
Bézier patches on almost toric surfaces

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Summary. The paper is devoted to the parametrization extension problem: given a loop composed of 3 (resp. 4) rational Bézier curves on a rational surface X , find a triangular (resp. tensor product) Bézier patch on X of optimal degree bounded by these curves. The constructive solution to the formulated problem is presented in details for cases where X is a sphere or a hyperbolic paraboloid. Then X is an almost toric surface the general solution is outlined using the universal rational parametrization theory. Also an open problem of a linear precision property for toric Bézier patches is discussed.

1 Introduction

Rational surfaces are widely used in geometric modeling in the form of rational Bézier patches, which have been tested in a variety of practical applications. On the other hand, potential of arbitrary rational surfaces are not fully realized, since in general it is difficult to control real rational parametrizations: they can have basepoints, they are not necessary one-to-one or even onto.

We focus on the *parametrization extension problem*: how to find Bézier patches of optimal degree on the given surface with given boundary curves. Our idea is to study all rational parametrizations of a given rational surface in order to find suitable Bézier patches on it. Here we restrict our considerations to the simple cases of quadric surfaces and then sketch the extension of this theory to a large important subclass of rational surfaces in $\mathbb{P}^3(\mathbb{R})$, so called *real almost toric surfaces*, that include many real rational surfaces that are popular in geometric modeling. The reason is that a constructive description of all rational parametrizations exactly for this class of surfaces was developed in terms of the Universal Rational Parametrization (URP) [2]. This is a natural extension of the results obtained by Dietz, Hoschek and Jüttler [6, 7] and announced by the author in [10].

In Section 2 the parametrization extension problem is discussed for the simplest case of Bézier patches in $\mathbb{P}^n(\mathbb{R})$. Section 3 is devoted to surfaces

$\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ as abstract versions of an oval and a double ruled quadrics. We survey the URP theory for real almost toric surfaces and sketch the solution to the parametrization extension problem for this class of surfaces in Section 4. The last Section 5 is devoted to an open problem of a linear precision property for toric Bézier patches.

2 Bézier curves and surface patches in $\mathbb{P}^n(\mathbb{R})$

2.1 Definitions

A real n -dimensional projective space $\mathbb{P}^n(\mathbb{R})$ is defined as a space of all lines in $\mathbb{R}^{n+1}$ going through the origin $0 = (0, \dots, 0)$. Alternatively $\mathbb{P}^n(\mathbb{R})$ can be regarded as a quotient of $\mathbb{R}^{n+1} \setminus 0$ with respect to the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus 0$ action $\lambda \cdot (x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n)$. This means that $\mathbb{P}^n(\mathbb{R})$ is a factor set of $\mathbb{R}^{n+1} \setminus 0$ with respect to the following equivalence relation: $x \sim x'$ if there exists such $\lambda \in \mathbb{R}^*$ that $x' = \lambda \cdot x$ (denote also $x'/x = \lambda$). There is a natural projection

$$\Pi: \mathbb{R}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n(\mathbb{R}), \quad (2.1)$$

which maps $(x_0, \dots, x_n)$ to its equivalence class $[x_0, \dots, x_n]$. These x_i are homogeneous coordinates of a corresponding point in $\mathbb{P}^n(\mathbb{R})$.

Complex projective spaces $\mathbb{P}^d(\mathbb{C})$ are similarly defined: just change all appearances of $\mathbb{R}$ to $\mathbb{C}$ (complex numbers) in the previous definition. In particular, $\mathbb{C}^* = \mathbb{C} \setminus 0$ denotes the multiplicative group of non-zero complex numbers.

Let us denote a closed interval $[0, 1] \subset \mathbb{R}$ by I , and define a standard affine triangle $\triangle$ and a unit square $\square$ as follows

$$\triangle = \{(s, t) \in \mathbb{R}^2 \mid s \geq 0, t \geq 0, s + t \leq 1\}, \quad (2.2)$$

$$\square = \{(s, t) \in \mathbb{R}^2 \mid 0 \leq s \leq 1, 0 \leq t \leq 1\}. \quad (2.3)$$

We consider a rational Bézier curve $c: I \rightarrow \mathbb{P}^n(\mathbb{R})$, a Bézier rectangular (=tensor-product) patch $c: \square \rightarrow \mathbb{P}^n(\mathbb{R})$ or a Bézier triangular patch $c: \triangle \rightarrow \mathbb{P}^n(\mathbb{R})$ in a real projective space $\mathbb{P}^n(\mathbb{R})$ as a composition of the polynomial map γ and the projection Π :

$$D \xrightarrow{\gamma} \mathbb{R}^{n+1} \setminus 0 \xrightarrow{\Pi} \mathbb{P}^n(\mathbb{R}), \quad D = I, \square, \triangle. \quad (2.4)$$

Here γ is a polynomial Bézier curve $\gamma(t)$, a Bézier triangle or a Bézier rectangle $\gamma(s, t)$ defined as linear combinations of homogeneous control points $p_i, p_{ij}, p_{ijk} \in \mathbb{R}^{n+1}$

$$\sum_{i=0}^d p_i B_i^d(t), \quad \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} p_{ij} B_i^{d_1}(s) B_j^{d_2}(t), \quad \sum_{i+j+k=d} p_{ijk} B_{ijk}^d(s, t) \quad (2.5)$$

with coefficients expressed in terms of Bernstein polynomials

$$B_i^d(t) = \binom{d}{i} (1-t)^{d-i} t^i, \quad B_{ijk}^d(s, t) = \frac{d!}{i!j!k!} (1-s-t)^i s^j t^k. \quad (2.6)$$

In all these cases polynomial maps $\gamma : D \rightarrow \mathbb{R}^{n+1}$ are *liftings* of rational maps $c : D \rightarrow \mathbb{P}^n(\mathbb{R})$ in the sense that $\gamma \circ \Pi = c$.

Definition 1. Let γ and γ' be two polynomial maps.

- (i) γ and γ' are called *equivalent* $\gamma \sim \gamma'$ if $\Pi \circ \gamma = \Pi \circ \gamma'$.
- (ii) γ is called an *irreducible* if $\gcd(\gamma_0, \dots, \gamma_n) = 1$ for all coordinates $(\gamma_0, \dots, \gamma_n)$ of γ .

Lemma 2. Let γ and γ' be two polynomial maps. If $\gamma \sim \gamma'$ and γ is irreducible then $\gamma' = \lambda \cdot \gamma$ for some polynomial $\lambda = \lambda(t)$ (i.e. polynomial map $\lambda : D \rightarrow \mathbb{R}^*$ in the group $\mathbb{R}^*$).

Proof. For every t vectors $\gamma(t)$ and $\gamma'(t)$ are proportional, and a function $\lambda(t) = \gamma'_i(t)/\gamma_i(t)$ is defined for some their non-zero coordinates. So λ is a rational function, and it can be represented in an irreducible form $\lambda = p/q$, where p and q are some mutually prime polynomials. Hence $p\gamma_i = q\gamma'_i$, and q divides γ_i for all i . Since γ is irreducible, q is constant.

Corollary 3. If $\gamma \sim \gamma'$ and both polynomial maps γ and γ' are irreducible then $\gamma' = \lambda \cdot \gamma$, for some constant $\lambda \in \mathbb{R}^*$. Therefore, for any rational map $c : D \rightarrow \mathbb{P}^n(\mathbb{R})$ there exists the unique irreducible lifting $\tilde{c} : D \rightarrow \mathbb{R}^{n+1}$ up to constant multiplier in $\mathbb{R}^*$.

