Preface

In 1973 F. Black and M. Scholes published their pathbreaking paper [BS 73] on option pricing. The key idea — attributed to R. Merton in a footnote of the Black-Scholes paper — is the use of trading in continuous time and the notion of arbitrage. The simple and economically very convincing *“principle of no-arbitrage”* allows one to derive, in certain mathematical models of financial markets (such as the Samuelson model, [S 65], nowadays also referred to as the “Black-Scholes” model, based on geometric Brownian motion), unique prices for options and other contingent claims.

This remarkable achievement by F. Black, M. Scholes and R. Merton had a profound effect on financial markets and it shifted the paradigm of dealing with financial risks towards the use of quite sophisticated mathematical models.

It was in the late seventies that the central role of no-arbitrage arguments was crystallised in three seminal papers by M. Harrison, D. Kreps and S. Pliska ([HK 79], [HP 81], [K 81]) They considered a general framework, which allows a systematic study of different models of financial markets. The Black-Scholes model is just one, obviously very important, example embedded into the framework of a general theory. A basic insight of these papers was the intimate relation between no-arbitrage arguments on one hand, and martingale theory on the other hand. This relation is the theme of the *“Fundamental Theorem of Asset Pricing”* (this name was given by Ph. Dybvig and S. Ross [DR 87]), which is not just a single theorem but rather a general principle to relate no-arbitrage with martingale theory. Loosely speaking, it states that a mathematical model of a financial market is free of arbitrage if and only if it is a martingale under an equivalent probability measure; once this basic relation is established, one can quickly deduce precise information on the pricing and hedging of contingent claims such as options. In fact, the relation to martingale theory and stochastic integration opens the gates to the application of a powerful mathematical theory.

The mathematical challenge is to turn this general principle into precise theorems. This was first established by M. Harrison and S. Pliska in [HP 81] for the case of finite probability spaces. The typical example of a model based on a finite probability space is the "binomial" model, also known as the "Cox-Ross-Rubinstein" model in finance.

Clearly, the assumption of finite Ω is very restrictive and does not even apply to the very first examples of the theory, such as the Black-Scholes model or the much older model considered by L. Bachelier [B 00] in 1900, namely just Brownian motion. Hence the question of establishing theorems applying to more general situations than just finite probability spaces Ω remained open.

Starting with the work of D. Kreps [K 81], a long line of research of increasingly general — and mathematically rigorous — versions of the "Fundamental Theorem of Asset Pricing" was achieved in the past two decades. It turned out that this task was mathematically quite challenging and to the benefit of both theories which it links. As far as the financial aspect is concerned, it helped to develop a deeper understanding of the notions of arbitrage, trading strategies, etc., which turned out to be crucial for several applications, such as for the development of a dynamic duality theory of portfolio optimisation (compare, e.g., the survey paper [S 01a]). Furthermore, it also was fruitful for the purely mathematical aspects of stochastic integration theory, leading in the nineties to a renaissance of this theory, which had originally flourished in the sixties and seventies.

It would go beyond the framework of this preface to give an account of the many contributors to this development. We refer, e.g., to the papers [DS 94] and [DS 98], which are reprinted in Chapters 9 and 14.

In these two papers the present authors obtained a version of the "Fundamental Theorem of Asset Pricing", pertaining to general $\mathbb{R}^d$-valued semi-martingales. The arguments are quite technical. Many colleagues have asked us to provide a more accessible approach to these results as well as to several other of our related papers on Mathematical Finance, which are scattered through various journals. The idea for such a book already started in 1993 and 1994 when we visited the Department of Mathematics of Tokyo University and gave a series of lectures there.

Following the example of M. Yor [Y 01] and the advice of C. Byrne of Springer-Verlag, we finally decided to reprint updated versions of seven of our papers on Mathematical Finance, accompanied by a guided tour through the theory. This guided tour provides the background and the motivation for these research papers, hopefully making them more accessible to a broader audience.

The present book therefore is organised as follows. Part I contains the "guided tour" which is divided into eight chapters. In the introductory chapter we present, as we did before in a note in the Notices of the American Mathematical Society [DS 04], the theme of the Fundamental Theorem of Asset Pricing in a nutshell. This chapter is very informal and should serve mainly to build up some economic intuition.

In Chapter 2 we then start to present things in a mathematically rigourous way. In order to keep the technicalities as simple as possible we first restrict ourselves to the case of finite probability spaces Ω. This implies that all the function spaces $L^p(\Omega, \mathcal{F}, \mathbf{P})$ are finite-dimensional, thus reducing the functional analytic delicacies to simple linear algebra. In this chapter, which presents the theory of pricing and hedging of contingent claims in the framework of finite probability spaces, we follow closely the Saint Flour lectures given by the second author [S 03].

