Remarks on Free Entropy Dimension

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Summary. We prove a technical result, showing that the existence of a closable unbounded dual system in the sense of Voiculescu is equivalent to the finiteness of free Fisher information. This approach allows one to give a purely operator-algebraic proof of the computation of the non-microstates free entropy dimension for generators of groups carried out in an earlier joint work with I. Mineyev [4]. The same technique also works for finite-dimensional algebras.

We also show that Voiculescu's question of semi-continuity of free entropy dimension, as stated, admits a counterexample. We state a modified version of the question, which avoids the counterexample, but answering which in the affirmative would still imply the non-isomorphism of free group factors.

Introduction

Free entropy dimension was introduced by Voiculescu [7, 8, 9] both in the context of his microstates and non-microstates free entropy. We refer the reader to the survey [11] for a list of properties as well as applications of this quantity in the theory of von Neumann algebras.

The purpose of this note is to discuss several technical aspects related to estimates for free entropy dimension.

The first deals with existence of "Dual Systems of operators", which were considered by Voiculescu [9] in connection with the properties of the difference quotient derivation, which is at the heart of the non-microstates definition of free entropy. We prove that if one considers dual systems of closed unbounded operators (as opposed to bounded operators as in [9]), then existence of a dual system becomes equivalent to finiteness of free Fisher information. Using these ideas allows one to give a purely operator-algebraic proof of the expression for the free entropy dimension of a set of generators of a group algebra in terms of the L^2 Betti numbers of the group [4], clarifying the reason for why the equality holds in the group case. We also point out that for the same

*Research supported by the National Science Foundation.

reason one is able to express the non-microstates free entropy dimension of an n-tuple of generators of a finite-dimensional von Neumann algebra in terms of its L^2 Betti numbers. In particular, the microstates and non-microstates free entropy dimension is the same in this case.

The second aspect deals with the question of semi-continuity of free entropy dimension, as formulated by Voiculescu in [7, 8]. We point out that a counterexample exists to the question of semi-continuity, as stated. However, the possibility that the free entropy dimension is independent of the choice of generators of a von Neumann algebra is not ruled out by the counterexample.

1 Unbounded Dual Systems and Derivations

1.1 Non-Commutative Difference Quotients and Dual Systems

Let $X_1, \ldots, X_n$ be an n-tuple of self-adjoint elements in a tracial von Neumann algebra M. In [9], Voiculescu considered the densely defined derivations ∂_j defined on the polynomial algebra $\mathbb{C}(X_1, \ldots, X_n)$ generated by $X_1, \ldots, X_n$ and with values in $L^2(M) \otimes L^2(M) \cong HS(L^2(M))$, the space of Hilbert-Schmidt operators on $L^2(M)$. If we denote by $P_1 : L^2(M) \to L^2(M)$ the orthogonal projection onto the trace vector 1, then the derivations ∂_j are determined by the requirement that $\partial_j(X_i) = \delta_{ij} P_1$.

Voiculescu showed that if $\partial_j^*(P_1)$ exists, then ∂_j is closable. This is of interest because the existence of $\partial_j^*(P_1)$, $j = 1, \ldots, n$ is equivalent to finiteness of the free Fisher information of $X_1, \ldots, X_n$ [9].

Also in [9], Voiculescu introduced the notion of a "dual system" to $X_1, \ldots, X_n$. In his definition, such a dual system consists of an n tuple of operators $Y_1, \ldots, Y_n$, so that $[Y_i, X_j] = \delta_{ij} P_1$, where Although Voiculescu required that the operators Y_j be anti-self-adjoint, it will be more convenient to drop this requirement. However, this is not a big difference, since if $(Y_1', \ldots, Y_n')$ is another dual system, then $[Y_i - Y_i', X_j] = 0$ for all i, j, and so $Y_i - Y_i'$ belongs to the commutant of $W^*(X_1, \ldots, X_n)$.

Note that the existence of a dual system is equivalent to the requirement that the derivations $\partial_j : \mathbb{C}(X_1, \ldots, X_n) \to HS \subset B(L^2(M))$ are inner as derivations into $B(L^2(M))$. In particular, Voiculescu showed that if a dual system exists, then $\partial_j : L^2(M) \to HS$ are actually closable, and $\partial_j^*(P_1)$ is given by $(Y_j - JY_j^* J)1$. However, the existence of a dual system is a stronger requirement than the existence of $\partial_j^*(P_1)$.