2.2 Parametrization extension problem

Let X be one of the previous spaces where some kind of curves in polynomial representation make sense, i.e. $X = \mathbb{R}^n, \mathbb{R}^{n+1} \setminus 0, \mathbb{P}^n(\mathbb{R}), \mathbb{R}^*, \mathbb{P}^n(\mathbb{C})$, any rational surface in $\mathbb{P}^n(\mathbb{R})$, etc.

Definition 4. A *chain* of curves $\langle c_i \rangle_0^k = \langle c_0, \dots, c_k \rangle$ in X is a sequence $c_i : I \rightarrow X$ such that $c_i(1) = c_{i+1}(0)$, $i = 0, \dots, k-1$. The chain $\langle c_i \rangle_0^k$ is *closed* if $c_0(0) = c_k(1)$.

Example 5. Let $\sigma : \triangle \rightarrow X$ be a Bézier triangle of degree d . Then there is a natural closed chain of Bézier curves of degree d called its boundary $\partial\sigma = \langle \partial_0\sigma, \partial_1\sigma, \partial_2\sigma \rangle$: $\partial_0\sigma(t) = \sigma(t, 1-t)$, $\partial_1\sigma(t) = \sigma(1-t, 0)$, $\partial_2\sigma(t) = \sigma(0, t)$.

Example 6. Let $\sigma : \square \rightarrow X$ be a Bézier quadrangle of bi-degree (d_1, d_2) . Then there is a natural closed chain of Bézier curves of degrees d_1, d_2, d_1, d_2 called its boundary $\partial\sigma = \langle \partial_0\sigma, \partial_1\sigma, \partial_2\sigma, \partial_3\sigma \rangle$: $\partial_0\sigma(t) = \sigma(t, 0)$, $\partial_1\sigma(t) = \sigma(1, t)$, $\partial_2\sigma(t) = \sigma(1-t, 1)$, $\partial_3\sigma(t) = \sigma(0, 1-t)$.

Now we are ready to formulate the *parametrization extension problem*.

Problem 7. For a given closed chain $\langle c_i \rangle_0^k$ of curves in X ($k = 2, 3$) find a Bézier patch $c : D \rightarrow \mathbb{P}^n(\mathbb{R})$ ($D = \triangle, \square$) such that $\partial c = \langle c_i \rangle_0^k$. The minimal possible parametrization degree would be desirable.

When X is an affine space $\mathbb{R}^n$ or $\mathbb{R}^n \setminus 0$ a solution of Problem 7 for polynomial patches is almost obvious. Let us start from the first non-trivial case $X = \mathbb{P}^n(\mathbb{R})$ in this section.

Any solution to Problem 7 $c : D \rightarrow \mathbb{P}^n(\mathbb{R})$ will have an irreducible lifting $\tilde{c} : D \rightarrow \mathbb{R}^{n+1}$. The boundary $\partial \tilde{c}$ should be closed (not necessary irreducible) lifting of the given chain $\langle c_i \rangle_0^k$. This is the necessary condition for existence of the solution.

Therefore we consider the following smaller problem first: how to find closed liftings of closed chains of curves. From Corollary 3 it follows that any chain has a unique lifting of irreducible curves in $\mathbb{R}^{n+1} \setminus 0$ with a given beginning point. Unfortunately, a lifting of a closed chain is not necessary closed.

Let $\langle c_i \rangle_0^k$ be a closed chain of Bézier curves in $\mathbb{P}^n(\mathbb{R})$, and let $\langle \gamma_i \rangle_0^k$ be a chain of arbitrary liftings of c_i , i.e. $\Pi \circ \gamma_i = c_i$, for all $i = 0, \dots, k$. Define a nonzero number

$$\Omega(\langle \gamma_i \rangle_0^k) = \frac{\gamma_0(0)}{\gamma_k(1)} \in \mathbb{R}^*. \quad (2.7)$$

It is clear that any proportional lifting $\langle \gamma'_i \rangle_0^k$, $\gamma'_i = \lambda \cdot \gamma_i$, gives the same number $\Omega(\langle \gamma'_i \rangle_0^k) = \Omega(\langle \gamma_i \rangle_0^k)$.

Suppose all $\tilde{c}_i$ are irreducible liftings of c_i for $i = 0, \dots, k$. Then the definition $\Omega(\langle c_i \rangle_0^k) := \Omega(\langle \tilde{c}_i \rangle_0^k)$ is justified by Corollary 3. In fact $\Omega(\langle c_i \rangle_0^k)$ is an obstruction to the existence of a closed irreducible lifting in the following sense.

Lemma 8. (i) A closed chain $\langle c_i \rangle$ of Bézier curves in $\mathbb{P}^n(\mathbb{R})$ has closed irreducible lifting to $\mathbb{R}^{n+1} \setminus 0$ if and only if $\Omega(\langle c_i \rangle) = 1$.

(ii) All closed liftings of $\langle c_i \rangle$ are in 1-1 correspondence with chains in the group $\mathbb{R}^*$ connecting 1 with $\Omega(\langle c_i \rangle)$.

Proof. The part (i) is clear, so we will check only (ii). Fix an irreducible lifting $\langle \gamma_i \rangle$ of a closed chain $\langle c_i \rangle$ in $\mathbb{P}^n(\mathbb{R})$. For any closed lifting $\langle \gamma'_i \rangle$ of the same chain with the same initial point $\gamma'_0(0) = \gamma_0(0)$ define a chain $\langle \lambda_i \rangle = \langle \gamma'_i / \gamma_i \rangle$ in the group $\mathbb{R}^*$. Since $\lambda_0(0) = \gamma'_0(0) / \gamma_0(0) = 1$, $\lambda_k(1) = \gamma'_k(1) / \gamma_k(1) = \Omega(\langle c_i \rangle)$, the chain $\langle \lambda_i \rangle$ connects 1 and $\Omega(\langle c_i \rangle)$. It is clear that inverse correspondence is defined by the formula $\langle \gamma'_i \rangle = \langle \lambda_i \cdot \gamma_i \rangle$.

Now one can solve the parametrization extension problem for a Bézier triangle in $\mathbb{P}^n(\mathbb{R})$ as stated the following theorem (the Bézier rectangle case is similar).

Theorem 9. Let $\langle c_0, c_1, c_2 \rangle$ be a closed chain of curves of degrees $\deg c_i = d_i$, $d = d_0 \geq d_1 \geq d_2$, in $\mathbb{P}^n(\mathbb{R})$. Then a Bézier triangle $c : \triangle \rightarrow \mathbb{P}^n(\mathbb{R})$ of the minimal degree such that $\partial c = \langle c_i \rangle$ can be found or its non-existence can be proved in two steps:

- (1) Find a closed lifting $\langle \gamma_i \rangle$ of the minimal degree d' of the chain $\langle c_i \rangle$ in $\mathbb{R}^{n+1} \setminus 0$. There are three cases depending on $\Omega = \Omega \langle c_i \rangle \in \mathbb{R}^*$
 - (i) if $\Omega < 0$ then such a lifting does not exist;
 - (ii) if $\Omega = 1$ then $d' = d$;
 - (iii) if $\Omega > 0$ and $\Omega \neq 1$ then $d' = d$, if $d_2 \leq d-1$, and $d' = d+1$, otherwise;
- (2) Find a Bézier triangle $\sigma : \Delta \rightarrow \mathbb{R}^{n+1} \setminus 0$ with a boundary $\partial\sigma = \langle \gamma_i \rangle$. Its existence and degree d'' depends on the dimension n and the position of the origin $0 \in \mathbb{R}^{n+1}$:
 - ($n = 1$): the *sufficient* condition for the existence: 0 does not belong to a convex hull of all control points of all curves γ_i ; then $d'' = d'$;
 - ($n > 1$): the triangle σ exists, $d'' = d'$, except the case $0 \in \sigma(\Delta)$ and $d' = 1, 2$ when $d'' = 3$.

