = -\lambda_1; \quad \lambda_0 \geq 0, \; \lambda_1 \geq 0, \; \lambda_0^2 + \lambda_1^2 = 1 \; .$$

This implies that $p_1(t)$ is a linear nondecreasing function. The corresponding Hamilton-Pontryagin function is equal to

$$H\big(x(t), p(t+h), u(t)\big) = p_1(t+h)u(t) + \text{terms not depending on } u \; .$$

Examining the latter expression and taking into account that the optimal controls are equal to $\bar{u}(t) = 1$ for all t but $t = 0.5$, we conclude that the approximate maximum condition (6.85) holds only if $p_1(t)$ is either nonnegative or tends to zero everywhere except $t = 0.5$. Observe that $p_1(t) \equiv 0$ yields $\lambda_1 = \lambda_2 = 0$, which contradicts the nontriviality condition. Hence $p_1(t)$ must be positive away from $t = 0$. Therefore a sequence of controls having a point of switch not tending to zero as $h \downarrow 0$ *cannot* satisfy the approximate maximum condition at this point. This shows that the AMP *doesn't hold* for the sequence of optimal controls to the problems (P_N) built above. $\triangle$

Many examples of this type can be constructed based on the above idea, which essentially means the following. Take a continuous-time problem with active inequality constraints and *nonconvex* admissible velocity sets $f(x, U, t)$. It often happens that after the discretization the "former" optimal control becomes not feasible in discrete approximations, and the "new" optimal control in the sequence of discrete-time problems has a *singular point of switch* (thus making the sequence of optimal controls not proper), where the approximate maximum condition is not satisfied.

The next example shows that the AMP may be violated for *proper* sequences of optimal controls to discrete approximation problems for continuous-time systems with *equality* endpoint constraints if such constraints are *not perturbed consistently* with the step of discretization.

Example 6.61 (AMP may not hold with no consistent perturbations of equality constraints). *There is a two-dimensional linear control problem with a linear endpoint constraint of the equality type such that a proper sequence of optimal controls to its discrete approximations doesn't satisfy the AMP without consistent constraint perturbations.*

Proof. Consider first the following optimal control problem for a two-dimensional system with an endpoint constraint of the equality type:

$$\begin{cases} \text{minimize } \; \varphi_0\big(x(1)\big) := x_2(1) \; \text{ subject to} \\[2ex] \dot{x} = u, \quad t \in T := [0,1], \;\; x(0) = 0 \; , \\[2ex] u(t) \in U := \Big\{(0,0), \; (0,-1), \; (1, -\sqrt{2}), \; (-\sqrt{2}, -3)\Big\} \; , \\[2ex] \varphi_1\big(x(1)\big) := x_1(1) = 0 \; , \end{cases}$$

where $x = (x_1, x_2) \in I\!R^2$ and $u = (u_2, u_2) \in I\!R^2$. One can see that this linear problem is as standard and simple as possible with the only exception regarding the *nonconvexity* of the control region U.

Construct a sequence of discrete approximation problems (P_N) in the standard way of Theorem 6.59 by taking *zero perturbation* of the endpoint constraint, i.e., with $\xi_N = 0$. Thus we have:

$$\begin{cases} \text{minimize } \varphi_0\big(x_N(1)\big) = x_{2N}(1) \text{ subject to} \\ x_N(t+h_N) = x_N(t) + h_N u_N(t), \quad x_N(0) = 0 \in I\!R^2 , \\ u(t) \in U, \quad t \in T_N := \big\{0, h_N, \ldots, 1-h_N\big\} , \\ \varphi_1\big(x_N(1)\big) = x_{1N}(1) = 0 \text{ with } h_N = N^{-1}, \quad N \in I\!N . \end{cases}$$

It is easy to check that the only optimal solutions to problems (P_N) are

$$\bar{u}_N(t) = (0, -1), \quad \bar{x}_N(t) = (0, -t) \text{ for all } t \in T_N, \quad N \in I\!N ,$$

which give the minimal value of the cost functional $\bar{J}_N = -1$. Note that the sequence $\{\bar{u}_N(\cdot)\}$ is obviously *proper* in the sense of Definition 6.58. The corresponding trajectories $p_N(\cdot)$ of the adjoint system (6.78) satisfying the transversality condition (6.93) are

$$p_N(t) = (-\lambda_{1N}, -\lambda_{0N}) \text{ for all } t \in T_N \cup \{1\} ,$$

where the sign and nontriviality conditions (6.92) for the multipliers $(\lambda_{0N}, \lambda_{1N})$ are written as

$$\lambda_{0N} \geq 0, \quad \lambda_{0N}^2 + \lambda_{1N}^2 = 1 \text{ whenever } N \in I\!N .$$

Furthermore, for each $N \in I\!N$ the Hamilton-Pontryagin function in the discrete-time system computed along $\bar{x}_N(\cdot)$ and the corresponding adjoint trajectory $p_N(\cdot)$ reduces to

$$H_N(u, t) = -\lambda_{1N} u_1 - \lambda_{0N} u_2, \quad t \in T_N ,$$

that gives $H_N(\bar{u}_N) = \lambda_{0N}$ for the optimal control.

Let us justify the estimate

$$\delta_N := \max\big\{H_N(u)\big|\; u \in U\big\} - H_N(\bar{u}_N) \geq 1 \text{ for all } N \in I\!N ,$$

which shows that the approximate maximum condition (6.85) is violated in the above sequence of problems (P_N). To proceed, consider the two possible cases for the multipliers $(\lambda_{0N}, \lambda_{1N})$:

(a) $\lambda_{0N} \geq 0, \;\; \lambda_{1N} \geq 0, \;\; \lambda_{0N}^2 + \lambda_{1N}^2 = 1;$

(b) $\lambda_{0N} \geq 0, \;\; \lambda_{1N} < 0, \;\; \lambda_{0N}^2 + \lambda_{1N}^2 = 1.$

In case (a) we have that

$$\delta_N = \lambda_{1N}\sqrt{2} + 3\lambda_{0N} - \lambda_{0N} \geq \sqrt{2}\big(\lambda_{1N} + \lambda_{0N}\big) \geq \sqrt{2}\ ,$$

while case (b) allows the estimate

$$\delta_N \geq |\lambda_{1N}| + \sqrt{2}\lambda_{0N} - \lambda_{0N} \geq (\sqrt{2}-1)(|\lambda_{1N}| + \lambda_{0N}) \geq \sqrt{2} - 1\ .$$

Thus the AMP doesn't hold in the sequence of discrete approximation problems under consideration. $\triangle$

We can observe from the above discussion that the failure of the AMP in Example 6.61 is due to the fact that the equality constraint is not perturbed (or *not sufficiently perturbed*) in the process of discrete approximation, while the optimal value of the cost functional is *not stable* with respect to such perturbations. Indeed, any control $u_N(t)$ equal to either $(1,-2)$ or $(-\sqrt{2},-3)$ at some $t \in T_N$ and giving the value $J_N[u_N] < -1$ to the cost functional is *not feasible* for the constraint $x_{1N}(1) = 0$, being however feasible for appropriate perturbations of this constraint. On the other hand, these very points of U provide the *maximum* to the Hamilton-Pontryagin function. Such a situation occurs in the discrete-time systems of Example 6.61 due to the *incommensurability* of irrational numbers in the control set U and just the rational mesh T_N for all $N \in N$. Of course, this is *not possible* in *continuous-time* systems by the completeness of real numbers.

6.4.5 Approximate Maximum Principle under Endpoint Constraints: Proofs and Applications

After all the discussions above, let us start proving Theorem 6.59. We split the proof into *three major steps* including two lemmas of independent interest, which contribute to our understanding of an appropriate counterpart of the *hidden convexity for discrete approximations*. Then we derive an *upper subdifferential* extension of the AMP to constrained problems with inequality constraint described by uniformly upper subdifferential functions. Finally, we present some typical applications of the AMP to discrete-time (with small stepsize) and continuous-time systems.

Let $u_N(t) \in U$ for all $t \in T_N$ as $N \in I\!N$. Given an integer number r with $1 \leq r \leq N-1$, we define needle-type variations of the control $u_N(\cdot)$ as follows. Consider a set of parameters $\{\theta_j(N), v_j(N)\}_{j=1}^r$, where $v_j(N) \in U$ and where $\theta_j(N)$ are integers satisfying

$$0 \leq \theta_j(N) \leq N-1 \ \text{ with } \ \theta_j(N) \neq \theta_i(N) \ \text{ if } \ j \neq i\ .$$

Denoting $\tau_{\theta_j(N)} := a + \theta_j(N)h_N$, we call

$$\widetilde{u}_N(t) := \begin{cases} v_j(N), & t = \tau_{\theta_j(N)}\ , \\ u_N(t), & t \in T_N,\ \ t \neq \tau_{\theta_j(N)},\ \ j = 1,\ldots,r\ , \end{cases} \tag{6.94}$$

the *r-needle variation* of the control $u_N(\cdot)$ with the parameters $\{\theta_j(N), v_j(N)\}$. When $r = 1$, control (6.94) is a (single) *needle* variation of $u_N(\cdot)$, while it is a *multineedle* variation of $u_N(\cdot)$ for $r > 1$. The variations introduced are discrete-time counterparts of the corresponding needle-type variations (6.71) and (6.72) of continuous-time controls, being however essentially different from the latter especially in the multineedle case.

Let $\widetilde{x}_N(\cdot)$ be the trajectory of the finite-difference system

$$x_N(t+h_N) = x_N(t) + h_N f\big(x_N(t), u_N(t), t\big), \quad x_N(a) = x_0 \,, \tag{6.95}$$

corresponding to the control variation $\widetilde{u}_N(\cdot)$ with the parameters $\{\theta_j(N), v_j(N)\}$; in what follows we usually skip indicating their dependence on N. Then the difference $\widetilde{x}_N(\cdot) - x_N(\cdot)$ is denoted by $\Delta^r_{\{\theta_j,v_j\}} x_N(\cdot)$ for $r > 1$ and by $\Delta_{\theta,v} x_N(\cdot)$ for $r = 1$; it is called for convenience the *multineedle* (or *r-needle*) and the (single) *needle trajectory increment*, respectively. We speak about the corresponding *endpoint increments* when $t = b$.

Our first intention is to establish relationships between *integer combinations* of endpoint trajectory increments generated by *single needle* variations of the reference controls $u_N(\cdot)$ as $N \to \infty$ and some *multineedle* endpoint trajectory increments. The result derived below can be essentially viewed as an *approximate finite-difference* analog of the *hidden convexity* property crucial for continuous-time systems.

Let $\{u_N(t)\}$, $t \in T_N$, be the reference control sequence, and let $(\theta_j(N), v_j(N))$ be parameters of *single* needle variations of $u_N(\cdot)$ for each $j = 1, \ldots, p$, where p is a natural number independent of N. Given nonnegative integers m_j as $j = 1, \ldots, p$ also independent of N, consider the corresponding needle trajectory increments $\Delta_{\theta_j,v_j} x_N(b)$ and denote them by $\Delta_{\theta,v,j} x(b)$ for simplicity. Form the *integer combination*

$$\Delta_N(p, m_j) := \sum_{j=1}^{p} m_j \Delta_{\theta,v,j} x_N(b)$$

of the (single) needle trajectory increments for each $N = p, p+1, \ldots$ and show that it can be represented, up to a *small quantity* of order $o(h_N)$, as a *multineedle* variation of the reference control.

Lemma 6.62 (integer combinations of needle trajectory increments). *Let $\{u_N(\cdot)\}$, $N \in I\!N$, be a proper sequence of reference controls, let $p \in I\!N$ and $m_j \in I\!N \cup \{0\}$ for $j = 1, \ldots, p$ be independent of N, and let $\big(\theta_j(N), v_j(N)\big)$, $j = 1, \ldots, p$, be parameters of (single) needle variations. Then there are $r \in I\!N$ independent of N and parameters $\{\widetilde{\theta}_j(N), \widetilde{v}_j(N)\}_{j=1}^r$, of r-needle variations of type (6.94) such that*

$$\Delta_N(p, m_j) = \Delta^r_{\{\widetilde{\theta}_j,\widetilde{v}_j\}} x_N(b) + o(h_N) \quad \text{as} \quad N \to \infty \,.$$

for the corresponding endpoint trajectory increments.

Proof. First we obtain convenient representation of endpoint trajectory increments generated by needle and multineedle variations of the reference controls, which are *not* required to form a proper sequence in this setting. Recall the above notation for matrix products and denote by $K>0$ a common uniform norm bound of f and $\nabla_x f$ along $\{u_N(\cdot), x_N(\cdot)\}$, which exists due to the standing assumptions formulated in Subsect. 6.4.3. Note that, for applications to the main theorems, below but not in this lemma, we actually need the uniform boundedness along the reference sequence of *optimal* solutions to (P_N).

We start with *single* needle variations generated by parameters $\big(\theta(N), v(N)\big)$. It immediately follows from (6.95) and the smoothness of f in x that

$$\Delta_{\theta,v} x_N(\tau_i) = 0, \quad i = 0, \ldots, \theta \ ,$$

$$\Delta_{\theta,v} x_N(\tau_{\theta+1}) = h_N \big[f\big(x_N(\tau_\theta), v, \tau_\theta\big) - f\big(x_N(\tau_\theta), u_N(\tau_\theta), \tau_\theta\big)\big] =: h_N y \ ,$$

$$\begin{aligned}\Delta_{\theta,v} x_N(\tau_{\theta+2}) = h_N \big[I + h_N \nabla_x f\big(x_N(\tau_{\theta+1}), u_N(\tau_{\theta+1}), \tau_{\theta+1}\big)\big] y \\ + h_N\, o\big(\|\Delta_{\theta,v} x_N(\tau_{\theta+1})\|\big) \ .\end{aligned}$$

Then we easily have by induction that

$$\begin{aligned}&\Delta_{\theta,v} x_N(b) = h_N \Big\{ \prod_{\theta+1}^{i=N-1} \Big[I + h_N \nabla_x f\big(x_N(\tau_i), u_N(\tau_i), \tau_i\big)\Big]\Big\} y \\ &+ h_N \sum_{k=\theta+2}^{N-1} \Big\{ \prod_{k}^{i=N-1} \Big[I + h_N \nabla_x f\big(x_N(\tau_i), u_N(\tau_i), \tau_i\big)\Big]\Big\}\, o\big(\|\Delta_{\theta,v}(\tau_{k-1})\|\big) \\ &+ h_N\, o\big(\|\Delta_{\theta,v} x_N(\tau_{N-1})\|\big) \ .\end{aligned}$$

Observe from (6.95) and the assumptions made that $\Delta_{\theta,v} x_N(t) = O(h_N)$ for all $t \in T_N$ uniformly in N. Thus given any $\varepsilon > 0$, there is $N_\varepsilon \in I\!N$ such that

$$\big\| o\big(\|\Delta_{\theta,v} x_N(\tau_k)\|\big)\big\| \le \varepsilon h_N, \quad k = \theta+2, \ldots, N-1, \ \ N \ge N_\varepsilon \ ,$$

which implies the estimate

$$\begin{aligned}&\Big\| \sum_{k=\theta+2}^{N-1} \Big\{ \prod_{k}^{i=N-1} \Big(I + h_N \nabla_x f\big(x_N(\tau_i), u_N(\tau_i), \tau_i\big)\Big)\Big\}\, o\big(\|\Delta_{\theta,v} x_N(\tau_{k-1})\|\big)\Big\| \\ &\le \varepsilon h_N \sum_{k=\theta+2}^{N-1} \prod_{k}^{i=N-1} (1 + h_N K)^i \le \frac{\varepsilon}{K} \exp\big(K(b-a)\big) \ .\end{aligned}$$

Combining this with the above formula for $\Delta_{\theta,v}x_N(b)$, we arrive at the efficient representation

$$\begin{aligned}\Delta_{\theta,v}x_N(b) = h_N\Big\{\prod_{\theta+1}^{i=N-1}\Big[I + h_N\nabla_x f\big(x_N(\tau_i),u_N(\tau_i),\tau_i\big)\Big]\Big\}y \\ +o(h_N) \ \text{ as } \ N\to\infty\end{aligned} \tag{6.96}$$

for the endpoint trajectory increments generated by *single needle* variations of the reference controls, where $o(h_N)/h_N \to 0$ independently of the needle parameters $\theta=\theta(N)$ and $v=v(N)$ as $N\to\infty$.

Consider now endpoint trajectory increments generated by *multineedle* variations (6.74) with parameters $\{\theta_j(N),v_j(N)\}_{j=1}^r$. Similarly to (6.96) we derive the following representation:

$$\begin{aligned}\Delta^r_{\{\theta_j,v_j\}}x_N(b) = h_N\Big\{\sum_{j=1}^r\Big[\prod_{\theta_j+1}^{i=N-1}\Big(I + h_N\nabla_x f\big(x_N(\tau_i),u_N(\tau_i),\tau_i\big)\Big)\Big]y_j\Big\} \\ +o(h_N) \ \text{ as } \ N\to\infty\ ,\end{aligned} \tag{6.97}$$

where $o(h_N)$ is independent of $\{\theta_j(N),v_j(N)\}$ but depends on the number r of varying points, and where

$$y_j := f\big(x_N(\tau_{\theta_j}),v_j,\tau_{\theta_j}\big) - f\big(x_N(\tau_{\theta_j}),u_N(\tau_{\theta_j}),\tau_{\theta_j}\big) \ \text{ for } \ j=1,\ldots,r\ .$$

Next we assume that the control sequence $\{u_N(\cdot)\}$ is *proper* and justify the main relationship formulated in this lemma. Without loss of generality, suppose that the mesh points

$$\tau_{\theta_j(N)} := a+\theta_j(N)h_N,\quad j=1,\ldots,p\ ,$$

converge to some numbers $\bar\tau_j\in[a,b]$, $j=1,\ldots,p$, as $N\to\infty$. First we examine the case of

$$\bar\tau_i\ne\bar\tau_j \ \text{ for } \ i\ne j, \quad \text{and} \ \ \bar\tau_j\ne b \ \text{ whenever } \ i,j\in\{1,\ldots,p\}\ . \tag{6.98}$$

Given the parameters of the integer combination $\Delta_N(p,m_j)$, for each $N\ge p$, we take the number $r:=m_1+\ldots+m_p$ independent of N and consider the endpoint trajectory increment $\Delta^r_{\{\widetilde\theta_{jq},\widetilde v_{jq}\}}x_N(b)$ generated by the multineedle control variation

$$\widetilde u_N(t) := \begin{cases} v_j(N) & \text{if } t=\tau_{\theta_j+q}(N)\ ,\\ u_N(t) & \text{if } t\ne\tau_{\theta_j+q}(N),\quad t\in T_N\ ,\end{cases} \tag{6.99}$$

whenever $j=1,\ldots,p$ and $q=0,\ldots,m_j-1$ with

$$\widetilde{\theta}_{jq}(N) := \theta_j(N) + q \;\text{ and }\; \widetilde{v}_{jq}(N) := v_j(N) \;\text{ for all }\; j, q \;.$$

By assumptions (6.98) these multineedle control variations are well defined for all large N. Employing representation (6.97) of the corresponding endpoint increments, we have

$$\Delta^r_{\{\widetilde{\theta}_{iq},\widetilde{v}_{jq}\}} x_N(b) = h_N \Big\{ \sum_{j=1}^{p} \sum_{q=1}^{m_j} \Big[\prod_{\theta_j+q}^{i=N-1} \Big(I + h_N \nabla_x f\big(x_N(\tau_i), u_N(\tau_i), \tau_i\big)\Big)\Big] y_{jq-1} \Big\}$$

$$+ o(h_N) \;\text{ as }\; N \to \infty$$

with a uniform estimate of $o(h_N)$ and with

$$y_{jq} := f\big(x_N(\tau_{\theta_j+q}), v_j, \tau_{\theta_j+q}\big) - f\big(x_N(\tau_{\theta_j+q}), u_N(\tau_{\theta_j+q}), \tau_{\theta_j+q}\big) \;.$$

By the *properness* of $\{u_N(\cdot)\}$ and the continuity of f with respect to all its variables we get $y_{j\theta} - y_{j0} \to 0$ as $N \to \infty$, which implies the representation

$$\Delta^r_{\{\widetilde{\theta}_{jq},\widetilde{v}_{jq}\}} x_N(b) = \Big\{ \sum_{j=1}^{p} m_j \prod_{\theta_j+1}^{i=N-1} \Big[I + h_N \nabla_x f\big(x_N(\tau_i), u_N(\tau_i), \tau_i\big)\Big] y_j \Big\}$$

$$+ o(h_N) \;\text{ as }\; N \to \infty \;,$$

where y_j are defined in (6.97). Comparing the latter representation with formula (6.96) for the endpoint trajectory increment generated by *single* needle variations with the parameters $\big(\theta_j(N), v_j(N)\big)$ as $j = 1, \ldots, p$ and taking into account the expression for $\Delta_N(p, m_j)$, we arrive at the conclusion of the lemma under the above requirements (6.98) on the limiting point $\bar{\tau}_j$.

Suppose now that these requirements are not fulfilled. It is sufficient to examine the following two extreme cases:

(a) $\bar{\tau}_1 = \bar{\tau}_2 = \ldots = \bar{\tau}_p \neq b$,

(b) $\bar{\tau}_1 = \bar{\tau}_2 = \ldots = \bar{\tau}_p = b$,

which being combined with (6.98) cover all the possible locations of the limiting points $\bar{\tau}_j$ in $[a, b]$. Let us present the corresponding modifications of the multineedle variations (6.99) in both cases (a) and (b), which lead to the conclusion of the lemma similarly to the arguments above.

