LMI-based Gain Scheduled Controller Synthesis for a Class of Linear Parameter Varying Systems

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This paper presents a novel method for constructing controllers for a class of single-input multiple-output (SIMO) linear parameter varying (LPV) systems. This class of systems encompasses many physical systems, in particular systems where individual components vary with time, and is therefore of significant practical relevance to control designers. The control design presented in this paper has the properties that the system matrix of the closed loop is multi-affine in the various scalar parameters, and that the resulting controller ensures a certain degree of stability for the closed loop even when the parameters are varying, with the degree of stability related directly to a bound on the average rate of allowable parameter variations. Thus, if knowledge of the parameter variations is available, the conservativeness of the design can be kept at a minimum. The construction of the controller is formulated as a standard linear time-invariant (LTI) design combined with a set of linear matrix inequalities, which can be solved efficiently with software tools. The design procedure is illustrated by a numerical example.

1 Introduction

Many nonlinear systems can be formulated within the framework of linear parameter varying (LPV) systems and, as a consequence, LPV systems have received much attention in the control research community within the last two decades.

In general, LPV systems are systems that can be described by linear transfer functions or state space realizations, but where certain parameters in the description are non-constant. Obviously, such a description is quite general; it captures, for instance, many nonlinear systems that can be modeled using linearised dynamics, but

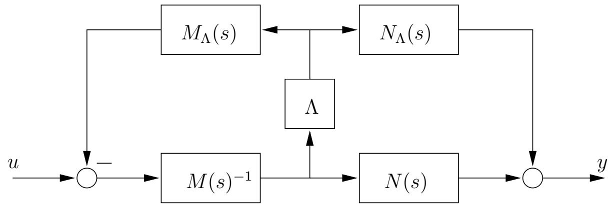

Fig. 1. *Perturbed plant represented using coprime factorizations.*

where the parameters vary as the operating point changes. Many examples of such systems can be found in the literature; [1] describes a method for formulating LPV models of propagation phenomena and shows two examples involving temperature control for a plate cooled by convection and mass flow rate regulation in a pneumatic network. [2] contains an example of gain scheduled $\mathcal{H}_\infty$ control design for the pitch axis of a (simplified) LPV model of a missile. [3] shows how to write the Van der Pol equation in LPV form; and so forth.

Due to the close relation to LTI plants, it is natural to want to use gain scheduling techniques to control LPV plants. Care must be exercised, however, when designing gain scheduled controllers that do not explicitly take into account the rate of parameter variations for LPV plants, since a gain scheduled closed-loop system can become unstable unless the scheduling happens "sufficiently slowly" [4]. It is thus desirable to take knowledge of the rate of parameter variations into account in the design, since otherwise one may be forced to introduce conservativeness by allowing arbitrarily fast variations. [5] provides a general survey of gain scheduling techniques and also lists references to various applications.

1.1 Systems of Interest

In this paper, we are interested in LPV plants where the parameter dependence can be written using a matrix fractional description, and where the parameter dependence terms are affine or in fact *multi-affine* in the parameters.

Specifically, we look at SIMO plants that permit a description of the form

$$P(\lambda, s) = (N(s) + N_\Lambda(s)\Lambda)(M(s) + M_\Lambda(s)\Lambda)^{-1} \tag{1}$$

where $N(s), N_\Lambda(s), M(s)$ and $M_\Lambda(s)$ are appropriately chosen stable transfer function matrices and $N(s), M(s)$ are coprime factors over $\mathcal{RH}_\infty$ of $P(0, s)$, and Λ is a vector multi-linear in some scalar λ_i, thus

$$\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_1\lambda_2, \ldots, \lambda_{r-1}\lambda_r, \ldots, \lambda_1\lambda_2\ldots\lambda_r]^T$$

with $\lambda = [\lambda_1, \ldots, \lambda_r]^T \in \lambda_{\text{box}} = \{-\bar{\lambda}_i \le \lambda_i \le \bar{\lambda}_i, i = 1, \ldots, r\}$. The term multi-affine allows products of different λ_i, but no powers. The vector λ describes the variation from a specific 'central' or 'nominal' value of the LPV system. This configuration is illustrated in Figure 1, from which it is clear that with $\lambda = 0$, the system is described by the 'nominal' plant $P(0, s) = N(s)M(s)^{-1}$.

Reference [6] pointed out that many physical linear systems with variations in individual physical parameters could be modelled in this way; essentially, the only restriction is that the varying components must not allow cross-couplings of energy storage. Examples of systems with multi-affine dependence on parameters thus include electric circuits where individual components such as resistors, capacitors and inductors vary, corresponding mechanical systems (masses, friction coefficients, inertias etc.), thermo-dynamical systems (fluid storage, valves etc.), and so forth[3].

Such systems often arise as a selection of linearizations of a nonlinear plant in various operating points, but may also appear from applications of system identification or from first-principles modeling (see e.g. [7]). Note also that parameter variations in physical systems may not be symmetrical about a central value of zero in the way implied by the definition of λ_{box} above; however, it is always possible, through a simple variable substitution, to rewrite the system such that the parameter variation becomes symmetrical.

We are going to embed the problem of designing controllers to stabilise the plant set of (1) within the problem of designing controllers to stabilise a larger set of plants. Denote the 2^r corners of λ_{box} by λ_{cor}. Thus $\lambda \in \lambda_{\text{cor}}$ if and only if $\lambda_i = \pm\bar{\lambda}_i$ for each i. Call the set of such λ values λ_{cori}, $i \in \{1, \ldots, 2^r\}$. Each corner value of λ gives rise to a plant $P(\lambda_{\text{cori}}, s)$. It is not hard to see, and indeed argued in [8], that for a generic value of Λ generated by a generic $\lambda \in \lambda_{box}$, there exists a set of nonnegative weights δ_i summing to 1 such that

$$N(s) + N_\Lambda(s)\Lambda = \sum_{i=1}^{2^r} \delta_i [N(s) + N_\Lambda(s)\Lambda_{cori}] \tag{2}$$

Indeed, we review this below, where it is also pointed out that the same set of δ_i yield

$$M(s) + M_\Lambda(s)\Lambda = \sum_{i=1}^{2^r} \delta_i [M(s) + M_\Lambda(s)\Lambda_{cori}] \tag{3}$$

[Here, Λ_{cori} denotes the value of Λ corresponding to λ_{cori}]. So we cover all plants of the set (1) by considering all plants of the form

$$\bar{P}(\delta, s) = \{\sum_{i=1}^{2^r} \delta_i [N(s) + N_\Lambda(s)\Lambda_{cori}]\}\{\sum_{i=1}^{2^r} \delta_i [M(s) + M_\Lambda(s)\Lambda_{cori}]\}^{-1}$$
$$\delta_i \geq 0 \quad \sum \delta_i = 1 \tag{4}$$

Call the set (1) $\mathcal{S}_\lambda$ and call the set just defined $\mathcal{S}_\delta$. There holds $\mathcal{S}_\lambda \subseteq \mathcal{S}_\delta$. We will actually explain how to design controllers for the set $\mathcal{S}_\delta$.

[3] Note that sometimes the entry corresponding to an individual λ_i may correspond to the inverse of a physical component value rather than the component value itself; this is of course not relevant in terms of stability etc., since e.g., the behavior of an electric circuit may be equally well described by a resistor's conductance as by its resistance.

The construction of the δ_i from a particular λ can be made independent of $N_\Lambda(s), M_\Lambda(s)$ etc, as indicated in the following argument, referred to in [9], which is a minor development of [8]. Let $g(\lambda_1, \lambda_2, \ldots, \lambda_r)$ be any multi-affine function of the λ_i, with $-1 \le \lambda_i \le 1$.Then successive use of the multi-affine property yields:

$$g(\lambda_1, \lambda_2, \ldots, \lambda_r) = (1/2)(1+\lambda_1)g(1, \lambda_2, \ldots, \lambda_r)+(1/2)(1-\lambda_1)g(-1, \lambda_2, \ldots, \lambda_r) \tag{5}$$

$$\begin{aligned} = (1/2)(1+\lambda_1)[(1/2(1+\lambda_2)g(1,1,\lambda_3,\ldots,\lambda_r) \\ + (1/2)(1-\lambda_2)g(1,-1,\lambda_3,\ldots,\lambda_r)] \\ + (1/2)(1-\lambda_1)[(1/2(1+\lambda_2)g(-1,1,\lambda_3,\ldots,\lambda_r) \\ + (1/2)(1-\lambda_2)g(-1,-1,\lambda_3,\ldots,\lambda_r)] \end{aligned}$$

$$\begin{aligned} = \sum_{i_j \in 1,2, j=1,\ldots,r} (1/2^r)[1+(-1)^{i_1}\lambda_1][1+(-1)^{i_2}\lambda_2] \\ \ldots [1+(-1)^{i_r}\lambda_r] g((-1)^{i_1}, (-1)^{i_2}, \ldots, (-1)^{i_r}) \end{aligned} \tag{6}$$

The δ_i are evidently the quantities $(1/2^r)[1+(-1)^{i_1}\lambda_1][1+(-1)^{i_2}\lambda_2] \ldots [1+(-1)^{i_r}\lambda_r]$. We shall describe this decomposition as the *standard convex representation*.