Finally, if both steps are successful, $c = \Pi \circ \sigma$ is the solution.

2.3 Reparametrizations of Bézier curves

Definition 10. Call two curves $c, c' : I \rightarrow \mathbb{P}^n(\mathbb{R})$ similar if they have the same endpoints and $c(I) = c'(I)$, i.e. they have the same images and orientations.

There is a useful method how to obtain a new polynomial Bézier curve from an old one with the same initial point and a similar projection:

$$\Phi_\rho^d : \sum_{i=0}^d p_i B_i^d(t) \mapsto \sum_{i=0}^d (\rho^i p_i) B_i^d(t), \quad \rho \in \mathbb{R}^*. \quad (2.8)$$

In fact $\Pi \circ \Phi_\rho^d \gamma$ is a composition of $\Pi \circ \gamma$ and some projective transformation of the domain $I = [0, 1]$ fixed on the endpoints (see details in [15]). Note that one endpoint is fixed $\Phi_\rho^d \gamma(0) = \gamma(0)$ and the other $\Phi_\rho^d \gamma(1) = \rho^d \cdot \gamma(1)$ depends on degree d and the parameter ρ .

In practice it is useful to relax requirements on the boundary curves and to formulate the following variant of Problem 7.

Problem 11. For a given closed chain $\langle c_i \rangle_0^k$ of curves in X ($k = 2, 3$) find a Bézier patch $c : D \rightarrow \mathbb{P}^n(\mathbb{R})$ ($D = \Delta, \square$) such that the chain ∂c is similar to $\langle c_i \rangle_0^k$. The minimal possible parametrization degree would be desirable.

This reparametrization trick can help to find better solutions. For example, we can skip the step (1)(iii) in Theorem 9, since the obstruction $\Omega > 0$ can be reduced to the unit 1 after the transformation Φ_ρ^d , $\rho = \Omega^{1/d}$.

3 Bézier patches in $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$

One can study Bézier patches also in more general spaces $\mathbb{P}^n(\mathbb{C})$ or $\mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^k(\mathbb{R})$ using the similar approach. The cases $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ are most

interesting, since they are directly related to oval and double ruled quadrics in $\mathbb{P}^3(\mathbb{R})$.

Define Bézier curves and surface patches on $\mathbb{P}^1(\mathbb{C})$ as compositions

$$D \xrightarrow{\gamma} \mathbb{C}^2 \setminus 0 \xrightarrow{\Pi} \mathbb{P}^1(\mathbb{C}), \quad D = \mathbf{I}, \square, \triangle, \quad (3.1)$$

where the map $\gamma : D \rightarrow \mathbb{C}^2 \setminus 0$ is polynomial with complex coefficients. We call γ a lifting of a rational map $c : D \rightarrow \mathbb{P}^1(\mathbb{C})$ if $\gamma \circ \Pi = c$.

Now we can repeat the similar definitions and lemmas as in Section 2.

Definition 12. Let γ and γ' be two complex polynomial maps.

- (i) γ and γ' are called *S-equivalent* $\gamma \sim_S \gamma'$ if $\Pi \circ \gamma = \Pi \circ \gamma'$.
- (ii) γ is called *S-irreducible* if $\gcd(\gamma_0, \gamma_1) = 1$ for complex coordinates (γ_0, γ_1) of γ .

Lemma 13. Let γ and γ' be two complex polynomial maps. If $\gamma \sim_S \gamma'$ and γ is *S-irreducible* then $\gamma' = \lambda \cdot \gamma$ for some complex polynomial $\lambda = \lambda(t)$ (i.e. polynomial map $\lambda : D \rightarrow \mathbb{C}^*$ in the group $\mathbb{C}^*$).

Corollary 14. If $\gamma \sim \gamma'$ and both complex polynomial maps γ and γ' are *S-irreducible* then $\gamma' = \lambda \cdot \gamma$, for some constant $\lambda \in \mathbb{C}^*$. Therefore, for any rational map $c : D \rightarrow \mathbb{P}^1(\mathbb{C})$ there exists the unique *S-irreducible lifting* $\tilde{c} : D \rightarrow \mathbb{C}^2$ up to constant multiplier in $\mathbb{C}^*$.

Let $\langle c_i \rangle_0^k$ be a closed chain of Bézier curves in $\mathbb{P}^1(\mathbb{C})$, and let $\langle \gamma_i \rangle_0^k$ be a chain of arbitrary liftings of c_i . Then we define a nonzero complex number

$$\Omega(\langle \gamma_i \rangle_0^k) = \frac{\gamma_0(0)}{\gamma_k(1)} \in \mathbb{C}^*. \quad (3.2)$$

Similarly using the *S-irreducible lifting* we define (cf. Corollary 14) the obstruction $\Omega\langle c_i \rangle_0^k := \Omega\langle \tilde{c}_i \rangle_0^k$. The following lemma can be proved similarly to Lemma 8.

Lemma 15. (i) A closed chain $\langle c_i \rangle$ of Bézier curves in $\mathbb{P}^1(\mathbb{C})$ has closed *S-irreducible lifting* to $\mathbb{C}^2 \setminus 0$ if and only if $\Omega\langle c_i \rangle = 1$.
(ii) All closed liftings of $\langle c_i \rangle$ are in 1-1 correspondence with chains in the group $\mathbb{C}^*$ connecting 1 with $\Omega\langle c_i \rangle$.

A unit sphere S with the equation $x_0^2 = x_1^2 + x_2^2 + x_3^2$ in $\mathbb{P}^3(\mathbb{R})$ can be parametrized by $\mathbb{P}^1(\mathbb{C})$ using the map

$$P_S(z_0, z_1) = (z_0 \bar{z}_0 + z_1 \bar{z}_1, \operatorname{Re}(2\bar{z}_0 z_1), \operatorname{Im}(2\bar{z}_0 z_1), z_0 \bar{z}_0 - z_1 \bar{z}_1), \quad (3.3)$$

which is homogeneous in the following sense $P_S(\lambda z_0, \lambda z_1) = |\lambda|^2 P_S(z_0, z_1)$.

Finally we use the key observation that follows from Section 4: there is 1-1 correspondence $\gamma \mapsto P_S \circ \gamma$ between *S-irreducible* maps $\gamma : D \rightarrow \mathbb{C}^2 \setminus 0$ and such irreducible maps $\gamma' : D \rightarrow \mathbb{R}^4 \setminus 0$, that the image $\Pi(\gamma'(D))$ is contained on the sphere $S \subset \mathbb{P}^3(\mathbb{R})$. Hence we can apply obtained results on $\mathbb{P}^1(\mathbb{C})$ to the sphere case, using the simple formula for degrees $\deg(P_S \circ \gamma) = 2 \deg \gamma$.