To proceed in case (a), reorder $\big(\theta_j(N), v_j(N)\big)$ as $j = 1, \ldots, p$ in such a way that $\theta_1 < \ldots < \theta_p$ (assuming that all θ_j are different without loss of generality) and identify for convenience the indexes θ_j with the corresponding mesh points τ_{θ_j}. Then construct the variations of $u_N(\cdot)$ at the points θ_1, $\theta_1 + 1$,…,$\theta_1 + m_1 - 1$ as in (6.99). Assuming that the control variations corresponding to the parameters (θ_i, v_i) as $1 \leq i \leq p - 1$ have been already built, construct them for (θ_{i+1}, v_{i+1}). Denote by θ_0 the greatest point among those of $\{\theta_j\}$ at which we have built the control variations. If $\theta_0 < \theta_{i+1}$, construct variations of $u_N(\cdot)$ at $\theta_{i+1}, \theta_{i+1}+1, \ldots, \theta_{i+1}+m_{i+1}$ as in (6.99). If $\theta_0 \geq \theta_{i+1}$, construct variations of the same type at $\theta_0+1, \ldots, \theta_0+m_{i+1}$. One can check the multineedle variations built in this way ensure the fulfillment of the lemma conclusion in case (a).

In case (b) we proceed by reordering $\big(\theta_j(N), v_j(N)\big)$ as $j = 1, \ldots, p$ so that $\theta_1 > \theta_2 > \ldots > \theta_p$ and then construct the corresponding multineedle variations of $u_N(\cdot)$ symmetrically to case (a), i.e., from the right to the left. In this way we complete the proof of the lemma. $\triangle$

The next result gives a *sequential* finite-difference analog of Lemma 6.44 and may be treated as a certain *approximate* (not exact/limiting) manifestation of the hidden convexity in discrete approximation problems, with no using the abstraction of time continuity. To proceed, we need to distinguish between *essential* and *inessential* inequality constraints in the process of discrete approximation important in what follows.

Definition 6.63 (essential and inessential inequality constraints for finite-difference systems). *The inequality endpoint constraint*

$$\varphi_i\big(x_N(b)\big) \leq \gamma_{iN} \quad \textit{with some} \;\; i \in \{1, \ldots, m\}$$

is ESSENTIAL *for a sequence of feasible solutions* $\{u_N(\cdot), x_N(\cdot)\}$ *to problems* (P_N) *along a subsequence of natural numbers* $\mathcal{M} \subset I\!N$ *if*

$$\varphi_i\big(x_N(b)\big) - \gamma_{iN} = O(h_N) \quad \textit{as} \;\; h_N \to \infty\ ,$$

i.e., there is a real number $K_i \geq 0$ *such that*

$$-K_i h_N \leq \varphi_i\big(x_N(b)\big) - \gamma_{iN} \leq 0 \quad \textit{as} \;\; N \to \infty, \quad N \in \mathcal{M}\ .$$

This constraint is INESSENTIAL *for the sequence* $\{u_N(\cdot), x_N(\cdot)\}$ *along* $\mathcal{M}$ *if whenever* $K > 0$ *there is* $N_0 \in I\!N$ *such that*

$$\varphi_i\big(x_N(b)\big) - \gamma_{iN} \leq -K h_N \quad \textit{for all} \;\; N \geq N_0, \quad N \in \mathcal{M}\ .$$

The notion of essential constraints in sequences of discrete approximations corresponds to the notion of *active* constraints in nonparametric optimization problems. Without loss of generality, suppose that for the sequence of optimal solutions $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ to the parametric problems (P_N) under consideration the *first* $l \in \{1, \ldots, m\}$ inequality constraints are *essential* while the other $m-l$ constraints are inessential along *all* natural numbers, i.e., with $\mathcal{M} = I\!N$.

Given optimal solutions $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ to problems (P_N) as $N \in I\!N$, we form the *linearized image set*

$$S_N := \big\{(y_0, \ldots, y_l) \in I\!R^{l+1} \big|\; y_i = \big\langle \nabla\varphi_i\big(\bar{x}_N(b)\big), \Delta_{\theta,v}\bar{x}_N(b)\big\rangle\big\} \qquad (6.100)$$

generated by inner products involving the gradients of the cost and *essential* inequality constraint functions and the endpoint trajectory increments corresponding to *all* the *single needle* variations of the optimal controls. Our goal is to show that the sequence $\{\operatorname{co} S_N\}$ of the convex hulls of sets (6.100) can be *shifted* by some quantities of *order* $o(h_N)$ as $h_N \to 0$ so that the resulting sets don't intersect the convex set of *forbidden points* in R^{l+1} given by

$$I\!R^{l+1}_{<} := \big\{(y_0,\ \ldots, y_l) \in I\!R^{l+1}\big|\ y_i < 0 \ \text{ for all } \ i = 0, \ldots, l\big\}\ .$$

Lemma 6.64 (hidden convexity and primal optimality conditions in discrete approximation problems with inequality constraints). *Let* $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ *be a sequence of optimal solutions to problems* (P_N) *with* $\varphi_i = 0$ *as* $i = m+1, \ldots, m+r$ *(no perturbed equality constraints). In addition to the standing assumptions, suppose that the endpoint functions* φ_i *are continuously differentiable around the limiting point(s) of* $\{\bar{x}_N(\cdot)\}$ *for all* $i = 0, \ldots, m$*. Assume also that the control sequence* $\{\bar{u}_N(\cdot)\}$ *is proper and that the first* $l \in \{1, \ldots, m\}$ *inequality constraints are essential for* $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ *while the other are inessential for these solutions. Then there is a sequence of* $(l+1)$*-dimensional quantities of order* $o(h_N)$ *as* $h_N \to 0$ *such that*

$$\big(\operatorname{co} S_N + o(h_N)\big) \cap I\!R^{l+1}_{<} = \emptyset \ \ \textit{for all large} \ \ N \in I\!N\ . \tag{6.101}$$

Proof. For each N and fixed $r \in I\!N$ independent of N, consider an endpoint trajectory increment $\Delta^r_{\{\theta_j, v_j\}}\bar{x}_N(b)$ generated by a *multineedle* variation of the optimal control $\bar{u}_N(\cdot)$, where $\{\theta_j(N), v_j(N)\}^r_j$ are the variation parameters in (6.94). Form a sequence of the vectors

$$y_N = (y_{N0}, \ldots, y_{Nl}) \in I\!R^{l+1} \ \text{ with } \ y_{Ni} := \big\langle \nabla\varphi_i\big(\bar{x}_N(b)\big),\, \Delta^r_{\{\theta_j, v_j\}}\bar{x}_N(b)\big\rangle$$

and show that there is a sequence of $(l+1)$-dimensional quantities of order $o(h_N)$ as $h_N \to 0$ such that

$$y_N + o(h_N) \notin I\!R^{l+1}_{<} \ \text{ as } N \to \infty\ . \tag{6.102}$$

Indeed, it follows from representation (6.97) and the assumptions made that

$$\big\|\Delta^r_{\{\theta_j, v_j\}}\bar{x}_N(b)\big\| \le \mu h_N \ \text{ for all } \ t \in T_N \ \text{ and } \ N \in I\!N\ ,$$

where $\mu > 0$ depends on r but not on $\{\theta_j(N), v_j(N)\}^r_{j=1}$. By optimality of $\bar{x}_N(\cdot)$ in problems (P_N) with no perturbed equality constraints, for each $N \in I\!N$ there is an index $i_0(N) \in \{0, \ldots, m\}$ such that

$$\varphi_{i_0}\big(\bar{x}_N(b) + \Delta^r_{\{\theta_j, v_j\}}\bar{x}_N(b)\big) - \varphi_{i_0}\big(\bar{x}_N(b)\big) \ge 0\ .$$

Since only the first l inequality constraints are essential for $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$, the latter inequality holds for some $i_0 \in \{0, \ldots, l\}$ whenever N is sufficiently large. Consider the numbers

$$\delta_N := \max_{0 \le i \le l} \sup\Big\{\ \big|\varphi_i\big(\bar{x}_N(b) + \Delta x\big) - \varphi_i\big(\bar{x}_N(b)\big)$$

$$-\big\langle \nabla\varphi_i\big(\bar{x}_N(b)\big), \Delta x\big\rangle\big|\ \Big|\ \|\Delta x\| \le \mu h_N\Big\}$$

for which $\delta_N/h_N \to 0$ as $N \to \infty$ uniformly with respect to variations due to the smoothness of φ_i assumed. This implies that

$$y_{Ni_0} + \delta_N \geq 0 \ \text{ as } \ N \to \infty \ ,$$

which justifies (6.102) with the quantities $o(h_N) := (0, \ldots, \delta_N, \ldots, 0) \in I\!R^{l+1}$, where δ_N appears at the $i_0(N)$-th position.

Our next goal is to obtain an analog of estimate (6.102) for *convex combinations* of endpoint trajectory increments generated by *single needle* variations of the optimal controls. In the case of such *integer combinations*, the corresponding analog of (6.102) follows directly from this estimate due to the preceding Lemma 6.62. Let us show that the case of convex combinations can be actually reduced to the integer one.

Consider a sequence of parameters $\big(\theta_j(N), v_j(N)\big)$, $j = 1, \ldots, p$, generating single needle variations of the optimal controls $\{\bar{u}_N(\cdot)\}$ with some $p \in I\!N$ and then define the *convex combinations*

$$\begin{aligned} &y_{Ni}(p,\alpha) := \sum_{j=1}^{p} \alpha_j(N)\Big\langle \nabla\varphi_i\big(\bar{x}_N(b)\big), \Delta_{\theta,v,j}\bar{x}_N(b)\Big\rangle \ , \\ &\text{as } \ \alpha_j(N) \geq 0, \ \ \alpha_1(N) + \ldots + \alpha_p(N) = 1, \quad i = 0, \ldots, l \ . \end{aligned} \tag{6.103}$$

Fixing (p,α) in the above combinations and taking $y_N(p,\alpha) \in I\!R^{l+1}$ with the components $y_{Ni}(p,\alpha)$, suppose that there is a number $N_0 \in I\!N$ such that

$$y_N(p,\alpha) \in I\!R^{l+1}_{<} \ \text{ whenever } \ N \geq N_0 \ .$$

Let us now show that for each natural number $N \geq N_0$ there is an index $i_0 = i_0(N) \in \{0, \ldots, l\}$ for which

$$0 > y_{Ni_0}(p,\alpha) = o(h_N) \ \text{ as } \ h_N \to \infty \ . \tag{6.104}$$

Assuming the contrary, we find a subsequence $\mathcal{M} \subset I\!N$ such that

$$\lim_{N\to\infty} \frac{y_{Ni}(p,\alpha)}{h_N} := \beta_i < 0 \ \text{ as } \ N \in \mathcal{M} \ \text{ for all } \ i = 0, \ldots, l \ .$$

Suppose without loss of generality that $\mathcal{M} = \{p, \ldots, p+1, \ldots\}$, that $\beta_i > -\infty$, and that the sequence $\{\alpha_j(N)\}$ converges to some $\alpha_j^0 \in I\!R$ as $N \to \infty$ for each $j = 1, \ldots, p$. Given $\nu > 0$, define p integers k_j by

$$k_j = k_j(\nu) := \left[\frac{\alpha_j^0}{\nu}\right] \ \text{ for all } \ j = 1, \ldots, p$$

and form the *integer combinations* $y_{Ni}(p,k)$ by

$$y_{Ni}(p,k) := \frac{y_{Ni}(p,\alpha^0)}{\nu} + \sum_{j=1}^{p}\Big(k_j - \frac{\alpha_j^0}{\nu}\Big)\Big\langle \nabla\varphi_i\big(\bar{x}_N(b)\big), \Delta_{\theta,v,j}\bar{x}_N(b)\Big\rangle$$

as $i = 0,\ldots,l$, where $k := (k_1,\ldots,k_p)$ and $\alpha^0 := (\alpha_1^0,\ldots\alpha_l^0)$.

Let $\mu > 0$ be the constant selected (with $r = 1$) in the proof of (6.102), and let $\kappa > 0$ be a uniform norm bound for all $\varphi_i\big(\bar{x}_N(b)\big)$ and $\nabla\varphi_i\big(\bar{x}_N(b)\big)$ as $i = 0,\ldots,l$. Choose $i_1 \in \{0,\ldots,l\}$ and define $\nu > 0$ so that

$$|\beta_{i_1}| = \min_{0\le i\le l}|\beta_i| \quad \text{and } \nu := \frac{\beta_{i_1}}{\beta_{i_1} - p\kappa\mu}\ .$$

Then we have the estimates

$$\lim_{N\to\infty}\frac{y_{Ni}(p,k)}{h_N} \le \beta_i - \frac{\beta_i\mu\kappa p}{\beta_{i_1}} + \mu\kappa p \le \beta_i < 0 \ \text{ whenever } \ i = 0,\ldots,l\ ,$$

which clearly contradict (6.102) by Lemma 6.62 on the representation of integer combinations of endpoint trajectory increments generated by (single) control variations. This proves (6.104).

Finally, we justify the required relationships (6.101). There is nothing to prove when $\mathrm{co}\, S_N \cap I\!R^{l+1}_< = \emptyset$ for all large $N \in I\!N$. Suppose that $\mathrm{co}\, S_N \cap I\!R^{l+1}_< \ne \emptyset$ along a subsequence $\{N\}$, which we put equal to the whole set $I\!N$ of natural numbers without loss of generality. For each $N \in I\!N$ define

$$\sigma_N := -\inf\Big\{\max_{0\le i\le l} y_i\Big|\ y = (y_0,\ldots,y_l) \in \mathrm{co}\, S_N \cap I\!R^{l+1}_-\Big\}\ ,$$

where the infimum is achieved at some $y_N \in I\!R^{l+1}_<$ under the assumptions made. Invoking the classical Carathéodory theorem, represent y_N in the convex combination form (6.103) with $p = l+2$. Employing now (6.104), we find an index $i_0 = i_0(N)$ such that

$$\sigma_N = -\max\big\{y_{Ni}\big|\ i = 0,\ \ldots, l\big\} \le -y_{Ni_0} = o(h_N) \ \text{ as } \ N \to \infty\ ,$$

which implies (6.101) with the $(l+1)$-dimensional shift $o(h_N) := (\sigma_N,\ldots,\sigma_N)$ and thus ends the proof of the lemma. △

Completing the proof of Theorem 6.59. Now we have all the major ingredients to complete the proof of the theorem. Let us start with the case when only the perturbed *inequality constraints* are present in problems (P_N), i.e., $\varphi_i = 0$ for $i = m+1,\ldots,m+r$. Since we suppose without loss of generality that the first $l \le m$ inequality constraints are *essential* for the sequence of optimal solutions $\{\bar{u}_N(\cdot),\bar{x}_N(\cdot)\}$, while the remaining $m-l$ inequality constraints are inessential for this sequence, it gives by Definition 6.63 that

$$\varphi_i\big(\bar{x}_N(b)\big) - \gamma_{iN} = O(h_N) \ \text{ as } \ N \to \infty \ \text{ for } \ i = 1,\ldots,l\ .$$

Employing Lemma 6.64 and the classical *separation theorem* for the convex sets in (6.101), we find a sequence of unit vectors $(\lambda_{0N},\ldots,\lambda_{lN}) \in I\!R^{l+1}$ that separate these sets. Taking into account the structures of the sets in (6.101), one easily has that

$$\lambda_{iN} \geq 0 \text{ for all } i = 0, \ldots, l, \quad \lambda_{0N}^2 + \ldots + \lambda_{lN}^2 = 1, \text{ and}$$

$$\sum_{i=0}^{l} \lambda_{iN} \Big\langle \nabla\varphi_i\big(\bar{x}_N(b)\big), \Delta_{\theta,v}\bar{x}_N(b)\Big\rangle + o(h_N) \geq 0 \text{ as } N \to \infty$$

for any (single) needle variations of the optimal controls with parameters $\big(\theta(N), v(N)\big)$. Putting now

$$\lambda_{iN} := 0 \text{ for } i = l+1, \ldots, m \text{ as } N \to \infty$$

and proceeding similarly to the proof of Theorem 6.50 for free-endpoint problems, we get as N becomes sufficiently large that

$$h_N\Big[H\big(\bar{x}_N(t), p_N(t+h_N), v, t\big) - H\big(\bar{x}_N(t), p_N(t+h_N), \bar{u}_N(t), t\big)\Big] + o(h_N) \leq 0$$

for all $v \in U$ and $t \in T_N$, where each $p_N(\cdot)$ satisfies the adjoint system (6.86) with the transversality condition (6.93) and where $\lambda_{0N}, \ldots, \lambda_{mN}$ obviously obey conditions (6.91) and (6.92) for the inequality constrained problems (P_N) under consideration. The above Hamiltonian inequality directly implies, arguing by contradiction as in the proof of Theorem 6.50, the approximate maximum condition (6.85). This completes the proof of the theorem in the case of problems (P_N) with inequality constraints.

Consider now the general case of (P_N) when the *perturbed equality constraints* are present as well. Each of the constraints $\big|\varphi_{iN}\big(x_N(b)\big)\big| \leq \xi_{iN}$ can be obviously split into the two inequality constraints

$$\varphi_{iN}^+\big(x_N(b)\big) := \varphi_i\big(x_N(b)\big) - \xi_{iN} \leq 0\,,$$

$$\varphi_{iN}^-\big(x_N(b)\big) := -\varphi_i\big(x_N(b)\big) - \xi_{iN} \leq 0$$

for $i = m+1, \ldots, m+r$. Let us show that if *one* of these constraints is *essential* for $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ along some subsequence $\mathcal{M} \subset I\!N$, then the *other* is *inessential* along the same subsequence under the *consistency condition* (6.80). Indeed, suppose for definiteness that the constraint $\varphi_{iN}^+\big(\bar{x}_N(b)\big) \leq 0$ is essential for some $i \in \{m+1, \ldots, m+r\}$ along $\mathcal{M}$. Then by (6.80) we have

$$\varphi_{iN}^-\big(\bar{x}_N(b)\big) = -\varphi_i\big(\bar{x}_N(b)\big) + \xi_{iN} - 2\xi_{iN} = -\varphi_{iN}^+\big(\bar{x}_N(b)\big) - 2\xi_{iN} \leq Kh_N$$

as $N \in \mathcal{M}$ for any $K > 0$, which means that the constraint $\varphi_{iN}^-\big(\bar{x}_N(b)\big) \leq 0$ is inessential. Applying in this way the inequality case of the theorem proved above, we find multipliers λ_{iN}^+ and λ_{iN}^- satisfying

$$\lambda_{iN}^+ \cdot \lambda_{iN}^- = 0 \text{ for } i = m+1, \ldots, m+r \text{ as } N \to \infty\,.$$

Putting finally

$$\lambda_{iN} := \lambda_{iN}^{+} - \lambda_{iN}^{-}, \quad i = m+1, \ldots, m+r \; ,$$

we complete the proof of the theorem. △

Remark 6.65 (AMP for control problem with constraints at both endpoints and at intermediate points of trajectories). The approach developed above allows us to derive necessary optimality conditions in the AMP form for more general discrete approximation problems of the type (P_N) with the cost function $\varphi_0\big(x_N(a), x_N(b)\big)$ and the constraints

$$\varphi_i\big(x_N(a), x_N(b)\big) \le \gamma_{iN}, \quad i = 1, \ldots, m \; ,$$

$$\big|\varphi_i\big(x_N(a), x_N(b)\big)\big| \le \xi_{iN}, \quad i = m+1, \; \ldots, m+r \; ,$$

imposed at both endpoints of feasible trajectories. The AMP holds for such problems, under the same assumptions on the initial data as in Theorems 6.50 and 6.59, with the additional *approximate* transversality condition at the *left endpoints* of optimal trajectories given by

$$p_N(a) = \sum_{i=0}^{m+r} \lambda_{iN} \nabla_{x_0} \varphi_i\big(\bar{x}_N(a), \bar{x}_N(b)\big) \; ,$$

where $\nabla_{x_a}\varphi_i$ stands for the partial derivatives of the functions $\varphi_i(x_a, x_b)$ at the optimal endpoints.

Similar results can be derived for analogs of problems (P_N) with the objective $\varphi_0 = \varphi(x_a, x_\tau, x_b)$ and *intermediate state constraints* of the type

$$\varphi_i\big(x_N(a), x_N(\tau), x_N(b)\big) \le \gamma_{iN}, \quad i = 1, \ldots, m \; ,$$

$$\big|\varphi_i\big(x_N(a), x_N(\tau_N), x_N(b)\big)\big| \le \xi_{iN}, \quad i = m+1, \ldots, m+r \; ,$$

where $\tau_N \in T_N$ is an intermediate point of the mesh. The AMP obtained for such problems involves the additional *exact* condition of the *jump* type:

$$p_N(\tau_N + h_N) - p_N(\tau_N) = \sum_{i=0}^{m+r} \lambda_{iN} \nabla_{x_\tau} \varphi_i\big(\bar{x}_N(a), \bar{x}_N(\tau_N), \bar{x}_N(b)\big)$$

$$- h_N \nabla_x H\big(\bar{x}_N(\tau_N), p_N(\tau_N + h_N), \bar{u}_N(\tau_N), \tau_N\big) \; .$$

Note that in this case the adjoint system (6.86) is required to hold for $p_N(\cdot)$ at points $t \in T_N \setminus \tau_N$.

Next we present an extension of Theorem 6.59 to *nonsmooth* problems (P_N), where the cost and inequality constraint functions φ_i, $i = 0, \ldots, m$, are assumed to be *uniformly upper subdifferentiable*. In this case the transversality conditions are obtained in the *upper subdifferential* form.

