1 Topological Foundations

1.1 Manifolds and Differentiable Manifolds

Definition 1.1.1   A manifold of dimension n is a connected Hausdorff space M for which every point has a neighbourhood U that is homeomorphic to an open subset V of $\mathbb{R}^n$. Such a homeomorphism

$$f : U \to V$$

is called a (coordinate) chart.
An atlas is a family of charts $\{U_\alpha, f_\alpha\}$ for which the U_α constitute an open covering of M.

Remarks.   –   The condition that M is Hausdorff means that any two distinct points of M have disjoint neighbourhoods.
–   A point $p \in U_\alpha$ is uniquely determined by $f_\alpha(p)$ and will often be identified with $f_\alpha(p)$. And we may even omit the index α, and call the components of $f(p) \in \mathbb{R}^n$ the coordinates of p.
–   We shall be mainly interested in the case $n = 2$. A manifold of dimension 2 is usually called a surface.

Definition 1.1.2   An atlas $\{U_\alpha, f_\alpha\}$ on a manifold is called differentiable if all chart transitions

$$f_\beta \circ f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

are differentiable of class C^∞ (in case $U_\alpha \cap U_\beta \neq \emptyset$).
A chart is called compatible with a differentiable atlas if adding this chart to the atlas yields again a differentiable atlas. Taking all charts compatible with a given differentiable atlas yields a differentiable structure. A differentiable manifold of dimension d is a manifold of dimension d together with a differentiable structure.

Remark.   One could impose a weaker differentiability condition than C^∞.

Definition 1.1.3   A continuous map $h : M \to M'$ between differentiable manifolds M and M' with charts $\{U_\alpha, f_\alpha\}$ and $\{U'_\alpha, f'_\alpha\}$ is said to be differentiable if all the maps $f'_\beta \circ h \circ f_\alpha^{-1}$ are differentiable (of class C^∞) wherever they are defined.

If h is a homeomorphism and if both h and h^{-1} are differentiable, then h is called a diffeomorphism.

Examples.   1) The sphere

$$S^n := \{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$$

is a differentiable manifold of dimension n. Charts can be given as follows:

On $U_1 := S^n \setminus \{(0,\ldots,0,1)\}$, we set

$$f_1(x_1,\ldots,x_{n+1}) := (f_1^1(x_1,\ldots,x_{n+1}),\ldots,f_1^n(x_1,\ldots,x_{n+1}))$$
$$:= \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}} \right),$$

and on $U_2 := S^n \setminus \{(0,\ldots,0,-1)\}$

$$f_2(x_1,\ldots,x_{n+1}) := (f_2^1,\ldots,f_2^n)$$
$$:= \left(\frac{x_1}{1+x_{n+1}}, \ldots, \frac{x_n}{1+x_{n+1}} \right).$$

2) Let $w_1, w_2 \in \mathbb{C}\setminus\{0\}$, $\frac{w_1}{w_2} \notin \mathbb{R}$. We call $z_1, z_2 \in \mathbb{C}$ equivalent if there exist $m, n \in \mathbb{Z}$ such that

$$z_1 - z_2 = nw_1 + mw_2.$$

Let π be the projection which maps $z \in \mathbb{C}$ to its equivalence class. The torus $T := \pi(\mathbb{C})$ can then be made a differentiable manifold (of dimension two) as follows: Let $\Delta_\alpha \subset \mathbb{C}$ be an open set of which no two points are equivalent. Then we set

$$U_\alpha := \pi(\Delta_\alpha) \quad \text{and} \quad f_\alpha := (\pi \mid \Delta_\alpha)^{-1}.$$

3) Note that the manifolds of both foregoing examples are compact. Naturally, there exist non-compact manifolds. The simplest example is $\mathbb{R}^n$. Generally, every open subset of a (differentiable) manifold is again a (differentiable) manifold.

Exercises for § 1.1

1) Show that the dimension of a differentiable manifold is uniquely determined. (This requires to prove that if M_1 and M_2 are differentiable manifolds, and $f : M_1 \to M_2$ is a diffeomorphism, meaning that f is invertible and both f and f^{-1} are differentiable, then dimension $M_1 =$ dimension M_2).

2) Generalize the construction of example 2 following Definition 1.1.3 to define an n-dimensional real torus through an appropriate equivalence relation on $\mathbb{R}^n$. Try also to define a complex n-dimensional torus via an equivalence relation on $\mathbb{C}^n$ (of course, this torus then will have $2n$ (real) dimensions). Examples of such complex tori will be encountered in § 5.3 as Jacobian varieties.

1.2 Homotopy of Maps. The Fundamental Group

For the considerations of this section, no differentiability is needed, so that the manifolds and maps which occur need not to be differentiable.

Definition 1.2.1 Two continuous maps $f_1, f_2 : S \to M$ between manifolds S and M are homotopic, if there exists a continuous map

$$F : S \times [0,1] \to M$$

with

$$F|_{S \times \{0\}} = f_1,$$
$$F|_{S \times \{1\}} = f_2.$$

We write: $f_1 \approx f_2$.

In what follows, we need to consider curves in M (or paths - we use the two words interchangeably); these are continuous maps $g : [0,1] \to M$. We define the notion of homotopy of curves with the same end-points:

Definition 1.2.2 Let $g_i : [0,1] \to M$, $i = 1, 2$, be curves with

$$g_1(0) = g_2(0) = p_0,$$
$$g_1(1) = g_2(1) = p_1.$$

We say that g_1 and g_2 are homotopic, if there exists a continuous map

$$G : [0,1] \times [0,1] \to M$$

such that

$$G|_{\{0\} \times [0,1]} = p_0, \quad G|_{\{1\} \times [0,1]} = p_1,$$
$$G|_{[0,1] \times \{0\}} = g_1, \quad G|_{[0,1] \times \{1\}} = g_2.$$

We again write: $g_1 \approx g_2$.