Theorem 16. *Let $\langle c_0, c_1, c_2 \rangle$ be a closed chain of curves of degrees $\deg c_i = 2d_i$, $d = d_0 \geq d_1 \geq d_2$, on the sphere S^2 . Then there exists a Bézier triangle c with ∂c equal (or similar) to this chain such that at least $\deg c = 2d'$, where d' depends on $\Omega = \Omega\langle c_0, c_1, c_2 \rangle \in \mathbb{C}^*$ and d_i according to the following cases (except a few degenerated cases when $d = 1, 2$):*

- (i) $\Omega = 1$ then $d' = d$;
- (ii) $\Omega \in \mathbb{R}_{>0}$, $\Omega \neq 1$ then $d' = d$ (with reparametrization), otherwise see (iii);
- (iii) $\Omega \in \mathbb{C} \setminus \mathbb{R}$ then $d' = d$ if $d_2 \leq d - 1$, otherwise $d' = d + 1$;
- (iv) $\Omega \in \mathbb{R}_{<0}$ then $d' = d$ if $d_2 \leq d - 2$ or $d_1 \leq d - 1$, otherwise $d' = d + 1$;

Proof. By Lemma 15 we need to find a chain in $\mathbb{C}^*$ connecting 1 and Ω in $\mathbb{C}^*$. The case (i) is obvious. In cases (ii) (without reparametrization) and (iii) we can connect 1 and Ω just by the line $(1 - t) \cdot 1 + t \cdot \Omega$. In the case (iv) we have to make a detour round the complex zero via an arc of some parabola or two line segments.

Consider reparametrizations of spherical curves. For any polynomial curve $\gamma: \mathbb{I} \rightarrow \mathbb{C}^2 \setminus 0$ the following formula can be easily verified

$$P_S \circ \Phi_\rho^d(\gamma) = \Phi_\rho^{2d}(P_S \circ \gamma). \quad (3.4)$$

Therefore reparametrization paths coincide with lines in $\mathbb{C}^*$ that go through the origin. So, if $\Omega \in \mathbb{R}_{>0}$ then we can move it to 1 using reparametrization of any curve γ_i .

Finally we have built a lifting $\langle \gamma'_i \rangle$ of degree d' in $\mathbb{C}^2 \setminus 0$ of the given chain $\langle c_i \rangle$ on the sphere S . Now we can almost always fill it by a Bézier triangle σ (i.e. $\partial \sigma = \langle \gamma'_i \rangle$) of the same degree d' . Then put $c = \Pi \circ P_S \circ \sigma$, $\deg c = 2d'$. There are just a few degenerated cases when the triangle σ is going through the origin $0 \in \mathbb{C}^2$ (cf. Theorem 9, Step (2) then $n = 1$).

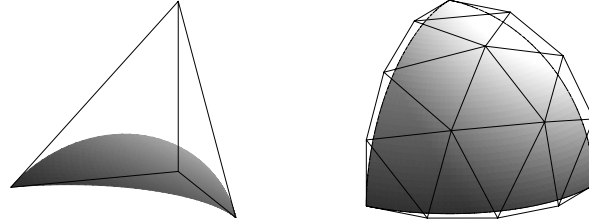


Fig. 1. A biangle and a symmetric Bézier triangle on the sphere.

Example 17. The obstruction Ω has a clear geometrical meaning. Consider a biangle patch on the sphere bounded by two circular arcs c_1 and c_2 as in Fig. 1(left). Add one constant curve $*$: $t \mapsto c_1(0)$ and obtain a closed chain $\langle *, c_1, c_2 \rangle$. Calculate $\Omega = \Omega\langle *, c_1, c_2 \rangle$, and suppose that $|\Omega| = 1$ (if not then

reparametrize c_2 according to (3.4) with $d = 1$ and $\rho = \sqrt{|\Omega|}$, i.e. $\Omega = e^{i\varphi}$. An angle φ is nothing else than an angle of the biangle. See [9] for details.

Example 18. Suppose we would like to find a symmetric Bézier representation of an octant of the sphere as in Fig. 1(right). It is easy to calculate S -irreducible representations for all three circular sides (as quadratic Bézier curves with weights: $1, 1/\sqrt{2}, 1$). The obstruction Ω is equal to $e^{i\varphi}$, $\varphi = \pi/4$. Indeed, for a spherical triangle with circular edges φ means a half of a difference between sum of all three angles and π (cf. [6]). Taking into account the symmetry requirement we build the chain of three lines connecting points $1, e^{i\pi/12}, e^{i\pi/6}$ and $e^{i\pi/4} = \Omega$ in $\mathbb{C}^*$. In Fig. 1(right) we see the resulting quartic Bézier triangle with its control net.

Let us consider the case of $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ now. Define Bézier curves and surface patches on $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ as compositions

$$D \xrightarrow{\gamma} (\mathbb{R}^2 \setminus 0) \times (\mathbb{R}^2 \setminus 0) \xrightarrow{\Pi^2} \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}), \quad D = I, \square, \triangle, \quad (3.5)$$

where the map $\gamma : D \rightarrow (\mathbb{R}^2 \setminus 0) \times (\mathbb{R}^2 \setminus 0)$ is in fact a pair $(\gamma^{(0)}, \gamma^{(1)})$ of polynomial maps $\gamma^{(i)} : D \rightarrow \mathbb{R}^2 \setminus 0$, $i = 0, 1$. We call γ a lifting of a rational map $c : D \rightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ if $\Pi^2 \circ \gamma = c$.

Now we can repeat similar definitions and lemmas as in case of $\mathbb{P}^1(\mathbb{C})$.

Definition 19. Let γ and γ' be two polynomial maps $D \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$.

- (i) γ and γ' are called *H-equivalent* $\gamma \sim_H \gamma'$ if $\Pi^2 \circ \gamma = \Pi^2 \circ \gamma'$.
- (ii) $\gamma = (\gamma_0, \gamma_1)$ is called *H-irreducible* if γ_0 and γ_1 are irreducible as polynomial maps $D \rightarrow \mathbb{R}^2$.

The group $\mathbb{R}^* \times \mathbb{R}^*$ acts on $(\mathbb{R}^2 \setminus 0) \times (\mathbb{R}^2 \setminus 0)$ by the formula $(\lambda, \mu) \cdot (s_0, s_1, t_0, t_1) = (\lambda s_0, \lambda s_1, \mu t_0, \mu t_1)$.

Lemma 20. Let γ and γ' be two polynomial maps $D \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$. If $\gamma \sim_H \gamma'$ and γ is *H-irreducible* then $\gamma' = (\lambda, \mu) \cdot \gamma$ for some polynomial pair $\lambda(t), \mu(t)$ (i.e. a polynomial map $\lambda : D \rightarrow \mathbb{R}^* \times \mathbb{R}^*$ in the group $\mathbb{R}^* \times \mathbb{R}^*$).

Corollary 21. If $\gamma \sim_H \gamma'$ and both polynomial maps γ and γ' are *H-irreducible* then $\gamma' = \lambda \cdot \gamma$, for some constant $(\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}^*$. Therefore, for any rational map $c : D \rightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ there exists the unique *H-irreducible lifting* $\tilde{c} : D \rightarrow (\mathbb{R}^2 \setminus 0) \times (\mathbb{R}^2 \setminus 0)$ up to constant multiplier in $\mathbb{R}^* \times \mathbb{R}^*$.