Theorem 6.66 (AMP for constrained nonsmooth problems with upper subdifferential transversality conditions). *Let* $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ *be optimal solutions to problems* (P_N) *for* $N \in I\!N$ *under all the assumptions of Theorem 6.59 except for the smoothness of* φ_i *for* $i = 0, \ldots, m$. *Instead we assume that these functions are uniformly upper subdifferentiable around the limiting point(s) of* $\{\bar{x}_N(b)\}$. *Then for any sequences of upper subgradients* $x^*_{iN} \in \widehat{\partial}^+\varphi_i\big(\bar{x}_N(b)\big)$, $i = 0, \ldots, m$, *there are numbers* $\big\{\lambda_{iN} \big| \, i = 0, \ldots, m + r\big\}$ *such that all the conditions* (6.85), (6.86), (6.91), *and* (6.92) *hold with*

$$p_N(b) = -\sum_{i=0}^{m} \lambda_{iN} x^*_{iN} - \sum_{i=m+1}^{m+r} \lambda_{iN} \nabla\varphi_i\big(\bar{x}_N(b)\big) \, .$$

Proof. Given $x^*_{iN} \in \widehat{\partial}^+\varphi_i\big(\bar{x}_N(b)\big)$ for $i = 0, \ldots, m$ and $N \in I\!N$, construct a nonsmooth counterpart of the set S_N in (6.100) by

$$S_N := \big\{(y_0, \ldots, y_l) \in I\!R^{l+1} \big| \, y_i = \big\langle x^*_{iN}, \Delta_{\theta,v}\bar{x}_N(b)\big\rangle\big\} \, .$$

Then we get an analog of Lemma 6.64 with a similar proof. The only difference is that instead of the equalities

$$\varphi_i\big(\bar{x}_N(b) + \Delta x\big) - \varphi_i\big(\bar{x}_N(b)\big) - \big\langle \nabla\varphi_i\big(\bar{x}_N(b)\big), \Delta x\big\rangle + o\big(\|\Delta x\|\big) = 0$$

used in the proof of Lemma 6.64 in the smooth case, we now arrive at the same conclusion based on the inequalities

$$\varphi_i\big(\bar{x}_N(b) + \Delta x\big) - \varphi_i\big(\bar{x}_N(b)\big) - \big\langle x^*_{iN}, \Delta x\big\rangle + o\big(\|\Delta x\|\big) \le 0$$

that are due to the uniform upper subdifferentiability of φ_i for $i = 0, \ldots, l$. The separation theorem applied to the above convex sets gives

$$\sum_{i=0}^{l} \big\langle x^*_{iN}, \Delta_{\theta,v}\bar{x}_N(b)\big\rangle + o(h_N) \ge 0 \, ,$$

which leads to the approximate maximum principle with the upper subdifferential transversality conditions similarly to the proof of Theorem 6.59. △

Remark 6.67 (suboptimality conditions for continuous-time systems via discrete approximations). The results on the fulfillment of the AMP in discrete approximation problems obtained above allow us to derive *suboptimality* conditions for *continuous-time* systems in the form of a certain *ε-maximum principle*. We have discussed in Subsect. 5.1.4 the importance of suboptimality conditions for the theory and applications of optimization problems, especially in the framework of infinite-dimensional spaces. The results and discussions of Subsect. 5.1.4 mostly concern problems of mathematical programming with functional constraints. In optimal control of continuous-time systems (even with finite-dimensional state spaces) suboptimality conditions are of great demand due to the well-known fact that *optimal solutions* *often fail to exist* in systems with *nonconvex* velocities. In such cases "almost necessary conditions" for "almost optimal" (suboptimal) solutions provide a substantial information about optimization problems that is crucial from both qualitative and quantitative/numerical viewpoints.

It follows from the above results on the *value stability* of discrete approximations (see Theorem 6.14 in Subsect. 6.1.4) that, given any $\varepsilon > 0$, optimal solutions $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ to the discrete approximation problems (P_N) considered in this subsection allow us to construct ε-*optimal solutions* $\{u_\varepsilon(\cdot), x_\varepsilon(\cdot)\}$ to the corresponding continuous-time counterpart (P) satisfying

$$\varphi_0\big(x_\varepsilon(b)\big) \le \inf J[x,u] + \varepsilon \quad \text{with}$$

$$\varphi_i\big(x_\varepsilon(b)\big) \le \varepsilon, \quad i = 1,\ldots,m, \quad \big|\varphi_i\big(x_\varepsilon(b)\big)\big| \le \varepsilon, \quad i = m+1,\ldots,m+r\ .$$

Moreover, ε-optimal controls to the continuous-time problem (P) may always be chosen to be *piecewise constant* on $[a,b]$.

Using now the necessary optimality conditions for the discrete approximation problems (P_N) provided by Theorem 6.59 in the AMP form, we arrive at the following ε-*maximum principle* for *suboptimal solutions* to (P): there are multipliers $(\lambda_0,\ldots,\lambda_{m+r}) \in I\!R^{m+r}$ satisfying

$$\lambda_i \ge 0 \ \text{ for } \ i = 0,\ldots,m, \quad \lambda_0^2 + \ldots + \lambda_{m+r}^2 = 1\ ,$$

$$\big|\lambda_i \varphi_i\big(x_\varepsilon(b)\big)\big| \le \varepsilon \ \text{ for } \ i = 1,\ldots,m\ ,$$

and such that, whenever $u \in U$ and $t \in [a,b]$, one has

$$H\big(x_\varepsilon(t), p_\varepsilon(t), u_\varepsilon(t), t\big) \ge H\big(x_\varepsilon(t), p_\varepsilon(t), u, t\big) - \varepsilon\ ,$$

where $p_\varepsilon(\cdot)$ is the corresponding trajectory of the adjoint system

$$\dot{p} = -\nabla H\big(x_\varepsilon(t), p, u_\varepsilon(t), t\big), \quad t \in [a,b]\ ,$$

with the transversality condition

$$p_\varepsilon(b) = -\sum_{i=0}^{m+r} \nabla\varphi_i\big(x_\varepsilon(b)\big)\ .$$

Similar results hold for continuous-time problems with *intermediate state constraints* imposed at some points $\tau_j \in (a,b)$ and also for problems with endpoint constraints at both $t = a$ and $t = b$; cf. Remark 6.65. In the latter case we get an ε-*transversality condition* at $t = a$ given by

$$\Big|p_\varepsilon(a) - \sum_{i=0}^{m+r} \lambda_i \nabla_{x_a}\varphi_i\big(x_\varepsilon(a), x_\varepsilon(b)\big)\Big| \le \varepsilon\ .$$

Note, however, that the *upper subdifferential* form of the AMP in Theorem 6.66 is *not* suitable to induce a similar suboptimality result for continuous-time systems, since the Fréchet upper subdifferential $\widehat{\partial}^+\varphi(\cdot)$ doesn't generally have the required *continuity* property for nonsmooth functions.

To conclude this subsection, we illustrate the application of the AMP to optimizing constrained discrete-time systems with small stepsizes of discretization. First observe from the proof of Theorem 6.50 (and the one for Theorem 6.59) that the *difference in values* of the cost and constraint functions between optimal controls $\bar{u}_N(\cdot)$ to problems (P_N) and controls $u_N(\cdot)$ *maximizing* the Hamilton-Pontryagin function $H\big(\bar{x}_N(t), p_N(t), \cdot, t\big)$ over $u \in U$ is of *order* $o(h_N)$ as $N \to \infty$. This means in fact that the application of the *approximate* maximum principle to optimization of discrete-time systems with small stepsizes h_N leads to practically the *same effects* as in the case of its *exact* counterpart, the discrete maximum principle. Taking this into account, we now use the AMP to solve discrete approximation problems arising in optimization of some chemical processes.

Example 6.68 (application of the AMP to optimization of catalyst replacement). Consider the following optimal control problem (P) for a two-dimensional continuous-time system that appears in the catalyst replacement modeling; see, e.g., Fan and Wang [426]:

$$\begin{cases} \text{minimize} \;\; J[u,x] = \varphi_0\big(x(1)\big) := x_1(1) \;\; \text{subject to} \\[1ex] \dot{x}_1 = -u_1(u_1+u_2), \;\; \dot{x}_2 = u_1, \;\; x_1(0) = x_2(0) = 0, \;\; t \in T := [0,1]\,, \\[1ex] u(t) = \big(u_1(t), u_2(t)\big) \in U := \big\{(u_1,u_2) \in I\!R^2 \big|\; 0 \le u_1, u_2 \le 2\big\}\,, \\[1ex] \varphi_1\big(x(1)\big) := x_2(1) \le \gamma_N\,. \end{cases}$$

To solve this problem numerically, construct a sequence of its discrete approximation problems (P_N):

$$\begin{cases} \text{minimize} \;\; J_N[u_N, x_N] := \varphi_0\big(x_N(1)\big) = x_{1N}(1) \;\; \text{subject to} \\[1ex] x_{1N}(t+h_N) = x_{1N}(t) - h_N u_{1N}(t)\big[u_{1N}(t) + u_{2N}(t)\big], \;\; x_{1N}(0) = 0\,, \\[1ex] x_{2N}(t+h_N) = x_{2N}(t) + h_N u_{1N}(t), \;\; x_{2N}(0) = 0, \;\; h_N := N^{-1}\,, \\[1ex] 0 \le u_{1N}(t) \le 2, \;\; 0 \le u_{2N}(t) \le 2, \;\; t \in T_N := \{0, h_N, \ldots, 1-h_N\}\,, \\[1ex] \varphi_1\big(x_N(1)\big) = x_{2N}(1) \le 0 \;\; \text{as} \;\; N \to \infty\,. \end{cases}$$

Since the sets of "admissible velocities" $f(x,U,t)$ in (P_N) are *not convex*, the (exact) *discrete maximum principle cannot be applied* to find optimal controls for these problems. Let us use for this purpose the *approximate maximum principle* justified in Theorem 6.59.

For each $N \in I\!N$ the corresponding trajectory $p_N(t) = \big(p_{1N}(t), p_{2N}(t)\big)$ of the adjoint system (6.86) with the transversality condition (6.93) is

$$p_{1N}(t) = -\lambda_{0N}, \quad p_{2N}(t) = -\lambda_{1N} \;\; \text{whenever} \;\; t \in T_N\,,$$

while the Hamilton-Pontryagin function along this trajectory is given by

$$H_N(u,t) = u_1\big(\lambda_{0N}u_1 + \lambda_{0N}u_2 - \lambda_{1N}\big), \quad t \in T_N \ .$$

Let us determine controls $\widehat{u}_N(t) = (\widehat{u}_{1N}(t), \widehat{u}_{2N}(t))$ that maximize the Hamilton-Pontryagin function over the control region U. One can easily see by the normalization condition in (6.92) that such controls maximize the function

$$H_\lambda(u_1,u_2) := u_1\big(\lambda u_1 + \lambda u_2 - \sqrt{1-\lambda^2}\big) \ \text{ over } \ (u_1,u_2) \in U$$

as $\lambda \in (0,1)$. It is not hard to compute, taking into account the structure of the control set U, that the maximizing controls $\widehat{u}_N(\cdot)$ are as follows depending on the values of the parameter $\lambda \in (0,1)$:

(a) if $\lambda > 1/\sqrt{17}$, then $\widehat{u}_{1N}(t) = 2$, $\widehat{u}_{2N}(t) = 2$ for all $t \in T_N$;

(b) if $\lambda < 1/\sqrt{17}$, then $\widehat{u}_{1N}(t) = 0$, $\widehat{u}_{2N}(t) \in [0,2]$ for all $t \in T_N$;

(c) if $\lambda = 1/\sqrt{17}$, then for each $t \in T_N$ one has either $\widehat{u}_{1N}(t) = \widehat{u}_{2N}(t) = 2$, or $\widehat{u}_{1N}(t) = 0$ and $\widehat{u}_{2N}(t) \in [0,2]$.

We can directly check that the controls $\widehat{u}_N(\cdot)$ in case (a) are not feasible for (P_N), since the corresponding trajectories $\widehat{x}_N(\cdot)$ don't satisfy the endpoint constraint. In case (b) the controls $\widehat{u}_N(\cdot)$ are far from optimality, since $J_N[\widehat{u}_N, \widehat{x}_N] = 0$ while $\inf J[u_N, x_N] \le -1$. In case (c) the controls $\widehat{u}_N(\cdot)$ are feasible for (P_N) provided that the number of points $t \in T_N$ at which $\widehat{u}_{1N}(t) = 2$ and $\widehat{u}_{2N}(t) = 2$ is not greater than $[N/2]$ as $N \in I\!N$. By Theorem 6.59 and the discussion right before this example we conclude that optimal controls $\bar{u}_N(\cdot)$ to (P_N) (which always exist) may be either feasible ones $\widehat{u}_N(\cdot)$ in case (c) satisfying the properness condition, or those for which the values of the cost and constraint functions are different from $\varphi_0\big(\widehat{x}_N(b)\big)$ and $\varphi_1\big(\widehat{x}_N(b)\big)$ by quantities of order $o(h_N)$ as $N \to \infty$.

Thus the AMP allows us to efficiently describe the collection of all feasible controls to (P_N) that are suspicious to optimality. Based on this information, we can finally determine from the structure of problems (P_N) that optimal solutions to the sequence of these problems are given by the controls

$$\begin{cases} \bar{u}_{1N}(t) = \bar{u}_{2N}(t) = 2 & \text{if } t \text{ is the } [\gamma_N/2h_N]\text{-th point of } T_N \ , \\[2ex] \bar{u}_{1N}(t) = 0, \ \ \bar{u}_{2N}(t) \in [0,2] & \text{for all other } t \in T_N \ . \end{cases}$$

This completely solves the problems under consideration.

6.4.6 Control Systems with Delays and of Neutral Type

The last subsection of this section is devoted to the extension of the AMP in the *upper subdifferential* form to finite-difference approximations of *time-delay* controls systems with smooth dynamics. For brevity we present results only for free-endpoint problems. The main theorem of this subsection provides a generalization of Theorem 6.50 in the case of delay problems; the corresponding extension of Theorems 6.59 and 6.66 can be derived similarly. On the other hand, we show at the end of this subsection that the AMP may *not hold* for discrete approximations of smooth functional-differential systems of *neutral type* that contain time-delays not only in state variables but in velocity variables as well.

We begin with the following continuous-time problem (D) with a single time delay in the state variable:

$$
\begin{cases}
\text{minimize } J[u,x] := \varphi\big(x(b)\big) \text{ subject to} \\[2ex]
\dot{x}(t) = f\big(x(t), x(t-\theta), u(t), t\big) \text{ a.e. } t \in [a,b]\,, \\[2ex]
x(t) = c(t), \quad t \in [a-\theta, a]\,, \\[2ex]
u(t) \in U \text{ a.e. } t \in [a,b]
\end{cases}
$$

over measurable controls $u\colon [a,b] \to U$ and the corresponding absolutely continuous trajectories $x\colon [a,b] \to X$ of the delay system, where $\theta > 0$ is a constant time-delay, and where $c\colon [a-\theta, a] \to X$ is a given function defining the initial "tail" condition that is needed to start the delay system; see Remark 6.40, where the results on the maximum principle for such problems have been discussed. Now our goal is to derive an appropriate version of the AMP for discrete approximation of the delay problem (D).

Let us build discrete approximations of (D) based on the Euler finite-difference replacement of the derivative. In the case of time-delay systems we need to ensure that the point $t-\theta$ belongs to the discrete grid whenever t does. It can be achieved by defining the discretization step as $h_N := \dfrac{\theta}{N}$ in contrast to $h_N = \dfrac{b-a}{N}$ for the non-delay problems (P_N^0) considered in Subsect. 6.4.3. In such a scheme the length of the time interval $b-a$ is generally *no longer commensurable* with the discretization step h_N. Define the grid T_N on the main time interval $[a,b]$ by

$$
T_N := \big\{a, a+h_N, \ldots, b-\widetilde{h}_N - h_N\big\} \text{ with}
$$

$$
h_N := \frac{\theta}{N} \text{ and } \widetilde{h}_N := b-a-h_N\Big[\frac{b-a}{h_N}\Big]
$$

and consider the following sequence of finite-difference approximation problems (D_N) with discrete time delays:

$$\begin{cases} \text{minimize } J[u_N, x_N] := \varphi\big(x_N(b)\big) \text{ subject to} \\ x_N(t+h_N) = x_N(t) + h_N f\big(x_N(t), x_N(t-Nh_N), u_N(t), t\big),\ t \in T_N\ , \\ x_N(b) = x_N(b-\widetilde{h}_N) + \widetilde{h}_N f\big(x_N(b-\widetilde{h}_N), u_N(b-\widetilde{h}_N), b-\widetilde{h}_N\big)\ , \\ x_N(t) = c(t), \quad t \in T_{0N} := \big\{a-\theta, a-\theta+h_N, \ldots, a\big\}\ , \\ u_N(t) \in U, \quad t \in T_N\ . \end{cases}$$

To derive the AMP for the sequence of problems (D_N), we reduce these problems to those *without delays* and employ the results of Theorem 6.57, where the *standing assumptions* are similar to the ones formulated in Subsect. 6.4.3 involving now the additional state variable y in $f(x,y,u,t)$ together with x. For convenience we introduce the following notation:

$$z_N(t) := \big(x_N(t), x_N(t-\theta)\big), \quad \bar{z}_N(t) := \big(\bar{x}_N(t), \bar{x}_N(t-\theta)\big)\ ,$$

$$f(z_N, u_N, t) := f\big(x_N(t), x_N(t-\theta), u_N(t), t\big)\ ,$$

$$f(\bar{z}_N, u_N, t) := f\big(\bar{x}_N(t), \bar{x}_N(t-\theta), u_N(t), t\big)$$

in which terms the *adjoint system* to (D_N) is written as

$$p_N(t) = p_N(t+h_N) + h_N \nabla_x f(\bar{z}_N, \bar{u}_N, t)^* p_N(t+h_N)$$
$$+ h_N \nabla_y f(\bar{z}_N, \bar{u}_N, t+\theta)^* p_N(t+\theta+h_N) \quad \text{for } t \in T_N\ ,$$

$$p_N(b-\widetilde{h}_N) = p_N(b) + \widetilde{h}_N \nabla_x f(\bar{z}_N, \bar{u}_N, b-\widetilde{h}_N)^* p_N(b)$$

along the optimal processes $\big\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\big\}$ to the delay problems (D_N) for each $N \in I\!N$. Introducing the corresponding *Hamilton-Pontryagin function*

$$H(x_N, y_N, p_N, u, t) := \begin{cases} \big\langle p_N(t+h_N), f(x_N, y_N, u, t)\big\rangle & \text{if } t \in T_N\ , \\ \big\langle p_N(t), f(x_N, y_N, u, t-\widetilde{h}_N)\big\rangle & \text{if } t = b-\widetilde{h}_N \end{cases}$$

with $y_N(t) := x_N(t-\theta)$, we rewrite the adjoint system as

$$p_N(t) = p_N(t+h_N) + h_N \Big[\nabla_x H(\bar{z}_N, p_N, \bar{u}_N, t) + \nabla_y H(\bar{z}_N, p_N, \bar{u}_N, t+\theta)\Big]$$

when $t \in T_N$ and

$$p_N(b-\widetilde{h}_N) = p_N(b) + \widetilde{h}_N \nabla_x H(\bar{z}_N, p_N, \bar{u}_N, b-\widetilde{h}_N)$$

at the "incommensurable" point. Then we have the following result on the fulfillment of the AMP for time-delay discrete approximations.

Theorem 6.69 (AMP for delay systems). *Let the pairs* $\{\bar{u}_N(\cdot), \bar{x}_N(\cdot)\}$ *be optimal to problems* (D_N)*. In addition to the standing assumptions, suppose that the cost function* φ *is uniformly upper subdifferentiable around the limiting point(s) of the sequence* $\{\bar{x}_N(b)\}$, $N \in I\!N$*. Then for every sequence of upper subgradients* $x_N^* \in \mathcal{D}^+\varphi(\bar{x}_N(b))$ *the approximate maximum condition*

$$H(\bar{z}_N, p_N, \bar{u}_N, t) = \max_{u \in U} H(\bar{z}_N, p_N, u, t) + \varepsilon(t, h_N), \quad t \in \widetilde{T}_N := T_N \cup \{b - \widetilde{h}_N\},$$

is fulfilled, where $\varepsilon(t, h_N) \to 0$ *as* $h_N \to 0$ *uniformly in* $t \in \widetilde{T}_N$*, and where* $p_N(\cdot)$ *satisfies the transversality relations*

$$p_N(b) = -x_N^*, \qquad p_N(t) = 0 \ \text{ as } \ t > b \,. \tag{6.105}$$

Furthermore, we can take any $x^* \in \widehat{\partial}^+\varphi(\bar{x}_N(b))$ *in* (6.105) *if* X *is reflexive and* φ *is continuous around the limiting point(s) of* $\{\bar{x}_N(b)\}$.

Proof. We reduce the delay discrete approximation problems to those with *no delay* (but with the *incommensurability* between $b - a$ and h_N) by the following *multistep procedure.* Denote

$$y_{1N}(t) := x_N(t - h_N), \quad t \in \{a + 2h_N, \ldots, b - \widetilde{h}_N\} \,,$$

$$y_{1N}(t) := c_N(t - h_N), \quad t \in \{a - \theta + h_N, \ldots, a + h_N\} \,,$$

$$y_{2N}(t) := y_{1N}(t - h_N), \quad t \in \{a - \theta + 2h_N, \ldots, b - \widetilde{h}_N\} \,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_{NN}(t) := y_{N-1,N}(t - h_N), \quad t \in \{a, \ldots, b - \widetilde{h}_N\} \,,$$

and observe that the values of $y_{1N}(b), \ldots, y_{NN}(b)$ can be defined arbitrarily, since they don't enter either the adjoint system or the cost function. To match the setup of Theorem 6.57, define

$$y_{1N}(b) := x_N(b - \widetilde{h}_N), \ y_{2N}(b) := y_{1N}(b - \widetilde{h}_N), \ldots, y_{NN}(b) := y_{N-1,N}(b - \widetilde{h}_N) \,.$$

After the change of variables we have

$$y_{NN}(t) = \begin{cases} x_N(t - \theta), \ t \in \{a + \theta + h_N, \ldots, b - \widetilde{h}_N\} \,, \\[2ex] c(t - \theta), \quad t \in \{a, \ldots, a + \theta\} \,. \end{cases}$$

The original system in (D_N) is thereby reduced, for each $N \in I\!N$, to the following non-delay system of dimension $I\!R^{(N+1)n}$:

$$\begin{cases} s_N(t + h_N) = s_N(t) + h_N g(s_N, u_N, t), \quad t \in T_N \,, \\[2ex] s_N(b) = s_N(b - \widetilde{h}_N) + \widetilde{h}_N g(s_N, u_N, b - \widetilde{h}_N) \end{cases}$$

with the state vector $s_N(t) := \big(x_N(t), y_{1N}(t), \ldots, y_{NN}(t)\big)$ and the "velocity" mapping $g(s_N, u_N, t)$ given by

$$g\big(s_N(t), u_N(t), t\big) := \begin{pmatrix} f\big(x_N(t), y_{NN}(t), u_N(t), t\big) \\ \\ \dfrac{x_N(t) - y_{1N}(t)}{h_N} \\ \\ \ldots\ldots \\ \dfrac{y_{N-1,N}(t) - y_{NN}(t)}{h_N} \end{pmatrix},$$

where h_N should be replaced by $\widetilde{h}_N$ for $t = b - \widetilde{h}_N$ in the last formula.