Thus the homotopy must keep the endpoints fixed.
For example, any two curves $g_1, g_2 : [0,1] \to \mathbb{R}^n$ with the same end-points are homotopic, namely via the homotopy

$$G(t,s) := (1-s)\,g_1(t) + s\,g_2(t).$$

Furthermore, two paths which are reparametrisations of each other are homotopic:
if $\tau : [0,1] \to [0,1]$ is continuous and strictly increasing $g_2(t) = g_1(\tau(t))$, we can set

$$G(t,s) := g_1\big((1-s)t + s\,\tau(t)\big).$$

The homotopy class of a map f (or a curve g) is the equivalence class consisting of all maps homotopic to f (or all paths with the same end-points, homotopic to g); we denote it by $\{f\}$ (resp. $\{g\}$). In particular, as we have just seen, the homotopy class of g does not change under reparametrisation.

Definition 1.2.3   Let $g_1, g_2 : [0,1] \to M$ be curves with

$$g_1(1) = g_2(0)$$

(i.e. the terminal point of g_1 is the initial point of g_2). Then the product $g_2 g_1 := g$ is defined by

$$g(t) := \begin{cases} g_1(2t) & \text{for } t \in [0, \tfrac{1}{2}] \\ g_2(2t-1) & \text{for } t \in [\tfrac{1}{2}, 1]. \end{cases}$$

It follows from the definition that $g_1 \approx g_1{'}, g_2 \approx g_2{'}$ implies

$$g_2 g_1 \approx g_2' g_1'.$$

Thus the homotopy class of $g_1 g_2$ depends only on the homotopy classes of g_1 and g_2; we can therefore define a multiplication of homotopy classes as well, namely by

$$\{g_1\} \cdot \{g_2\} = \{g_1 g_2\}.$$

Definition 1.2.4   For any $p_0 \in M$, the fundamental group $\pi_1(M, p_0)$ is the group of homotopy classes of paths $g : [0,1] \to M$ with $g(0) = g(1) = p_0$, i.e. closed paths with p_0 as initial and terminal point.

To justify this definition, we must show that, for closed paths with the same initial and terminal point, the multiplication of homotopy classes does in fact define a group:

Theorem 1.2.1 $\pi_1(M, p_0)$ *is a group with respect to the operation of multiplication of homotopy classes. The identity element is the class of the constant path* $g_0 \equiv p_0$.

Proof.   Since all the paths have the same initial and terminal point, the product of two homotopy classes is always defined. It is clear that the class of the constant path g_0 acts as the identity element, and that the product is associative. The inverse of a path g is given by the same path described in the opposite direction:

$$g^{-1}(t) := g(1 - t), \qquad t \in [0, 1].$$

We then have

$$\{g^{-1}\} \cdot \{g\} = 1 \quad \text{(the identity element).}$$

A homotopy of g_0 with $g^{-1} \cdot g$ is given e.g. by

$$g(t, s) := \begin{cases} g(2st), & t \in [0, \frac{1}{2}] \\ g^{-1}(1 + 2s(t - 1)) = g(2s(1 - t)), & t \in [\frac{1}{2}, 1]. \end{cases} \qquad \square$$

Remark. In the sequal, we shall often write g in place of $\{g\}$, hoping that this will not confuse the reader.

Lemma 1.2.1 *For any $p_0, p_1 \in M$, the groups $\pi_1(M, p_0)$ and $\pi_1(M, p_1)$ are isomorphic.*

Proof. We pick a curve γ with $\gamma(0) = p_0$, $\gamma(1) = p_1$. The map sending a path g with $g(0) = g(1) = p_1$ to the path $\gamma^{-1}g\gamma$ induces a map

$$\pi_1(M, p_1) \to \pi_1(M, p_0).$$

This map is an isomorphism of groups. $\qquad \square$

Definition 1.2.5 The abstract group $\pi_1(M)$ defined in view of Lemma 1.2.1 is called the fundamental group of M.

Remark. It is important to observe that the isomorphism between $\pi_1(M, p_0)$ and $\pi_1(M, p_1)$ constructed in Lemma 1.2.1 is not canonical, since it depends on the choice of the path γ.
A different path not homotopic to γ could give rise to a different isomorphism.
In particular, consider the case $p_0 = p_1$, so that $\gamma \in \pi_1(M, p_0)$. Then conjugation by γ

$$g \mapsto \gamma^{-1}g\gamma$$

is in general a non-trivial automorphism of $\pi_1(M, p_0)$.

Definition 1.2.6 We say that M is simply-connected if $\pi_1(M) = \{1\}$.

Lemma 1.2.2 *If M is simply-connected, then any two paths g_1, g_2 in M with*
$g_1(0) = g_2(0)$ and $g_1(1) = g_2(1)$ are homotopic.

This follows easily from the definitions. $\qquad \square$

Example 1 $\mathbb{R}^n$ is simply-connected, so is S^n for $n \geq 2$ (Exercise).

Definition 1.2.7 A path $g : [0, 1] \to M$ with $g(0) = g(1) = p_0$ which is homotopic to the constant path $g_0(t) \equiv p_0$ is called null-homotopic.

Remark.   This is generally accepted terminology although it might be more appropriate to call such a path one-homotopic as the neutral element of our group is denoted by 1.

Finally, we have:

Lemma 1.2.3   *Let $f : M \to N$ be a continuous map, and $q_0 := f(p_0)$. Then f induces a homomorphism*

$$f_* : \pi_1(M, p_0) \to \pi_1(N, q_0)$$

of fundamental groups.