Let $\langle c_i \rangle_0^k$ be a closed chain of Bézier curves in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$, and let $\langle \gamma_i \rangle_0^k$ be a chain of arbitrary liftings of c_i . Then we define a pair of real nonzero numbers

$$\Omega(\langle \gamma_i \rangle_0^k) = \left(\frac{\gamma_0^{(0)}(0)}{\gamma_k^{(0)}(1)}, \frac{\gamma_0^{(1)}(0)}{\gamma_k^{(1)}(1)} \right) \in \mathbb{R}^* \times \mathbb{R}^*. \quad (3.6)$$

Similarly using the *H-irreducible lifting* we define the obstruction $\Omega\langle c_i \rangle_0^k := \Omega\langle \tilde{c}_i \rangle_0^k$ (cf. Corollary 21). The following lemma is a consequence of Lemma 8.

Lemma 22. (i) A closed chain $\langle c_i \rangle$ of Bézier curves in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ has closed H -irreducible lifting to $(\mathbb{R}^2 \setminus 0) \times (\mathbb{R}^2 \setminus 0)$ if and only if $\Omega\langle c_i \rangle = 1$.
 (ii) All closed liftings of $\langle c_i \rangle$ are in 1–1 correspondence with chains in the group $\mathbb{R}^* \times \mathbb{R}^*$ connecting 1 with $\Omega\langle c_i \rangle$.

A hyperbolic paraboloid $H: x_0x_3 = x_1x_2$ in $\mathbb{P}^3(\mathbb{R})$ can be parametrized by $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ using the map

$$P_H(s_0, s_1, t_0, t_1) = (s_0t_0, s_1t_0, s_0t_1, s_1t_1), \quad (3.7)$$

which is bi-homogeneous $P_H(\lambda s_0, \lambda s_1, \mu t_0, \mu t_1) = \lambda\mu P_H(s_0, s_1, t_0, t_1)$.

Finally we use the key observation that follows from Section 4: there is 1–1 correspondence $\gamma \mapsto P_H \circ \gamma$ between H -irreducible maps $\gamma: D \rightarrow (\mathbb{R}^2 \setminus 0) \times (\mathbb{R}^2 \setminus 0)$ and such irreducible maps $\gamma': D \rightarrow \mathbb{R}^4 \setminus 0$, that the image $\Pi(\gamma'(D))$ is contained on the hyperbolic paraboloid $H \subset \mathbb{P}^3(\mathbb{R})$. Hence we can apply the results obtained in case of $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ to this case as well, using the formula for degrees $\deg(P_S \circ \gamma) = \deg \gamma^{(0)} + \deg \gamma^{(1)}$.

Theorem 23. Let $\langle c_0, c_1, c_2 \rangle$ be a closed chain of curves on H of bi-degrees

$$(\deg_0 c_i, \deg_1 c_i) = (d_{i0}, d_{i1}), \quad d_* = (d_{*0}, d_{*1}) = (\max_i d_{i0}, \max_i d_{i1}).$$

Then a triangle $c: \triangle \rightarrow H \subset \mathbb{P}^3(\mathbb{R})$ of the minimal degree d' such that ∂c is equal (or similar) to $\langle c_i \rangle$ can be found or its non-existence can be proved in two steps:

- (1) Find a closed lifting $\langle \gamma_i \rangle$ of the minimal bi-degree $d' = (d'_0, d'_1)$ of the chain $\langle c_i \rangle$ in $(\mathbb{R}^2 \setminus 0)^2$ (after possible reparametrization). There are three cases depending on the obstruction $(\Omega_0, \Omega_1) = \Omega = \Omega\langle c_0, c_1, c_2 \rangle \in (\mathbb{R}^*)^2$
 - (i) if $\Omega_0 < 0$ or $\Omega_1 < 0$ then such a lifting does not exist;
 - (ii) if $\Omega = (1, 1)$ then $d' = d_*$;
 - (iii) if not all d_i are equal to d_* or $d_{*1} \log \Omega_0 = d_{*0} \log \Omega_1$ then $d' = d_*$, otherwise $d' = (d_{*0} + 1, d_{*1})$ or $d' = (d_{*0}, d_{*1} + 1)$.
- (2) Find a polynomial Bézier triangle σ in $(\mathbb{R}^2 \setminus 0)^2$ with a boundary $\partial \sigma = \langle \gamma_i \rangle$, i.e. find a pair of such triangles $(\sigma^{(0)}, \sigma^{(1)})$ with a respective pair of boundaries in $\mathbb{R}^2 \setminus 0$. This is exactly the step (2) in Theorem 9 when $n = 1$.

Finally, if both steps are successful then put $c = \Pi_H \circ \sigma$.

Proof. Consider only the most different from Theorem 9 case (iii) of the step (1). The key idea is to reparametrize curves c_i and to simplify the obstruction Ω . For a curve $\gamma = (\gamma^{(0)}, \gamma^{(1)})$ in $(\mathbb{R}^2 \setminus 0)^2$ with a bi-degree (d_0, d_1) reparametrization of $P_H \circ \gamma$ can be calculated by a formula

$$P_H \circ (\Phi_\rho^{d_0}(\gamma^{(0)}), \Phi_\rho^{d_1}(\gamma^{(1)})) = \Phi_\rho^{d_0+d_1}(P_H \circ \gamma).$$

Hence, the reparametrization of an H -irreducible lifting $\tilde{c}_i = (\tilde{c}_{i0}, \tilde{c}_{i1})$ with different values of the parameter ρ traces a curve

$$\rho \mapsto (\rho^{-d_{i0}} \Omega_0, \rho^{-d_{i1}} \Omega_1) \in \mathbb{R}^* \times \mathbb{R}^*.$$

In logarithmic coordinates $(\log, \log): (\mathbb{R}^*)^2 \rightarrow \mathbb{R}^2$ such curve converts to a line going through a point $(\log \mathcal{O}_0, \log \mathcal{O}_1)$ with a direction vector $d_i = (d_{i0}, d_{i1})$. Now our goal is to reach the origin $(0, 0) = (\log 1, \log 1)$ when we are allowed to travel only in given fixed directions d_i , $i = 0, 1, 2$. Distinguish two cases: all d_i are equal to d_* or not. In the latter case we can find at least two different directions $d_* = (d_{*0}, d_{*1})$ and, for example, $(d_{*0} - 1, d_{*1})$ or $(d_{*0}, d_{*1} - 1)$ after possible degree raising (we skip trivial case when d_{*0} or d_{*1} is zero). It is sufficient for our success. In the former case there is only one direction $d_* = (d_{*0}, d_{*1})$. So we can reach the origin if and only if $d_{*1} \log \Omega_0 = d_{*0} \log \Omega_1$. Otherwise, we move until we intersect any of the coordinate axes and then apply Theorem 9, step (1).

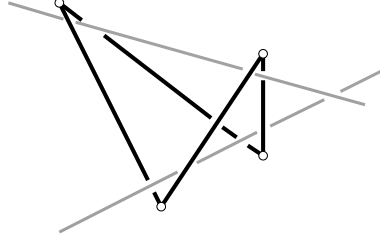


Fig. 2. A closed chain of line segments in $\mathbb{P}^3(\mathbb{R}) \setminus (L_1 \cup L_2)$.

Remark 24. The step (2) in Theorem 23 we have only sufficient conditions for existence. In practice this works in most cases. Homotopical triviality of the closed chain is an obvious necessary condition but it is not sufficient in general for given degrees. Consider for example a closed linear chain in $X = (\mathbb{R}^2 \setminus 0) \times (\mathbb{R}^2 \setminus 0)$ that is visualized in Fig. 2. In order to show the picture in 3D we reduced dimension 4 to 3 by including X in $\mathbb{R}^4 \setminus 0$ and then projecting to $\mathbb{P}^3(\mathbb{R})$. The complement $\mathbb{P}^3(\mathbb{R}) \setminus \Pi(X)$ is a union of two lines L_1 and L_2 shown in grey. Our goal is to fill the contour with Bézier rectangular patch of bi-degree $(1, 1)$ at the same time avoiding intersections with lines. Any such patch should be contained in the convex hull of control points, i.e., in the tetrahedron. Unfortunately, this is even topologically impossible: the contour is contractible in $\mathbb{P}^3(\mathbb{R}) \setminus (L_1 \cup L_2)$, but it is non-contractible in the affine (i.e. visible) part $\mathbb{R}^3 \setminus (L_1 \cup L_2)$.