Now, in order to treat the controller synthesis problem and furthermore be able to handle time-varying parameter values without losing stability guarantees, we will appeal to two different (though somewhat related) approaches for dealing with LPV systems. The first approach relates directly to affine parameter dependencies, but does not consider time variation. This idea was developed in [9], which considered strict positive realness of families of transfer function matrices and [10], which provided a convex parameterization of all *fixed* controllers that simultaneously stabilise a parameter-dependent system for any permissible fixed values of the parameters. The second idea was developed in [11] and relates the rate of parameter variation for time varying systems to degree of stability. We will briefly review the latter ideas in the following section.

1.2 Robust Stability Results for LPV Systems

Reference [10] addressed the synthesis of robust controllers for linear time-invariant systems. The uncertain real parameters were assumed to appear linearly in the closed loop characteristic polynomial, and the parameters were assumed to lie in a bounded set, i.e., a description that matched the setup in (1). The paper proceeded to give a convex parameterization of all controllers that stabilise $P(\lambda, s)$ for *fixed frozen* values of λ. The stability proof, which was directly inspired by [9], relied on showing the equivalence between the stability of a certain transfer function involving the parameters in an affine way and the existence of a frequency-dependent (stable, positive-real) multiplier π such that the inverse of the transfer function multiplied by π has positive real part for all frequencies. If a π can be found, it is then possible to compute a controller that is independent of λ and that stabilises P for any value of λ. However, no stability guarantees were given if the parameters are allowed to vary.

In the method presented here, the controller is allowed to depend on the time varying parameters; in fact, some of the parameters in the controller will be tracking the parameters in the plant directly. This offers the prospect of better performance, or ability to deal with a larger uncertainty set.

A significant contribution of this paper is to give a theoretical justification for being able to tolerate time varying parameters in a setup somewhat similar to the one given in [10]. To this end, we will appeal to the second approach to dealing with systems that are affine in the parameters mentioned in the previous subsection.

Recall that an unforced linear time varying (LTV) system

$$\dot{\chi}(t) = \mathcal{A}(\lambda(t))\chi(t) \tag{7}$$

where $\chi \in \mathbb{R}^n$ is the state vector and $\lambda \in \lambda_{\text{box}} \subset \mathbb{R}^r$ is a parameter vector describing the variation in the system parameters at time t, is said to be *exponentially asymptotically stable with degree of stability* $\gamma > 0$ if there exist real scalars $c, \alpha > 0$ such that for all $\chi(t_0)$ and $t \geq t_0$, we have

$$\|\chi(t)\| e^{\gamma(t-t_0)} \leq c\|\chi(t_0)\| e^{-\alpha(t-t_0)}$$

Reference [11] dealt with systems of this form for which the existence can be demonstrated of a quadratic Lyapunov function $\chi^T P(\lambda)\chi$ with $P(\lambda) = P^T(\lambda)$ positive definite, multi-affine in λ, and such that

$$\mathcal{A}(\lambda)^T P(\lambda) + P(\lambda)\mathcal{A}(\lambda) < -2\sigma P(\lambda) \tag{8}$$

[Existence of such $P(\lambda)$ is demonstrated in [11] for the case where $\mathcal{A}(\lambda) = A + h(\lambda)g^T$ with $h(\lambda) \in \mathcal{R}^n$ affine in the elements of λ; we will however not appeal to this part of reference [11]]. Equation (8) ensures that for any frozen value of λ, the eigenvalues of $\mathcal{A}(\lambda)$ lie to the left of $Re[s] = -\sigma$. Propositions 5.1 and 5.2 of reference [11] proceed to quantify the stability of the system (7) in terms of the permissible rate of change of parameter variation.

Theorem 1. *For $\lambda \in \lambda_{box}$, let $\mathcal{A}(\lambda)$ be a set of $n \times n$ matrices such that there exists a positive definite symmetric $P(\lambda)$, multi-affine in λ, satisfying* (8) *for some $\sigma > 0$. Then the time-varying system* (7) *is exponentially asymptotically stable with degree of stability $\gamma \in [0, \sigma)$ if there exist $T > 0$ and $\epsilon_{1i}, \epsilon_{2i} > 0, i = 1, \ldots, r$ such that for all $i = 1, \ldots, r$ and all $t > 0$ we have*

$$\lambda_i(t) \in [-\bar{\lambda}_i + \epsilon_{1i}, \bar{\lambda}_i - \epsilon_{2i}] \tag{9}$$

and

$$\sup_{t\geq 0} \int_t^{t+T} \sum_{i=1}^{r} \left|\frac{d \ln \frac{\lambda_i(\tau)+\bar{\lambda}_i}{\bar{\lambda}_i - \lambda_i(\tau)}}{d\tau}\right| d\tau < 4(\sigma - \gamma) \tag{10}$$

Proof. See [11].

This theorem implies that if it is possible to synthesise controllers for an LPV plant such that the closed loop is σ-Hurwitz for all (frozen) values of λ, and an associated quadratic Lyapunov function can be found with multi-affine dependence on λ, then we will have stability guarantees for the time-varying closed-loop system provided known bounds on the average rate of variation of λ are observed. It is important to note here that (10) is a bound on the *average* rate of parameter variation, and that abrupt parameter variations such as steps are allowed, as long as they occur sufficiently infrequently.

1.3 Contribution of This Paper

The contribution of this paper is thus to provide a method for constructing controllers for such systems with the property that the $\mathcal{A}$-matrix of the closed loop has the crucial property set out in Theorem 1. The construction of these controllers is formulated by means of a set of linear matrix inequalities, which can be solved efficiently with readily available software tools. If these matrix inequalities are feasible, the resulting controller ensures a certain degree of stability for the closed loop with any allowed constant parameter vector, and we can then quantify the rate at which the parameter variations are allowed to occur if certain extra conditions are satisfied, and build this knowledge into the controller. This provides a potential for less conservative design than other LFT-based methods in the sense that the resulting gain scheduled controller does not have to take unreasonably fast parameter variations into account.

1.4 Notation

Throughout the paper, 0 and I indicate zero and identity elements of appropriate vector spaces (of compatible dimension). $\mathcal{RH}_\infty^{p\times m}$ denotes the space of all proper, real rational stable transfer matrices mapping m-dimensional input signals to p-dimensional output signals. A system $P(s)$ is σ-Hurwitz if $P(s-\sigma) \in \mathcal{RH}_\infty^{p\times m}$.

We will also use the notion of a Laguerre basis (see e.g., [12]). If a transfer function matrix $P(s)$ with m inputs and p outputs can be written as

$$P(s) = \breve{P} B_\tau^k(s) \tag{11}$$

where

$$\breve{P} = \begin{bmatrix} \breve{P}_0 \; \breve{P}_1 \; \cdots \; \breve{P}_k \end{bmatrix} \quad \in \mathbb{R}^{p\times(k+1)m},$$
$$B_\tau^k(j\omega) = \left[I_m \; \left(\frac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right) I_m \; \left(\frac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right)^2 I_m \; \cdots \; \left(\frac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right)^k I_m \right]^T,$$

for some $\tau \in \mathbb{R}_+$ and $k \in \mathbb{Z}_+$, then $P(s)$ obviously belongs to $\mathcal{RH}_\infty^{p\times m}$. Finally, $\mathrm{Co}\{\cdot\}$ denotes the convex hull of $(\cdot)$.

1.5 Outline of paper

The outline of the rest of this paper is as follows. In the following section we provide an overview of the systems we are considering and discuss how to handle the parameter dependencies in a matrix fractional description of the plant and controller. Then, in Section 3, we formulate a finite set of linear matrix inequalities, which, if feasible, provides a controller that not only stabilises the plant under consideration for all possible parameter values, but also preserves the appropriate parameter dependency for the closed loop characteristic polynomial allowing application of Theorem 1. In Section 4 we provide an explicit implementation of our proposed design procedure. Finally, we present a numerical example illustrating the method in Section 5 and make some closing remarks in Section 6.

2 LPV Framework

As stated in the introduction, we will employ a matrix fractional description of the plant with parameter variations and demonstrate the existence of a set of synthesis LMIs which, if feasible, will provide a controller that stabilises the LPV system for all permissible values of λ, and which allows λ to vary with time. As will be shown, the controller constructed in this way depends on λ and the closed-loop characteristic polynomial of the controller-plant interconnection will have multi-affine dependency on λ.