Proof.   If $g_1 \approx g_2$, then we also have $f(g_1) \approx f(g_2)$, since f is continuous. Thus we obtain a well-defined map between fundamental groups. Clearly,

$$f(g_2^{-1} \cdot g_1) \approx (f(g_2))^{-1} \cdot f(g_1). \qquad \square$$

Exercises for § 1.2

– Show that $\mathbb{R}^n$ is simply connected, and so is S^n for $n \geq 2$.
– Determine the fundamental group of S^1.
  Outline of the solution:
  Let $S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} \in \mathbb{C};$ with $\theta \in \mathbb{R}, 0 \leq \theta \leq 2\pi\}$.
  Then paths γ_n in $\pi_1(S^1, 1)$ are given by

$$t \mapsto e^{2\pi i n t} \qquad (\, t \in [0, 1])$$

  for each $n \in \mathbb{Z}$.
  Show that γ_n and γ_m are not homotopic for $n \neq m$ and that on the other hand each $\gamma \in \pi_1(S^1, 1)$ is homotopic to some γ_n.
– Having solved 2), determine the fundamental group of a torus (as defined in example 2) after Def. 1.1.3). After having read § 1.3, you will know an argument that gives the result immediately.

1.3 Coverings

Definition 1.3.1   Let M' and M be manifolds. A map $\pi : M' \to M$ is said to be a local homeomorphism if each $x \in M'$ has a neighbourhood U such that $\pi(U)$ is open in M and $\pi \mid U$ is a homeomorphism (onto $\pi(U)$).

If M is a differentiable manifold with charts $\{U_\alpha, f_\alpha\}$, and $\pi : M' \to M$ a local homeomorphism, then we can introduce charts $\{V_\beta, g_\beta\}$ on M' by requiring that $\pi \mid V_\beta$ be a homeomorphism and that all $f_\alpha \circ \pi \circ g_\beta^{-1}$ be diffeomorphisms whenever they are defined. In this way, M' too becomes a differentiable manifold: the differentiable structure of M can be pulled back to M'. π then becomes a local diffeomorphism.

Definition 1.3.2 A local homeomorphism $\pi : M' \to M$ is called a covering if each $x \in M$ has a (connected) neighbourhood V such that every connected component of $\pi^{-1}(V)$ is mapped by π homeomorphically onto V. (If π is clear from the context, we sometimes also call M' a covering of M.)

Remarks. 1) In the literature on Complex Analysis, often a local homeomorphism is already referred to as a covering. A covering in the sense of Definition 1.3.2 is then called a perfect, or unlimited, covering.
2) The preceding definitions are still meaningful if M' and M are just topological spaces instead of manifolds.

Lemma 1.3.1 *If $\pi : M' \to M$ is a covering, then each point of M is covered the same number of times, i.e. $\pi^{-1}(x)$ has the same number of elements for each $x \in M$.*

Proof. Let $n \in \mathbb{N}$. Then one easily sees that the set of points in M with precisely n inverse images is both open and closed in M. Since M is connected, this set is either empty or all of M. Thus either there is an $n \in \mathbb{N}$ for which this set is all of M, or every point of M has infinitely many inverse images.
$\square$

Theorem 1.3.1 *Let $\pi : M' \to M$ be a covering, S a simply-connected manifold, and $f : S \to M$ a continuous map. Then there exists a continuous $f' : S \to M'$ with*

$$\pi \circ f' = f.$$

Definition 1.3.3 An f' as in the above theorem is called a lift of f.

Remark. Lifts are typically not unique.

We also say in this case that the diagram

$$
\begin{array}{ccc}
 & & M' \\
 & \stackrel{f'}{\nearrow} & \downarrow \pi \\
S & \stackrel{}{\underset{f}{\longrightarrow}} & M
\end{array}
$$

is commutative. For the proof of Theorem 1.3.1, we shall first prove two lemmas.

Lemma 1.3.2 *Let $\pi : M' \to M$ be a covering, $p_0 \in M$, $p_0' \in \pi^{-1}(p_0)$, and $g : [0,1] \to M$ a curve with $g(0) = p_0$. Then g can be lifted (as in Def. 1.3.3) to a curve $g' : [0,1] \to M'$ with $g'(0) = p_0'$, so that*

$$\pi \circ g' = g.$$

Further, g' is uniquely determined by the choice of its initial point p_0'.

Proof. Let
$$T := \{t \in [0, 1] : \ g|[0, t] \text{ can be lifted to a unique curve } g'|[0, t] \text{ with } g'(0) = p_0'\}.$$
We have $0 \in T$, hence $T \neq \emptyset$.

If $t \in T$, we choose a neighbourhood V of $g(t)$ as in Definition 1.3.2, so that π maps each component of $\pi^{-1}(V)$ homeomorphically onto V. Let V' denote the component of $\pi^{-1}(V)$ containing $g'(t)$. We can choose $\tau > 0$ so small that $g([t, t + \tau]) \subset V$. It is then clear that g' can be uniquely extended as a lift of g to $[t, t + \tau]$, since $\pi : V' \to V$ is a homeomorphism. This proves that T is open in [0,1].