4 General case: real almost toric surfaces

Here we just survey the theory of universal rational parametrizations in case of almost toric surfaces with canonical real structures for simplicity. We adapt

for our needs a part of more general results obtained for more general toric varieties [2] (also [13] for non-canonical real structures).

Let $\mathbb{Z}^2$ be the lattice of points with integer coordinates in the plane $\mathbb{R}^2$ and $\Delta \subset \mathbb{R}^2$ be a *lattice polygon*, i.e. a convex polygon with vertices in the lattice. Let the edges of the polygon Δ be on the lines $L_i(x) = 0$, $i = 1, \dots, r$, defined by the affine forms $L_i(x) = \langle n_i, x \rangle + a_i$, where $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathbb{Z}^2$, the normals n_i of edges are inward oriented shortest vectors with integer coordinates. Define the rational map (where $q_m \in \mathbb{C}^{d+1}$ are called control points):

$$P_\Delta(u_1, \dots, u_r) = \sum_{m \in \Delta \cap \mathbb{Z}^2} q_m u_1^{L_1(m)} \dots u_r^{L_r(m)}, \quad (4.1)$$

Suppose that the control points q_m are linearly independent. Then P_Δ defines a rational map to $\mathbb{P}^N(\mathbb{R})$, $N = \#(\Delta \cap \mathbb{Z}^2) - 1$.

Definition 25. A toric surface X_Δ is a subset in $\mathbb{P}^N(\mathbb{C})$ which is parametrized by a rational map P_Δ with linearly independent control points.

In this case it is well known that an implicit degree of X_Δ in $\mathbb{P}^N(\mathbb{C})$ is equal to the normalized area $\text{Area}(\Delta)$ of the polygon Δ , i.e. twice its usual euclidean area (cf. [3]). Also P_Δ is a rational map $\mathbb{C}^r \dashrightarrow \mathbb{P}^N(\mathbb{C})$ which is well defined outside of the *exceptional set* $Z = P_\Delta^{-1}(0) \subset \mathbb{C}^r$, $0 \in \mathbb{C}^{N+1}$.

In the general case P_Δ can be decomposed $\mathbb{C}^r \rightarrow \mathbb{P}^N(\mathbb{C}) \rightarrow \mathbb{P}^d(\mathbb{C})$, where the last map is some central projection $\pi : \mathbb{P}^N(\mathbb{C}) \rightarrow \mathbb{P}^d(\mathbb{C})$.

Definition 26. The image $X = \pi X_\Delta \subset \mathbb{P}^d(\mathbb{C})$ of the toric surface X_Δ is called *almost toric* if their implicit degrees coincide $\deg X = \deg X_\Delta$, i.e. $\deg X$ is equal to the normalized area $\text{Area}(\Delta)$ of the lattice polygon Δ .

We define a *real almost toric surface* as a real part $\mathbb{R}_c X$ of some complex almost toric surface X with respect to certain real structure c . The canonical real structure means that all coefficients must be real. References for non-canonical real structures (NCRS) are [13, 11].

Fix a lattice polygon $\Delta \subset \mathbb{R}^2$ bounded by lines L_i with normals n_i , $i = 1, \dots, r$, and the rational map P_Δ well defined on $\mathbb{C}^r \setminus Z$.

Then a subgroup $G \subset (\mathbb{C}^*)^r$, defined by two equations $\prod_{i=0}^r \lambda_i^{\langle n_i, e_j \rangle} = 1$ ($\{e_j\}$, $j = 1, 2$, is a basis of $\mathbb{Z}^2$) naturally acts on the space $\mathbb{C}^r \setminus Z$. The exceptional set Z , the group G and the group action depends only on the collection of normals $\{n_1, \dots, n_r\}$ which encode all the information about the *normal fan* Σ of the polygon Δ (see details in [1]). A fan Σ is *regular* if all pairs of adjacent normals $\{n_i, n_{i+1}\}$, $i = 1, \dots, r-1$, and $\{n_r, n_0\}$ define a basis in $\mathbb{Z}^2$. An *abstract* toric surface X_Σ is defined as a factor $(\mathbb{C}^r \setminus Z)/G$ by the action of the group G . X_Σ is non-singular if and only if Σ is regular.

The map P_Δ is homogeneous in the following sense: if $(\lambda_1, \dots, \lambda_r) \in G$ then $P_\Delta(\lambda_1 u_1, \dots, \lambda_r u_r) = \theta_\Delta(\lambda) P_\Delta(u_1, \dots, u_r)$, $\theta_\Delta(\lambda) = \prod_{i=1}^r \lambda^{a_i}$.

We say that a polynomial map $f = (f_1, \dots, f_r) : \mathbb{C}^k \rightarrow \mathbb{C}^r$ is Σ -irreducible if $\gcd(f_{i_1}, \dots, f_{i_s}) = 1$ whenever no vertex of Δ is incident with its edges defined by the lines $L_{i_1}, \dots, L_{i_s}$ for all subsets $\{i_1, \dots, i_s\} \subset \{1, \dots, r\}$.

The following theorem is valid also for the singular case if the fan Σ is substituted with its regular refinement $\tilde{\Sigma}$ [2].

Theorem 27. *Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an almost toric surface, defined as an image of P_Δ associated with a lattice polygon Δ with a regular fan Σ as above. Denote by $C(X) = \Pi^{-1}X \subset \mathbb{C}^{d+1}$ the cone over X , and by $\text{Sing } X$ the singular locus of X . Then P_Δ is a **universal rational parametrization** of X in the following sense:*

- (1) *for any Σ -irreducible map $f : \mathbb{C}^k \rightarrow \mathbb{C}^r$ the composition $P_\Delta \circ f$ is an irreducible map $\mathbb{C}^k \rightarrow C(X) \subset \mathbb{C}^{d+1}$;*
- (2) *conversely, given any irreducible map $g : \mathbb{C}^k \rightarrow CX \subset \mathbb{C}^{d+1}$ such that its image $\Pi(g(\mathbb{C}^k))$ in X is not contained in $\text{Sing } X$, there is a Σ -irreducible $f : \mathbb{C}^k \rightarrow \mathbb{C}^r$ such that $g = P_\Delta \circ f$.*
- (3) *If f and f' are Σ -irreducible, then $P_\Delta \circ f = P_\Delta \circ f'$ if and only if $f' = \lambda \cdot f$ for some $\lambda \in G$, $\theta_\Delta(\lambda) = 1$.*

The following corollary (cf. [2, Theorem 5.1]) will be most important for our applications.

Corollary 28. *Any polynomial map $h : \mathbb{C}^k \rightarrow \mathbb{C}^r$ can be represented in the form $h = g \cdot f$, where $f : \mathbb{C}^k \rightarrow \mathbb{C}^r$ is Σ -irreducible polynomial map and $g : \mathbb{C}^k \rightarrow G$ is in general a rational map to the group G .*

Indeed, call maps g in the group G admissible w.r.t a polynomial map f if $h = g \cdot f$ is also a polynomial map (cf. [2, Example 5.2]).