Let us apply Theorem 6.57 to the problem of minimizing the same functional as in (D_N) over the feasible pairs $\{u_N(\cdot), s_N(\cdot)\}$ of the obtained non-delay system. The adjoint system in this problem, with respect to the new adjoint variable $q \in I\!R^{(N+1)n}$, has the form

$$\begin{cases} q_N(t) = q_N(t + h_N) + h_N \nabla_s g(\bar{s}_N, \bar{u}_N, t)^* q(t + h_N), \quad t \in T_N , \\ \\ q_N(b - \widetilde{h}_N) = q_N(b) + \widetilde{h}_N \nabla_s g(\bar{s}_N, \bar{u}_N, b - \widetilde{h}_N)^* q_N(b) \end{cases}$$

with the transversality condition

$$q_N(b) = -(x_N^*, 0, \ldots, 0) \;\text{ for }\; x_N^* \in \mathcal{D}^+\varphi\big(\bar{x}_N(b)\big) ,$$

which reduces to $x_N^* \in \widehat{\partial}^+\varphi\big(\bar{x}_N(b)\big)$ when X is reflexive and φ is continuous. Taking into account the above relationship between g and f and performing elementary calculations, we express the operator $\nabla_s g^*$ via $\nabla_x f^*$ and $\nabla_y f^*$ and arrive at the transversality relations (6.105) for the first component $p_N(\cdot)$ of the adjoint trajectory $q_N(\cdot)$. Furthermore, one gets the relationship

$$\begin{aligned} \widetilde{H}(\bar{s}_N, q_N, u, t) &= \big\langle q_N(t + h_N), g(\bar{s}_N, u, t)\big\rangle \\ &= \big\langle p_N(t + h_N), f(\bar{z}_N, u, t)\big\rangle + r(\bar{s}_N, q_N, h_N, t) \\ &= H(\bar{z}_N, p_N, u, t) + r(\bar{s}_N, q_N, h_N, t), \quad t \in T_N , \end{aligned}$$

and similarly for $t = b - \widetilde{h}_N$, between the Hamilton-Pontryagin functions of the non-delay and original delay systems considered above, where the remainder $r(\bar{s}_N, q_N, h_N, t)$ doesn't depend on u. Applying now the approximate maximum condition from Theorem 6.57 to the non-delay system, we complete the proof of the theorem. △

To conclude this section, we consider optimal control problems for finite-difference approximations of the so-called *functional-differential systems of neutral type* (cf. also Sect. 7.1) given by

$$\dot{x}(t) = f\big(x(t), x(t-\theta), \dot{x}(t-\theta), u(t), t\big), \quad u(t) \in U, \quad \text{a.e. } t \in [a,b]\,,$$

which contain time-delays not only in state but also in *velocity* variables. A finite-difference counterpart of such systems with the stepsize h and with the grid $T := \{a, a+h, \ldots, b-h\}$ is

$$x(t+h) = x(t) + hf\Big(x(t), x(t-\theta), \frac{x(t-\theta+h) - x(t-\theta)}{h}, u(t), t\Big)$$

as $u(t) \in U$ for $t \in T$, and the adjoint system is given by

$$p(t) = p(t+h) + h\nabla_x f(\bar{v}, \bar{u}, t)^* p(t+h) + h\nabla_y f(\bar{v}, \bar{u}, t+\theta)^* p(t+\theta+h)$$

$$+h\nabla_z f(\bar{v}, \bar{u}, t+\theta-h)^* p(t+\theta) - h\nabla_z f(\bar{v}, \bar{u}, t+\theta)^* p(t+\theta+h)$$

for $t \in T$, where $\{\bar{u}(\cdot), \bar{x}(\cdot)\}$ is an optimal solution to the neutral analog of problem (D_N), and where

$$\bar{v}(t) := \Big(\bar{x}(t), \bar{x}(t-\theta), \frac{\bar{x}(t-\theta+h) - \bar{x}(t-\theta)}{h}\Big), \quad t \in T\,.$$

The following example shows that the AMP is *not generally fulfilled* for finite-difference neutral systems, in contrast to ordinary and delay ones, even in the case of *smooth* cost functions.

Example 6.70 (AMP may not hold for neutral systems). *There is a two-dimensional control problem of minimizing a linear function over a smooth neutral system with no endpoint constraints such that some sequence of optimal controls to discrete approximations doesn't satisfy the approximative maximum principle regardless of the stepsize and a mesh point.*

Proof. Consider the following parametric family of discrete optimal control problems for neutral systems with the parameter $h > 0$:

$$\begin{cases} \text{minimize } J[u, x_1, x_2] := x_2(2) \ \text{ subject to} \\[2mm] x_1(t+h) = x_1(t) + hu(t), \quad t \in T := \big\{0, h, \ldots, 2-h\big\}\,, \\[2mm] x_2(t+h) = x_2(t) + h\Big(\dfrac{x_1(t-1+h) - x_1(t-1)}{h}\Big)^2 - hu^2(t), \quad t \in T\,, \\[2mm] x_1(t) \equiv x_2(t) \equiv 0, \quad t \in T_0 := \{-1, \ldots, 0\}\,, \\[2mm] |u(t)| \le 1, \quad t \in T\,. \end{cases}$$

It is easy to see that

$$x_2(1) = -h\sum_{t=0}^{1-h} u^2(t) \quad \text{and}$$

$$x_2(2) = x_2(1) + h\sum_{t=1}^{2-h}\Big(\frac{x_1(t-1+h) - x_1(t-1)}{h}\Big)^2 - h\sum_{t=1}^{2-h}u^2(t)$$

$$= -h\sum_{t=0}^{1-h}u^2(t) + h\sum_{t=0}^{1-h}u^2(t) - h\sum_{t=1}^{2-h}u^2(t) = -h\sum_{t=1}^{2-h}u^2(t)\,.$$

Thus the control

$$\bar u(t) = \begin{cases} 0,\ t \in \{0,\ldots,1-h\}\,, \\ 1,\ t \in \{1,\ldots,2-h\}\,, \end{cases}$$

is an optimal control to the problems under consideration for any h. The corresponding trajectory is

$$\bar x_1(t) = \begin{cases} 0, & t \in \{0,\ldots,1-h\}\,, \\ t-1, & t \in \{1,\ldots,2-h\}; \end{cases} \qquad \bar x_2(t) = \begin{cases} 0, & t \in \{0,\ldots,1-h\}\,, \\ -t+1, & t \in \{1,\ldots,2-h\}\,. \end{cases}$$

Computing the partial derivatives of the "velocity" mapping f in the above system, we get

$$\nabla_x f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \nabla_y f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and}$$

$$\nabla_z f(t+1) = \frac{1}{h}\begin{pmatrix} 0 & 0 \\ 2(x_1(t+h) - x_1(t)) & 0 \end{pmatrix}\,.$$

Hence the adjoint system reduces to

$$p_1(t) = p_1(t+h) + 2\big(\bar x_1(t) - \bar x_1(t-h)\big)\,p_2(t+1)$$

$$-2\big(\bar x_1(t+h) - \bar x_1(t)\big)\,p_2(t+1+h), \quad t \in \{0,\ldots,2-h\}\,,$$

with $p_2(t) \equiv \text{const}$ and with the transversality conditions

$$p_1(2) = 0, \quad p_2(2) = -1; \quad p_1(t) = p_2(t) = 0 \ \text{ for } \ t > 2\,.$$

The solution of this system is

$$p_1(t) \equiv 0, \quad p_2(t) \equiv -1 \ \text{ for all } \ t \in \{0,\ldots,2-h\}\,.$$

Thus the Hamilton-Pontryagin function along the optimal solution is

$$H(t,\bar x_1,\bar x_2,p_1,p_2,u) = p_2(t+h)\Big\{\Big(\frac{x_1(t-1+h) - x_1(t-1)}{h}\Big)^2 - u^2\Big\}$$

$$+p_1(t+h)u = u^2, \quad t \in \{0,\ldots,1-h\}\,.$$

This shows that the optimal control $\bar{u}(t) = 0$ *doesn't* provide the approximate maximum to the Hamilton-Pontryagin function regardless of h and mesh points $t \in \{0, \ldots, 1-h\}$. Note at the same time that *another sequence* of optimal controls with $\bar{u}(t) = 1$ for all $t \in \{0, \ldots, 2-h\}$ *does* satisfy the *exact* discrete maximum principle regardless of h. △

6.5 Commentary to Chap. 6

6.5.1. Calculus of Variations and Optimal Control. Chapter 6 is devoted to problems of *dynamic optimization*. This name conventionally reflects the fact that some initial data of a given optimization problem *evolve in time*. The origin of such problems goes back to the classical *calculus of variations*, which was in the beginning of all infinite-dimensional analysis; we refer the reader to the seminal contributions by Euler [411], Lagrange [737], Hamilton [548], Jacobi [625], Mayer [859], Weierstrass [1326], Bolza [130], Tonelli [1260], Carathéodory [222], and Bliss [119] (with his famous Chicago school) among other developments the most influential for the topics considered in this book.

The theory of *optimal control* for *ordinary differential equations* (ODE), which has been well recognized as a modern counterpart of the classical calculus of variations, distinguishes from its predecessor by, first of all, the presence of *hard/pointwise* constraints on control functions generating system trajectories (often called admissible arcs) via the *evolution ODE systems*

$$\dot{x} = f(x, u, t), \quad u(t) \in U, \quad t \in [a, b], \quad x \in I\!R^n \; . \tag{6.106}$$

Such control constraints given by sets U of a rather *irregular nature*, which appeared already in the very first problems of optimal control arisen from practical applications, have been a permanent source of *intrinsic nonsmoothness* in optimal control theory and have eventually *motivated* the development of many crucial aspects of modern *variational analysis* and *generalized differentiation*.

As mentioned in Subsect. 1.4.1, the fundamental result of optimal control theory widely known as the *Pontryagin maximum principle* (PMP) [1102], which was formulated by Pontryagin and then was proved by Gamkrelidze [494] for linear systems and by Boltyanskii [124] for problems with nonlinear *smooth* dynamics, has played a major role in developing modern variational analysis. It is interesting to observe that the first attempt [129] in formulating the maximum principle—as a *sufficient* condition for *local* optimality—was *wrong*; see the papers by Boltyanskii [128] and Gamkrelidze [498] for (rather different) historical accounts in the discovery of the maximum principle. In these papers and also in the book by Hestenes [565] and in the survey paper by McShane [865], the reader can find various discussions on the relationships between the maximum principle and the preceding results obtained in the

Chicago school on the calculus of variations and in the theory and applications of automatic control; see also the excellent survey by Gabasov and Kirillova [487]. Probably the closest predecessors to optimal control theory were nonstandard variational problems and results developed for *optimal systems* of linear *automatic control*, in particular, the so-called "theorem on n-intervals" by Feldbaum [440] and the "bang-bang principle" by Bellman, Glicksberg and Gross [95].

Although analogs of many elements in both formulation and proof of the PMP can be found in the calculus of variations (particularly *needle variations* employed by McShane [860], which actually go back to Weierstrass [1326] and his necessary optimality condition for strong minimizers; *tangential convex approximations* and the usage of *convex separation* as in McShane [860]; *canonical* variables and a modified *Hamiltonian* function, etc.), the discovery of the PMP and its proof came as a *surprise* ("sensation" in Pshenichnyi's wording [1106]). It is difficult to overestimate the impact and role of the PMP in the development of modern variational analysis. We refer the reader to [7, 32, 105, 124, 218, 235, 255, 370, 485, 486, 497, 504, 539, 565, 618, 801, 863, 865, 877, 1002, 1106, 1239, 1289, 1315, 1351] for more results and discussions on the relationships between optimal control, the calculus of variations, and mathematical programming.

It seems that among the *most significant new contributions* of the PMP in comparison with the classical calculus of variations was the discovery (by Pontryagin) of the *adjoint system* to (6.106) given by

$$\dot{p} = -\frac{\partial f(\bar{x}, \bar{u}, t)}{\partial x}^{*} p = -\nabla_x H(\bar{x}, p, \bar{u}, t) , \tag{6.107}$$

via the *Hamilton-Pontryagin function*

$$H(x, p, u, t) := \langle p, f(x, u, t) \rangle, \quad p \in I\!R^n , \tag{6.108}$$

computed along the optimal process $(\bar{x}, \bar{u})$, in which terms the crucial pointwise *maximum condition* was written as

$$H\big(\bar{x}(t), p(t), \bar{u}(t), t\big) = \max_{u \in U} H\big(\bar{x}(t), p(t), u, t\big) \quad \text{a.e.} \tag{6.109}$$

It has been recognized, after the discovery of the PMP, that the maximum condition (6.109) is an optimal control counterpart of the *Weierstrass's excess function condition* for *strong minimizers* in the calculus of variations.

6.5.2. Differential Inclusions. A notable disadvantage of the original optimal control model (6.106) is that it doesn't cover problems with *state-dependent* control sets $U = U(x)$ important for both the theory and applications. Problems of this class, as well as of other significant classes in control and dynamic optimization, can be naturally written in the form of *differential inclusions*

$$\dot{x} \in F(x,t), \quad x \in I\!R^n\ , \tag{6.110}$$

which actually go back to the classes of set-valued differential equations studies (not from the control viewpoint) in the 1930s as "contingent equations" by Marchaud [850] and "paratingent equations" by Zaremba [1355]; see also Nagumo [990] and Ważewski [1325] for early developments. Control systems (6.106) equivalently reduce to the differential inclusion form (6.110) by the so-called "Filippov implicit function lemma" [449], which is in fact a result on measurable selections of set-valued mappings; see, e.g., Castaing and Valadier [229] and Rockafellar and Wets [1165] for more references and discussions.

Observe that control systems governed by differential inclusions (6.110) are significantly *more complicated* in comparison with the classical ones (6.106) due to, e.g., the impossibility of employing standard needle variations to derive optimality conditions. Moreover, systems (6.110) *explicitly* reveal the *intrinsic nonsmoothness* inherent even in classical optimal control via, first of all, *hard* control constraints of the type $u(t) \in U$, particularly given by finite sets like $U = \{0, 1\}$ that are typical in automatic control applications. This phenomenon is somehow hidden in the PMP for systems (6.106) of *smooth dynamics* due to using the Hamilton-Pontryagin function (6.108) differentiable in the state-costate variables (x, p). Another manifestation of nonsmoothness in optimal control is provided by the *Hamiltonian* function

$$\mathcal{H}(x,p,t) := \sup\big\{\langle p, v\rangle\big|\ v \in F(x,t)\big\} \tag{6.111}$$

for the differential inclusion (6.110), which corresponds to the "true" Hamiltonian

$$\mathcal{H}(x,p,t) := \sup\big\{H(x,p,u,t)\big|\ u \in U\big\}$$

for the standard/parameterized control systems (6.106). These generalized Hamiltonians can be viewed as control counterparts of the classical Hamiltonian in problems of the calculus of variations and mechanics associated (via the *Legendre transform* if the latter is well-defined) with the Lagrangian, i.e., integrand under minimization.

6.5.3. Optimality Conditions for Smooth or Graph-Convex Differential Inclusions. Nonsmoothness is a *characteristic feature* of the Hamiltonian (6.111) and its above implementation for control systems (6.106); a smooth behavior occurs only under some quite restrictive assumptions. However, the first necessary optimality conditions for control problems governed by differential inclusions were obtained (under the name of "support principle") by Boltyanskii [125] assuming the *smoothness* of (6.111) in the state variable; see also the related papers by Fedorenko [438, 439], Boltyanskii [127], Blagodatskikh [117], Blagodatskikh and Filippov [118] with other (mostly Russian) references therein.

In [1143, 1144, 1145], Rockafellar derived necessary (and sufficient) optimality condition applied to differential inclusions (6.110) under more reasonable assumptions of the *graph-convexity* for $F(\cdot, t)$. In fact, Rockafellar considered a more general framework of the (fully) convex *generalized problem of Bolza*:

$$\text{minimize } \varphi\big(x(a), x(b)\big) + \int_a^b \vartheta\big(x(t), \dot{x}(t), t\big)\, dt \;, \tag{6.112}$$

where, in contrast to the classical Bolza problem [130] and the preceding Mayer problem [859] with $\vartheta = 0$, the functions φ and ϑ may be *extended-real-valued*, i.e., (6.112) particularly incorporates the differential inclusion model (6.110) via the indicator function $\vartheta(x, v, t) := \delta\big((x, v); \operatorname{gph} F(\cdot, t)\big)$. The convexity assumption on $\vartheta(x, v, t)$ in both variables (x, v) made in [1143, 1144, 1145] implies that the Hamiltonian (6.111) associated with the differential inclusion (6.110) is convex in p and concave in x, so it is *subdifferentiable* as a *saddle function* with respect to (x, p) in the sense of convex analysis. Using the machinery of convex analysis in infinite-dimensional spaces, Rockafellar obtained necessary and sufficient conditions for optimal solutions $\bar{x}(\cdot)$ to the convex generalized problem of Bolza and thus for convex-graph differential inclusions via the *generalized Hamiltonian equation* [1145] called also the *Hamiltonian condition/inclusion*

$$\big(-\dot{p}(t), \dot{\bar{x}}(t)\big) \in \partial\mathcal{H}\big(\bar{x}(t), p(t), t\big) \quad \text{a.e.} \;, \tag{6.113}$$

where $\partial\mathcal{H}$ stands for the subdifferential of the Hamiltonian function $\mathcal{H}(x, p, t)$ with respect to (x, p). If $\mathcal{H}(x, p, t)$ happens to be differentiable with respect to x and p, inclusion (6.113) reduces to the classical Hamiltonian system

$$\dot{\bar{x}}(t) = \nabla_p\mathcal{H}\big(\bar{x}(t), p(t), t\big) \;\text{ and }\; -\dot{p}(t) = \nabla_x\mathcal{H}\big(\bar{x}(t), p(t), t\big) \;.$$

Somewhat different (while mostly equivalent) results for optimization problems governed by *convex-graph* differential inclusions were later obtained by Halkin [542], Berliocchi and Lasry [107], and Pshenichnyi [1107, 1109].

6.5.4. Clarke's Euler-Lagrange Condition. Observe that although the graph-convexity assumption on $F(\cdot, t)$ is more reasonable in comparison with the smoothness requirement on the Hamiltonian, it is still rather restrictive. In particular, for standard control systems (6.106) this assumption actually reduces to the *linearity* of $f(\cdot, \cdot, t)$ and the convexity of U; see Rockafellar [1143]. A crucial step from *fully convex*, or "biconvex" in Halkin's terminology, problems (i.e., those for which the integrand in (6.112) in convex in *both* (x, v) variables) to problems involving the *convexity* only in the *velocity* variable v, which corresponds to the *convex-valuedness* of $F(x, t)$ in the differential inclusion framework (6.110), was made by Clarke in his pioneering work in the 1970s starting with his dissertation [243].

The initial point for Clarke [243, 245] was the Bolza-type problem (6.112) with *finite* (moreover Lipschitzian) integrand/Lagrangian $\vartheta(\cdot,\cdot,t)$ considered without any smoothness and convexity assumptions on the integrand ϑ as well as on the l.s.c. endpoint function φ, which was allowed to be extended-real-valued. The main necessary optimality condition was obtained in the *Euler-Lagrange form*

$$\big(\dot p(t), p(t)\big) \in \partial_C \vartheta\big(\bar x(t), \dot{\bar x}(t), t\big) \ \text{ a.e.} \tag{6.114}$$

via Clarke's generalized gradient of $\vartheta(\cdot,\cdot,t)$ in (6.114). Inclusion (6.114) gets back the classical Euler-Lagrange equation if $\vartheta(x,v,t)$ is smooth in (x,v); it reduces to the Euler-Lagrange inclusion obtained by Rockafellar [1143] if ϑ is convex in both x and v variables. Furthermore, Clarke's proof of (6.114) in [243, 245] was based on *reducing* the nonconvex Bolza problem under consideration to the *fully convex* problem comprehensively studied by Rockafellar. The *convex-valuedness* of Clarke's generalized gradient and its duality relationship with his generalized directional derivative played a *major role* in the possibility to accomplish the latter reduction and thus in the whole proof of (6.114).

Based on the Euler-Lagrange condition (6.114) for finite Lagrangians, Clarke obtained [247] its counterpart

$$\big(\dot p(t), p(t)\big) \in N_C\big((\bar x(t), \dot{\bar x}(t)); \operatorname{gph} F(t)\big) \quad \text{a.e.} \tag{6.115}$$

for *Lipschitzian and bounded* differential inclusions (6.110) via his normal cone to the graph of $F = F(\cdot,t)$. Then he derived [248] the Euler-Lagrange inclusion (6.114) for the generalized Bolza problem (6.112), where ϑ was assumed to be extended-real-valued and epi-Lipschitzian in (x,v). The most notable and restrictive assumption imposed in [247, 248] was the *calmness* condition similar to that discussed in Subsect. 5.5.16 for problems of mathematical programming. This is a kind of constraint qualification/regularity requirement, which ensures the normal form of necessary optimality conditions and holds, in particular, when the endpoint function φ is locally Lipschitzian in either variable; the latter however excludes the corresponding endpoint constraints. Note that the calmness requirement allowed Clarke to avoid formally the convexity assumption on ϑ even in v, while the convexity property was actually present in [247, 248] due to the "admissible relaxation" provided by calmness; see also [246] for a detailed study of these relationships. Moreover, as mentioned in [248, p. 683], "...the [bi]convex case [developed by Rockafellar] lies at the heart of the proof of our result."