Suppose now that $(t_n) \subset T$, and $t_n \to t_0 \in [0, 1]$. We choose a neighbourhood V of $g(t_0)$ as before. Then there exists $n_0 \in \mathbb{N}$ with $g([t_{n_0}, t_0]) \subset V$. We let V' be the component of $\pi^{-1}(V)$ containing $g'(t_{n_0})$. We can extend g' to $[t_{n_0}, t_0]$. Hence $t_0 \in T$, so that T is also closed. Thus $T = [0, 1]$. $\qquad\qquad\square$

Lemma 1.3.3 *Let $\pi : M' \to M$ be a covering, and $\Gamma : [0, 1] \times [0, 1] \to M$ a homotopy between the paths $\gamma_0 := \Gamma(\cdot, 0)$ and $\gamma_1 := \Gamma(\cdot, 1)$ with fixed end points $p_0 = \gamma_0(0) = \gamma_1(0)$ and $p_1 = \gamma_0(1) = \gamma_1(1)$. Let $p_0' \in \pi^{-1}(p_0)$.*
Then Γ can be lifted to a homotopy $\Gamma' : [0, 1] \times [0, 1] \to M'$ with initial point p_0' (i.e. $\Gamma'(0, s) = p_0'$ for all $s \in [0, 1]$); thus $\pi \circ \Gamma' = \Gamma$. In particular, the lifted paths γ_0' and γ_1' with initial point p_0' have the same terminal point $p_1' \in \pi^{-1}(p_1)$, and are homotopic.

Proof. Each path $\Gamma(\cdot, s)$ can be lifted to a path γ_s' with initial point p_0' by Lemma 1.3.2. We set
$$\Gamma'(t, s) := \gamma_s'(t),$$
and we must show that Γ is continuous.

Let $\Sigma := \{(t, s) \in [0, 1] \times [0, 1] : \ \Gamma' \text{ is continuous at } (t, s)\}$. We first take a neighbourhood U' of p_0' such that $\pi : U' \to U$ is a homeomorphism onto a neighbourhood U of p_0; let $\varphi : U \to U'$ be its inverse. Since $\Gamma(\{0\} \times [0, 1]) = p_0$ and Γ is continuous, there exists an $\varepsilon > 0$ such that $\Gamma([0, \varepsilon] \times [0, 1]) \subset U'$. By the uniqueness assertion of Lemma 1.3.2, we have

$$\gamma_s' \mid [0, \varepsilon] = \varphi \circ \gamma_s \mid [0, \varepsilon]$$

for all $s \in [0, 1]$. Hence

$$\Gamma' = \varphi \circ \Gamma \text{ on } [0, \varepsilon] \times [0, 1].$$

In particular, $(0, 0) \in \Sigma$.

Now let $(t_0, s_0) \in \Sigma$. We choose a neighbourhood U' of $\Gamma'(t_0, s_0)$ for which $\pi : U' \to U$ is a homeomorphism onto a neighbourhood U of $\Gamma(t_0, s_0)$; we denote its inverse again by $\varphi : U \to U'$.

Since Γ' is continuous at (s_0, t_0), we have $\Gamma'(t, s) \in U'$ for $|t - t_0| < \varepsilon$, $|s - s_0| < \varepsilon$ if $\varepsilon > 0$ is small enough. By the uniqueness of lifting we again have

$$\gamma'_s(t) = \varphi \circ \gamma_s(t) \text{ for } |t - t_0|, |s - s_0| < \varepsilon,$$

so that

$$\Gamma' = \varphi \circ \Gamma \text{ on } \{|t - t_0| < \varepsilon\} \times \{|s - s_0| < \varepsilon\}.$$

In particular, Γ' is continuous in a neighbourhood of (t_0, s_0). Thus Σ is open. The proof that Σ is closed is similar. It follows that $\Sigma = [0,1] \times [0,1]$, i.e. Γ' is continuous.

Since $\Gamma(\{1\} \times [0,1]) = p_1$ and $\pi \circ \Gamma' = \Gamma$, we must have $\Gamma'(\{1\} \times [0,1]) \subset \pi^{-1}(p_1)$. But $\pi^{-1}(p_1)$ is discrete since π is a covering and $\Gamma'(\{1\} \times [0,1])$ is connected, hence the latter must reduce to a single point.

Thus, all the curves γ'_s have the same end point. □

Proof of Theorem 1.3.1 We pick a $y_0 \in S$, put $p_0 := f(y_0)$, and choose a $p'_0 \in \pi^{-1}(p_0)$.

For any $y \in S$, we can find a path $\gamma : [0,1] \to S$ with $\gamma(0) = y_0, \gamma(1) = y$. By Lemma 1.3.2, the path $g := f \circ \gamma$ can be lifted to a path g' starting at p'_0. We set $f'(y) := g'(1)$. Since S is simply-connected, any two paths γ_1 and γ_2 in S with $\gamma_1(0) = \gamma_2(0) = y_0$ and $\gamma_1(1) = \gamma_2(1) = y$ are homotopic. Hence $f(\gamma_1)$ and $f(\gamma_2)$ are also homotopic, since f is continuous. Thus, it follows from Lemma 1.3.3 that the point $f'(y)$ obtained above is independent of the choice of the path γ joining y_0 to y_1. The continuity of f' can be proved exactly as in the proof of Lemma 1.3.3. □

Corollary 1.3.1 *Let $\pi' : M' \to M$ be a covering, $g : [0,1] \to M$ a curve with $g(0) = g(1) = p_0$, and $g' : [0,1] \to M'$ a lift of g. Suppose g is homotopic to the constant curve $\gamma(t) \equiv p_0$. Then g' is closed and homotopic to the constant curve.*

Proof. This follows directly from Lemma 1.3.2. □

Definition 1.3.4 Let $\pi_1 : M'_1 \to M$ and $\pi_2 : M'_2 \to M$ be two coverings. We say that (π_2, M'_2) dominates (π_1, M'_1) if there exists a covering $\pi_{21} : M'_2 \to M'_1$ such that $\pi_2 = \pi_1 \circ \pi_{21}$. The two coverings are said to be equivalent if there exists a homeomorphism $\pi_{21} : M'_2 \to M'_1$ such that $\pi_2 = \pi_1 \circ \pi_{21}$.