Now we switch to real coefficients and polygonal domains $D = I, \Delta, \square \subset \mathbb{R}^2$ as in Sections 2 and 3. Consider a real almost toric surface X and its parametrization $\Pi_\Delta = \Pi \circ P_\Delta : \mathbb{R}^r \setminus Z \rightarrow X \subset \mathbb{P}^d(\mathbb{R})$. Bézier curves and surface patches on X are rational maps $c : D \rightarrow X$ that can be lifted to some polynomial maps $\gamma : D \rightarrow \mathbb{R}^r \setminus Z$, i.e. $c = \Pi_\Delta \circ \gamma$.

Let $\langle c_i \rangle_0^k$ be a closed chain of Bézier curves on X , and let $\langle \tilde{c}_i \rangle_0^k$ be a chain of Σ -irreducible liftings of c_i . Since it is unique up to the action of G , we can define the obstruction $\Omega(\langle c_i \rangle_0^k) = \tilde{c}_0(0)/\tilde{c}_k(1) \in G$. The following lemma is similar to Lemma 8.

Lemma 29. (i) *A closed chain $\langle c_i \rangle$ of Bézier curves in X has closed Σ -irreducible lifting to $\mathbb{R}^r \setminus Z$ if and only if $\Omega\langle c_i \rangle = 1$.*
(ii) *All closed liftings of $\langle c_i \rangle$ are in 1-1 correspondence with admissible chains in the group G connecting 1 with $\Omega\langle c_i \rangle$.*

Now we are ready to sketch a parametrization extension algorithm which should solve Problems 7 and 11 for the most general case of a real almost toric surface $X \subset \mathbb{P}^d(\mathbb{R})$ parametrized by $\Pi_\Delta = P_\Delta \circ \Pi : \mathbb{R}^r \setminus Z \rightarrow X$.

4.1 Parametrization extension algorithm

Input: a closed chain of curves $\langle c_i \rangle$ on X .

Output: a Bézier patch $c : D \rightarrow X$ of minimal degree such that $\partial c = \langle c_i \rangle$.

- (1) lift the boundary chain $\langle c_i \rangle$ to Σ -irreducible chain $\langle \gamma_i \rangle$ in the space $\mathbb{R}^s \setminus Z$, and calculate $\Omega = \Omega \langle \gamma_i \rangle$; IF $\Omega = 1$ THEN go to (4);
- (2) simplify $\Omega := \Phi \cdot \Omega$ by using reparametrization paths of curves γ_i ; IF $\Omega = 1$ THEN go to (4);
- (3) find a closed chain of *admissible* curves $\langle g_i \rangle$ in the group G connecting 1 and Ω , and calculate the closed lifting $\langle \gamma_i \rangle$, $\gamma_i := g_i \cdot \gamma_i$;
- (4) fill this closed contour by some Bézier patch $\sigma : D \rightarrow \mathbb{R}^s \setminus Z$, i.e. $\partial \sigma = \langle \gamma_i \rangle$;
- (5) calculate a Bézier patch $c : D \rightarrow X$ as the projection of σ : $c = \Pi_\Delta \circ \sigma$.

4.2 Examples of real almost toric surfaces

A class of real almost toric surfaces includes many rational surfaces that are popular in geometric modeling:

- quadric surfaces in $\mathbb{P}^3(\mathbb{R})$:
 - double ruled quadric, in particular a hyperbolic paraboloid H (Section 3);
 - oval quadric (with NCRS), in particular a sphere S (Section 3) [13], a quadratic biangle patch [9].
 - quadratic cone (singular case) (see [13])
- toric Bézier patches (with general control points) [12], in particular:
 - Bézier triangular and rectangular patches (images of $\mathbb{P}^2(\mathbb{R})$ and $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$);
 - cones over arbitrary rational curves (singular case) [13];
 - *all* ruled surfaces $X \subset \mathbb{P}^3(\mathbb{R})$: according to [16, Theorem 3], there are two directrix curves on X of degrees d_1 and d_2 , $d_1 + d_2 = \deg X$, and the corresponding lattice trapezoid has the same normalized area;
- other cases with NCRS:
 - quartic ring, 1-horn and 2-horn Dupin cyclides [11], in particular a torus [13];
 - surfaces of revolution with conical directrices [11];
 - a pear shaped surface of degree 6 [11];
 - higher degree biangle patches with general control points [9].

5 Linear precision for toric Bézier patches

In Section 2 a rational Bézier patch is defined as a map $c : D \rightarrow \mathbb{P}^n(\mathbb{R})$ that is defined on a polygon $D = \triangle, \square$ in an affine plane $\mathbb{R}^2$ and takes its values in $\mathbb{P}^n(\mathbb{R})$. The map $c : D \rightarrow \mathbb{P}^n(\mathbb{R})$ can be considered as defined on the whole affine plane $\mathbb{R}^2$ as well. From the algebraic geometric point of view it is more

natural to consider projective extensions of Bézier patches. Perhaps for the first time this idea was used in geometric modeling in [5].

Let us look at the details of this construction. We substitute the Bernstein polynomials $B_i^d(t)$ and $B_{ijk}^d(s, t)$ (2.6) by their homogeneous versions

$$\tilde{B}_i^d(t_0, t_1) = \binom{d}{i} t_0^{d-i} t_1^i, \quad \tilde{B}_{ijk}^d(t_0, t_1, t_2) = \frac{d!}{i!j!k!} t_0^i t_1^j t_2^k, \quad (5.1)$$

into formulas (2.5) in the definition for polynomial Bézier patches. Then the collections of variables (t_0, t_1) and (t_0, t_1, t_2) are treated as homogeneous coordinates on $\mathbb{P}^1(\mathbb{R})$ and $\mathbb{P}^2(\mathbb{R})$, respectively. Finally we obtain yet other decompositions of Bézier patches:

- $D = \triangle$: a Bézier triangular patch is the composition $\triangle \subset \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^n(\mathbb{R})$, where coordinates (s, t) of the affine triangle $\triangle$ goes to homogeneous coordinates $[1 - s - t, s, t] \in \mathbb{P}^2(\mathbb{R})$.
- $D = \square$: a Bézier rectangular patch is the composition $\square \subset \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}^n(\mathbb{R})$, where $(s, t) \in \square$ goes to $([1 - s, s], [1 - t, t]) \in \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$.

Notice, that in Section 4 $\mathbb{P}^2(\mathbb{R})$ and $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ were identified as particular cases of toric surfaces. Now we are ready to generalize these classical Bézier constructions to more general toric surfaces.

Consider a toric map P_Δ (see (30)) associated with an arbitrary lattice polygon $\Delta \subset \mathbb{R}^2$, bounded by lines $L_i(t) = 0$, $i = 1, \dots, r$, and define the inclusion

$$L : \Delta \rightarrow \mathbb{R}^r, \quad t \mapsto (L_1(t), \dots, L_r(t)). \quad (5.2)$$

The composition $T_\Delta = \Pi \circ P_\Delta \circ L$ defines a toric generalization of Bézier patches. Recall that Π is a natural projection $\mathbb{R}^{n+1} \rightarrow \mathbb{P}^n(\mathbb{R})$.