2.1 Plant and Controller fractional descriptions

Let P be a SIMO plant of the form (1), i.e., mapping a single input signal to $p \geq 1$ output signals; we assume the parameter dependence allows us to find stable transfer functions $N(s) \in \mathcal{RH}_\infty^{p\times 1}, N_\Lambda(s) \in \mathcal{RH}_\infty^{p\times l}, M(s) \in \mathcal{RH}_\infty^{1\times 1}$ and $M_\Lambda(s) \in \mathcal{RH}_\infty^{1\times l}$ such that

$$P(\lambda, s) = (N(s) + N_\Lambda(s)\Lambda)(M(s) + M_\Lambda(s)\Lambda)^{-1} \tag{12}$$

where

$$\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_1\lambda_2, \ldots, \lambda_{r-1}\lambda_r, \ldots, \lambda_1\lambda_2\ldots\lambda_r]^T$$

with $\lambda = [\lambda_1, \ldots, \lambda_r]^T \in \lambda_{\text{box}} = \{-\bar{\lambda}_i \leq \lambda_i \leq \bar{\lambda}_i, i = 1, \ldots, r\}$. One can check that $l = 2^r - 1$. This configuration is illustrated in Figure 1, from which it is clear that with $\lambda = 0$ we have the nominal LTI plant $P(0, s) = N(s)M(s)^{-1}$.

Specifically, $N(s)$ and $M(s)$ are chosen to be right coprime factorizations of $P(0, s)$, while $N_\Lambda(s)$ and $M_\Lambda(s)$ are chosen such that $N(s) + N_\Lambda(s)\Lambda$ and $M(s) + M_\Lambda(s)\Lambda$ represent a right coprime factorization of $P(\lambda, s)$ over $\mathcal{RH}_\infty$ for almost all $\lambda \in \lambda_{\text{box}}$. There is a nontrivial assumption buried here. Note though that a continuity argument shows that the coprimeness existing at $\lambda = 0$ persists for sufficiently small values of λ. So the assumption is harmless in a suitably small region. Also, in case $p > 1$, one could argue generically that the zeros of the scalar denominator are never

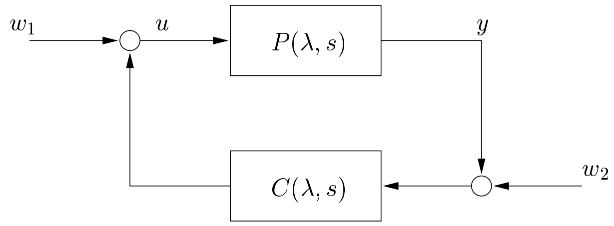

Fig. 2. *Closed loop of plant and controller.*

likely to be simultaneously zeros of all entries of the vector numerator in (12), even if as λ varies, these zeros may move backwards and forwards along the real axis in the complex plane.

We shall further assume that N_Λ and M_Λ are strictly proper; if the form of $P(\lambda, s)$ arises as described in Section 1.1 or reference [6], this will occur. Using this matrix fractional description, we define the following transfer function

$$G(\lambda, s) = \begin{bmatrix} N(s) + N_\Lambda(s)\Lambda \\ M(s) + M_\Lambda(s)\Lambda \end{bmatrix} \quad \in \mathcal{RH}_\infty^{(p+1)\times 1}. \tag{13}$$

We now postulate a controller of basically the same structure as the plant, i.e.,

$$C(\lambda, s) = (\tilde{V}(s) + \Lambda^T \tilde{V}_\Lambda(s))^{-1}(\tilde{U}(s) + \Lambda^T \tilde{U}_\Lambda(s)) \tag{14}$$

where $\tilde{U}(s) \in \mathcal{RH}_\infty^{1\times p}$, $\tilde{V}(s) \in \mathcal{RH}_\infty^{1\times 1}$, $\tilde{U}_\Lambda(s) \in \mathcal{RH}_\infty^{l\times p}$ and $\tilde{V}_\Lambda(s) \in \mathcal{RH}_\infty^{l\times 1}$ are chosen such that $\tilde{U}(s), \tilde{V}(s)$ are left coprime factors over $\mathcal{RH}_\infty$ of the nominal controller $C(0, s)$ and $\tilde{U}(s) + \Lambda^T \tilde{U}_\Lambda(s)$ and $\tilde{V}(s) + \Lambda^T \tilde{V}_\Lambda(s)$ represent a left coprime factorization of $C(\lambda, s)$ over $\mathcal{RH}_\infty$ for almost all $\lambda \in \lambda_{box}$. In fact, $C(0, s) = \tilde{V}(s)^{-1}\tilde{U}(s)$ should be designed (using any standard method) to stabilise the nominal plant $P(0, s) = N(s)M(s)^{-1}$ and indeed achieve other specified performance objectives. $C(0, s)$ might for instance be designed to provide certain stability margins etc.

Analogously with G, we introduce the transfer function

$$\tilde{K}(\lambda, s) = \begin{bmatrix} -\tilde{U}(s) - \Lambda^T \tilde{U}_\Lambda(s) & \tilde{V}(s) + \Lambda^T \tilde{V}_\Lambda(s) \end{bmatrix} \quad \in \mathcal{RH}_\infty^{1\times (p+1)} \tag{15}$$

corresponding to the controller $C(\lambda, s)$.

2.2 Internal Stability and Multi-affine Parameter Dependence

The controller is placed in closed loop with the plant as depicted in Figure 2. The internal stability of the closed loop can then be determined by examining the collection of transfer functions[4]

[4] Note that, actually, $H(P, C)$ is the transfer function from $[-w_2, w_1]^T$ to $[y, u]^T$.

$$\begin{aligned} H(P,C) &= \begin{bmatrix} P(\lambda,s) \\ I \end{bmatrix} (I - C(\lambda,s)P(\lambda,s))^{-1} \begin{bmatrix} -C(\lambda,s) & I \end{bmatrix} \\ &= G(\tilde{K}G)^{-1}\tilde{K} \end{aligned} \tag{16}$$

Inserting the plant and controller factorizations in this expression yields

$$\begin{aligned} H(P,C) &= G\left(\begin{bmatrix} -\tilde{U} - \Lambda^T\tilde{U}_\Lambda & \tilde{V} + \Lambda^T\tilde{V}_\Lambda \end{bmatrix} \begin{bmatrix} N + N_\Lambda\Lambda \\ M + M_\Lambda\Lambda \end{bmatrix} \right)^{-1} \tilde{K} \\ &= G\Big((\tilde{V}M - \tilde{U}N) + \Lambda^T(\tilde{V}_\Lambda M - \tilde{U}_\Lambda N) + (\tilde{V}M_\Lambda - \tilde{U}N_\Lambda)\Lambda \\ &\quad + \Lambda^T(\tilde{V}_\Lambda M_\Lambda - \tilde{U}_\Lambda N_\Lambda)\Lambda \Big)^{-1} \tilde{K} \end{aligned} \tag{17}$$

where the dependencies on s have been suppressed in the notation for the sake of brevity.

Note that the inverted term in the above expression is a scalar function of the entries of Λ and thus of λ; this term (without the inverse) plays the role of the characteristic polynomial of the closed loop transfer matrix of the controlled system including the λ-dependence. Since both G and $\tilde{K}$ are stable transfer functions, the zeros of this "characteristic polynomial" determine the stability of the closed loop, as any unstable poles must occur here. In order to yield the desired multi-affine dependence on λ of the closed-loop characteristic polynomial, this scalar denominator must be affine in Λ (cf. [6]), which is equivalent to enforcing $\tilde{V}_\Lambda(s)M_\Lambda(s) - \tilde{U}_\Lambda(s)N_\Lambda(s)$ to be skew-symmetric. Since $N_\Lambda(s)$ and $M_\Lambda(s)$ are given from the plant description, it is clear that we must choose $\tilde{U}_\Lambda(s)$ and $\tilde{V}_\Lambda(s)$ to achieve this.