Let $\pi : M' \to M$ be a covering, $p_0 \in M$, $p'_0 \in \pi^{-1}(p_0)$, $g : [0,1] \to M$ a path with $g(0) = g(1) = p_0$, and $g' : [0,1] \to M'$ the lift of g with $g'(0) = p'_0$. By Corollary 1.3.1, if g is null-homotopic, then g' is closed and null-homotopic.

Lemma 1.3.4 $G_\pi := \{\{g\} : g' \text{ is closed}\}$ *is a subgroup of* $\pi_1(M, p_0)$.

Proof. If $\{g_1\}, \{g_2\}$ lie in G_π, so do $\{g_1^{-1}\}$ and $\{g_1 g_2\}$. □

The G_π defined above depends on the choice of $p'_0 \in \pi^{-1}(p_0)$, hence we denote it by $G_\pi(p'_0)$ when we want to be precise. If p''_0 is another point of $\pi^{-1}(p_0)$, and γ' is a path from p'_0 to p''_0, then $\gamma := \pi(\gamma')$ is a closed path at p_0.

If g is a closed path at p_0, then the lift of g starting at p_0' is closed precisely when the lift of $\gamma g \gamma^{-1}$ starting at p_0'' is closed. Hence

$$G_\pi(p_0'') = \{\gamma\} \cdot G_\pi(p_0') \cdot \{\gamma^{-1}\}.$$

Thus $G_\pi(p_0')$ and $G_\pi(p_0'')$ are conjugate subgroups of $\pi_1(M, p_0)$. Conversely, every subgroup conjugate to $G_\pi(p_0')$ can be obtained in this way. It is also easy to see that equivalent coverings lead to the same conjugacy class of subgroups of $\pi_1(M, p_0)$.

Theorem 1.3.2 $\pi_1(M')$ *is isomorphic to* G_π, *and we obtain in this way a bijective correspondence between conjugacy classes of subgroups of* $\pi_1(M)$ *and equivalence classes of coverings* $\pi : M' \to M$.

Proof. Let $\gamma' \in \pi_1(M', p_0')$, and $\gamma := \pi(\gamma')$. Since γ' is closed, we have $\gamma \in G_\pi$; also, being a continuous map, π maps homotopic curves to homotopic curves, so that we obtain a map

$$\pi_* : \pi_1(M', p_0') \to G_\pi(p_0).$$

This map is a group homomorphism by Lemma 1.2.3, surjective by the definition of G_π, and injective since, by Corollary 1.3.1, γ' is null-homotopic if γ is. Thus π_* is an isomorphism. As already noted, the conjugacy class of G_π depends only on the equivalence class of $\pi : M' \to M$. Conversely, given a subgroup G of $\pi_1(M, p_0)$, we now want to construct a corresponding covering $\pi : M' \to M$. As a set, M' will be the set of all equivalence classes $[\gamma]$ of paths γ in M with $\gamma(0) = p_0$, two paths γ_1 and γ_2 being regarded as equivalent if $\gamma_1(1) = \gamma_2(1)$ and $\{\gamma_1 \gamma_2^{-1}\} \in G$. The map $\pi : M' \to M$ is defined by

$$\pi([\gamma]) = \gamma(1).$$

We wish to make M' a manifold in such a way that $\pi : M' \to M$ is a covering. Let $\{U_\alpha, f_\alpha\}$ be the charts for M. By covering the U_α by smaller open sets if necessary, we may assume that all the U_α are homeomorphic to the ball $\{x \in \mathbb{R}^n : \|x\| < 1\}$. Let $q_0 \in U_\alpha$, and $q_0' = [\gamma_0] \in \pi^{-1}(q_0)$. For any $q \in U_\alpha$, we can find a path $g : [0,1] \to U_\alpha$ with $g(0) = q_0, g(1) = q$. Then $[g\gamma_0]$ depends on q_0 and q, but not on g. Let $U_\alpha'(q_0')$ be the subset of M' consisting of all such $[g\gamma]$. Then $\pi : U_\alpha'(q_0') \to U_\alpha$ is bijective, and we declare $\{U_\alpha'(q_0'), f_\alpha \circ \pi\}$ as the charts for M'.
Let us show that, if $p_1' \neq p_2', \pi(p_1') = \pi(p_2')$, and $p_1' \in U_\alpha'(q_1'), p_2' \in U_\beta'(q_2')$,

$$U_\alpha'(q_1') \cap U_\beta'(q_2') = \emptyset. \qquad (1.3.1)$$

Thus, let $p_1' = [g'\gamma], p_2' = [g''\gamma_2]$, where $\gamma_1(0) = \gamma_2(0) = p_0$, $\gamma_1(1) = q_1$, and $\gamma_2(1) = q_2$. Then $\gamma_2^{-1} g''^{-1} g' \gamma_1$ is closed, and does not lie in G_π. If now q is any point of $U_\alpha'(q_1') \cap U_\beta'(q_2')$, then q has two representations $[h'\gamma_1]$ and $[h'\gamma_2]$, with $\gamma_2^{-1} g''^{-1} g' \gamma_1 \in G_\pi$. However, the U_α are simply connected, hence

$h''^{-1}h' \approx g''^{-1}g'$, so we get $\gamma_2^{-1}g''^{-1}g'\gamma_1 \in G_\pi$, a contradiction. This proves (1.3.1). If now $r_1' \neq r_2' \in M'$ with $\pi(r_1') = \pi(r_2')$, it is obvious that r_1' and r_2' have disjoint neighbourhoods. If on the other hand $\pi(r_1') = \pi(r_2')$, this follows from (1.3.1), so that M' is a Hausdorff space.