The most serious *drawback* of the Euler-Lagrange inclusion in form (6.115), fully recognized only later, is that it involves the Clarke normal cone to the *graph* of $F(\cdot,t)$ from (6.110), which happens to be a *linear subspace* of dimension $d \ge n$ whenever F is *graphically Lipschitzian* near the optimal solution; see Subsect. 1.4.4 for more discussions. Due to this property, the set on the right-hand side of (6.115) may be *too large* to provide an adequate information on adjoint arcs $p(\cdot)$ in many situations important for the theory and applications.

6.5.5. Clarke's Hamiltonian Condition. Besides the Euler-Lagrange condition (6.115), Clarke also established necessary optimality conditions for the generalized Bolza problem and thus for Lipschitzian differential inclusions in the following *Hamiltonian form*:

$$\big(-\dot{p}(t),\dot{\bar{x}}(t)\big)\in\partial_C\mathcal{H}\big(\bar{x}(t),p(t),t\big)\quad\text{a.e.}\tag{6.116}$$

involving his generalized gradient of the Hamiltonian function in both (x,p) variables. The first Hamiltonian results were obtained under the calmness assumption [253, 255] and then without this and other constraint qualifications [256].

Note that, in the absence of regularity/normality assumptions, the validity of the Hamiltonian condition (6.116) was established only for *convex-valued* differential inclusions (which corresponds to the convexity in v of the Lagrangian in the generalized Bolza form); the derivation of (6.116) without convexity originally presented in [251] was incorrect in the proof of Claim on p. 262 therein related to the convexification procedure. Similar approach based on employing the Ekeland variational principle worked nevertheless for proving Clarke's extension [250] of the Pontryagin maximum principle for *nonsmooth* optimal control systems of type (6.106). A long-standing conjecture about the validity of the Hamiltonian necessary optimality condition (6.116) without the above convexity assumption, which resisted the efforts of several authors, has been recently resolved by Clarke [261] for Lipschitzian and bounded differential inclusions by applying *Stegall's variational principle* [1224] instead of Ekeland's one in the framework of his proof. Observe that, in contrast to the classical smooth case and to the fully convex case of Rockafellar, Clarke's Euler-Lagrange condition (6.115) and Hamiltonian condition (6.116) are *not equivalent* even in simple situations. Moreover, they don't follow from each other being truly *independent*; see examples and discussions in Kaśkosz and Lojasiewicz [667] and in Loewen and Rockafellar [805].

It was not even clear till the work by Loewen and Rockafellar [804] whether one could find a *common adjoint arc* $p(\cdot)$ satisfying both Euler-Lagrange condition (6.115) and Hamiltonian condition (6.116) simultaneously. The affirmative answer was given in [804] for *convex-valued* and Lipschitzian differential inclusions with no assumption of calmness or normality. Note that in this case both conditions (6.115) and (6.116) automatically imply the *Weierstrass-Pontryagin maximum condition*

$$\big\langle p(t),\dot{\bar{x}}(t)\big\rangle=\mathcal{H}\big(\bar{x}(t),p(t),t\big)\quad\text{a.e.}\tag{6.117}$$

We refer the reader to [254, 255, 256, 267, 268, 272, 273, 274, 276, 595, 666, 667, 803, 804, 808, 1178, 1291, 1292] and the bibliographies therein for extensions and modifications of necessary optimality conditions of the Euler-Lagrange and Hamiltonian types obtained in terms of Clarke's generalized differential constructions for various problems of dynamic optimization and optimal control.

6.5.6. Transversality Conditions. Necessary optimality conditions in problems of dynamic optimization include, besides dynamic relations of the type discussed above (Euler-Lagrange, Hamiltonian, Weierstrass-Pontryagin), also endpoint relations on adjoint trajectories called *transversality conditions*. They are expressed via appropriate (generalized) differential constructions for cost and constraint functions depending on endpoints of state trajectories. Note that endpoint constraints on $(x(a), x(b))$ can be implicitly included in the endpoint cost function φ if it is assumed to be extended-real-valued as in the generalized problem of Bolza (6.112). However, typically such constraints are given explicitly in the form

$$\big(x(a), x(b)\big) \in \Omega \subset I\!\!R^n \ , \tag{6.118}$$

where the constraint/target set Ω may be specified in some functional form by, e.g., equalities and inequalities with real-valued (often Lipschitzian) functions.

In the afore-mentioned publications by Clarke and his followers concerning minimization of Lipschitzian cost functions φ as in (6.112) subject to endpoint constraints of type (6.118), the transversality conditions were derived in the form

$$\big(p(a), -p(b)\big) \in \lambda\partial_C\varphi\big(\bar{x}(a), \bar{x}(b)\big) + N_C\big((\bar{x}(a), \bar{x}(b)); \Omega\big) \tag{6.119}$$

with $\lambda \geq 0$ via Clarke's generalized gradient of φ and his normal cone to Ω at the optimal endpoints $(\bar{x}(a), \bar{x}(b))$. When φ and Ω happen to be convex, the transversality inclusion (6.119) reduces to that obtained earlier by Rockafellar [1143]. Note that the normal form $\lambda = 1$ holds under the calmness assumption and that a proper counterpart of (6.119) is expressed in terms of Clarke's normal cone to the epigraph of $\varphi + \delta(\cdot; \Omega)$ if φ is merely l.s.c. around $(\bar{x}(a), \bar{x}(b))$.

Transversality conditions in the significantly more *advanced form*

$$\big(p(a), -p(b)\big) \in \lambda\partial\varphi\big(\bar{x}(a), \bar{x}(b)\big) + N\big((\bar{x}(a), \bar{x}(b)); \Omega\big) \tag{6.120}$$

were first established by Mordukhovich in the mid-1970s via his basic/limiting normal cone and subdifferential: in [887] for time optimal control problems and in [889, 892] for other classes of problems in optimal control and dynamic optimization involving ODE control systems (6.106) and differential inclusions (6.110); see also [717, 897, 900, 901, 902, 904]. These results were obtained by the *method of metric approximations*, which was actually the driving force to introduce the nonconvex-valued normal cone and subdifferential in [887]; more comments and discussions were given in Subsects. 1.4.5 and 2.6.1.

It seems that the transversality conditions in form (6.120) didn't get a proper attention in the Western literature before Mordukhovich's talk at the

Montreal workshop (February 1989) and the publication of Clarke's second book [257], where these conditions were mentioned in footnotes with the reference to Mordukhovich; see Subsect. 1.4.8. However, even after that many papers (see, e.g., those listed in Subsect. 1.4.8) still continued using transversality conditions in form (6.119) instead of the advanced one (6.120).

Nevertheless, it has been eventually recognized the possibility to justify the advanced transversality conditions (6.120) in *any* investigated setting of dynamic optimization. We particularly refer the reader to the publications [33, 40, 93, 113, 258, 260, 261, 264, 265, 275, 443, 444, 506, 605, 611, 616, 801, 805, 806, 807, 845, 847, 878, 880, 914, 915, 916, 921, 932, 955, 959, 970, 971, 973, 974, 976, 1021, 1022, 1074, 1075, 1076, 1077, 1078, 1079, 1080, 1118, 1161, 1162, 1176, 1179, 1211, 1215, 1216, 1233, 1289, 1293, 1294, 1295, 1372], which clearly demonstrated this for various problems of the calculus of variations and optimal control of ordinary differential systems and their distributed-parameter counterparts.

6.5.7. Extended Euler-Lagrange Conditions for Convex-Valued Differential Inclusions. The usage of the nonconvex normal cone from [887] in the framework of *dynamic* optimality conditions for differential inclusions was initiated in the 1980 paper by Mordukhovich [892] for the problem of minimizing the cost function $\varphi(x(a), x(b))$ over absolutely continuous trajectories for the *convex-valued*, bounded, and Lipschitzian (in x) differential inclusion (6.110) subject to the endpoint constraints (6.118). Given an optimal solution $\bar{x}(\cdot)$ to this problem, a dynamic necessary optimality condition was obtained in [892] in the form

$$\begin{aligned}\big(\dot{p}(t), \dot{\bar{x}}(t)\big) \in \operatorname{co}\Big\{(u, v) \in I\!R^{2n}\Big|\big(u, p(t)\big) \in N\big((\bar{x}(t), v); \operatorname{gph} F(t)\big)\,,\\ v \in M\big(\bar{x}(t), p(t), t\big)\Big\}\ \text{ a.e. }\ t \in [a, b]\end{aligned} \tag{6.121}$$

with the *argmaximum sets* $M(x, p, t)$ defined by

$$M(x, p, t) := \big\{v \in F(x, t)\big|\ \langle p, v\rangle = \mathcal{H}(x, p, t)\big\}$$

and the transversality inclusion (6.120) held when φ is locally Lipschitzian. If the argmaximum set $M(\bar{x}(t), p(t), t)$ is a *singleton* for a.e. $t \in [a, b]$ (it happens, in particular, when the velocity set $F(\bar{x}(t), t)$ is *strictly convex* almost everywhere), condition (6.121) reduces to

$$\big(\dot{p}(t), \dot{\bar{x}}(t)\big) \in \operatorname{co}\Big\{(u, v)\Big|\ \big(u, p(t)\big) \in N\big((\bar{x}(t), \dot{\bar{x}}(t)); \operatorname{gph} F(t)\big)\Big\}\ \text{ a.e.} \tag{6.122}$$

It is worth mentioning that these results were derived in [892] with *no* calmness and/or any other qualification conditions by using the method of *discrete approximations*; see Subsect. 6.5.12 for more discussions on this technique.

Observe that in contrast to Clarke's Euler-Lagrange condition (6.115) requiring the *full convexification* of the basic normal cone (since $N_C = \operatorname{clco} N$), both conditions (6.121) and (6.122) involve only a *partial* convexification, which allows us to avoid troubles with the subspace property of the Clarke normal cone to graphical sets.

Condition (6.122) obviously implies the Euler-Lagrange condition in Clarke's form (6.115); it is easy to find examples when (6.122) is strictly better. This is however not the case regarding the comparison between (6.115) and (6.121) when the velocity sets $F(x,t)$ are not strictly convex. Indeed, there are examples in Loewen and Rockafellar [805] showing that these two necessary optimality conditions are generally *independent*. Moreover, it has been subsequently proved by Ioffe [603] and Rockafellar [1162] (as the two complementary implications) that Mordukhovich's initial version of the Euler-Lagrange condition (6.121) for convex-valued differential inclusions happens to be *equivalent* to Clarke's Hamiltonian condition (6.116).

We refer the reader to other publications by Mordukhovich [901, 902, 908] containing the developments of condition (6.121), and thus of (6.122) in the case of strictly convex velocity sets, for various dynamic optimization problems involving convex-valued (or relaxed) differential inclusions; in particular, for problems with free time, intermediate state constraints, Bolza-type functionals, etc. Developing then the discrete approximation techniques of [892, 901, 902, 908], Smirnov [1215] established the validity of the refined Euler-Lagrange condition (6.122) for (*not strictly*) convex-valued, Lipschitzian, bounded, and autonomous differential inclusions by reduction them in fact to the strictly convex case.

Further results in this direction were obtained by Loewen and Rockafellar [805] for convex-valued and *unbounded* differential inclusions of type (6.110), with the replacement of the standard Lipschitzian property of $F(\cdot,t)$ for bounded inclusions by its "integrable sub-Lipschitzian" counterpart in the unbounded case. They derived the Euler-Lagrange condition in the advanced form (6.122) emphasizing that "two simple themes underlie our approach: *truncation* and *strict convexity*." The latter means that they developed an efficient technique allowing them to reduce the general case under consideration to bounded and Lipschitzian differential inclusions, for which condition (6.121) held and agreed with the refined one (6.122). Note that the *convexity* assumption on the sets $F(x,t)$ played a crucial role in the technique developed in [805]. The two subsequent papers by Loewen and Rockafellar [806, 807] contained extensions of these results to the generalized problem of Bolza with state constraints and free time. It is worth mentioning that in [806] the general Bolza case with an extended-real-valued integrand/Lagrangian in (6.112) was reduced under mild "epi-continuity" and growth assumptions to a Mayer problem for an unbounded differential inclusion satisfying the "integrable sub-Lipschitzian" property of [805]; moreover, the *coderivative criterion* for Lipschitz-like behavior established by Mordukhovich [909] (see Theorem 4.10) served as a key technical ingredient in justifying the possibility of such a reduction.

At this point we observe that the Euler-Lagrange inclusion (6.122) can be equivalently written in the *coderivative form*

$$\dot{p}(t) \in \mathrm{co} D_x^* F\big(\bar{x}(t), \dot{\bar{x}}(t), t\big)\big(-p(t)\big) \quad \text{a.e.}\,, \tag{6.123}$$

which was actually the *original motivation* for introducing the coderivative construction in [892] (as the *adjoint mapping* to F) to describe adjoint systems in optimal control problems governed by discrete-time and differential inclusions. Since the coderivative reduces to the *adjoint Jacobian* for smooth single-valued mappings, relation (6.123) can be viewed as an appropriate extension of the *adjoint system* (6.107) to generalized control processes governed by differential inclusions. Note that the Hamiltonian form of necessary optimality conditions as in (6.113) *doesn't offer* such an extension in the nonsmooth setting. Besides an intrinsic esthetic value, form (6.123) carries a powerful *technical component* allowing us to employ comprehensive coderivative calculus and dual characterizations of Lipschitzian and related properties to the study of many issues in control theory for differential inclusions, particularly those concerning limiting processes; see, e.g., the above proofs of the major results presented in Sects. 6.1 and 6.2 of this book.

6.5.8. Extended Euler-Lagrange and Weierstrass-Pontryagin Conditions for Nonconvex-Valued Differential Inclusions. As mentioned, the results discussed in Subsect. 6.5.7 (as well as the previous versions reviewed in Subsect. 6.5.6) were derived under the *convexity* hypothesis imposed on the velocity sets $F(x,t)$ of differential inclusions in the absence of calmness-like assumptions. Necessary optimality conditions for *nonconvex*-valued (while Lipschitzian and bounded) differential inclusions with endpoint constraints involving the *extended Euler-Lagrange* condition (6.123) were first established by Mordukhovich [915] without any constraint qualifications. Observe that the Euler-Lagrange condition in Clarke's *fully convexified* form (6.115) was previously obtained by Kaśkosz and Lojasiewicz [667] for boundary trajectories of nonconvex, bounded, and Lipschitzian differential inclusions. In [915], the reader can find the corresponding version of the extended Euler-Lagrange condition (6.123) for the Bolza problem (6.112) with a finite nonconvex integrand over nonconvex differential inclusions, while another paper by Mordukhovich [916] concerned problems with free time.

The *Weierstrass-Pontryagin maximum condition* (6.117) doesn't play an independent role for convex-valued differential inclusions, since it follows automatically from any version of the Euler-Lagrange conditions discussed above. This is *no* longer true in the nonconvex setting for which the maximum condition was not established in the afore-mentioned papers [667, 915]. Nevertheless, it was asserted in [915, Remark 7.6] that the methods developed therein would allow us to prove (6.117) accompanying the refined Euler-Lagrange condition (6.123) if the classical Weierstrass necessary condition would be established for strong minimizers of the Bolza problem with *finite* Lagrangian and free endpoints without imposing any smoothness and/or convexity assumptions. The latter task was first accomplished by Ioffe and Rockafellar [616] who derived the counterpart

$$\dot{p}(t) \in \operatorname{co}\left\{u \in I\!R^n \,\middle|\, \big(u, p(t)\big) \in \partial\vartheta\big(\bar{x}(t), \dot{\bar{x}}(t), t\big)\right\} \text{ a.e.} \tag{6.124}$$

of the extended Euler-Lagrange condition (6.123) accompanied by the classical Weierstrass condition, valid for all $v \in I\!R^n$ and a.e. t,

$$\vartheta\big(\bar{x}(t), v, t\big) \geq \vartheta\big(\bar{x}(t), \dot{\bar{x}}(t), t\big) + \langle p(t), v - \dot{\bar{x}}(t)\rangle \tag{6.125}$$

for the *nonconvex* Bolza problem (6.112) with the finite (real-valued) integrand ϑ.

Based on Ioffe-Rockafellar's result and the techniques of [915], Mordukhovich derived in [914] the Euler-Lagrange condition (6.123) accompanied by the Weierstrass-Pontryagin maximum condition (6.117) for nonconvex differential inclusions under the *boundedness* and Lipschitzian assumptions on F with respect to x. More general results of this type were then obtained in the concurrent papers by Ioffe [604] and Vinter and Zheng [1294] who derived, by different techniques, the extended Euler-Lagrange (6.123) and Weierstrass-Pontryagin (6.117) necessary optimality conditions for nonconvex and *unbounded* differential inclusions under the *integrable sub-Lipschitzian* assumption by Loewen and Rockafellar [805]. It is interesting to observe that Vinter and Zheng [1294] gave another proof of Ioffe-Rockafellar's results (6.124) and (6.125) for problems with finite Lagrangians based on their reduction to optimal control problems for systems with *smooth dynamics* and *nonsmooth endpoint constraints* employing to them the version of the maximum principle with transversality conditions (6.120) originally obtained in the 1976 paper by Mordukhovich [916]. We also refer the reader to the subsequent papers by Vinter and Zheng [1295, 1296, 1297] for appropriate versions of the extended Euler-Lagrange and Weierstrass-Pontryagin conditions to problems with state constraints and free time, and also to their applications. Furthermore, Rampazzo and Vinter [1118] generalized these results for nonconvex differential inclusions with the so-called *degenerated* state constraints providing nondegenerate necessary optimality conditions for problems in which endpoints may belong to the boundary of state constraints, and so the standard necessary conditions convey no useful information. See also Arutyunov and Aseev [33], Ferreira, Fontes and Vinter [443] with the references therein for previous results concerning degenerate control problems.

Quite recently, Clarke [260, 261] derived necessary optimality conditions in the extended Euler-Lagrange form (6.123) accompanied by the Weierstrass-Pontryagin maximum condition (6.117) for nonconvex and unbounded differential inclusions under *fairly weak* (probably minimal) assumptions on the initial data. In the process of proof, he developed a delicate and powerful technique involving *smooth variational principles* and *decoupling* machinery that allowed him to reduce these conditions under the weak assumptions made to the settings already known and discussed above. The conditions derived in [260, 261] also incorporated a novel *stratified* feature in which both the assumptions and conclusions were formulated relative to a *prescribed* radius function. They also gave rise to new forms of the so-called "hybrid maximum principle" for optimal control problems with cost integrands of a very general nature while with the smooth underlying dynamics.

Note that in certain special situations potentially *stronger* versions of the extended Euler-Lagrange condition can be obtained for minimizing nonconvex and nonsmooth integral functionals of the calculus of variations and related problems. To this end we refer the reader to the papers by Ambrosio, Ascenzi and Buttazzo [17], Marcelli [845, 846], and Marcelli, Outkine and Sytchev [847], where some versions of the Euler-Lagrange conditions via the *subdifferential of convex analysis* were derived for *nonconvex* problems with some special structures. The results of this type are heavily based on *relaxation techniques* particularly involving the Lyapunov convexity theorem [822] and its various extensions and modifications.

6.5.9. Dualization and Extended Hamiltonian Formalism. In Subsects. 6.5.5 and 6.5.7 we have discussed some relationships between the previous versions of the Euler-Lagrange and Hamiltonian optimality conditions for differential inclusions and for the generalized problem of Bolza. Recall that, in contrast to the classical smooth and fully convex cases, Clarke's versions of the Euler-Lagrange (6.115) and Hamiltonian (6.116) conditions are *not equivalent* even in simple settings, while his Hamiltonian condition happens to be *equivalent* to the early Mordukhovich's version of the Euler-Lagrange condition (6.121) for convex-valued differential inclusions. What about an appropriate *Hamiltonian* counterpart of the *extended* Euler-Lagrange condition written as (6.122), or equivalently as (6.123), for differential inclusions and as (6.124) and the problem of Bolza in the *absence of strict convexity*?

This question was first investigated by Rockafellar [1162] in the general framework of the *Legendre-Fenchel transform* (or the *conjugacy correspondence*) of convex analysis defined by the classical formula

$$\vartheta^*(x,p)=\sup_{v\in I\!\!R^n}\big\{\langle p,v\rangle-\vartheta(x,v)\big\}\ . \tag{6.126}$$

It is well known from convex analysis [1142] that for any proper, *convex*, and l.s.c. function $\vartheta(x,\cdot)\colon I\!\!R^n\to\overline{I\!\!R}$ the conjugate function $\vartheta^*(x,\cdot)$ enjoys the same properties on $I\!\!R^n$ satisfying moreover the symmetric *biconjugacy* relationship

$$\vartheta(x,v)=\sup_{p\in I\!\!R^n}\big\{\langle p,v\rangle-\vartheta^*(x,p)\big\}\ .$$

The question stated and resolved by Rockafellar [1162] was about relationships between *basic subgradients* of the functions $\vartheta(x,v)$ and $\vartheta^*(x,p)$ with respect to their *both* variables. Under a certain "epi-continuity" assumption, which automatically holds when either ϑ or ϑ^* is locally Lipschitzian around the reference point, it was established in [1162] the following relationship for the *convex hulls*:

$$\operatorname{co}\left\{u \in I\!R^n \,\middle|\, (u, p) \in \partial\vartheta(x, v)\right\} = -\operatorname{co}\left\{u \in I\!R^n \,\middle|\, (u, v) \in \partial\vartheta^*(x, p)\right\}. \quad (6.127)$$

For the case corresponding to differential inclusions, with $\vartheta(x, v) = \delta((x, v); \operatorname{gph} F)$, the relationships (6.127) reduces to

$$\operatorname{co}\left\{u \in R^n \,\middle|\, (u, p) \in N\big((x, v); \operatorname{gph} F\big)\right\} = \operatorname{co}\left\{u \in I\!R^n \,\middle|\, (-u, v) \in \partial\mathcal{H}(x, p)\right\}$$

by taking into account (6.126) and the Hamiltonian construction (6.111). The proof of the *Rockafellar dualization theorem* (6.127) given in [1162] was rather involved based on advanced tools of convex analysis in finite dimensions including Moreau-Yosida's approximation techniques, Wijsman's epi-continuity theorem, Attouch's theorem on convergence of subgradients, etc.