Towards this end, let

$$F = \begin{bmatrix} N_\Lambda \\ M_\Lambda \end{bmatrix} \in \mathcal{RH}_\infty^{(p+1)\times l} \tag{18}$$

and suppose the normal rank of F is ρ. When $\rho < p+1$, F has a nontrivial left kernel of dimension $q = p + 1 - \rho$. We pick a basis for this kernel $F^\perp \in \mathcal{RH}_\infty^{q\times(p+1)}$ that has full row normal rank and satisfies $F^\perp(s)F(s) = 0\ \forall s \in \mathbb{C}$ (such an $F^\perp$ always exists when $\rho < p + 1$ and this is easily seen via a McMillan decomposition of F). We shall further require that $F^\perp$ is strictly proper. This is a mild restriction, but appears necessary for the results to follow. The case $\rho < p+1$ is guaranteed to arise when $l < p + 1$, a situation which corresponds to having at least as many outputs as scalar parameters. The case may arise even if $l \geq p + 1$, but it is not generic. If $p = 1$, i.e. we are working with SISO systems; then $\rho = 1$ when there is one scalar parameter only. When $\rho = p + 1$, there is of course no nontrivial left kernel for F; for consistency of notation however, we let $q = 1$ and set $F^\perp = 0 \in \mathcal{RH}_\infty^{q\times(p+1)}$. Then a simple choice

$$\begin{bmatrix} -\tilde{U}_\Lambda(s) & \tilde{V}_\Lambda(s) \end{bmatrix} = \alpha(s)F^\perp(s) + F^T(s)\beta(s) \tag{19}$$

with an arbitrary $\alpha \in \mathcal{RH}_\infty^{l\times q}$ and an arbitrary $\beta \in \mathcal{RH}_\infty^{(p+1)\times(p+1)}$ that satisfies $\beta = -\beta^T$ (i.e. is skew-symmetric) does the job, since $\tilde{V}_\Lambda(s)M_\Lambda(s) - \tilde{U}_\Lambda(s)N_\Lambda(s)$ is clearly skew-symmetric and hence $\Lambda^T(\tilde{V}_\Lambda(s)M_\Lambda(s) - \tilde{U}_\Lambda(s)N_\Lambda(s))\Lambda = 0 \; \forall s \in \mathbb{C}$. Moreover, some lines of linear algebra shows that (19) includes all possible choices which ensure the quadratic form is zero. Notice for future reference that $\tilde{U}_\Lambda(s)$ and $\tilde{V}_\Lambda(s)$ are necessarily strictly proper.

For notational convenience, we define

$$\Phi(s) = \tilde{V}(s)M_\Lambda(s) - \tilde{U}(s)N_\Lambda(s) \tag{20}$$

Inserting this expression in equation (17), we arrive at

$$H(P,C) = G(\lambda,s)\Big[[\tilde{V}(s)M(s) - \tilde{U}(s)N(s)] + \Phi(s)\Lambda + \Lambda^T[\alpha(s)F^\perp(s) + F^T(s)\beta(s)]G(0,s)\Big]^{-1}\tilde{K}(\lambda,s).$$

Note that there is no loss of generality in forcing the nominal controller coprime factors $\tilde{V}$ and $\tilde{U}$ to satisfy the Bezout identity $\tilde{V}(s)M(s) - \tilde{U}(s)N(s) = 1$, as can be seen from the following argument. Consider $\tilde{V}(s)M(s) - \tilde{U}(s)N(s) = Z(s) \neq 1$, where $Z(s), Z(s)^{-1} \in \mathcal{RH}_\infty$ since the nominal controller is assumed to stabilise the nominal plant. Then it will be possible to extract a common factor $Z(s)^{-1}$ on the right of $(\tilde{K}G)^{-1}$, which cancels the common factor $Z(s)$ that can be extracted on the left of $\tilde{K}(\lambda,s)$.

3 Controller Synthesis

We are now ready to state our main results. We first present a controller synthesis result, and then show that the corresponding controller-plant interconnection has an appropriate multi-affine parameter dependence that allows us to invoke Theorem 1 in order to deal with time variations.

3.1 Parameter-dependent controller construction

The synthesis result presented in the following theorem relates the existence of a parameter-dependent controller to the feasibility of a set of linear matrix inequalities.

Theorem 2. *Consider the plant $P(\lambda,s)$ given in (12) where*

$$\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_1\lambda_2, \ldots, \lambda_{r-1}\lambda_r, \ldots, \lambda_1\lambda_2\ldots\lambda_r]^T, l = 2^r - 1,$$

and the parameter vector λ belongs to the polytope $\lambda_{\text{box}} = \{\lambda \in \mathbb{R}^r : -\bar{\lambda}_i \leq \lambda_i \leq \bar{\lambda}_i, i = 1,\ldots,r\}$, and assume that the fractional description in (12) is coprime for all $\lambda \in \lambda_{box}$, with $N_\Lambda(s)$ and $M_\Lambda(s)$ strictly proper. Assume furthermore that a nominal LTI controller $C(0,s)$ that stabilises $P(0,s)$ has been found and let $C(0,s) = \tilde{V}^{-1}(s)\tilde{U}(s)$ be a left coprime factorisation over $\mathcal{RH}_\infty$ such that $\tilde{U}, \tilde{V}$ satisfy the Bezout identity $\tilde{V}(s)M(s) - \tilde{U}(s)N(s) = 1$.

Then there exists a controller $C(\lambda, s)$ of the form (14), where $\tilde{U}_\Lambda(s)$ and $\tilde{V}_\Lambda(s)$ are chosen in accordance with (18), (19) and the associated remarks, that stabilises $P(\lambda, s)$ for all (frozen) $\lambda \in \lambda_{box}$ if there exist integers $N_1, N_2, N_3 \in \mathbb{Z}_+$, constants $\tau_1, \tau_2, \tau_3 \in \mathbb{R}_+$ and matrices $\breve{\Pi} \in \mathbb{R}^{1\times(N_1+1)}, \breve{\Gamma} \in \mathbb{R}^{l\times q(N_2+1)}, \breve{\Xi}_i = -\breve{\Xi}_i^T \in \mathbb{R}^{(p+1)\times(p+1)}$ for $i \in \{0, 1, \ldots, N_3\}$ that satisfy

$$\begin{aligned} Re\Big\{ &[1 + \Phi(j\omega)\Lambda]\breve{\Pi} B_{\tau_1}^{N_1}(j\omega) + \Lambda^T[\breve{\Gamma} B_{\tau_2}^{N_2}(j\omega) F^\perp(j\omega) \\ &+ F^T(j\omega) \left[\breve{\Xi}_0 \; \breve{\Xi}_1 \; \ldots \; \breve{\Xi}_{N_3}\right] B_{\tau_3}^{N_3}(j\omega)]G(0, j\omega)\Big\} > 0 \end{aligned} \quad (21)$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$ and all $\Lambda = \Lambda_{\text{cori}}$, where Λ_{cori} denotes the value assumed by Λ when $\lambda \in \lambda_{\text{cori}}$, i.e. λ assumes a value at a corner of λ_{box}, and where $G, F, F^\perp, \Phi$ are all defined in Section 2.

Proof. We shall first establish that (21) is a necessary and sufficient condition for the stability of a closed loop formed from one of the set of plants $\mathcal{S}_\delta$, and an associated controller. The set $\mathcal{S}_\lambda$ of plants (12) is a subset of $\mathcal{S}_\delta$, and thus establishing the claim for $\mathcal{S}_\delta$ will yield the result for $\mathcal{S}_\lambda$. To establish the claim in relation to $\mathcal{S}_\delta$, we shall link a series of equivalent statements.

Consider the plant set introduced in Section 1 defined by

$$\bar{P}(\delta, s) = \{\sum_{i=1}^{2^r} \delta_i[N(s) + N_\Lambda(s)\Lambda_{cori}]\}\{\sum_{i=1}^{2^r} \delta_i[M(s) + M_\Lambda(s)\Lambda_{cori}]\}^{-1}$$
$$\delta_i \geq 0 \quad \sum \delta_i = 1 \quad (22)$$

Consider also a corresponding set of controllers:

$$\bar{C}(\delta, s) = \{\sum_{i=1}^{2^r} \delta_i[\tilde{V}(s) + \Lambda_{cori}^T \tilde{V}_\Lambda(s)]\}^{-1}\{\sum_{i=1}^{2^r} \delta_i[\tilde{U}(s) + \Lambda_{cori}^T \tilde{V}_\Lambda(s)]\}$$
$$\delta_i \geq 0 \quad \sum \delta_i = 1 \quad (23)$$

where $[-\tilde{U}_\Lambda(s)\ \tilde{V}_\Lambda(s)]$ is chosen as described in the Theorem hypothesis.

We assert now the equivalence of the following statements.

(a) there exists a controller $\bar{C}(\delta, s)$ of the form (23), where $\tilde{U}_\Lambda(s)$ and $\tilde{V}_\Lambda(s)$ are chosen in accordance with (18), (19) and the associated remarks, that stabilises $\bar{P}(\delta, s)$ in (22) for all (frozen) δ_i with $\delta_i \geq 0$ and $\sum \delta_i = 1$.

(b) $\exists \alpha \in \mathcal{RH}_\infty^{l\times q}$ and $\beta \in \mathcal{RH}_\infty^{(p+1)\times(p+1)}$ that satisfies $\beta = -\beta^T$ such that

$$H(\bar{P},\bar{C}) = G(\lambda,s)\Big[1+\Phi(s)\left(\sum_{i=1}^{2^r}\delta_i\Lambda_{\text{cori}}\right)+ \left(\sum_{i=1}^{2^r}\delta_i\Lambda_{\text{cori}}^T\right)[\alpha(s)F^{\perp}(s)+F^T(s)\beta(s)]G(0,s)\Big]^{-1}\tilde{K}(\lambda,s)\in\mathcal{RH}_\infty \quad \forall\delta_i\geq 0, \sum\delta_i=1,$$

where $q, G, \tilde{K}, F, F^{\perp}, \Phi$ are all defined in Section 2. This is merely a restatement of (a) above and the algebra for massaging $H(P,C)$ into the form given here was discussed in Section 2.