It also follows from (1.3.1) that two distinct sets $U_\alpha'(q_1')$, $U_\alpha'(q_2')$ are disjoint. Hence the $U_\alpha'(q')$ are the connected components of $\pi^{-1}(U_\alpha)$ and π maps each of them homeomorphically onto U_α. It follows that $\pi : M' \to M$ is a covering. It remains only to show the covering $\pi : M' \to M$ we have constructed has $G_\pi = G$. Let then $p_0' = [1]$, and $\gamma : [0,1] \to M$ a closed path at p_0. Then the lift γ' of γ starting at p_0' is given by $\gamma'(t) = [\gamma \mid [0,t]]$. Hence γ' is closed precisely when $\gamma \in G$. $\square$

Corollary 1.3.2 *If M is simply connected, then every covering $M' \to M$ is a homeomorphism.*

Proof.   Since $\pi_1(M) = \{1\}$, the only subgroup is $\{1\}$ itself. This subgroup corresponds to the identity covering id $: M \to M$. From Theorem 1.3.2 we conclude that M' then is homeomorphic to M. $\square$

Corollary 1.3.3 *If $G = \{1\}$, and $\pi : \widetilde{M} \to M$ the corresponding covering, then $\pi_1(\widetilde{M}) = \{1\}$, and a path $\widetilde{\gamma}$ in $\widetilde{M}$ is closed precisely when $\pi(\widetilde{\gamma})$ is closed and null-homotopic.*
If $\pi_1(M) = \{1\}$, then $\widetilde{M} = M$.

Definition 1.3.5 The covering $\widetilde{M}$ of M with $\pi_1(\widetilde{M}) = \{1\}$ - which exists by Corollary 1.3.2 - is called the universal covering of M.

Theorem 1.3.3 *Let $f : M \to N$ be a continuous map, and $\pi : \widetilde{M} \to M$, $\pi' : \widetilde{N} \to N$ the universal coverings. Then there exists a lift $\widetilde{f} : \widetilde{M} \to \widetilde{N}$, i.e. a continuous map such that the diagram*

$$
\begin{array}{ccc}
\widetilde{M} & \overset{\widetilde{f}}{\longrightarrow} & \widetilde{N} \\
\pi \downarrow & & \downarrow \pi' \\
M & \overset{f}{\longrightarrow} & N
\end{array}
$$

is commutative (so that $f \circ \pi = \pi' \circ \widetilde{f}$).

Proof.   This follows from Theorem 1.3.1, applied to $f \circ \pi$. $\square$

Definition 1.3.6 Let $\pi : M' \to M$ be a local homeomorphism. Then a homeomorphism $\varphi : M' \to M'$ is called a covering transformation if $\pi \circ \varphi = \pi$. The covering transformations form a group H_π.

Lemma 1.3.5 *If $\varphi \neq$ Id is a covering transformation, then φ has no fixed point.*

Proof.   Let $\Sigma := \{p \in M' : \varphi(p) = p\}$ be the set of fixed points of φ. Let $p \in \Sigma$, and U' a neighbourhood of p such that $\pi : U' \to U := \pi(U')$ is a homeomorphism. Let $V' \subset U'$ be a neighbourhood of p with $\varphi(V') \subset U'$. For $q \in V'$, we have $\pi(\varphi(q)) = \pi(q) \in U$, hence $\varphi(q) = q$, since both q and $\varphi(q)$ lie in U'. Thus Σ is open. Since Σ is obviously closed, we must have $\Sigma = \emptyset$ or $\Sigma = M'$. In the latter case, $\varphi = \text{Id}$.   □

It follows in particular from Lemma 1.3.5 that two covering transformations φ_1 and φ_2 with $\varphi_1(p) = \varphi_2(p)$ for one $p \in M'$ must be identical. We recall from group theory:

Definition 1.3.7   Let $G \subset H$ be groups. Then $N(G) = \{g \in H : g^{-1}Gg = G\}$ is called the normaliser of G in H. G is called a normal subgroup of H if $N(G) = H$.

Theorem 1.3.4   *For any covering $\pi : M' \to M$, the group of covering transformations H_π is isomorphic to $N(G_\pi)/G_\pi$. Thus, if $\pi : \tilde{M} \to M$ is the universal covering of M, then*

$$H_\pi \approx \pi_1(M) \quad (\text{``}\approx\text{'' means isomorphic as groups}).$$

Proof.   We choose a base point $p_0 \in M$ and a $p_0' \in \pi^{-1}(p_0)$. Let $\gamma \in N(G_\pi(p_0))$. For any $p' \in M'$, let $g' : [0,1] \to M'$ be a path joining p_0' to p'. We put with $g := \pi(g')$

$$\varphi_\gamma(p') = (g\gamma)'(1).$$

If g_1' is another path in M' from p_0' to p', then $g_1^{-1}g \in G_\pi$, hence $\gamma^{-1}g_1^{-1}g\gamma \in G_\pi$, since $\gamma \in N(G_\pi)$. Thus $(g_1\gamma)'(1) = (g\gamma)'(1)$, i.e. the definition of $\varphi_\gamma(p')$ does not depend on the choice of g'. We have

$$\pi(\varphi_\gamma(p')) = \pi((g\gamma)'(1)) = \pi(g'(1)) = \pi(p'),$$

so that φ_γ is a covering transformation. Also,

$$\varphi_{\gamma_2\gamma_1}(p_0') = (\gamma_2\gamma_1)'(1) = \varphi_{\gamma_2} \circ \varphi_{\gamma_1}(p_0'),$$

hence $\varphi_{\gamma_2\gamma_1} = \varphi_{\gamma_2} \circ \varphi_{\gamma_1}$ by Lemma (1.3.5), and

$$\varphi_\gamma = \text{Id} \Longleftrightarrow \varphi_\gamma(p_0') = p_0' \quad (\text{by Lemma } (1.3.5))$$
$$\Longleftrightarrow \gamma'(1) = p_0' \Longleftrightarrow \gamma \in G_\pi.$$