In view of (6.127), the advanced/*extended Hamiltonian* form *equivalent* to the extended Euler-Lagrange condition (6.123) for *convex*-valued differential inclusions reads as follows:

$$\dot{p}(t) \in \operatorname{co}\left\{u \in I\!R^n \,\middle|\, \big(-u, \dot{\bar{x}}(t)\big) \in \partial\mathcal{H}\big(\bar{x}(t), p(t), t\big)\right\} \quad \text{a.e.} \qquad (6.128)$$

The same form of the extended Hamiltonian condition holds true for the generalized Bolza problem (6.112), with the Hamiltonian defined accordingly as the conjugate of the Lagrangian integrand $\vartheta(x, p, t)$ in the velocity variable v. The elaboration of the assumptions needed for the fulfillment of the associated Euler-Lagrange condition (6.124) together with the equivalent Hamiltonian form (6.128) in the framework of the generalized problem of Bolza with the integrand $\vartheta(x, v, t)$ *convex* in v was given by Loewen and Rockafellar [806]; see the corresponding discussions on the extended Euler-Lagrange condition in Subsect. 6.5.7, presented right before (6.123), which can now be equally relate to the Hamiltonian condition (6.128) due to Rockafellar's dualization result (6.127).

In [604], Ioffe established the *inclusion* "$\subset$" in (6.127) under significantly weaker assumptions in comparison with those in Rockafellar [1162], while still under the *convexity* of $\vartheta(x, \cdot)$. Employing this result, he justified necessary optimality conditions in both Euler-Lagrange (6.123) and Hamiltonian (6.128) forms for *convex*-valued and *unbounded* differential inclusions with the replacement of the "integrable sub-Lipschitzian" property as in Loewen and Rockafellar [806] by the more general Lipschitz-like (Aubin's "pseudo-Lipschitzian") property of $F(\cdot, t)$. Observe that Ioffe's proof clearly reveals the *pivoting role* of the Euler-Lagrange condition (6.123) in nonsmooth optimal control, which holds with *no* convexity assumptions (see Subsect. 6.5.8) and directly implies the extended Hamiltonian condition (6.128) for convex-valued problems. Note to this end that the validity of the latter Hamiltonian inclusion (6.128) for *nonconvex* problems is still an *open question*, even for bounded and Lipschitzian differential inclusions.

Another proof of the inclusion "$\subset$" in Rockafellar's dualization theorem (6.127) under about the same hypotheses as in [1162] was later given by Bessis, Ledyaev and Vinter [113] (see also Sect. 7.6 in Vinter's book [1289]). The proof of [113, 1289] employed not Moreau-Yosida's approximations as in [604, 1162] but more direct and conventional (while rather involved) techniques of *proximal analysis*.

6.5.10. Other Techniques and Results in Nonsmooth Optimal Control. It is worth mentioning that, as shown by Ioffe [604], the advanced Euler-Lagrange formalism for *nonconvex* differential inclusions discussed in Subsect. 6.5.8 easily implies a *nonsmooth* version of the *Pontryagin maximum principle* for parameterized control systems of type (6.106) with the adjoint equation

$$-\dot{p}(t) \in \big[J_x f\big(\bar{x}(t), \bar{u}(t), t\big)\big]^* p(t) \;\text{ a.e.} \tag{6.129}$$

written via Clarke's generalized Jacobian $J_x f$ of f with respect to x. Recall that the *generalized Jacobian* [252, 255] of a Lipschitzian mapping $f\colon I\!\!R^n \to I\!\!R^m$ is defined as the *convex hull* of the classical Jacobian $m \times n$ matrices at points $x_k \to \bar{x}$; the latter set is nonempty and compact by the fundamental Rademacher's theorem [1114]. Such a nonsmooth maximum principle involving the adjoint equation (6.129) was first obtained by Clarke [250, 255] directly for control systems (6.106) based on approximation procedures via *Ekeland's variational principle*. Note also that Ioffe [604] deduced the maximum principle in the somewhat more advanced form suggested by Kaśkosz and Lojasiewicz [666] for parameterized families of *vector fields* from the extended Euler-Lagrange formalism for differential inclusions.

Probably the very *first extension* of the Pontryagin maximum principle to nonsmooth control systems was published by Kugushev [722] who employed a certain constructive technique to approximate the given nonsmooth system by a sequence of *smooth* ones. However, he didn't described efficiently the resulting set of "subgradients" that appeared in this procedure. Other early results on the nonsmooth maximum principle for systems (6.106) were independently obtained by Warga [1316, 1317, 1321] (starting with the end of 1973) using some smooth approximation technique of the *mollifier* type and his *derivate containers* for mappings $f\colon I\!\!R^n \to I\!\!R^m$. The latter objects, which are *not uniquely* defined, give more precise results than Clarke's generalized Jacobian in some settings of variational analysis, optimization, and control. However, the *convex hull* of any derivate container provides no more information than the generalized Jacobian (as shown in [1320]), and thus the adjoint system in form (6.129) subsumes that of Warga [1316].

Warga's approach to derive necessary optimality and controllability conditions was extended by Zhu [1370] to nonconvex *differential inclusions* satisfying, besides the standard assumptions of boundedness and Lipschitz continuity, also requirements on the existence of some *local selections*, which were incorporated in the optimality conditions obtained in [1370]. An obvious drawback of such and related conditions (see, e.g., Tuan [1273]) is the absence of *any analytic mechanism* for obtaining required selections, even in the case of convex-valued inclusions. Similar remarks on the possibility to constructively verify assumptions and conclusions explicitly involving certain *auxiliary objects* of approximation and linearization types can be equally addressed to some other necessary optimality conditions for nonsmooth optimal control and variational problems obtained particularly by Frankowska [464, 465, 468] and by Polovinkin and Smirnov [1094, 1095]; cf. also Ahmed and Xiang [6] for problems involving infinite-dimensional differential inclusions.

Note that there is another direction in the theory of necessary optimality conditions for differential inclusions, developed mostly in the Russian school, that aims to derive results for differential inclusions by limiting procedures from the Pontryagin maximum principle for *smooth* optimal control problems involving systems of type (6.106). In this way, using different kinds of *smooth approximations*, some interesting results mainly related to those already known in the theory of *convex-valued* differential inclusions were obtained by Arutyunov, Aseev and Blagodatskikh [34], Aseev [39, 40, 41], and Milyutin [875, 876]; the latter paper was the last work by Alexei Alexeevich Milyutin submitted and published after his death.

On the other way of development, new results for *nonsmooth* control systems (6.106) different from Clarke's version of the nonsmooth maximum principle with the adjoint equation (6.129) were obtained by de Pinho, Vinter, and their collaborators using an appropriate approximation of control systems by differential inclusions with the help of Ekeland's variational principle. These results are described via *joint* subgradients of the Hamilton-Pontryagin function (6.108), called sometimes the *unmaximized Hamiltonian*, in the (x, p, u) variables. The first result of this type was derived by de Pinho and Vinter [1078] for standard optimal control problems with endpoint constraints under the name of "Euler-Lagrange inclusion," which didn't seem to be in accordance with the real essence of this condition. Then the name has been appropriately changed, and the results of this type were labeled as necessary optimality conditions for nonsmooth control systems involving the *unmaximized Hamiltonian inclusion* (UHI); see [1076] for more discussions. The subsequent papers of these authors and their collaborators [1074, 1075, 1076, 1077, 1079, 1080] contained various extensions of the UHI type results to optimal control problems with state constraints, with mixed constraints on control and state variables, with algebraic-differential constraints, etc. The results of this type are particularly efficient for *weak minimizers*; cf. also the related paper by Páles and Zeidan [1036]. One of the strongest advantages (as well as the original motivation) of the UHI formalism in comparison with Clarke's version of the nonsmooth maximum principle is that the possibility to get *necessary and suf-*

ficient conditions for optimality in nonsmooth *convex* control problems, which is not the case for Clarke's formalism (6.129).

6.5.11. Dual versus Primal Methods in Optimal Control. Observe that the majority of techniques developed for optimization of *differential inclusions* don't employ the *method of variations* and its modifications that lie at the heart of the classical calculus of variations and optimal control dealing with parameterized control systems of type (6.106). Perhaps the most significant technical reason for this in the context of differential inclusions (6.110) relates to the fact that the method of variations based on the comparison between the given optimal solution and its small (in some sense) local variations doesn't fit well to the very nature of the dynamic constraints $\dot{x} \in F(x)$ and also of control constraints of the type $u \in U(x)$ with the *state-dependent* control region $U(x)$.

Alternative approaches to developing necessary optimality conditions for differential inclusions, as well as for constrained control systems of type (6.106), are based on certain *approximation/perturbation* procedures concerning the *whole problem* under consideration, not only its optimal solution. This may involve various approximations of dynamic optimization problems by those with *no* right-endpoint constraints (which are much easier to handle), exact penalization, decoupling, discrete approximations, etc.; see more details and discussions in Clarke [250, 255], Ioffe [604, 611], Mordukhovich [887, 915], Vinter [1289] with their references.

The techniques and results of the latter type lead to *subgradient-oriented* theories of necessary conditions in nonsmooth optimization and optimal control involving generalized differential constructions in *dual spaces* (normal cones, subdifferential, coderivatives). It seems that the strongest general results of this type are expressed in terms of our basic/limiting dual-space constructions, which *cannot* be generated by derivative-like objects in primal spaces (as tangent cones and directional derivatives) due to their *intrinsic nonconvexity*. This allows us to unify the results obtained in this direction under the name of *dual-space theory*.

On the other line of developments, approaches and results related to the method of variations and its modifications deal with variations and perturbations of optimal solutions in *primal spaces* involving various *tangential* approximations, particularly of reachable sets for control systems; see, e.g., the proof of the Pontryagin maximum principle in [1102] and the subsequent developments by Dubovitskii and Milyutin [370, 877], Halkin [539, 545], Neustadt [1001, 1002], Warga [1315, 1316], and others. We refer to results of this type as to *primal-space theory*. Note that this terminology is *not* in accordance with the one adopted by Vinter [1289, pp. 228–231].

Necessary optimality conditions for nonsmooth optimal control obtained in the dual-space and primal-space theories are generally *independent* from the viewpoints of treated *local minimizers*, employed *analytic machineries*, and imposed *assumptions* on the initial data. In more detail:

—–Types of local minima investigated by primal-space methods depend on the *variations* used, while dual-space methods deal with local minimizers defined *regardless* of variations.

—–Realizations and implementations of primal-space methods heavily depend on using *powerful tools* of nonlinear analysis (including open mapping and implicit function theorems and/or fixed-point results), while dual-space methods are *free* of this machinery employing instead more simple *penalty-type* techniques in finite dimensions as well as modern *variational principles* in infinite-dimensional settings.

—–Assumptions needed for approximation/perturbation techniques in dual-space theory require good behavior *around* points of minima (e.g., Lipschitzian properties and metric regularity), while primal-space techniques may produce results under *at-point* assumptions.

—–Primal-space methods for (smooth and nonsmooth) *constrained* optimization (including constrained optimal control) require finally the usage of *convex separation* for obtaining efficient results in eventually dual terms (Lagrange multipliers, adjoint trajectories, etc.), while dual-space methods *don't appeal* as a rule to convex separation theorems.

In Sect. 6.3, the reader can find some advanced results in the primal-space direction derived in the *conventional PMP form* and its *upper subdifferential* extension. The obtained results concern parameterized control systems of type (6.106) with *smooth dynamics* in *infinite-dimensional* spaces and endpoint equality and inequality constraints described by finitely many real-valued functions. However, these functions may be merely *Fréchet differentiable at* the reference optimal point, *not* even being *continuous* around it (the latter applies only to the functions describing the endpoint objective as well as inequality constraints); see more comments to the material of Sect. 6.3 presented below.

The most general results of the primal type in *nonsmooth* optimal control for *finite-dimensional* systems have been developed by Sussmann during the last decade; see [1235, 1236, 1237, 1238] and the references therein. He started [1235] with the remarkable result called the *Lojasiewicz refinement* of the maximum principle that came out of Lojasiewicz's idea formulated in the unpublished (and probably unfinished) paper [810]. This refinement consists of justifying a version of the PMP by assuming that the velocity mapping $f(x,u,t)$ in (6.106) is not $\mathcal{C}^1$ with respect to x for all $u \in U$ a.e. in t as in the classical PMP and not locally Lipschitzian in x for all $u \in U$ and a.e. t as in Clarke's nonsmooth version of the PMP under "minimal hypotheses" [250] but merely locally Lipschitzian in x *along the given optimal control* $u = \bar{u}(t)$ for a.e. t. A "weak differentiable" version of this result justifies the validity of the

PMP when $f(\cdot, \bar{u}(t), t)$ is *differentiable* (possibly not strictly differentiable) *at one point* $\bar{x}(t)$ along the optimal control $u = \bar{u}(t)$ for a.e. t.

Sussmann proved these results and their far-going generalizations in nonsmooth optimal control developing certain abstract versions of *needle variations* (crucial in the proof of the classical PMP) and *primal-space* constructions of generalized differentials. In the recent paper [38], Arutyunov and Vinter provided a simplified proof of the "weak differentiable" version in the Lojasiewicz refinement of the PMP based on the so-called "inner finite approximations" involving special needle-type variations of the reference optimal control $\bar{u}(\cdot)$ that *don't violate* endpoint constraints on trajectories. The idea of this finite approximation scheme goes back to Tikhomirov being published in [7], where it was applied to the classical PMP in smooth optimal control. Further results in this direction were derived by Shvartsman [1209] for nonsmooth control systems with state constraints.

6.5.12. The Method of Discrete Approximations. Section 6.1 is devoted to a thorough study of dynamic optimization problems in infinite-dimensional spaces by using the method of *discrete approximations*. Although our primary goal is to develop this method as a *vehicle* to derive necessary optimality conditions of the *extended Euler-Lagrange* type (6.123) for dynamic processes governed by nonconvex differential/evolution inclusions, we also present some results of *numerical value* for such processes that concern *well-posedness* and *convergence* issues for discrete approximations of evolution inclusions *with* and *without* optimization involved. It seems that neither necessary optimality conditions for *infinite-dimensional evolution inclusions* nor discrete approximations of such processes have been previously considered in the literature besides the author's recent paper [932], where some of the results obtained in this book were announced. They follow however a series of finite-dimensional developments; see below.

The method of discrete approximations for the study of continuous-time systems goes back to Euler [411] who developed it to establish the famous first-order necessary condition (known now as the Euler or Euler-Lagrange equation) for minimizing integral functionals in the one-dimensional calculus of variations. It is significant to note that Euler regarded the integral under minimization as an *infinite sum* and *didn't employ* limiting processes interpreting instead (via a geometric diagram) the differentials along the minimizing curve as *infinitesimal changes* in comparison with "broken lines," i.e., finite differences. Euler's derivation of the necessary optimality condition in *one equational form* for a "general" (at that time) problem of the calculus of variations signified a major theoretical achievement providing the synthesis of many special cases and examples appeared in the work of earlier researchers. It is worth mentioning that an approximation idea based on replacing a curve by broken lines was partly (and rather vaguely) used by Leibniz [757] in his solution of the brachistochrone problem in the very beginning of the calculus of variations.

Since that time, Euler's finite-difference method and its modifications have been widely employed in various areas of dynamic optimization and numerical analysis of differential systems, with mostly numerical emphasis that has become more significant in the computer era. There is an abundant literature devoted to different aspects of discrete approximations and their numerous applications; we refer the reader to [28, 98, 184, 185, 220, 221, 298, 299, 302, 303, 338, 343, 344, 345, 346, 347, 348, 349, 353, 354, 357, 358, 359, 367, 407, 425, 488, 520, 535, 542, 702, 721, 760, 761, 828, 831, 832, 890, 892, 900, 901, 902, 908, 915, 916, 941, 959, 973, 974, 976, 1012, 1061, 1062, 1086, 1107, 1109, 1215, 1175, 1216, 1280, 1282, 1283, 1284, 1301, 1333, 1379] and the bibliographies therein for representative publications related to dynamic optimization and control systems.

In Sect. 6.1 we extend to the general infinite-dimensional setting of nonconvex evolution/differential inclusions the basic constructions and results of the method of discrete approximations developed previously by Mordukhovich [915] for differential inclusions in finite-dimensional spaces; see also [890, 892, 901, 902, 908, 1107, 1109, 1215, 1216] and the comments below for the preceding work in this direction concerning convex-graph and convex-valued differential inclusions in finite dimensions.

The *underlying idea* and the *basic scheme* of the method of discrete approximations to derive necessary optimality conditions for variational problems involving differential inclusions contain the following *three major components*:

(i) to replace/approximate the original continuous-time variational problem by a *well-posed* sequence of discrete-time optimization problems whose optimal solutions *converge*, in a certain suitable sense, to some (or to the given) optimal solution for the original problem;

(ii) to derive necessary optimality conditions in discrete-time problems of dynamic optimization by reducing them to constrained problems of mathematical programming, which occur to be *intrinsically nonsmooth*, and then by employing *appropriate* tools of *generalized differentiation* with good *calculus*;

(iii) to establish *robust/pointbased* necessary optimality conditions for the original continuous-time dynamic optimization problem by *passing to the limit* from necessary conditions for its discrete approximations and by using the *convergence/stability* results obtained for the discrete approximation procedure together with the corresponding properties of the generalized differential constructions that ensure the required convergence of *adjoint* trajectories.

In Mordukhovich's paper [915], the described discrete approximation scheme was implemented for the general Bolza problem governed by *nonconvex* differential inclusions in *finite-dimensional* spaces; the extended Euler-Lagrange condition of the advanced type (6.123) was first established there in this way for nonconvex problems. The realization of each of the three steps (i)–(iii) listed above for evolution inclusions in *infinite dimensions* requires certain additional developments most of which happen to be significantly different from the finite-dimensional setting.

6.5.13. Discrete Approximations of Evolution Inclusions. The main aspects of the theory of differential inclusions of type (6.1) in infinite-dimensional spaces, called often *evolution inclusions*, are presented in the books by Deimling [314] and by Tolstonogov [1258], while much more is available for differential inclusions in finite dimensions; see, e.g., the books by Aubin and Cellina [50] and by Filippov [450] with the references therein. We follow Deimling [314] in Definition 6.1 of solutions to differential/evolution inclusions in Banach spaces. Note that it differs from Carathéodory solutions in finite dimensions (which go back to [222] in the case of differential equations) by the additional requirement on the validity of the *Newton-Leibniz formula* in terms of the Bochner integral; the latter is not automatic for absolutely continuous mappings with infinite-dimensional values. On the other hand, there is a *precise characterization* of Banach spaces, where the fulfillment of the Newton-Leibniz formula is *equivalent* to the absolute continuity: these are spaces with the *Radon-Nikodým property* (RNP) for which more details are available in the classical monographs by Bourgin [169] and by Diestel and Uhl [334]. The latter property is fundamental in functional analysis; in particular, its validity for the *dual space* X^* is *equivalent* to the *Asplund property* of X. This justifies another line of using the remarkable class of Asplund spaces in the book.

The principal result of Subsect. 6.1.1, Theorem 6.4, justifies a constructive algorithm to *strongly approximate* (in the norm of the Sobolev space $W^{1,2}([a,b];X)$ ensuring particularly the a.e. *pointwise* convergence with respect to *velocities*) of *any* given feasible trajectory for the Lipschitzian differential inclusion (6.1) in arbitrary Banach space X by extended trajectories of its *finite-difference* counterparts (6.3) obtained by using the standard Euler scheme. This result is an infinite-dimensional version of that by Mordukhovich [915, Theorem 3.1] (with just a little change in the proof) extending his previous constructions and results from [901, 902] and those from Smirnov's paper [1215]; see also [1216]. This theorem, besides its independent interest and numerical value to justify an efficient procedure for approximating the set of feasible solutions to a general differential inclusion *regardless of optimization*, provides the foundation for constructing *well-posed* discrete approximations of variational problems for continuous-time evolution systems.

Observe that we don't impose in Theorem 6.4 *any convexity* assumptions on the velocity sets $F(x,t)$ and realize the *proximal algorithm* based on the *projection of velocities* in (6.10). This distinguishes the *velocity approach* from more conventional results on discrete approximations of (convex-graph or convex-valued) differential inclusions involving projections of state vectors and ensuring merely the $\mathcal{C}([a,b];I\!R^n)$-norm convergence of trajectories; see, e.g., Pshenichnyi [1107, 1109] and the survey papers by Dontchev and Lempio [359] and by Lempio and Veliov [761]. We emphasize that the latter convergence doesn't allow us to deal with nonconvex inclusions (since the uniform convergence of trajectories corresponds to the *weak* convergence of derivatives and eventually requires the subsequent *convexification* by the Mazur weak closure theorem) and that the achievement of the a.e. *pointwise* convergence of derivatives/velocities plays a *crucial role* in the possibility to establish necessary optimality conditions for *nonconvex problems*.

Let us mention two recent developments on the convergence of discrete approximations in direction (i) listed in Subsect. 6.5.12. In [343], Donchev derived some extensions of the approximation and convergence results from the afore-mentioned paper [915] to finite-dimensional differential inclusions whose right-hand side mappings $F(x,t)$ satisfy the so-called *Kamke* condition with respect to x, where the standard Lipschitz modulus is replaced by a Kamke-type function. The latter property happens to be *generic* (in Baire's sense) in the class of all continuous multifunctions $F(\cdot,t)$. The other work is due to Mordukhovich and Pennanen [941] who established the *epi-convergence* of discrete approximations in the *generalized Bolza* framework under certain *convexity* and Lipschitzian assumptions.

6.5.14. Intermediate Local Minima. In Subsect. 6.2.2 we start studying the Bolza problem for constrained differential/evolution inclusions in Banach spaces following mainly the procedure developed by Mordukhovich [915] in finite dimensions, with some significant infinite-dimensional changes on which we comment below. Note that, in contrast to the generalized Bolza problem in form (6.13) with extended-real-valued functions φ and ϑ implicitly incorporating endpoint and dynamic constraints, we deal with such constraints *explicitly*, since the continuity and Lipschitzian assumptions imposed on φ and ϑ in the results obtained in Sect. 6.1 *exclude* in fact the infinite values of these functions.