(c) $\exists\alpha\in\mathcal{RH}_\infty^{l\times q}$ and $\beta\in\mathcal{RH}_\infty^{(p+1)\times(p+1)}$ that satisfies $\beta=-\beta^T$ such that

$$\Big[1+\Phi(s)\left(\sum_{i=1}^{2^r}\delta_i\Lambda_{\text{cori}}\right)+\left(\sum_{i=1}^{2^r}\delta_i\Lambda_{\text{cori}}^T\right)[\alpha(s)F^{\perp}(s)+F^T(s)\beta(s)]G(0,s)\Big]^{-1} \in\mathcal{RH}_\infty \quad \forall\delta_i\geq 0, \sum\delta_i=1.$$

Sufficiency is obvious since $G(\lambda,s), \tilde{K}(\lambda,s)\in\mathcal{RH}_\infty$. Necessity follows via the following argument: Since coprimeness of the fractional descriptions[5] of $P(\lambda,s)$ and $C(\lambda,s)$ means that $G(\lambda,s)$ and $\tilde{K}(\lambda,s)$ have full column and full row rank respectively at all $s\in\bar{\mathbb{C}}_+$, then there exist left and right stable inverses for $G(\lambda,s)$ and $\tilde{K}(\lambda,s)$ respectively.

(d) $\exists\pi\in\mathcal{RH}_\infty^{1\times 1}, \alpha\in\mathcal{RH}_\infty^{l\times q}$ and $\beta\in\mathcal{RH}_\infty^{(p+1)\times(p+1)}$ that satisfies $\beta=-\beta^T$ such that

$$\text{Re}\bigg\{\Big[1+\Phi(j\omega)\left(\sum_{i=1}^{2^r}\delta_i\Lambda_{\text{cori}}\right) + \left(\sum_{i=1}^{2^r}\delta_i\Lambda_{\text{cori}}^T\right)[\alpha(j\omega)F^{\perp}(j\omega)+F^T(j\omega)\beta(j\omega)]G(0,j\omega)\Big]\pi(j\omega)\bigg\}>0 \quad \forall\omega\in\mathbb{R}\cup\{\infty\}, \lambda\in\Lambda.$$

This equivalence is via [9].

(e) $\exists\pi\in\mathcal{RH}_\infty^{1\times 1}, \alpha\in\mathcal{RH}_\infty^{l\times q}$ and $\beta\in\mathcal{RH}_\infty^{(p+1)\times(p+1)}$ that satisfies $\beta=-\beta^T$ such that

[5] Notice that if the fractional description of the plant $P(\lambda,s)$ (resp. the controller $C(\lambda,s)$) is not coprime for some $\hat{\lambda}\in\Lambda$, then $G(\lambda,s)$ (resp. $\tilde{K}(\lambda,s)$) will drop in rank for some $s_0\in\bar{\mathbb{C}}_+$ and for $\hat{\lambda}$, and the interconnection of plant $P(\lambda,s)$ and controller $C(\lambda,s)$ will not be internally stable. Thus the presumption that the controller $C(\lambda,s)$ stabilises $P(\lambda,s)$ automatically requires coprimeness of the fractional description for all λ of interest.

$$\mathrm{Re}\Big\{\Big[1+\Phi(j\omega)\Lambda_{\text{cori}} + \Lambda_{\text{cori}}^T[\alpha(j\omega)F^{\perp}(j\omega)+F^T(j\omega)\beta(j\omega)]G(0,j\omega)\Big]\pi(j\omega)\Big\}>0$$
$$\forall\omega\in\mathbb{R}\cup\{\infty\}, i=1,\ldots,2^r,$$

Necessity is obvious since if condition (d) holds for all $\delta_i \geq 0, \sum \delta_i = 1$, then it is also satisfied for the corner points. On the other hand, sufficiency follows by taking a positive linear combination of the inequalities of condition (e).

(f) $\exists \pi \in \mathcal{RH}_\infty^{1\times 1}, \gamma \in \mathcal{RH}_\infty^{l\times q}$ and $\xi \in \mathcal{RH}_\infty^{(p+1)\times(p+1)}$ that satisfies $\xi = -\xi^T$ such that

$$\mathrm{Re}\Big\{[1+\Phi(j\omega)\Lambda_{\text{cori}}]\pi(j\omega) + \Lambda_{\text{cori}}^T[\gamma(j\omega)F^{\perp}(j\omega)+F^T(j\omega)\xi(j\omega)]G(0,j\omega)\Big\}>0$$
$$\forall\omega\in\mathbb{R}\cup\{\infty\}, i=1,\ldots,2^r.$$

Necessity is easy via a relabling of $\gamma = \alpha\pi$ and $\xi = \beta\pi$. Sufficiency follows via the following argument: Summing up the 2^r inequalities above yields

$$2^r\mathrm{Re}\{\pi(j\omega)\}>0 \quad \forall\omega\in\mathbb{R}\cup\{\infty\}$$

since all the other terms disappear due to the symmetry about zero of the polytope considered. Consequently, π is stable and strictly positive real, which implies it must be a unit in $\mathcal{RH}_\infty^{1\times 1}$. Hence, given any $\gamma \in \mathcal{RH}_\infty^{l\times q}$ and any $\xi \in \mathcal{RH}_\infty^{(p+1)\times(p+1)}$ that satisfies $\xi = -\xi^T$, it is possible to construct $\alpha = \gamma/\pi \in \mathcal{RH}_\infty^{l\times q}$ and $\beta = \xi/\pi \in \mathcal{RH}_\infty^{(p+1)\times(p+1)}$ that satisfies $\beta = -\beta^T$.

(g) there exist integers $N_1, N_2, N_3 \in \mathbb{Z}_+$, constants $\tau_1, \tau_2, \tau_3 \in \mathbb{R}_+$ and matrices $\breve{\Pi} \in \mathbb{R}^{1\times(N_1+1)}, \breve{\Gamma} \in \mathbb{R}^{l\times q(N_2+1)}, \breve{\Xi}_i = -\breve{\Xi}_i^T \in \mathbb{R}^{(p+1)\times(p+1)}$ for $i \in \{0,1,\ldots,N_3\}$ that satisfy

$$\mathrm{Re}\Big\{[1+\Phi(j\omega)\Lambda]\breve{\Pi}B_{\tau_1}^{N_1}(j\omega) + \Lambda^T[\breve{\Gamma}B_{\tau_2}^{N_2}(j\omega)F^{\perp}(j\omega) + F^T(j\omega)\left[\breve{\Xi}_0\ \breve{\Xi}_1\ \ldots\ \breve{\Xi}_{N_3}\right]B_{\tau_3}^{N_3}(j\omega)]G(0,j\omega)\Big\}>0$$

for all $\omega \in \mathbb{R}\cup\{\infty\}$ and all $\Lambda = \Lambda_{\text{cori}}$. Here, $B_{\tau_i}^{N_i}(s)$ ($i \in \{1,2,3\}$) denotes Laguerre basis matrices as defined in (11). The index m in (11) takes the values 1, q and $p+1$ respectively.
Sufficiency is trivially easy via a labelling $\pi = \breve{\Pi}B_{\tau_1}^{N_1}(s) \in \mathcal{RH}_\infty^{1\times 1}$, $\gamma = \breve{\Gamma}B_{\tau_2}^{N_2}(s) \in \mathcal{RH}_\infty^{l\times q}$ and $\xi = \left[\breve{\Xi}_0\ \breve{\Xi}_1\ \ldots\ \breve{\Xi}_{N_3}\right]B_{\tau_3}^{N_3}(s) \in \mathcal{RH}_\infty^{(p+1)\times(p+1)}$ and noting that the constructed ξ also satisfies $\xi = -\xi^T$ since $\breve{\Xi}_i$ is skew-symmetric for all $i \in \{0,1,\ldots,N_3\}$. Necessity on the other hand follows via the following argument: Since Laguerre parametrisations provide a uniform

approximation of objects in $\mathcal{RH}_\infty$ and furthermore since the inequalities in (f) are strict inequalities, it follows that there always exists sufficiently large N_i and sufficiently small τ_i ($i \in \{1,2,3\}$) to ensure that the truncation error does not alter the sign of the inequalities.