1.2 Dual Systems of Unbounded Operators

More generally, given an n-tuple $T = (T_1, \ldots, T_n) \in HS^n$, we may consider a derivation $\partial_T : \mathbb{C}(X_1, \ldots, X_n) \to HS$ determined by $\partial_T(X_j) = T_j$ [6]. The particular case of ∂_j corresponds to $T = (0, \ldots, P_1, \ldots, 0)$ (P_1 in j-th place).

Theorem 1. *Let $T \in HS^n$ and assume that $M = W^*(X_1, \ldots, X_n)$. The following are equivalent:*
(a) $\partial_T^*(P_1)$ exists;
(b) There exists a closable unbounded operator $Y : L^2(M) \to L^2(M)$, whose domain includes $\mathbb{C}(X_1, \ldots, X_n)$, so that $Y1 = 0$ and 1 belongs to the domain of Y^*, and so that $[Y, X_j] = T_j$.

Proof. Assume first that (b) holds. Let $\xi = (Y - JY^*J)1 = JY^*1$, which by assumptions on Y makes sense. Then for any polynomial $Q \in \mathbb{C}(X_1, \ldots, X_n)$,

$$\begin{aligned}
\langle \xi, P \rangle &= \langle (Y - JY^*J)1, Q1 \rangle \\
&= \langle [Y, Q]1, 1 \rangle \\
&= Tr(P_1[Y, Q]) \\
&= \langle P_1, [Y, Q] \rangle_{HS} \\
&= \langle P_1, \partial_T(Q) \rangle_{HS},
\end{aligned}$$

since the derivations $Q \mapsto \partial_T(Q)$ and $Q \mapsto [Y, Q]$ have the same values on generators and hence are equal on $\mathbb{C}(X_1, \ldots, X_n)$. But this means that $\xi = \partial_T^*(P_1)$.

Assume now that (a) holds. If we assume that $Y1 = 0$, then the equation $[Y, X_j] = T_j$ determines an operator $Y : \mathbb{C}(X_1, \ldots, X_n) \to L^2(M)$. Indeed, if Q is a polynomial in $X_1, \ldots, X_n$, then we have

$$Y(Q \cdot 1) = [Y, Q] \cdot 1 - Q(Y \cdot 1) = [Y, Q] \cdot 1 = \partial_T(Q) \cdot 1.$$

To show that the operator Y that we have thus defined is closable, it is sufficient to prove that a formal adjoint can be defined on a dense subset. We define Y^* on $Q \in \mathbb{C}(X_1, \ldots, X_n)$ by

$$Y^*(Q \cdot 1) = -(\partial_T(Q^*))^* \cdot 1 + \partial_T^*(P_1).$$

Hence $Y^* \cdot 1 = \partial_T^*(P_1)$ and Y^* satisfies $[Y^*, Q] = -(\partial_T(Q^*))^* = -[Y, Q^*]^*$.

It remains to check that $\langle YQ \cdot 1, R \cdot 1 \rangle = \langle Q \cdot 1, Y^*R \cdot 1 \rangle$, for all $Q, R \in \mathbb{C}(X_1, \ldots, X_n)$. We have:

$$\begin{aligned}
\langle YQ \cdot 1, R \cdot 1 \rangle &= \langle [Y, Q] \cdot 1, R \cdot 1 \rangle \\
&= \langle 1, -[Y^*, Q^*]R \cdot 1 \rangle \\
&= \langle 1, Q^*Y^*R \cdot 1 \rangle - \langle 1, Y^*Q^*R \cdot 1 \rangle \\
&= \langle Q \cdot 1, Y^*R \cdot 1 \rangle - \langle 1, Y^*Q^*R \cdot 1 \rangle.
\end{aligned}$$

Hence it remains to prove that $\langle 1, Y^*Q^*R \cdot 1 \rangle = 0$. To this end we write

$$\begin{aligned}
\langle 1, Y^*Q^*R \cdot 1 \rangle &= \langle 1, [Y^*, Q^*R] \cdot 1 \rangle - \langle 1, Q^*RY^* \cdot 1 \rangle \\
&= \langle [R^*Q, Y] \cdot 1, 1 \rangle - \langle R^*Q \cdot 1, Y^* \cdot 1 \rangle \\
&= Tr([R^*Q, Y]P_1) - \langle R^*Q \cdot 1, \partial_T^*(P_1) \rangle \\
&= \langle \partial_T(R^*Q), P_1 \rangle_{HS} - \langle \partial_T(R^*Q), P_1 \rangle_{HS} = 0.
\end{aligned}$$

Thus Y is closable.