The main attention in our study is paid to the notions of *intermediate local minima* of rank $p \in [0,\infty)$ (i.l.m.; see Definition 6.7) and its *relaxed* version (r.i.l.m.; see Definition 6.12). Both notions were introduced by Mordukhovich [915] and were later studied by Ioffe and Rockafellar [616], Ioffe [604], Vinter and Woodford [1293], Woodford [1331], Vinter and Zheng [1294, 1295, 1289], Vinter [1289], and Clarke [260, 261] for various dynamic optimization problems, mostly in the case of $p=1$, referred to as $W^{1,1}$ local minimizers.

Intermediate local minimizers occupy an *intermediate position* between the classical *weak* and *strong* minimizers for variational problems; that is where this name came from in [915]. Examples 6.8–6.10 show that these three major types of local minimizers may be different even in relatively simple problems of dynamic optimization problems involving particularly convex-valued, bounded, and Lipschitzian differential inclusions. Example 6.8 on the difference between weak and strong minimizers is classical going back to Weierstrass [1326]. The simplified version of Example 6.9 on the difference between weak and intermediate minimizers was presented in [915], while the full version of this example as well as of Example 6.10 are taken from Vinter and Woodford

[1293]. The latter paper and Woodford's dissertation [1331] contain also other examples illustrating the difference between these notions of local minima, particularly the difference between intermediate minimizers of *various ranks* for *convex* and *unbounded* differential inclusions in finite dimensions.

6.5.15. Relaxation Stability and Hidden Convexity. The remainder of Subsect. 6.1.2 presents the construction of the *relaxed* Bolza problem for differential inclusions together with the associated definition and discussions on *relaxation stability*. The idea of proper relaxation (or extension, generalization, regularization) plays a remarkable role in modern variational theory. In general terms, it goes back to Hilbert [567] stating in his famous 20th Problem that "*every problem in the calculus of variations has a solution provided that the word solution is suitably understood.*"

It was fully realized in the 1930s, independently by Bogolyubov [121] and by Young [1349, 1350] for one-dimensional problems of the calculus of variations who showed that adequate extensions of variational problems, which automatically ensure the existence of generalized optimal solutions and their approximations by "ordinary curves," could be achieved by a certain *convexification* with respect to *velocities*. In optimal control, this idea was independently developed by Gamkrelidze [495] and by Warga [1313]; in the latter paper the term "relaxation" was first introduced. Another term broadly used now for similar issues is "Young measures." We refer the reader to [3, 4, 25, 31, 50, 75, 212, 213, 231, 232, 235, 237, 246, 255, 308, 362, 401, 432, 450, 497, 527, 617, 618, 682, 704, 821, 823, 863, 886, 888, 901, 915, 1020, 1049, 1082, 1173, 1174, 1176, 1177, 1258, 1259, 1277, 1315, 1323, 1351] and the bibliographies therein for various relaxation results and their applications to problems of the calculus of variations, optimal control, and related topics.

In this book we follow the constructions developed in [915] for the Bolza problem involving finite-dimensional differential inclusions and employ the relaxation procedure *not* to ensure the existence of generalized solutions but to describe *limiting points* of optimal solutions to discrete approximation problems together with the minimizing functional values. To proceed in this way, the notion of *relaxation stability* formulated in (6.19) plays a crucial role. This property is typically *inherent* in *continuous-time* control systems and differential inclusions relating to their *hidden convexity*; see more discussions and sufficient conditions for relaxation stability presented in Subsect. 6.1.2 and the references therein. We specifically note the approximation property of Theorem 6.11 taken from the recent paper by De Blasi, Pianigiani and Tolstonogov [308], which is a manifestation of the hidden convexity in the framework of the general Bolza problem for infinite-dimensional differential inclusions. Observe also that, in a deep sense, the hidden convexity may be traced to the classical Lyapunov theorem on the range convexity of *nonatomic* vector measures [822] and to its Aumann's version [55] on set-valued integration; see Arkin and Levin [25] and Diestel and Uhl [334] for infinite-dimensional counterparts of such results. We also refer the reader to some other remarkable manifestations of the hidden convexity:

—–Estimates of the "duality gap" in nonconvex programming discovered by Ekeland [398] and then developed by Aubin and Ekeland [51]. These developments are strongly related to the classical Shapley-Folkman theorem in mathematical economics; see the book by Ekeland and Temam [401] for more details and discussions.

—–Convexity of the "nonlinear image of a small ball" recently discovered by Polyak [1098, 1100] who obtained various applications of this phenomenon to optimization, control, and related areas; see also Bobylev, Emel'yanov and Korovin [120] for further developments.

6.5.16. Convergence of Discrete Approximations. While the main attention in Subsect. 6.1.1 was paid to finite-difference approximations of differential/evolution inclusions with *no* optimization involved, the results of Subsect. 6.1.3 concern approximation issues for the *whole* variational problem of Bolza under consideration. This means that we aim to construct well-posed discrete approximations of the original Bolza problem (P) by sequences of discrete-time dynamic optimization problems in such a way that optimal solutions for discrete approximations converge, in a certain prescribed sense, to those for the continuous-time problem. In fact, we present *well-posedness/stability* results that justify the convergence of discrete approximations of the following *two types*:

(I) *Value convergence* ensuring the convergence of *optimal values* of the cost functionals in *constructively* built discrete approximation problems to the optimal value (infimum) of the cost functional in the original problem for which the existence of optimal solutions is not assumed.

(II) *Strong convergence* of optimal solutions for discrete-time problems to the *given* optimal solution for the original problem; the strong convergence is understood in the $W^{1,2}$-norm for piecewise linearly extended discrete trajectories.

Observe that the results of type (II) *explicitly* involve the given optimal solution (actually an *intermediate minimizer*) to the original problem. They are not constructive any more (from the numerical viewpoint) while justifying the way to derive necessary optimality conditions for continuous-time problems by using their discrete approximations (instead of, say, the method of variations, which is not applicable in this framework). The convergence results of type (II) obtained in Subsect. 6.1.3 are of the main interest for deriving necessary optimality conditions in Sect. 6.1 of this book (cf. also Sect. 7.1 for their counterparts concerning functional-differential control systems); they generally impose *milder* assumptions in comparison with those needed to prove the value convergence in (I).

Results of type (I) traditionally relate to *computational* methods in optimal control; they justify "direct" numerical techniques based on approximations of continuous-time control problems by sequences of finite-difference ones, which reduce to problems of *mathematical programming* in finite dimensions provided that state vectors in control systems are finite-dimensional. We are not familiar with any results in this directions for infinite-dimensional differential inclusions, even in the parameterized control form (6.106), besides those presented in Subsect. 6.1.3.

First results on value convergence for standard control systems (6.106) were probably obtained by Budak, Berkovich and Solovieva [184] and Cullum [302] in the late 1960s under rather restrictive assumptions; see also [185, 303, 407] for earlier developments. Then Mordukhovich [890] established the *equivalence* between the *value convergence* of discrete approximations and the *relaxation stability* for general control problems involving parameterized systems (6.106) provided *appropriate perturbations* of state/endpoint constraints *consistent* with the stepsize of discretization. These results were extended in [899, 901, 902] to Lipschitzian differential inclusions; cf. also related results in Dontchev [349] and Dontchev and Zolezzi [367]. Efficient estimates of *convergence rates*, not only with respect to cost functions but also with respect to controls and trajectories, were derived for systems of special structures by Hager [535], Malanowski [831], Dontchev [347], Dontchev and Hager [355], Veliov [1284], and others; see the surveys in [352, 359, 761] for more details and references.

Theorem 6.14 seems to be new even for finite-dimensional differential inclusions developing the corresponding methods and results from Mordukhovich [890, 899, 901]. Observe that the proof of this theorem and the related Theorem 6.13 are more technically involved in comparison with the finite-dimensional case based, besides other things, on the fundamental Dunford theorem ensuring the sequential weak compactness in $L^1([a,b];X)$ provided that both spaces X and X^* satisfy the Radon-Nikodým property, which is the case when *both* X and X^* are *Asplund*. As we remember, the Asplund structure plays a crucial role in the generalized differentiation theory developed in this book from the viewpoint *not related* to the RNP!

Theorem 6.13, which is what we actually need to implement the method of discrete approximations as a *vehicle* for deriving *necessary optimality conditions* for continuous-time systems (i.e., for "*theoretical*" vs. numerical applications) is an infinite-dimensional extension and a modification of Theorem 3.3 from Mordukhovich [915]. The difference between these two results (even in finite dimensions) concerns the way of approximating the original integral functional: we now adopt construction (6.20) instead of the simplified one (6.28) as in [915]. This modification allows us to deal with *measurable* integrands with respect to t that is important for applications in Sect. 6.2, where the integrand *must* be measurable.

Observe the importance of the last term in (6.20) and (6.28) approximating the derivative of the given intermediate minimizer $\bar{x}(\cdot)$. The presence of this term and the usage of the approximation result from Theorem 6.4 allow us to establish the *strong* (in the norm of $W^{1,2}([a,b];X)$) convergence of optimal solutions for the discrete approximation problems to the *given* local minimizer for the original one, which further leads to deriving necessary conditions of type (6.123) for continuous-time problems by passing to the limit from those for their discrete-time counterparts. Besides [915], this approximating term was previously used by Smirnov [1215] (see also his book [1216]) for the Mayer problem involving convex-valued, bounded, and autonomous differential inclusions in finite dimensions. The previous attempts to employ discrete approximations for deriving necessary optimality conditions in the Mayer framework of convex-valued or even convex-graph differential inclusions were able to ensure merely the uniform convergence of extended discrete trajectories to $\bar{x}(\cdot)$ by using an approximating term of the "state type"

$$\sum_{j=0}^{N-1} \|x_N(t_j) - \bar{x}(t_j)\|^2$$

with no derivative $\dot{\bar{x}}(\cdot)$ involved; cf. Halkin [542], Pshenichnyi [1107, 1109], and Mordukhovich [892, 901, 902].

6.5.17. Necessary Optimality Conditions for Discrete Approximations. After establishing the required *strong convergence/stability* of discrete approximations discussed above, the *second step* in realizing the strategy of this method to establish necessary optimality conditions for constrained differential inclusions is to derive *necessary conditions* for *discrete-time problems* formulated in Subsect. 6.1.3. We consider two forms of the discrete approximation problems:

—the "integral" form (P_N) involving the minimization of the cost functional (6.20) subject to the constraints (6.3), (6.21)–(6.23), and

—the "simplified" form $(\overline{P}_N)$ in which the other cost functional (6.28) is minimized under the same constraints.

As discussed, the only distinction between the two functionals (6.20) and (6.28) relates to different ways of approximating the integral functional in the original continuous-time Bolza problem (P): the integral type of (6.20) allows us to consider *measurable* integrands $\vartheta(x,v,\cdot)$ in (6.13), while the summation/simplified type of (6.28) requires the *a.e. continuity* assumption imposing on $\vartheta(x,v,\cdot)$. The reason to consider the latter simplified approximation is that the *summation form* in (6.28) makes it possible to obtain necessary optimality conditions for discrete-time and then for continuous-time problems in more general settings of *Asplund state spaces* X in comparison with the *reflexivity and separability* requirements needed in the case of the integral approximation as in (6.20). This is due to the more developed *subdifferential calculus* for *finite sums* vs. that for *integral functionals*; see below.

In Subsect. 6.1.4 we derived necessary optimality conditions for discrete-time dynamic optimization problems (P_N) and $(\overline{P}_N)$ as well as for their less structured counterpart (DP) called the *Bolza problem for discrete-time inclusions* in infinite dimensions. These problems are certainly of independent interest for discrete systems with fixed steps being important for many applications, particularly to models of economic dynamics; see, e.g., Dyukalov [379] and Dzalilov, Ivanov and Rubinov [380]. Furthermore, necessary optimality conditions for them provide, due to the convergence results of Subsect. 6.1.3, *suboptimality* conditions for the continuous-time Bolza problem under consideration. However, our main interest is to derive such necessary optimality conditions for (P_N) and $(\overline{P}_N)$, which are more convenient for *passing to the limit* in order to establish necessary optimality conditions for the Bolza problem involving infinite-dimensional differential inclusions.

The discrete-time dynamic optimization problems under consideration in Subsect. 6.1.4 can be reduced to the form of *constrained mathematical programming* (MP) given in (6.29). Problems (MP) appeared in this way have *two characteristic features* that distinguish them from other classes of constrained problems in mathematical programming:

(a) They involve *finitely many geometric constraints* the number of which tends to *infinity* when the stepsize of discrete approximations is decreasing to zero. It is worth mentioning that these geometric constraints are of the *graphical* type, which are generated by the discretized inclusions. The presence of such constraints makes the (MP) problem (6.29) *intrinsically nonsmooth* even for smooth functional data in (6.29) and in the generating problems (P_N), $(\overline{P}_N)$, and (P).

(b) If the original state space X is *infinite-dimensional*, the (MP) problem (6.29) unavoidably contains *operator constraints* of the equality type $f(x) = 0$, where the range space for f *cannot* be finite-dimensional. We know that such constraints are among the most difficult in optimization, even for smooth mappings f, which is actually the case for applications to the discrete-time problems under consideration.

The theory of necessary optimality conditions for mathematical programming problems of type (6.29) is available from Chap. 5, where we established necessary conditions in terms of the basic/limiting generalized differential constructions. The main conditions for problems of this type involving extended *Lagrange multipliers* are summarized in Proposition 6.16, where finitely many geometric constraints in (6.29) are incorporated via the *intersection rule* for the basic normal cone and the corresponding *SNC calculus* result in the framework of Asplund spaces. Employing these optimality conditions for (MP) together with *exact/pointwise* calculus rules developed for basic normals and subgradients, we arrive at necessary optimality conditions for the discrete Bolza problem (DP) governed by difference inclusions in the *extended Euler-Lagrange form* of Theorem 6.17. Note that the latter result doesn't impose *any* *convexity* and/or *Lipschitzian* assumptions on the discrete velocity sets $F_j(x)$. The conditions obtained in Theorem 6.17 give an Asplund space version of the finite-dimensional conditions from Mordukhovich [915, Theorem 5.2] under certain SNC requirements needed in infinite dimensions.

The *pointbased* necessary optimality conditions for the discrete Bolza problem (DP) obtained in Theorem 6.17 are important for its own sake and, furthermore, provide a sufficient ground for deriving necessary optimality conditions of the extended Euler-Lagrange type (6.123) for continuous-time problems in finite dimensions; see [915] for more details. However, it is *not* precisely the case in *infinite dimensions*, where the realization of this scheme requires *extra* SNC assumptions ensuring the fulfillment of the pointbased necessary optimality conditions in discrete approximations and then the passage to the limit from them as $N \to \infty$. These extra assumptions can be *avoided* by deriving *approximate/fuzzy* necessary conditions for discrete-time problems, instead of the pointbased ones as in Theorem 6.17. Such approximate optimality conditions are obtained in Theorems 6.19 and 6.20 for the discrete approximation problems $(\overline{P}_N)$ and (P_N), respectively.

The proofs of the afore-mentioned approximate optimality conditions are rather involved requiring, among other things, the usage of *fuzzy* calculus rules as well as *neighborhood* coderivative characterizations of *metric regularity* established by Mordukhovich and Shao [946]. Observe also a significant role of Lemma 6.18 extending to the case of *basic* subgradients the classical *Leibniz rule* on (*sub*)*differentiation under integral sign*. This is an auxiliary result for the proof of Theorem 6.20 allowing us to deal with *summable* integrands in (P) under discrete approximations of type (P_N), while the rule itself is certainly of independent interest. Its proof employs an infinite-dimensional extension of the Lyapunov-Aumann convexity theorem and the corresponding rule for Clarke's subgradients [255, Theorem 2.7.2], which is strongly based in turn on the generalized version of Leibniz's rule established by Ioffe and Levin [612] for subgradients of convex analysis.

6.5.18. Passing to the Limit from Discrete Approximations. In Subsect. 6.1.5 we accomplish the *third step* (labeled as (iii) in Subsect. 6.5.12) in the method of discrete approximations to derive necessary optimality conditions in the original Bolza problem (P) for differential inclusions. The primary goal at this step is to justify the passage to the limit from the obtained necessary conditions in the well-posed discrete approximation problems (P_N) and $(\overline{P}_N)$ and to describe efficiently the resulting necessary optimality conditions for the continuous-time problems that come out of this procedure. As we see, the resulting conditions occur to be those of the *extended Euler-Lagrange* type for *relaxed intermediate local minimizers* in (P) established in Theorems 6.21 and 6.22.

These major results of Subsect. 6.1.5 are somewhat different from each other, in both aspects of the assumptions made and of formulating the extended Euler-Lagrange inclusions in (6.44) and (6.47). The differences came from the corresponding results of Subsect. 6.1.4 for the two types of discrete approximation problems, $(\overline{P}_N)$ and (P_N), as well as from additional requirements needed for passing to the limit in the necessary optimality conditions for these problems.

Theorem 6.21, based on the limiting procedure from the simplified discrete approximations $(\overline{P}_N)$, is an infinite-dimensional generalization of that in Mordukhovich [915, Theorem 6.1] with involving the *extended normal cone* in (6.44). The usage of the basic normal cone in a similar setting of [915] was supported by certain technical hypotheses ensuring the *normal semicontinuity* formulated in Definition 5.69 and discussed after it. Theorem 6.22 is new even in finite dimensions.

One of the main concerns in passing to the limit from the discrete-time necessary optimality conditions in the proofs of both Theorem 6.21 and Theorem 6.22 is to justify appropriate convergences of *adjoint trajectories* and their *derivatives*. To establish the required convergence, we employ a dual *coderivative characterization* of *Lipschitzian behavior* for set-valued mappings used so often in this book; such criteria play a *crucial role* in accomplishing limiting procedures for adjoint systems associated with discrete-time and continuous-time inclusions in dynamic optimization problems described by Lipschitzian mappings.

The principal issue that distinguishes the necessary optimality conditions obtained for *infinite-dimensional* differential inclusions from their finite-dimensional counterparts is the presence of the SNC (actually *strong PSNC*) assumption on the constraint/target set Ω imposed in Theorems 6.21 and 6.22. Assumptions of this type are crucial for optimal control problems for infinite-dimensional evolution systems. In particular, it is well known that *no* analog of the Pontryagin maximum principle holds even for simple optimal control problems governed by the one-dimensional heat equation with a *singleton* target set $\Omega = \{x_1\}$ in Hilbert spaces, which is *never PSNC* in infinite dimensions. The first example of this type was given by Y. Egorov [393]. The reader can also consult with the books by Fattorini [432] and by Li and Yong [789] for more discussions involving the *finite codimension* property equivalent to the SNC one for convex sets; see Remark 6.25. Let us emphasize to this end the result of Corollary 6.24 justifying the extended Euler-Lagrange conditions for the Bolza problem (P) governed by evolution inclusions with *no explicit* (while *hidden*) *SNC/PSNC* assumptions on the constraint set Ω given by *finitely many* equalities and inequalities via Lipschitzian functions.

Lastly, we refer the reader to the recent papers by Mordukhovich and D. Wang [970, 971], where some counterparts of the above results are derived for optimal control problems governed by *semilinear unbounded evolution inclusions* that are particularly convenient for modeling *parabolic PDEs*; see Remark 6.26.

6.5.19. Euler-Lagrange and Maximum Conditions with No Relaxation. As seen, the extended Euler-Lagrange conditions established in

Sect. 6.1 by the method of discrete approximations apply to *relaxed* intermediate local minimizers for the Bolza problem governed by infinite-dimensional differential inclusions. The primary goal of Sect. 6.2 is to derive, based on the conditions obtained in Sect. 6.1 and involving additional variational techniques, refined results of the Euler-Lagrange type accompanied furthermore by the Weierstrass-Pontryagin maximum condition for *nonconvex* differential inclusions *without any relaxation*. The main result, for simplicity formulated in Theorem 6.27 in the case of the Mayer-type problem (P_M) with a fixed left endpoint and arbitrary geometric constraints imposed on right endpoints of trajectories, is new in infinite dimensions; its preceding finite-dimensional versions were discussed in Subsect. 6.5.8.

As in Sect. 6.1, the principal distinction between necessary conditions obtained in finite-dimensional and infinite-dimensional settings relates to the presence of *SNC requirements* unavoidable in infinite dimensions. On the other hand, the technical assumptions made in Theorem 6.27 are *different* from those imposed in Theorems 6.21 and 6.22. Observe also the more general forms (6.51) and (6.52) of the transversality conditions in Theorem 6.27 in comparison with the major results of Sect. 6.1 involving only Lipschitzian cost and constraint functions.

The proof of the pivoting Euler-Lagrange condition (6.49) for intermediate local minimizers to nonconvex problems with *no relaxation* is based, besides applying rather delicate calculus and convergence results of variational analysis, on *two perturbation/approximation* procedures allowing us to reduce the original problem (P_M) to the *unconstrained* (while nonsmooth and nonconvex) Bolza problem (6.55) with finite-valued data that are *Lipschitzian* in the state and velocity variables and *measurable* in t. Since any intermediate local minimizer for the latter problem is automatically a *relaxed* one, it can be treated by the necessary optimality conditions obtained in Theorem 6.22 via discrete approximations.

The first of the afore-mentioned perturbation techniques can be recognized as the *method of metric approximations* originally developed by Mordukhovich [887] to prove the maximum principle for finite-dimensional control problems with smooth dynamics and nonsmooth endpoint constraints by reducing them to free-endpoint problems. The second perturbation technique, involving the *Ekeland variational principle* and *penalization* of dynamic constraints, goes back to Clarke [251] in connections with his results on Hamiltonian and maximum conditions for nonsmooth control systems in finite dimensions. The *claim* in the proof of Theorem 6.27 is an infinite-dimensional extension of the corresponding result by Kaśkosz and Lojasiewicz [667] established there for *strong* minimizers (or *boundary* trajectories). Note the importance of the generalized differential results from Subsect. 1.3.3 for the *distance function* at *in-set* and *out-of-set* points to deal with approximating problems and also a crucial role of the *coderivative criterion* for Lipschitzian behavior that allows us to accomplish the convergence procedure in deriving the extended Euler-Lagrange and transversality inclusions of Theorem 6.27.