At this point, we have a necessary and sufficient condition for the controllers (23) to stabilise the corresponding plant in the set $\mathcal{S}_\delta$, see (22). Now recall from Section 1 that for any Λ generated by a generic $\lambda \in \lambda_{box}$, there exists a set of nonnegative weights δ_i summing to 1 such that

$$N(s) + N_\Lambda(s)\Lambda = \sum_{i=1}^{2^r} \delta_i [N(s) + N_\Lambda(s)\Lambda_{cori}] \tag{24}$$

This is the standard convex representation and there are three similar equations for $M(s)+M_\Lambda(s)\Lambda$ and the two fractional components of the controller. Recall also that the injective mapping from Λ to the δ_i is independent of $N_\Lambda(s)$ etc. It follows that the set of plants $P(\lambda, s)$ and controllers $C(\lambda, s),\ \lambda \in \lambda_{box}$, are respectively subsets of the set of plants $\bar{P}(\delta, s)$ and controllers $\bar{C}(\delta, s)$. Hence condition (g) is a sufficient condition guaranteeing the $C(\lambda, s)$ stabilise the corresponding $P(\lambda, s)$.

3.2 Incorporating a degree of stability in the frozen parameter design

As pointed out in the Introduction, Theorem 1 relates a certain degree of stability of a collection of frozen parameter systems to the degree of stability of a time-variable parameter system given a bound on the average rate of change of the parameters. In order to allow a reasonable amount of time-variation in the parameter λ, we have to ensure that the closed loop with frozen parameter values has a degree of stability of σ and that this property is expressible with a particular Lyapunov function. We shall deal with the Lyapunov function in the next subsection, and the degree of stability issue here. The results of Subsection 3.1 only ensure the closed-loop is stable with no extra degree of stability, i.e., that the closed-loop poles are only guaranteed to be in the left half of the complex plane for all $\lambda \in \lambda_{\text{box}}$. Fortunately, this difficulty is straightforward to address. We will first shift the given plant by an amount $\sigma > 0$ to the right in the complex plane, work with this σ-shifted version of the plant, design a controller for this shifted plant such that the plant/controller interconnection is stable for all permissible values of λ, and then shift the controller back to obtain the necessary degree of stability σ for the closed-loop interconnection.

In more detail, assume the original given plant that we wish to control is $\hat{P}(\lambda, s) \in \mathcal{R}^{p\times 1}$ and that the shifted plant, denoted by $P(\lambda, s)$, is defined by

$$\begin{aligned} P(\lambda, s) &:= \hat{P}(\lambda, s-\sigma) \\ &= (\hat{N}(s-\sigma) + \hat{N}_\Lambda(s-\sigma)\Lambda)(\hat{M}(s-\sigma) + \hat{M}_\Lambda(s-\sigma)\Lambda)^{-1} \end{aligned} \tag{25}$$

where $\hat{N}(s-\sigma) \in \mathcal{RH}_\infty^{p\times 1}, \hat{N}_\Lambda(s-\sigma) \in \mathcal{RH}_\infty^{p\times l}, \hat{M}(s-\sigma) \in \mathcal{RH}_\infty^{1\times 1}$ and $\hat{M}_\Lambda(s-\sigma) \in \mathcal{RH}_\infty^{1\times l}$ and $\sigma > 0$ is a real constant. Of course, (25) must be a coprime factorization for all λ of interest, which is a stronger condition than requiring $(\hat{N}(s) + \hat{N}_\Lambda(s)\Lambda)(\hat{M}(s) + \hat{M}_\Lambda(s)\Lambda)^{-1}$ being a coprime realization of $\hat{P}(\lambda, s)$. Design a controller $C(\lambda, s)$ for the shifted plant $P(\lambda, s)$ according to the results presented in the previous two subsections. This controller $C(\lambda, s)$ should then be shifted back to get the correct controller $\hat{C}(\lambda, s)$ that will actually be implemented in closed loop with the original given plant $\hat{P}(\lambda, s)$. This controller $\hat{C}(\lambda, s)$ is given by

$$\hat{C}(\lambda, s) := C(\lambda, s + \sigma).$$

Note that this σ-shift does not ruin the multi-affine parameter dependency appearing in the fractional descriptions of the plant and controller, nor in the closed-loop quantity required to have no zeros in the closed right half plane.

3.3 Securing a quadratic Lyapunov function with multi-affine parameter dependence

Theorem 1 is the key result from which one can conclude retention of stability even in the face of parameter variations. In this subsection we explain how the Lyapunov equation (8) can be satisfied with a multi-affine $P(\lambda)$. The argument is akin to that in [11]. With notation as arising in the proof of Theorem 2, define for $i = 1, 2, \ldots, 2^r$ strictly positive real functions $Z_i(s)$ by

$$Z_i(j\omega) = \Big[1 + \Phi(j\omega)\Lambda_{cori} + \Lambda_{cori}^T \left[\alpha(j\omega)F^{\perp}(j\omega) + F^T(j\omega)\beta(j\omega)\right] G(0, j\omega)\Big]\pi(j\omega) \quad (26)$$

where, without loss of generality, we assume $\pi(s)$ has been normalised to satisfy $\pi(\infty) = 1$. Suppose that each $Z_i(j\omega)$ has a minimal state-variable realization $1 + c_i^T(sI - A)^{-1}b$. [Recall that $\Phi(s)$ and $\alpha(s)F^{\perp}(s) + F^T(s)\beta(s)$ are strictly proper; this ensures $Z(\infty) = 1$]. We are implicitly ruling out the possibility of any Λ_{cori} being non-generic in the sense of giving rise to a pole-zero cancellation, or nonminimality for some i of the state-variable realization. As recalled in [11] by the Kalman-Yakubovic Lemma, there exist positive definite symmetric P_i and Q_i with

$$\begin{bmatrix} -P_iA - A^TP_i - Q_i & P_ib - c_i \\ (P_ib - c_i)^T & 2 \end{bmatrix} > 0 \quad (27)$$

Now we know that for any $\lambda \in \lambda_{box}$, the transfer function

$$Z(\lambda, j\omega) = [1 + \Phi(j\omega)\Lambda + \Lambda^T[\alpha(j\omega)F^{\perp}(j\omega) + F^T(j\omega)\beta(j\omega)]G(0, j\omega)]\pi(j\omega) \quad (28)$$

is expressible as a convex combination of the $Z_i(j\omega)$, and accordingly is also strictly positive real. Moreover, the standard convex representation in Section 1 implies the existence of a unique multi-affine $P(\lambda), Q(\lambda)$ and $c(\lambda)$ (expressible with the same weights as a convex linear combination of the respective corner values as for $Z(\lambda, j\omega)$) for which

$$Z(\lambda, j\omega) = 1 + c^T(\lambda)(j\omega I - A)^{-1}b \tag{29}$$

and

$$\begin{bmatrix} -P(\lambda)A - A^T P(\lambda) - Q(\lambda) & P(\lambda)b - c(\lambda) \\ (P(\lambda)b - c(\lambda))^T & 2 \end{bmatrix} > 0 \tag{30}$$

Of course, $P(\lambda), Q(\lambda)$ and $c(\lambda)$ assume the values P_i, Q_i and c_i at the corners of λ_{box}. Now when the shifting procedure of subsection 3.3 is used, it actually results that $Z_i(j\omega - \sigma)$ and $Z(\lambda, j\omega - \sigma)$ are strictly positive real, which means that

$$\begin{bmatrix} -P(\lambda)A - A^T P(\lambda) - Q(\lambda) - 2\sigma P(\lambda) & P(\lambda)b - c(\lambda) \\ (P(\lambda)b - c(\lambda))^T & 2 \end{bmatrix} > 0 \tag{31}$$

Performing a congruence transformation

$$\begin{bmatrix} I & c(\lambda) \\ 0 & I \end{bmatrix}$$

on inequality (31) yields inequality (32) below

$$\begin{bmatrix} -P(\lambda)[A - bc^T(\lambda)] - [A - bc^T(\lambda)]^T P(\lambda) - Q(\lambda) - 2\sigma P(\lambda) & P(\lambda)b + c(\lambda) \\ (P(\lambda)b + c(\lambda))^T & 2 \end{bmatrix} > 0 \tag{32}$$

which is associated with $X^{-1}(s) = 1 - c^T(sI - A + bc^T)^{-1}b$ being strictly positive real, due to $X(s) = 1 + c^T(sI - A)^{-1}b$ being strictly positive real.

From inequality (32), it is immediate that

$$[A - bc^T(\lambda)]^T P(\lambda) + P(\lambda)[A - bc^T(\lambda)] < -2\sigma P(\lambda) \tag{33}$$

Now the frozen closed-loop characteristic polynomial as derived in the proof of Theorem 2 is the numerator of

$$1 + \Phi(s)\Lambda + \Lambda^T[\alpha(s)F^{\perp}(s) + F^T(s)\beta(s)]G(0, s) \tag{34}$$

which is a factor of the numerator of

$$[1 + \Phi(s)\Lambda + \Lambda^T[\alpha(s)F^{\perp}(s) + F^T(s)\beta(s)]G(0, s)]\pi(s) = 1 + c^T(\lambda)(sI - A)^{-1}b \tag{35}$$

Thus the closed-loop characteristic polynomial is a factor of the characteristic polynomial of $A - bc^T(\lambda)$. Equation (33) above, Theorem 1 and the multi-affine character of $P(\lambda)$ then assure stability for time-varying λ, as summarised in the following theorem.