Corollary 2. *Let* $M = W^*(X_1, \dots, X_n)$. *Then* $\Phi^*(X_1, \dots, X_n) < +\infty$ *if and only if there exist unbounded essentially anti-symmetric operators* $Y_1, \dots, Y_n : L^2(M) \to L^2(M)$ *whose domain includes* $\mathbb{C}(X_1, \dots, X_n)$, *and which satisfy* $[Y_j, X_i] = \delta_{ji} P_1$.

Proof. A slight modification of the first part of the proof of Theorem 1 gives that if $Y_1, \dots, Y_n$ exist, then $\partial_j^*(P_1) = (Y_j - JY_jJ)1$ and hence $\Phi^*(X_1, \dots, X_n)$ (which is by definition $\sum_j \|\partial_j^*(P_1)\|_2^2$) is finite.

Conversely, if $\Phi^*(X_1, \dots, X_n) < +\infty$, then $\partial_j^*(P_1)$ exists for all j. Hence by Theorem 1, we obtain non-self adjoint closable unbounded operators $Y_1, \dots, Y_n$ so that the domains of $Y_j$and Y_j^* include $\mathbb{C}(X_1, \dots, X_n)$, and so that $[Y_j, X_i] = \delta_{ji} P_1$. Now since $X_j = X_j^*$ we also have $[Y_j^*, X_i] = -\delta_{ji} P_1^* = -\delta_{ji} P_1$. Hence if we set $\tilde{Y}_j = \frac{1}{2}(Y_j - Y_j^*)$, we obtain the desired operators.

2 Dual Systems and L^2 Cohomology

Let as before $X_1, \dots, X_n \in (M, \tau)$ be a family of self-adjoint elements.

In conjunctions with estimates on free entropy dimension [6, 4] and L^2 cohomology [2], it is interesting to consider the following spaces:

$$H_0 = \mathrm{cl}\,\{T = (T_1, \dots, T_n) \in HS^n : \\ \exists Y \in B(L^2(M)) \;\; [Y, X_j] = T_j\}.$$

Here cl refers to closure in the Hilbert-Schmidt topology. We also consider

$$H_1 = \text{span }\; \mathrm{cl}\,\{T = (T_1, \dots, T_n) \in HS^n : \exists Y = Y^* \text{ unbounded densely defined} \\ \text{with 1 in the domain of } Y,\; [Y, X_j] = T_j,\; j = 1, \dots, n\}.$$

Note that in particular, $H_0 \subset H_1$.

One has the following estimates [6, 2]:

$$\dim_{M \bar{\otimes} M^o} H_0 \le \delta^*(X_1, \dots, X_n) \le \delta^\star(X_1, \dots, X_n) \le \Delta(X_1, \dots, X_n).$$

The main result of this section is the following theorem, whose proof has similarities with the Sauvageot's theory of quantum Dirichlet forms [5]:

Theorem 3. $H_0 = H_1$.

Proof. It is sufficient to prove that H_0 is dense in H_1.

Let $T = (T_1, \dots, T_n) \in HS^n$ be such that $T_j^* = T_j = [\mathrm{i}A, X_j]$, $j = 1, \dots, n$, with $A = A^*$ a closed unbounded operator and 1 in the domain of A.

For each $0 < R < \infty$, let now $f_R : \mathbb{R} \to \mathbb{R}$ be a C^∞ function, so that

1. $f_R(x) = x$ for all $-R \le x \le R$;
2. $|f_R(x)| \le R + 1$ for all x;
3. the difference quotient $g_R(s,t) = \frac{f_R(s) - f_R(t)}{s - t}$ is uniformly bounded by 2.

Let $A_R = f_R(A)$ and let $T_j^{(R)} = [\mathrm{i}A_R, X_j]$. Note that for each R, $T^{(R)} = (T_1^{(R)}, \dots, T_n^{(R)}) \in H_0$. Hence it is sufficient to prove that $T^{(R)} \to T$ in Hilbert-Schmidt norm as $R \to \infty$.

Let $\mathcal{A} \cong L^\infty(\mathbb{R}, \mu)$ be the von Neumann algebra generated by the spectral projections of A; hence $A_R \in \mathcal{A}$ for all R. If we regard $L^2(M)$ as a module over A, then $HS = L^2(M)\bar{\otimes}L^2(M)$ is a bimodule over $\mathcal{A}$, and hence a module over $\mathcal{A}\bar{\otimes}\mathcal{A} \cong L^\infty(\mathbb{R}^2, \mu \times \mu)$ in such a way that if s, t are coordinates on $\mathbb{R}^2$, and $Q \in HS$, then $sQ = AQ$ and $tQ = QA$ (more precisely, for any bounded measurable function f, $f(s)Q = f(A)Q$ and $f(t)Q = Qf(A)$). In particular, we can identify, up to multiplicity, HS with $L^2(\mathbb{R}^2, \mu \times \mu)$.