In Chapter 3 we still consider only finite probability spaces and develop the basic duality theory for the optimisation of dynamic portfolios. We deal with the cases of complete as well as incomplete markets and illustrate these results by applying them to the cases of the binomial as well as the trinomial model.

In Chapter 4 we give an overview of the two basic continuous-time models, the "Bachelier" and the "Black-Scholes" models. These topics are of course standard and may be found in many textbooks on Mathematical Finance. Nevertheless we hope that some of the material, e.g., the comparison of Bachelier versus Black-Scholes, based on the data used by L. Bachelier in 1900, will be of interest to the initiated reader as well.

Thus Chapters 1–4 give expositions of basic topics of Mathematical Finance and are kept at an elementary technical level. From Chapter 5 on, the level of technical sophistication has to increase rather steeply in order to build a bridge to the original research papers. We systematically study the setting of general probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$. We start by presenting, in Chapter 5, D. Kreps' version of the Fundamental Theorem of Asset Pricing involving the notion of "No Free Lunch". In Chapter 6 we apply this theory to prove the Fundamental Theorem of Asset Pricing for the case of finite, discrete time (but using a probability space that is not necessarily finite). This is the theme of the Dalang-Morton-Willinger theorem [DMW 90]. For dimension $d \geq 2$, its proof is surprisingly tricky and is sometimes called the "100 meter sprint" of Mathematical Finance, as many authors have elaborated on different proofs of this result. We deal with this topic quite extensively, considering several different proofs of this theorem. In particular, we present a proof based on the notion of "measurably parameterised subsequences" of a sequence $(f_n)_{n=1}^{\infty}$ of functions. This technique, due to Y. Kabanov and C. Stricker [KS 01], seems at present to provide the easiest approach to a proof of the Dalang-Morton-Willinger theorem.

In Chapter 7 we give a quick overview of stochastic integration. Because of the general nature of the models we draw attention to general stochastic integration theory and therefore include processes with jumps. However, a systematic development of stochastic integration theory is beyond the scope of the present "guided tour". We suppose (at least from Chapter 7 onwards) that the reader is sufficiently familiar with this theory as presented in several beautiful textbooks (e.g., [P 90], [RY 91], [RW 00]). Nevertheless, we do highlight those aspects that are particularly important for the applications to Finance.

Finally, in Chapter 8, we discuss the proof of the Fundamental Theorem of Asset Pricing in its version obtained in [DS 94] and [DS 98]. These papers are reprinted in Chapters 9 and 14.

The main goal of our "guided tour" is to build up some intuitive insight into the Mathematics of Arbitrage. We have refrained from a logically well-ordered deductive approach; rather we have tried to pass from examples and special situations to the general theory. We did so at the cost of occasionally being somewhat incoherent, for instance when applying the theory with a degree of generality that has not yet been formally developed. A typical example is the discussion of the Bachelier and Black-Scholes models in Chapter 4, which is introduced before the formal development of the continuous time theory. This approach corresponds to our experience that the human mind works inductively rather than by logical deduction. We decided therefore on several occasions, e.g., in the introductory chapter, to jump right into the subject in order to build up the motivation for the subsequent theory, which will be formally developed only in later chapters.

In Part II we reproduce updated versions of the following papers. We have corrected a number of typographical errors and two mathematical inaccuracies (indicated by footnotes) pointed out to us over the past years by several colleagues. Here is the list of the papers.

Chapter 9: [DS 94] A General Version of the Fundamental Theorem of Asset Pricing
Chapter 10: [DS 98a] A Simple Counter-Example to Several Problems in the Theory of Asset Pricing
Chapter 11: [DS 95b] The No-Arbitrage Property under a Change of Numéraire
Chapter 12: [DS 95a] The Existence of Absolutely Continuous Local Martingale Measures
Chapter 13: [DS 97] The Banach Space of Workable Contingent Claims in Arbitrage Theory
Chapter 14: [DS 98] The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes
Chapter 15: [DS 99] A Compactness Principle for Bounded Sequences of Martingales with Applications

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This book owes much to many: in particular, we are deeply indebted to our many friends in the functional analysis, the probability, as well as the mathematical finance communities, from whom we have learned and benefitted over the years.

Zurich, November 2005, *Freddy Delbaen*
Vienna, November 2005 *Walter Schachermayer*

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