Theorem 3. *Consider a plant $\hat{P}(\lambda, s)$ and the associated σ-shifted plant $P(\lambda, s)$ as given in* (25)*. Assume that a controller of the form* (14) *that satisfies the suppositions of Theorem 2 has been synthesised for the σ-shifted plant $P(\lambda, s)$. Further, define*

$\hat{C}(\lambda, s) := C(\lambda, s+\sigma)$ *as the controller that is actually implemented with the original plant* $\hat{P}(\lambda, s)$*. Then the closed-loop interconnection of* $\hat{P}(\lambda, s)$ *and* $\hat{C}(\lambda, s)$ *is exponentially asymptotically stable with degree of stability* $\gamma \in [0, \sigma]$ *for all time-varying* $\lambda(t) \in \lambda_{box}$ *that satisfy conditions* (9) *and* (10) *of Theorem 1.*

4 Design Procedure

We will now provide an explicit procedure that can be followed in order to construct the LPV controller proposed above. We will assume that a description of the plant $\hat{P}(\lambda, s)$ with multi-affine parameter dependence of the form (1) is available. If the plant is described in state space, it can be rewritten in transfer matrix form by following the procedure in [6].

1. Rescale, if necessary, the polytope λ_{box} so that it is symmetric around $\lambda = 0$ and obtain $P(\lambda, s)$ by shifting $\hat{P}(\lambda, s)$ by $\sigma > 0$ as detailed in (25).
2. Design a nominal controller $C(0, s)$ for the nominal shifted plant $P(0, s)$ and calculate coprime factors $\tilde{U}(s)$ and $\tilde{V}(s)$ satisfying a Bezout identity with the nominal plant coprime factors $N(s)$ and $M(s)$.
3. Define $G(\lambda, s)$ as in (13), $F(s)$ as in (18) and construct $F^{\perp}(s)$ as described immediately after (18) ensuring $F^{\perp}(s)$ is strictly proper. Define $\Phi(s)$ as in (20) and $B_{\tau_i}^{N_i}(s)$ for $i \in \{1, 2, 3\}$ as in (11), with $m = 1, q, (p+1)$ respectively.
4. Set up the 2^r linear matrix inequalities (21) corresponding to every vertex in λ_{box} and specify an appropriate ω-grid in order to obtain a finite-dimensional problem (a set of LMIs). Attempt to solve (as is standard, see [13]) these LMIs for matrices $\breve{\Pi} \in \mathbb{R}^{1\times(N_1+1)}$, $\breve{\Gamma} \in \mathbb{R}^{l\times q(N_2+1)}$, $\breve{\Xi}_i = -\breve{\Xi}_i^T \in \mathbb{R}^{(p+1)\times(p+1)}$ for $i \in \{0, 1, \ldots, N_3\}$: start with small integers N_1, N_2, N_3 and large $\tau_1, \tau_2, \tau_3 > 0$, yielding a crude Laguerre parametrisation, and if infeasible, gradually increase N_1, N_2, N_3 and decrease $\tau_1, \tau_2, \tau_3 > 0$ until the LMIs can be solved.
5. Once matrices $\breve{\Pi} \in \mathbb{R}^{1\times(N_1+1)}$, $\breve{\Gamma} \in \mathbb{R}^{l\times q(N_2+1)}$, $\breve{\Xi}_i = -\breve{\Xi}_i^T \in \mathbb{R}^{(p+1)\times(p+1)}$ for $i \in \{0, 1, \ldots, N_3\}$ have been found, construct $\pi = \breve{\Pi} B_{\tau_1}^{N_1}(s) \in \mathcal{RH}_\infty^{1\times 1}$, $\gamma = \breve{\Gamma} B_{\tau_2}^{N_2}(s) \in \mathcal{RH}_\infty^{l\times q}$ and $\xi = \begin{bmatrix} \breve{\Xi}_0 \ \breve{\Xi}_1 \ \ldots \ \breve{\Xi}_{N_3} \end{bmatrix} B_{\tau_3}^{N_3}(s) \in \mathcal{RH}_\infty^{(p+1)\times(p+1)}$.
6. Define $\alpha = \gamma/\pi \in \mathcal{RH}_\infty^{l\times q}$ and $\beta = \xi/\pi \in \mathcal{RH}_\infty^{(p+1)\times(p+1)}$, and model reduce if necessary (one should check that the reduced quantities still satisfy the LMIs).
7. Compute
$$\begin{bmatrix} -\tilde{U}_\Lambda(s) \ \tilde{V}_\Lambda(s) \end{bmatrix} = \alpha(s) F^{\perp}(s) + F^T(s)\beta(s)$$
as described in (19) and construct $C(\lambda, s)$ as given in (14).
8. Shift the controller by σ to obtain the final controller $\hat{C}(\lambda, s) = C(\lambda, s+\sigma)$. The closed loop involving $\hat{C}(\lambda, s)$ and $\hat{P}(\lambda, s)$ will then have degree of stability $\sigma > 0$ for all (frozen) $\lambda \in \lambda_{\text{box}}$, and will also be guaranteed to be exponentially asymptotically stable with degree of stability $\gamma \in [0, \sigma)$ for all time-varying $\lambda(t) \in \lambda_{\text{box}}$ that satisfy conditions (9) and (10) in Theorem 1.

For a given (fixed) $\sigma > 0$, we can attempt to maximise $\bar{\lambda}_i$, without really worrying about the possible isolated points where we might lose coprimeness, as the existence of a solution to the LMIs is still sufficient for $H(P, C)$ to be stable. Maximization over each separate $\bar{\lambda}_i$ at a time (or all together, assuming $\bar{\lambda}_1 = \bar{\lambda}_2 = \cdots = \bar{\lambda}_l$) is a generalised eigenvalue (i.e., quasi-convex) problem, which can be easily solved using commercially available software. Consequently, one strategy is to first set the same value for all $\bar{\lambda}_i$ and optimise this value until the largest hyper-cube of feasible parameters is found. Then, $\bar{\lambda}_i$ is fixed for some i and the optimization is repeated for a hyper-cube of lower dimension, and so forth. By this strategy, eventually, a large polytope symmetric around $\lambda = 0$ for which the LMIs are satisfied will be found, allowing for correspondingly large parameter variations.

To find a good trade-off between σ and $\bar{\lambda}$, a bisection algorithm may be run for different values of σ, where the quasi-convex optimization strategy outlined above can be carried out for each chosen value of σ. One has to employ a bisection algorithm (or similar technique) to optimise σ because σ enters the expressions for $\Phi(s), F(s), F^{\perp}(s), G(0, s), \ldots$ in a highly nonlinear fashion.

5 Numerical Example

Here, we will illustrate the procedure outlined in the previous Section with a numerical example. Consider the unshifted LPV plant $\hat{P}(\lambda, s)$ given by

$$\hat{P}(\lambda, s) = \frac{\hat{n}(s)}{\hat{m}(s) + \lambda_1 \hat{\mu}_1(s) + \lambda_2 \hat{\mu}_2(s) + \lambda_1 \lambda_2 \hat{\mu}_{12}(s)}$$

where the nominal part is given as

$$\hat{P}(0, s) = \frac{\hat{n}(s)}{\hat{m}(s)} = \frac{1}{s^3 + 1.5s^2 + 2.5s - 0.1}$$

and the system variations are given by $\hat{\mu}_1(s) = 0.65s + 0.4225$, $\hat{\mu}_2(s) = 0.405s^2 + 0.5265s - 1.079$ and $\hat{\mu}_{12}(s) = 0.75s + 0.9875$. The parameters $\lambda = [\lambda_1 \ \lambda_2]^T$ are allowed to vary within $\lambda_{\text{box}} = [-1 \ ; \ 1]^2$.