The proof of the maximum condition (6.50) supplementing the extended Euler-Lagrange condition (6.49) in the nonconvex case is outlined but not fully presented in Subsect. 6.2.1, since it is technically involved while closely follows the line developed by Vinter and Zheng [1294] (see also Vinter's book [1289, Theorem 7.4.1]) for finite-dimensional differential inclusions; the reader can check all the details. Note that this proof is based on reducing the general Mayer problem for differential inclusions to an optimal control problem with *smooth dynamics* and *nonsmooth endpoint constraints* first treated by Mordukhovich [887] via his nonconvex/limiting normal cone; see Sect. 6.3 for related control problems and techniques in infinite-dimensional settings. It seems that the other available proofs of the maximum condition (6.50) in the Euler-Lagrange framework (6.49) given by Ioffe [598] and by Clarke [261] are restricted to the case of finite-dimensional state spaces.

6.5.20. Related Topics and Results in Optimal Control of Differential Inclusions. The variational methods developed in this book allow us to obtain extensions and counterparts of Theorem 6.27 in various settings partly discussed in Subsect. 6.2.2, which particularly include *upper subdifferential* conditions and *multiobjective* control problems; cf. also Zhu [1372], Bellaassali and Jourani [93], and Eisenhart [395] for related developments in multiobjective dynamic optimization concerning finite-dimensional control systems. It seems however that necessary optimality conditions of the *Hamiltonian* type as well as results on *local controllability* for differential inclusions require the *finite dimensionality* of state spaces; see more details and discussions in Remarks 6.32 and 6.33.

The examples given at the end of Subsect. 6.2.2 illustrate some characteristic features of the results obtained for differential inclusions and the relationships between them. Example 6.34 confirming that the *partial* convexification is *essential* for the validity of both Euler-Lagrange and Hamiltonian optimality conditions of the established extended type is due to Shvartsman (personal communication). Example 6.35 taken from Loewen and Rockafellar [805] shows that the *extended Euler-Lagrange* condition involving only the partial convexification is *strictly better* than the *Hamiltonian condition* in Clarke's fully convexified form even for Lipschitzian control systems with convex velocities. Finally, Example 6.36 given by Ioffe [604] demonstrates that the *partially convexified* Hamiltonian condition, which may *not* be equivalent to its Euler-Lagrange counterpart, also *strictly improves* the *fully convexified* Hamiltonian formalism in rather general settings.

6.5.21. Primal-Space Approach via the Increment Method. Section 6.3 concerns optimal control problems in the more traditional *parameterized* framework (6.61), involving however the *infinite-dimensional dynamics*. Even more, we impose in this section the *continuous differentiability/smoothness* assumption on the velocity function f with respect to the state variable x. Nevertheless, the results presented in Sect. 6.3 are different from those obtained in Sects. 6.1 and 6.2 for dynamic optimization problems governed by nonsmooth evolution inclusions at least in the following major aspects:

—–there are *no* additional geometric assumptions of the state space in question, which is an *arbitrary Banach* space;

—–the objective and (equality and inequality) endpoint constraint functions may *not* be *locally Lipschitzian*, even *not continuous* around the reference point in the case of those functions describing the objective and inequality constraints, while the resulting necessary optimality conditions are obtained in the *conventional PMP form*, whenever the functions are Fréchet differentiable *at* the point in question, and in its *upper subdifferential* extension for special classes of nonsmooth functions.

In contrast to the approximation/perturbation methods employed in Sects. 6.1 and 6.2, we now rely on the more conventional *primal-space* approach that goes back to the classical proof of the Pontryagin maximum principle [124, 1102] with subsequent significant developments in the route paved by Rozonoér [1180] for finite-dimensional control systems. There are *two major ingredients* of the employed primal-space techniques, the traces of which could be found in McShane's paper [860] on the calculus of variations: the usage of *needle variations* and the employment of *convex separation*. Both of these ingredients were crucial in the original proof of the maximum principle [124, 1102], while their clarifications and important modifications came later starting—in different directions—with the papers by Rozonoér [1180] and Dubovitskii and Milyutin [369, 370]; see also other references and discussions in Subsects. 1.4.1 and 6.5.1.

In the proof of the maximum principle formulated in Theorem 6.37 we mainly follow the line initiated in the three-part paper by Rozonoér [1180], who was probably the first to fully recognize a major variational role of the *free-endpoint* "terminal control" (i.e., Mayer) problem in the maximum principle and to develop the so-called *increment method* in proving the PMP for problems of this type employing needle variations. Endpoint constraints were then treated as in finite-dimensional nonlinear programming by using *convex separation* techniques related to the so-called *image space analysis*; cf. Plotnikov [1083], Gabasov and Kirillova [485], and the recent book by Giannessi [504]. A delicate derivation of the transversality conditions for control problems with *equality* endpoint constraints given by merely differentiable functions was developed by Halkin [545] based on the Brouwer fixed-point theorem.

The *upper subdifferential* conditions of the PMP obtained in Theorem 6.38 seems to be new even for finite-dimensional control systems. The closest conditions were derived in the recent book by Cannarsa and Sinestrari [217, Theorem 7.3.1] for free-endpoint control problems in finite dimensions under more restrictive assumptions, while somewhat related results were established by

Mordukhovich and Shvartsman [955, 956] for discrete-time systems and discrete approximations; see Section 6.4. Note that Fréchet upper subgradients (or "supergradients") of the *value function* were used in optimal control for *synthesis* problems via Hamilton-Jacobi equations; see, e.g., Subbotina [1231], Zhou [1366], Cannarsa and Frankowska [216], Cannarsa and Sinestrari [217], Frankowska [472], and their references.

6.5.22. Multineedle Variations and Convex Separation in Image Spaces. In the proof of Theorem 6.37 given in Subsects. 6.3.2–6.3.4 we mainly develop the scheme implemented by Gabasov and Kirillova [485] for finite-dimensional control systems under substantially more restrictive assumptions. As mentioned, the basic idea of the proof for the *free-endpoint* problem in Subsect. 6.3.2 goes back to Rozonoér [1180], while needle variations of *measurable* controls via the increment formula are treated as in Mordukhovich [887, 901]. The reader can find more recent developments on needle variations including their usage for higher-order necessary optimality conditions in the publications by Agrachev and Sachkov [2], Bianchini and Kawski [114], Krener [703], Ledzewicz and Schättler [756], Sussmann [1236, 1238], and in the references therein.

The proof of Theorem 6.37 in the presence of *endpoint constraints* is significantly more involved in comparison with that for the free-endpoint problem. Now it requires taking into account the *geometry* of *reachable sets* for dynamic control systems. The usage of *multineedle variations* occurs to be crucial in the constraint framework. It allows us to construct a *convex tangential* approximation of the reachable set in the *image space*, the dimension of which is equal to the number of endpoint constraints plus one of the cost function. Thus, although the control problem under consideration involves the *infinite-dimensional* dynamics/state space, the proof of the maximum principle relies on the *finite-dimensional convex separation*.

Observe that *no* SNC-type property is involved in Sect. 6.3 to obtain the required *pointbased* results as in the general settings of Sects. 6.1 and 6.2. In fact, the latter is in accordance with the results obtained in the preceding sections, where we observed that the SNC property of the constraint/target set was actually *automatic* in the case of *finitely* many endpoint constraints. This phenomenon relates to the *finite codimension* property of the constraint set, which readily *yields* the sequential normal compactness *unavoidable* in infinite dimensions. Note also that, as one can see from the proofs in Subsects. 6.3.3 and 6.3.4, the *convexity* of the underlying approximation set in the *image space* was reached due to the *continuity* of the time interval; this is yet another manifestation of the *hidden convexity* inherent in continuous-time control systems.

6.5.23. The Discrete Maximum Principle. Section 6.4 again concerns optimal control problems with *discrete time* as well as *discrete approximations* of continuous-time systems. However, now our agenda is completely different from that in Sect. 6.1, where discrete approximations were mostly used as the *driving force* to derive necessary optimality conditions for differential inclusions, although the results obtained therein for *discrete inclusions* are certainly of independent interest. Recall that in Subsect. 6.1.4 we established necessary optimality conditions of the *Euler-Lagrange type* for general (nonconvex and non-Lipschitzian) discrete inclusions by reducing them to nonsmooth mathematical programming with many geometric constraints. When the "discrete velocity" sets $F_j(x)$ are *convex*, the results obtained automatically imply the *maximum-type* conditions by the extremal property of coderivatives for convex-valued mappings from Theorem 1.34, which is actually due to the extremal form of the normal cone to convex sets. It is clear from the general viewpoint of nonsmooth analysis that a certain *convexity* is undoubtedly *needed* for such extremal-type representations. On the other hand, the Pontryagin maximum principle and its nonsmooth extensions hold for *continuous-time* control systems with *no explicit convexity assumptions.* As seen from the results and discussions of Sects. 6.1–6.3, this is due to the *hidden convexity* strongly inherent in the continuous-time dynamics.

Considering optimal control problems for discrete systems with *fixed* step-sizes, we don't have grounds to expect such maximum-type results in the absence of some convexity. Nevertheless, the exact analog of the Pontryagin maximum principle for discrete control problems was first obtained by Rozonoér [1180], under the name of the *discrete maximum principle*, for minimizing a linear function of the right endpoint $x(K)$ without any constraints on $x(K)$ over the discrete-time system

$$\begin{cases} x(t+1) = Ax(t) + b\big(u(t), t\big), \quad x(0) = x_0 \ , \\ \\ u(t) \in U, \quad t = 0, \ldots, K-1 \ , \end{cases} \tag{6.130}$$

with *no* convexity assumptions imposed. The proof of this result was based on the increment formula over needle variations of the reference optimal control at one point $t = \theta$, similarly to the continuous-time case but without involving of course a (nonexistent) interval of "small length." The latter result and its proof given by Rozonoér heavily depended on the specific structure of system (6.130) while probably creating a false impression that the discrete maximum principle might be valid for general nonlinear systems, at least for free-endpoint problems. Note that doubts about such a possibility were clearly expressed in [1180].

A number of papers, mostly in the Western literature, and the book by Fan and Wang [426] were published with incorrect proofs "justifying" that of the discrete maximum principle was necessary for optimality. The first explicit (rather involved) example on violating the discrete maximum principle was given by Butkovsky [208]. Many other examples in this direction, much simpler than the one from [208], can be found in the book by Gabasov and Kirillova [486]; see also the references therein.

Example 6.46 is taken from Mordukhovich [901]. Note that it describes a class of discrete control systems, where the *global minimum* (instead of maximum) condition holds under certain relationships between the initial data. Other examples from [901] show that the discrete maximum principle can be violated even for systems of type (6.130), *linear* in *both* state and control variables, with a nonlinear minimizing function and a nonconvex control set U. In this way we get *counterexamples* to the *conjecture* by Gabasov and Kirillova [486, Commentary to Chap. 5] (repeated later by several authors) on the relationship between the *validity* of the *discrete maximum principle* in discrete-time systems with sufficiently small stepsizes and the *existence of optimal solutions* for continuous-time systems. More striking counterexamples in this direction, showing that the existence of optimal controls in continuous-time systems doesn't imply the fulfillment of even an *approximate* analog of the maximum principle for discrete approximations, are given in Subsects. 6.4.3 and 6.4.4.

The first correct result on the validity of the discrete maximum principle for nonlinear control systems of the type

$$\begin{cases} x(t+1) = f\big(x(t), u(t), t\big), \quad x(0) = x_0 \ , \\ u(t) \in U, \quad t = 0, \ldots, K-1 \ , \end{cases} \tag{6.131}$$

was probably due to Halkin [540] who established it under the *convexity* of the admissible "velocity sets" $f(x, U, t)$; see also the books by Cannon, Cullum and Polak [218], Boltyanskii [127], and Propoi [1105] for further results and discussions in this direction. On the other hand, Gabasov and Kirillova [486] and Mordukhovich [901] singled out special classes of nonlinear free-endpoint control problems for which the discrete maximum principle holds with *no* convexity assumptions. Furthermore, Mordukhovich's book [901] contains the so-called *individual conditions* for the fulfillment of the discrete maximum principle that allow us to describe relationships between the *initial data* of nonconvex systems ensuring either validity or violation of the discrete maximum principle. In particular, these conditions *comprehensively* treat the situation in Example 6.46: the discrete maximum principle holds therein *if and only if* $\gamma \leq 0$ and $\eta \geq 0$.

6.5.24. Necessary Conditions for Free-Endpoint Discrete Parametric Systems. The previous discussions clearly illustrate the *gap* between the Pontryagin maximum principle for continuous-time systems and its discrete-time counterpart in the classical framework of optimal control, even for free-endpoint problems. Besides the striking theoretical value of this phenomenon, it may have a serious *numerical* impact signifying a possible *instability* of the PMP under computing, which inevitably requires the time discretization. Observe however that computer calculations deal not with fixed-step discrete systems of type (6.131) but with parametric *discrete approximation* systems of the type

$$x(t+h) = x(t) + hf\big(x(t), u(t), t\big) \quad \text{as} \quad h \downarrow 0\,, \tag{6.132}$$

where the stepsize h is a discretization parameter. Thus it is natural to consider necessary optimality conditions for control problems involving parametric systems (6.132) that themselves *depend on the parameter* h.

The first result in this direction was obtained by Gabasov and Kirillova [484, 486] who derived, under the name of "quasimaximum principle," necessary optimality conditions for *free-endpoint parametric* control problems governed by general discrete-time systems of the type

$$x(t+1) = f\big(x(t), u(t), t, h\big), \quad x \in I\!R^n, \quad h \in I\!R^m\,,$$

imposing rather standard smoothness while *no convexity* assumptions. Their result asserts, for any given $\varepsilon > 0$, the fulfillment of a certain ε-*maximum* condition over a part of the control region that depends on ε and h. Being specified to the discrete approximation systems (6.132), the ε-maximum condition is as close to the one in the Pontryagin maximum principle as smaller ε and h are. Similar results were subsequently derived for discrete approximations of nonconvex free-endpoint control problems in the books by Moiseev [884, 885] and by Ermoliev, Gulenko and Tzarenko [407]; see the aforementioned books and also those by Propoi [1105] and Evtushenko [412] for various discussions and applications of such results to numerical methods in optimal control for continuous-time and discrete-time systems.

The proof of the quasimaximum principle and the related results for free-endpoint problems of discrete approximation given in [484, 486, 884, 885, 407] were similar to each other being, in fact, similar to Rozonoér's proof of the PMP for continuous-time systems with no constraints on trajectories; compare, e.g., the proof of Theorem 6.37 in the unconstrained case of Subsect. 6.3.2 with the one for Theorem 6.50 in the smooth unconstrained case of Subsect. 6.4.3. All these proofs strongly exploited the *unconstrained* nature of the control problems under consideration involving cost increment formulas on *single needle variations* of optimal controls. The only difference between the continuous-time and finite-difference cases concerned the usage of a small discretization stepsize in the parametric family of discrete-time problems *instead* of a small length of needle variations in continuous-time systems. These proofs didn't provide any hint of the possibility to obtain an appropriate counterpart of the PMP for discrete approximations of optimal control problems with endpoint constraints, where some *finite-difference* counterpart of the *hidden convexity* and the geometry of reachable sets must play a crucial role.

6.5.25. The Approximate Maximum Principle for Constrained Discrete Approximations. Necessary optimality conditions in the form of the *approximate maximum principle* (AMP) for optimal control problems of discrete approximation (6.132) with *smooth dynamics* and *smooth endpoint constraints* were first announced by Mordukhovich in [891] and then were developed in the subsequent publications [942, 899, 900, 901, 903]. The final version for smooth control problems presented in Theorem 6.59 was established in [901, 903]; see also [906]. The proof of this major theorem given in Subsect. 6.4.5 goes along the *primal-space* direction, being however significantly different in crucial aspects from its continuous-time counterpart considered in Subsects. 6.3.3 and 6.3.4. There are *three key assumptions* under which we justify the AMP in Theorem 6.59:

——the *consistence* of perturbations of the *equality* constraints;

——the *properness* of the sequence of optimal controls;

——the *smoothness* of the initial data with respect to the state variables.

Each of these assumptions occurs to be *essential* for the validity of the AMP in discrete approximations of *nonconvex constrained* problems as demonstrated by counterexamples of Subsect. 6.4.4.

The crucial role of *consistent perturbations* of endpoint constraints for achieving the *stability* of discrete approximations, from both viewpoints of the *value convergence* and the *validity of the AMP*, has been realized by Mordukhovich since the very beginning of his study; see [890, 891]. Example 6.61 showing that the AMP may be violated if the endpoint equality constraints are not appropriately perturbed (must decrease *slower* than the discretization stepsize) is taken from Mordukhovich and Raketskii [942]; see also [901, 903].

Example 6.60, which is taken from Mordukhovich and Shvartsman [956], demonstrates the significance of the *properness* property along the reference optimal control sequence for the validity of the AMP in constrained nonconvex problems. This property is specific for discrete approximations, although it may be viewed as some analog of the *piecewise continuity*, or generally *Lebesgue points* of measurable controls, that are not of any restriction for continuous-time systems. Note that we *don't need* to impose the properness assumption to ensure the AMP in free-endpoint problems; see Theorem 6.50 and its proof.

6.5.26. Nonsmooth Versions of the Approximate Maximum Principle. One of the most *striking* features of the approximate maximum principle is its *sensitivity to nonsmoothness*. This is probably *the only* result on optimality conditions and related topics of variational analysis we are familiar with that doesn't have any conventional *lower subdifferential* (regarding minimization) extension to nonsmooth (even *convex*) settings. This is demonstrated by examples from the paper of Mordukhovich and Shvartsman [956] presented in Subsect. 6.4.3 for free-endpoint control problems.

On the other hand, the afore-mentioned paper [956] justifies a new form of the approximate maximum principle involving *upper subdifferential transversality conditions* for *free-endpoint* problems with nonsmooth cost functions (Theorem 6.50) and for constrained problems whose *inequality-type* endpoint constraints are described by nonsmooth functions (Theorem 6.66). The results obtained in this direction apply to a special class of nonsmooth functions called *uniformly upper subdifferentiable* in [956]. This class contains, besides smooth and concave functions, also *semiconcave* functions (see Subsect. 5.5.4) being actually closely connected with a localized version of "weakly concave" functions in the sense of Nurminskii [1017] who efficiently used them in numerical optimization. Theorem 6.49 seems to be new in reflexive spaces; some of its conclusions and related properties were established in [956, 1017] with different proofs in finite dimensions.

Theorem 6.50 on the AMP for free-endpoint problems gives an *infinite-dimensional* extension of the upper subdifferential result from Mordukhovich and Shvartsman [956], which smooth version [901] is actually equivalent to the "quasimaximum principle" by Gabasov and Kirillova [484, 486] established under somewhat more restrictive assumptions.

Observe that the *free-endpoint version* of the AMP in Theorem 6.50 doesn't fully follow from the constrained versions of Subsect. 6.4.4 in both smooth and nonsmooth settings. Besides the infinite dimensionality and the *absence* of the properness property for free-endpoint problems, there are *error estimates* of the rate $\varepsilon(t, h_N) = O(h_N)$ for the maximum condition (6.85) in Corollaries 6.52 and 6.53 valid for smooth and concave cost functions in arbitrary Banach spaces.

6.5.27. Applications of the Approximate Maximum Principle. At the end of Subsect. 6.4.5 we present two *typical applications* of the approximate maximum principle. The first one, described in Remark 6.67, follows the route from the paper by Gabasov, Kirillova and Mordukhovich [488] to derive *suboptimality* conditions for continuous-time systems by using the value convergence and necessary optimality conditions for discrete approximations.

Secondly, we consider a more *practical application* of using the approximate maximum principle to solve optimal control problems governed by discrete-time systems with sufficiently *small stepsizes*. Example 6.68 taken from Mordukhovich [901] concerns a (simplified) practical problem of *chemical engineering* described in the book by Fan and Wang [426]. The discrete maximum principle cannot be applied to find optimal solutions to this constrained nonconvex problem, although the authors of [426] mistakenly did it throughout their book and related papers. On the other hand, the application of the approximate maximum principle justified in Theorem 6.59 allows us to find optimal controls.

Other applications of the AMP for constrained discrete approximation problems were developed by Nitka-Styczen [1013, 1014, 1015] who considered the framework of *optimal periodic control* involving *equality* endpoint constraints. Based on the AMP machinery, she designed efficient numerical methods of solving such problems and applied them to practical problems arising in optimization of chemical, biotechnological, and ecological processes. Some of the models considered in [1015] are described by hereditary/delay control systems that require certain modifications of the formulation of the AMP given in [1015] and in Subsect. 6.4.6 of this book.

6.5.28. The Approximate Maximum Principle in Systems with Delays. The results presented in Subsect. 6.4.6 are taken from the paper by Mordukhovich and Shvartsman [956], with their direct extension to delay systems in *infinite-dimensional* spaces. Considering for simplicity only free-endpoint problems, we derive the AMP with *upper subdifferential* transversality conditions for nonlinear systems with *time-delays in state* variables. The proof of this result for delay systems is based on their reduction, following the approach by Warga [1315], to ordinary discrete-time systems with possible *incommensurability* between the length of the underlying time interval $b - a$ and the discretization stepsize h_N.

The final Example 6.70 of Subsect. 6.4.6 draws the reader's attention to a very interesting class of hereditary systems, called functional-differential systems of *neutral type*, that are significantly different from ordinary control systems and their extensions systems with delays only in state variables. Such systems, admitting *time-delays in velocity* variables, are considered in more details in Sect. 7.1; see also Commentary to Chap. 7. Example 6.70, which is a finite-difference adaptation of the continuous-time example from the book by Gabasov and Kirillova [485, Section 3.6], shows that there is *no* natural analog of the AMP held for *smooth free-endpoint* control problems governed by *finite-difference systems of neutral type*.

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