Thus, we have defined a homomorphism of $N(G_\pi)$ into H_π with kernel G_π. Now let $\varphi \in H_\pi$, and let $\gamma' : [0,1] \to M'$ be a path from p_0' to $\varphi(p_0')$. We set $\gamma := \pi(\gamma')$. Then $\{\gamma\} \in N(G_\pi)$, and $\varphi_\gamma(p_0') = \varphi(p_0')$. Hence $\varphi_\gamma = \varphi$ by Lemma 1.3.5. Hence our homomorphism is also surjective, and our assertion follows.   □

Corollary 1.3.4 *Let G be a normal subgroup of $\pi_1(M, p_0)$ and $\pi : M' \to M$ the covering corresponding to G according to Theorem 1.3.2. Let $p_0' \in \pi^{-1}(p_0)$. Then, for every $p_0'' \in \pi^{-1}(p_0)$, there exists precisely one covering transformation φ with $\varphi(p_0') = p_0''$. This φ corresponds (under the isomorphism of Theorem 1.3.4) to $\pi(\gamma') \in \pi_1(M, p_0)$, where γ' is any path from p_0' to p_0''.*

Remark. H_π operates properly discontinuously in the sense of Def. 2.4.1 below, and
$M = M'/H_\pi$ in the sense of Def. 2.4.2.

Example 2 We consider the torus T of Example 2) in § 1.1. By construction

$$\pi : \mathbb{C} \to T$$

is a covering. We have

$$\pi_1(\mathbb{C}) = \{1\}$$

as $\mathbb{C}$ ($= \mathbb{R}^2$ as a manifold) is simply connected, see the Example after Lemma 1.2.2.
Therefore

$$\pi : \mathbb{C} \to T$$

is the universal covering of T. The corresponding covering transformations are given by

$$z \mapsto z + nw_1 + mw_2$$

for $n, m \in \mathbb{Z}$. Thus, the group H_π of covering transformations is $\mathbb{Z}^2$. From Theorem 1.3.4, we therefore conclude

$$\pi_1(T) = \mathbb{Z}^2 \ .$$

Since $\mathbb{Z}^2$ is an abelian group, conjugate subgroups are identical and therefore the equivalence classes of coverings of T are in bijective correspondance with the subgroups of $\mathbb{Z}^2$, by Theorem 1.3.2.
Let us consider the subgroup

$$G_{p,q} := \{(pn, qm) : \ n, m \in \mathbb{Z}\} \quad \text{for given } p, q \in \mathbb{Z}\backslash\{0\}.$$

This group corresponds to the covering

$$\pi_{p,q} : T_{p,q} \to T$$

where $T_{p,q}$ is the torus generated by pw_1 and qw_2 (in the same way as our original torus T is generated by w_1, w_2). By Theorem 1.3.4, the group of covering transformations is $\mathbb{Z}^2/G_{p,q} = \mathbb{Z}_p \times \mathbb{Z}_q$. $(\alpha, \beta) \in \mathbb{Z}_p \times \mathbb{Z}_q$ operates on $T_{p,q}$ via

$$z \mapsto z + \alpha w_1 + \beta w_2$$

(here, we consider α as an element of $\{0, 1, \ldots, p-1\}$, β as an element of $\{0, 1, \ldots, q-1\}$ and the addition is the one induced from $\mathbb{C}$.)

Let us consider the subgroup

$$G := \{(n, 0) : n \in \mathbb{Z}\} \qquad \text{of } \mathbb{Z}^2.$$

The corresponding covering this time is a cylinder C constructed as follows: We call $z_1, z_2 \in \mathbb{C}$ equivalent if there exists $n \in \mathbb{Z}$ with

$$z_1 - z_2 = nw_1.$$

Let π' be the projection which maps $z \in \mathbb{C}$ to its equivalence class. $C := \pi'(\mathbb{C})$ then becomes a differentiable manifold as in the construction of T. The group of covering transformations is $\mathbb{Z}^2/G = \mathbb{Z}$, again by Theorem 1.3.4. $m \in \mathbb{Z}$ here operates on C by

$$z \mapsto z + mw_2,$$

with the addition induced from $\mathbb{C}$.

More generally, consider the subgroup

$$G_p := \{(pn, 0) : n \in \mathbb{Z}\} \qquad \text{for } p \in \mathbb{Z}\backslash\{0\}.$$

The corresponding covering now is the cylinder C_p generated by pw_1, and the group of covering transformations is

$$\mathbb{Z}^2/G_p = \mathbb{Z}_p \times \mathbb{Z}.$$

For $\alpha \in \mathbb{Z}_p$, $q \in \mathbb{Z}$, the operations on C_p is

$$z \mapsto z + \alpha w_1 + q w_2$$

as above, with α considered as an element of $\{0, 1, \ldots, p-1\}$.