The plant is now shifted by $\sigma = 0.65$ to

$$P(\lambda, s) = \frac{1}{s^3 - 0.45s^2 + 1.817s - 1.366 + \lambda_1 \mu_1(s) + \lambda_2 \mu_2(s) + \lambda_1 \lambda_2 \mu_{12}(s)}$$

where the new parameter variations are found to $\mu_1(s) = 0.65s$, $\mu_2(s) = 0.405s^2 - 1.25$ and $\mu_{12}(s) = 0.75s + 0.5$. Using standard methods, an observer-based state space compensator $C(0, s)$ is designed for the nominal system $P(0, s)$ to place the closed-loop poles in $s = -0.7$, $s = -1$ and $s = -1.2$ for the controller part and $s = -1.6$, $s = -2.0$ and $s = -2.4$ for the observer. The nominal controller stabilises the nominal plant as required in Theorem 2, but does not stabilise the plant for all values of $\lambda \in \lambda_{\text{box}}$, as can be seen from Figure 3 (note that the figure shows the closed loop of $\hat{P}(\lambda, s)$ and $\hat{C}(0, s)$, i.e., shifted back by σ). In other words, a

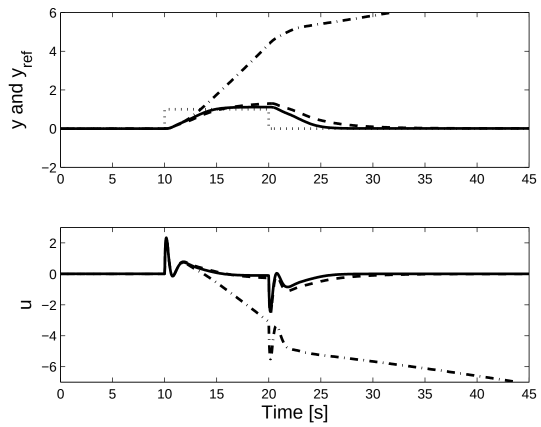

Fig. 3. *Simulation of $\hat{P}(\lambda, s)$ interconnected with the nominal controller $\hat{C}(0, s)$; top: reference and output, bottom: control signal. The parameters are fixed to $\lambda = [0, 0]^T$ (full line), $\lambda = [0.4, 0.5]^T$ (dashed line) and $\lambda = [-0.4, 0.5]^T$ (dash-dotted line). For the latter value of λ, the nominal controller fails to stabilise the system even in case of fixed parameters.*

parameter-dependent controller $\hat{C}(\lambda, s)$ is indeed required to stabilise the system for some values of $\lambda \in \lambda_{\text{box}}$, even in case of fixed parameters.

We thus define $\Lambda = [\lambda_1 \quad \lambda_2 \quad \lambda_1\lambda_2]^T$ and factorise the nominal plant and controller to satisfy the Bezout identity $\tilde{V}(s)M(s) - \tilde{U}(s)N(s) = 1$ as follows:

$$N(s) = \frac{1}{s^3 + 2.9s^2 + 2.74s + 0.84}$$
$$M(s) = \frac{s^3 - 0.45s^2 + 1.817s - 1.366}{s^3 + 2.9s^2 + 2.74s + 0.84}$$
$$\tilde{U}(s) = \frac{-36.67s^2 + 29.54s - 87.22}{s^3 + 6s^2 + 11.84s + 7.68}$$
$$\tilde{V}(s) = \frac{s^3 + 9.35s^2 + 34.37s + 59.13}{s^3 + 6s^2 + 11.84s + 7.68}$$

Using this factorization we obtain

$$F(s) = \begin{bmatrix} N_\Lambda(s) \\ M_\Lambda(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{0.65s}{s^3+2.9s^2+2.74s+0.84} & \frac{0.405s^2-1.25}{s^3+2.9s^2+2.74s+0.84} & \frac{0.75s+0.5}{s^3+2.9s^2+2.74s+0.84} \end{bmatrix}$$

It is straightforward to find an annihilator for this system, for instance $F^{\perp}(s) = \left[\frac{1}{s+1}\ 0\right]$.

We then compute $\Phi(j\omega)$ and define a grid over the $j\omega$ axis (in this case, equidistant in the interval from $\omega = 10^{-2} rad/s$ to $\omega = 10^5 rad/s$, along with

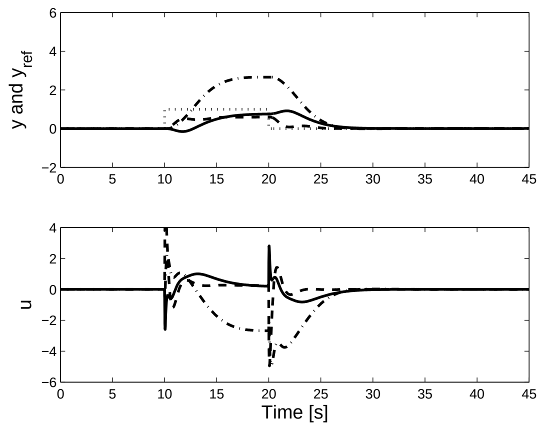

Fig. 4. *Simulation of $\hat{P}(\lambda, s)$ interconnected with the LPV controller $\hat{C}(\lambda, s)$; top: reference and output, bottom: control signal. The parameters are fixed to $\lambda = [1, 1]^T$ (full line), $\lambda = [1, -1]^T$ (dashed line) and $\lambda = [-0.4, 0.5]^T$ (dash-dotted line). The closed-loop system remains stable.*

infinity) and set up the synthesis LMIs (21) using Matlab's LMILAB toolbox. In this example, the LMIs were found to be feasible for $N_1 = 2$, $N_2 = N_3 = 1$ and $\tau_1 = \tau_2 = \tau_3 = 0.05$. We were able to compute the parameter-dependent factors in the controller as

$$\tilde{U}_\Lambda = \begin{bmatrix} \frac{306.5s^3+1.243\times10^4 s^2+6877s}{s^5+75.18s^4+1850s^3+4948s^2+4548s+1376} \\ \frac{191s^4+7746s^3+3695s^2-2.391\times10^4 s-1.322\times10^4}{s^5+75.18s^4+1850s^3+4948s^2+4548s+1376} \\ \frac{353.6s^3+1.458\times10^4 s^2+1.75\times10^4 s+5290}{s^5+75.18s^4+1850s^3+4948s^2+4548s+1376} \end{bmatrix}$$

$$\tilde{V}_\Lambda = 0$$

This controller is then shifted back by σ, yielding $\hat{C}(\lambda, s)$, which is implemented in closed loop with $\hat{P}(\lambda, s)$. The parameter-dependent controller is indeed able to stabilise the system for any fixed value of $\lambda \in \lambda_{box}$; Figure 4 shows a few examples hereof. Note the significant variations in closed-loop behavior caused by the different values of λ.

Furthermore, when λ is allowed to vary with time, for instance as

$$\lambda = \lambda(t) = \begin{bmatrix} \sin \tilde{\omega} t \\ \cos \tilde{\omega} t \end{bmatrix}$$

the parameter-varying controller is able to stabilise the system for slow parameter variations ($\tilde{\omega} < 0.6$), as can be seen from Figure 5. As can also be seen from the figure, the closed loop becomes unstable when the parameter variations become faster, which is to be expected when the frequency of variation becomes large in comparison with the degree of stability of the frozen-parameter nominal closed-loop system.

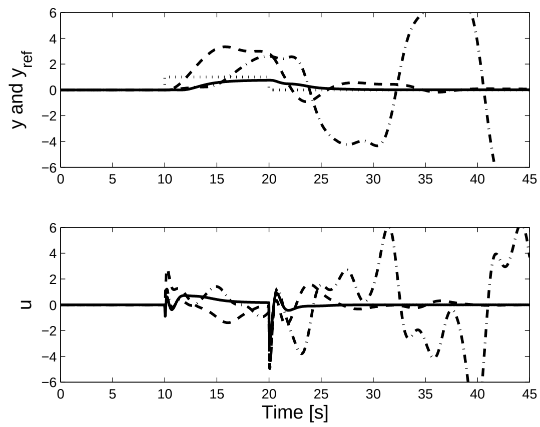

Fig. 5. *Simulation of $\hat{P}(\lambda, s)$ interconnected with the LPV controller $\hat{C}(\lambda, s)$; top: reference and output, bottom: control signal. The parameters vary according to $\lambda(t) = [\sin\tilde{\omega}t, \cos\tilde{\omega}t]^T$, with $\tilde{\omega} = 0.1$ (full line), $\tilde{\omega} = 0.5$ (dashed line) and $\tilde{\omega} = 0.75$ (dash-dotted line). The closed-loop system is unstable for fast parameter variations, but remains stable for slow variations.*

6 Discussion

In this paper we presented a novel procedure for design of controllers for LPV systems whose transfer functions display multi-affine parameter dependence. This class of systems encompasses many physical systems, and is therefore of significant practical relevance to control designers. The controller structure was chosen to reflect the structure of the plant, and the terms associated with parameter dependencies in the controller were chosen to remove terms that are not multi-affine in the parameters. The idea was to obtain a multi-affine dependence for the parameters of the closed loop A-matrix, and thus allow the designer to obtain guarantees on the extent of time variation in the parameters that can be allowed.

The problem of finding the controller parameter dependencies was cast as a finite-dimensional LMI problem. We outlined the procedure in a constructive manner for a particular choice of controllers and illustrated the feasibility of the method with a numerical example. With r parameters, we actually have 2^r different frequency domain conditions which have to be fulfilled.

As it stands, the method is limited to apply to single-input (but possibly multi-output) plants. This is due to the fact that some of the manipulations of $(\tilde{K}G)^{-1}$ rely on this quantity being scalar. This is a weakness of the current design and should be alleviated in future research if possible.

Finally, it should be noted that all the results presented deal with stability, not performance, i.e., we are not able to quantify loss of performance compared to the nominal designs, either due to time variations or due to the constrained way of handling the parameter dependence in the controller. This would be interesting to deal with in the future.

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