Definition 30 (cf. [12]). A toric Bézier patch $T_\Delta : \Delta \rightarrow \mathbb{P}^n(\mathbb{R})$ with homogeneous control points $q_m \in \mathbb{R}^{n+1}$, $m \in D \cap \mathbb{Z}^2$, is defined by the formula

$$T_\Delta(t) = \Pi \left(\sum_{m \in D \cap \mathbb{Z}^2} q_m L_1(t)^{L_1(m)} \dots L_r(t)^{L_r(m)} \right).$$

There is just one difference from [12, Definition 1]: the coefficients c_m are skipped. Here we include them in weights w_m of the corresponding control points $p_m \in \mathbb{R}^n$, i.e. $q_m = (w_m, w_m p_m) \in \mathbb{R}^{n+1}$.

Definition 31. A rational patch $c : D \rightarrow \mathbb{P}^n(\mathbb{R})$, $c(t) = \Pi(\sum_i (w_i, w_i p_i) F_i(t))$, $D \subset \mathbb{R}^2$, has the linear precision property if for some collection of control points $p_i \in \mathbb{R}^2$ and their weights $w_i \in \mathbb{R}$, it is an identity on the domain D .

Many important examples of rational surface patches have this linear precision property which is important for practice: Bézier triangular and rectangular surfaces, Wachspress patches [4], S-patches [14], M-patches [8], etc. Hence it is natural to raise the following problem.

Problem 32. Find out what toric Bézier patches or their modifications have the linear precision property.

Here by modification we mean different parametrizations of the same patch.

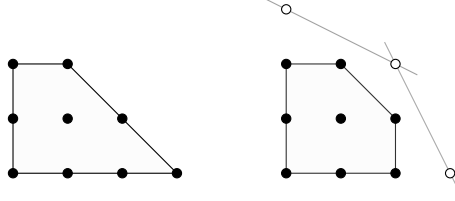


Fig. 3. Lattice polygons for Hirzebruch and pentagonal patches.

The following example is easily derived from Bézier rectangular patch using the degree elevation procedure.

Example 33. Consider the Hirzebruch patch associated with the lattice trapezoid Δ shown in Fig. 3 (left) and slightly modify the affine map L (see (5.2)) by taking $L'(s, t) = (t, 1 - s, 1 - t, s)$. Then $P_\Delta(L'(s, t))$ has linear precision with the following points p_m and weights w_m , $m \in \Delta \cap \mathbb{Z}^2$,

$$p_m = \begin{pmatrix} (0, 1) & (1, 1) \\ (0, 1/2) & (1/2, 1/2) & (1, 1/2) \\ (0, 0) & (1/3, 0) & (2/3, 0) & (1, 0) \end{pmatrix}, \quad w_m = \begin{pmatrix} 1 & 1 \\ 2 & 4 & 2 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

The natural domain D will be defined by inequalities $L'_i(s, t) \geq 0$, $i = 1, \dots, 4$, i.e. it will be a unit square $\square$.

Example 34. Consider a toric Bézier patch associated with a lattice pentagon Δ shown in Fig 3(right). Denote the bounding lines $L_1 = t$, $L_2 = 2 - s$, $L_3 = 3 - s - t$, $L_4 = 2 - t$, $L_5 = s$, and consider two additional lines: $H_1 = 3 - s - t/2$, $H_2 = 3 - s/2 - t$ (shown in grey). Define $L' = (L_1 H_2, L_2, L_3, L_4, L_5 H_1)$ and modify the patch T_Δ as follows $T'_\Delta = \Pi \circ P_\Delta \circ L'$ (with the same domain Δ). Then one can check linear precision by straightforward calculations with the following control points and weights:

$$p_m = \begin{pmatrix} (0, 2) & (1, 2) \\ (0, 6/5) & (8/7, 8/7) & (2, 1) \\ (0, 0) & (6/5, 0) & (2, 0) \end{pmatrix}, \quad w_m = \begin{pmatrix} 2 & 2 \\ 5 & 7 & 2 \\ 3 & 5 & 2 \end{pmatrix}.$$

It is interesting that the total parametrization degree of T'_Δ and T_Δ are the same.

This example was derived from the pentagonal M-patch [8]. It is a surprise that weights and coordinates of control points are such simple rational numbers.

Problem 32 is open even for a toric patch associated with the hexagon that has vertices $(0, 0)$, $(1, 0)$, $(2, 1)$, $(2, 2)$, $(1, 2)$, $(0, 1)$.

References

1. D. Cox: What is a toric variety? In: Goldman, R., Krasauskas, R. (eds.) Topics in algebraic geometry and geometric modeling. *Contemp. Math.* **334** (2003), 203–223.
2. D. Cox, R. Krasauskas, M. Mustăță: Universal rational parametrizations and toric varieties. In: Goldman, R., Krasauskas, R. (eds.) Topics in algebraic geometry and geometric modeling. *Contemp. Math.* **334** (2003), 241–265.
3. D. Cox, J. Little, D. O’Shea: Using algebraic geometry. GTM 185, Springer, Berlin (1998).
4. W. Dahmeh, H. P. Dikshit, A. Ojha: On Wachspress quadrilateral patches. *Comput. Aided Geom. Design* **17** (2000), 879–890.
5. T. DeRose: Rational Bezier curves and surfaces on projective domains. In: G. Farin, (ed.) NURBS for Curve and Surface Design, SIAM, Philadelphia, PA, 35–45 (1991).
6. R. Dietz, R., Hoschek, J., and Jüttler, B.: An algebraic approach to curves and surfaces on the sphere and other quadrics. *Comput. Aided Geom. Design* **10** (1993), 211–229.
7. Dietz, J. Hoschek, B. Jüttler: Rational patches on quadric surfaces. *Computer-aided Design*, **27** (1995), 27–40.
8. K. Karčiauskas: Rational M-patches and tensor-border patches. In: R. Goldman, R. Krasauskas (eds.), Topics in Algebraic Geometry and Geometric Modeling. *Contemp. Math.* **334** (2003), 101–128.
9. K. Karčiauskas, R. Krasauskas: Rational biangle surface patches. In: Proceedings of VI Intern. Conf. Central Europe on Computer Graphics and Visualization, Plzen (1998), 165–170.
10. R. Krasauskas: Universal parameterizations of some rational surfaces. In: A. Le Méhauté, C. Rabut, L. L. Schumaker (eds.), Curves and Surfaces with Applications in CAGD. Vanderbilt Univ. Press, Nashville (1997), 231–238.
11. R. Krasauskas: Shape of toric surfaces. In: R. Durikovic, S. Czanner (eds.), Proceedings of the Spring Conference on Computer Graphics SCCG 2001, IEEE (2002), 55–62.
12. R. Krasauskas: Toric surface patches. *Advances in Computational Mathematics*, **17** (2002), 89–113.
13. R. Krasauskas, M. Kazakevičiūtė: Universal rational parametrizations and spline curves on toric surfaces. In: Computational Methods for Algebraic Spline Surfaces, ESF Exploratory Workshop, Springer (2005), 213–231.
14. C. T. Loop, T. DeRose: A multisided generalization of Bézier surfaces. *ACM Transactions on Graphics*, **8** (1989), 204–234.
15. R. R. Patterson: Projective transformations of the parameter of a Bernstein–Bézier curve. *ACM Transactions on Graphics*, **4** (1985), 276–290.
16. M. Peternell, H. Pottmann, B. Ravani: On the computational geometry of ruled surfaces. *Computer-aided Design*, **13** (1999), 17–32.

Algebraic Geometry and Geometric Modeling

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2006, X, 252 p. 52 illus., 27 illus. in color., Hardcover

ISBN: 978-3-540-33274-9