It is not hard to see that then

$$[f(A), X_j] = g \cdot [A, X_j],$$

where g is the difference quotient $g(s,t) = (f(s) - f(t))/(s-t)$. Indeed, it is sufficient to verify this equation on vectors in $\mathbb{C}[X_1, \dots, X_n]$ for f a polynomial in A, in which case the result reduces to

$$[A^n, X_j] = \sum_{k=0}^{n-1} A^k [A, X_j] A^{n-k-1} = \frac{s^n - t^n}{s - t} \cdot [A, X_j].$$

It follows that

$$T_j^{(R)} = [A_R, X_j] = [f_R(A), X_j] = g_R(A) \cdot [A, X_j] = g_R(A) \cdot T_j.$$

Now, since $g_R(A)$ are bounded and $g_R(A) = 1$ on the square $-R \leq s, t \leq R$, it follows that multiplication operators $g_R(A)$ converge to 1 ultra-strongly as $R \to \infty$. Since HS is a multiple of $L^2(\mathbb{R}^2, \mu \times \mu)$, it follows that $g_R(A)T_j \to T_j$ in Hilbert-Schmidt norm. Hence $T^{(R)} \to T$ as $R \to \infty$.

As a corollary, we re-derive the main result of [4] (the difference is that we use Theorem 3 instead of the more combinatorial argument [4]; we sketch the proof to emphasize the exact point at which the fact that we are dealing with a group algebra becomes completely clear):

Corollary 4. *Let $X_1, \dots, X_n$ be generators of the group algebra $\mathbb{C}\Gamma$. Then $\delta^\star(X_1, \dots, X_n) = \delta^*(X_1, \dots, X_n) = \Delta(X_1, \dots, X_n) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$, where $\beta_j^{(2)}(\Gamma)$ are the L^2-Betti numbers of Γ.*

Proof. (Sketch). We first point out that in the preceding we could have worked with self-adjoint families $F = (X_1, \dots, X_n)$ rather than self-adjoint elements (all we ever needed was that $X \in F \Rightarrow X^* \in F$).

By [4], we may assume that $X_j \in \Gamma \subset \mathbb{C}\Gamma$, since the dimension of H_0 depends only on the pair $\mathbb{C}(X_1, \dots, X_n)$ and its trace.

Recall [2] that $\Delta(X_1, \dots, X_n) = \dim_{M\bar{\otimes}M^o} H_2$, where

$$H_2 = \{(T_1, \dots, T_n) \in HS : \exists Y^{(k)} \in HS \text{ s.t. } [Y^{(k)}, X_j] \to T_j \text{ weakly}\}.$$

By [2], $\Delta(X_1,\ldots,X_n)=\beta_1^{(2)}(\Gamma)-\beta_0^{(2)}(\Gamma)+1$; moreover, from the proof we see that in the group case,

$$H_2=\mathrm{cl}\{MXM\},$$

where

$$X=\{(T_1X_1,\ldots,T_nX_n):T_j\in\ell^2(\Gamma),\\ T_j \text{ is the value of some } \ell^2 \text{ group cocycle on } X_j\},$$

and where we think of $\ell^2(\Gamma)\subset HS$ as "diagonal operators" by sending a sequence $(a_\gamma)_{\gamma\in\Gamma}\in\ell^2(\Gamma)$ to the Hilbert-Schmidt operator $\sum a_\gamma P_\gamma$, where P_γ is the rank 1 projection onto the subspace spanned by the delta function supported on γ.

Let $\mathfrak{F}$ be the space of all functions on Γ. Since the group cohomology $H^1(\Gamma;\mathfrak{F}(\Gamma))$ is clearly trivial, it follows that if c is any ℓ^2-cocycle on Γ, then $c(X_j)=f(X_j)-f(e)$, for some $f\in\mathfrak{F}$. Hence

$$X=\{([f,X_1],\ldots,[f,X_n]):f\in\mathfrak{F}\}\cap HS^n.$$

Since *every element of* $\mathfrak{F}$ *is automatically an essentially self-adjoint operator on* $\ell^2(\Gamma)$, whose domain includes $\mathbb{C}\Gamma$ we obtain that

$$MXM\subset H_1.$$

In particular, $H_2\subset H_1$. Hence

$$\dim_{M\bar{\otimes}M^o}H_2=\beta_1^{(2)}(\Gamma)-\beta_0^{(2)}(\Gamma)+1\le\dim_{M\bar{\otimes}M^o}H_1\le\dim_{M\bar{\otimes}M_0}H_2,$$

which forces $H_1=H_2$. Since $H_0=H_1$, we get that in the following equation

$$\dim_{M\bar{\otimes}M^o}H_0\le\delta^*\le\delta^\star\le\dim_{M\bar{\otimes}M^o}H_2=\beta_1^{(2)}(\Gamma)-\beta_0^{(1)}(\Gamma)+1$$

all inequalities are forced to be equalities, which gives the result.