Exercises for § 1.3

1) Determine all equivalence classes of coverings of a torus and their covering transformations.
2) Construct a manifold M with a (nontrivial) covering map $\pi : S^3 \to M$. Hint: The group $SO(4)$ operates on S^3 considered as the unit sphere in $\mathbb{R}^4$. Find a discrete subgroup Γ of $SO(4)$ for which no $\gamma \in \Gamma\backslash\{\text{identity}\}$ has a fixed point on S^3.
3) Let

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{Z}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 3, \ ad - bc = 1 \right\}$$

operate on

$$H := \{z = x + iy \in \mathbb{C}, \; y > 0\}$$

via

$$z \mapsto \frac{az + b}{cz + d}.$$

Show that if $\gamma \in \Gamma$ is different from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then γ has no fixed points in H. Interpret Γ as the group of covering transformations associated with a manifold H/Γ and a covering $\pi : H \to H/\Gamma$. Construct different coverings of H/Γ associated with conjugacy classes of subgroups of Γ.

1.4 Global Continuation of Functions on Simply-Connected Manifolds

Later on, in §2.2, we shall need the following lemma. The reader might wish to read §2.2 before the present one, in order to understand the motivation for this lemma.

Lemma 1.4.1 *Let M be a simply connected manifold, and $\{U_\alpha\}$ an open covering of M, assume that all the U_α are connected. Suppose given on each U_α a family F_α of functions (not satisfying $F_\alpha = \emptyset$ for all α) with the following properties: i) if $f_\alpha \in F_\alpha$, $f_\beta \in F_\beta$ and $V_{\alpha\beta}$ is a component of $U_\alpha \cap U_\beta$, then*

$$f_\alpha \equiv f_\beta \text{ in a neighbourhood of some } p \in V_{\alpha\beta}$$

implies

$$f_\alpha \equiv f_\beta \text{ on } V_{\alpha\beta};$$

ii) if $f_\alpha \in F_\alpha$ and $V_{\alpha\beta}$ is a component of $U_\alpha \cap U_\beta$, then there exists a function $f_\beta \in F_\beta$ with

$$f_\alpha \equiv f_\beta \text{ on } V_{\alpha\beta}.$$

Then there exists a function f on M such that $f_{|U_\alpha} \in F_\alpha$ for all α. Indeed, given $f_{\alpha_0} \in F_{\alpha_0}$, there exists a unique such f with $f_{|U_{\alpha_0}} = f$.

Proof.   We consider the set of all pairs (p, f) with $p \in U_\alpha$, $f \in F_\alpha$ (α arbitrary).
We set

$$(p, f) \sim (q, g) \iff p = q \text{ and } f = g \text{ in some neighbourhood of } p.$$

Let $[p, f]$ be the equivalence class of (p, f), and M^* the set of such equivalence classes; define $\pi : M^* \to M$ by $\pi([p, f]) = p$.
For $f_\alpha \in F_\alpha$, let $U'(\alpha, f_\alpha) := \{[p, f_\alpha] : \; p \in U_\alpha\}$. Then $\pi : U'(\alpha, f_\alpha) \to U_\alpha$ is bijective. By (i), $\pi(U'(\alpha, f_\alpha) \cap U'(\beta, f_\beta))$ is a union of connected components

of $U_\alpha \cap U_\beta$, hence open in M. Thus the $U'(\alpha, f_\alpha)$ define a topology on M^*. ($\Omega \subset U'(\alpha, f_\alpha)$ is by definition open, if $\pi(\Omega) \subset U_\alpha$ is open. An arbitrary $\Omega \in M^*$ is open if $\Omega \cap U'(\alpha, f_\alpha)$ is open for each α.) This topology is Hausdorff by (i).

Now let M' be a connected component of M^*. We assert that $\pi : M' \to M$ is a covering. To see this, let $p^* = (p, f) \in \pi^{-1}(U_\alpha)$, i.e. $\pi(p^*) = p \in U_\alpha$. By definition of M^*, there is a β such that $p \in U_\beta$ and $f \in F_\beta$. Thus $p \in U_\alpha \cap U_\beta$. By (ii), there exists $g \in F_\alpha$ with $f(p) = g(p)$. Thus $p^* \in U'(\alpha, g)$. Conversely, each $U'(\alpha, g)$ is contained in $\pi^{-1}(U_\alpha)$. Hence

$$\pi^{-1}(U_\alpha) = \bigcup_{f_\alpha \in F_\alpha} U'(\alpha, f_\alpha).$$

The $U'(\alpha, f_\alpha)$ are open, and connected because they are homeomorphic to the U_α under π. By (i), for distinct $f_\alpha^1, f_\alpha^2 \in F_\alpha$, we have $U'(\alpha, f_\alpha^1) \cap U'(\alpha, f_\alpha^2) = \emptyset$. Hence the $U'(\alpha, f_\alpha)$ are the connected components of $\pi^{-1}(U_\alpha)$, and those of them which are contained in M' are the components of $M' \cap \pi^{-1}(U_\alpha)$. It follows that $\pi : M' \to M$ is a covering.

But M is simply connected by assumption, hence $\pi : M' \to M$ is a homeomorphism by Corollary 1.3.2.. Hence each $\pi^{-1}(U_\alpha)$ is a single $U'(\alpha, f_\alpha)$, $f_\alpha \in F_\alpha$. If $U_\alpha \cap U_\beta \neq \emptyset$, we must have $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$, so that there is a well-defined function f on M with

$$f_{|U_\alpha} = f_\alpha \in F_\alpha \text{ for all } \alpha,$$

using (ii). If $f_{\alpha_0} \in F_{\alpha_0}$ is prescribed, we choose M' as the connected component of M^* containing $U'(\alpha_0, f_0)$, so that the f obtained above satisfies $f_{|U_{\alpha_0}} = f_{\alpha_0}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. Constructions of the above kind (the space M^* with its topology) arise frequently in complex analysis under the name "Sheaf Theory". For our purposes, the above lemma is sufficient, so there is no need to introduce these general concepts here.

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