Corollary 5. *Let (M,τ) be a finite-dimensional algebra, and let $X_1,\ldots,X_n$ be any of its self-adjoint generators. Then $\delta^*(X_1,\ldots,X_n)=\delta^\star(X_1,\ldots,X_n)=\Delta(X_1,\ldots,X_n)=1-\beta_0(M,\tau)=\delta_0(X_1,\ldots,X_n)$.*

Proof. As in the proof of the last corollary, we have the inequalities

$$\dim_{M\bar{\otimes}M^o}H_0\le\delta^*\le\delta^\star\le\dim_{M\bar{\otimes}M^o}H_2,$$

where

$$H_2=\{(T_1,\ldots,T_n)\in HS:\exists Y^{(k)}\in HS \text{ s.t. } [Y^{(k)},X_j]\to T_j \text{ weakly}\}.$$

Since $L^2(M)$ is finite-dimensional, there is no difference between weak and norm convergence; hence H_2 is in the (norm) closure of $\{(T_1,\ldots,T_n):\exists Y\in$

HS s.t. $T_j = [Y, X_j]\} \subset H_0$; since H_0 is closed, we get that $H_0 = H_1$ and so all inequalities become equalities. Moreover,

$$\dim_{M\bar{\otimes}M^o} H_2 = \Delta(X_1, \ldots, X_n) = 1 - \beta_0(M, \tau)$$

(see [2]).

Comparing the values of $1 - \beta_0(M, \tau)$ with the computations in [3] gives $\delta_0(X_1, \ldots, X_n) = \delta^*(X_1, \ldots, X_n)$.

3 Some Remarks on Semi-Continuity of Free Dimension

In [7, 8], Voiculescu asked the question of whether the free dimension δ satisfies the following semi-continuity property. Let $X_j^{(k)}, X_j \in (M, \tau)$ be self-adjoint variables, $j = 1, \ldots, n$, $k = 1, 2, \ldots$, and assume that $X_j^{(k)} \to X_j$ strongly, $\sup_k \|X_j^{(k)}\| < \infty$. Then is it true that

$$\liminf_k \delta(X_1^{(k)}, \ldots, X_n^{(k)}) \geq \delta(X_1, \ldots, X_n)?$$

As shown in [7, 8], a positive answer to this question (or a number of related questions, where δ is replaced by some modification, such as δ_0, δ^*, etc.) implies non-isomorphism of free group factors. In the case of δ_0. a positive answer would imply that the value of δ_0 is independent of the choice of generators of a von Neumann algebra.

Although this question is very natural from the geometric standpoint, we give a counterexample, which shows that some additional assumptions on the sequence $X_j^{(k)}$ are necessary. Fortunately, the kinds of properties of δ that would be required to prove the non-isomorphism of free group factors are not ruled out by this counterexample (see Question 8).

We first need a lemma.

Lemma 6. *Let $X_1, \ldots, X_n$ be any generators of the group algebra of the free group $\mathbb{F}_k$. Then $\delta_0(X_1, \ldots, X_n) = \delta(X_1, \ldots, X_n) = \delta^*(X_1, \ldots, X_n) = \delta^\star(X_1, \ldots, X_n) = k$.*

Proof. Note that by [1] we always have

$$\delta_0(X_1, \ldots, X_n) \leq \delta(X_1, \ldots, X_n) \leq \delta^*(X_1, \ldots, X_n) \leq \delta^\star(X_1, \ldots, X_n);$$

furthermore, by [4], $\delta^\star(X_1, \ldots, X_n) = k$. Since δ_0 is an algebraic invariant [10], $\delta_0(X_1, \ldots, X_n) = \delta_0(U_1, \ldots, U_k)$, where $U_1, \ldots, U_k$ are the free group generators. Then by [8], $\delta_0(U_1, \ldots, U_k) = k$. This forces equalities throughout.

Example 7. Let u, v be two free generators of $\mathbb{F}_2$, and consider the map $\phi : \mathbb{F}_2 \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ given by $\phi(u) = \phi(v) = 1$. The kernel of this map is a subgroup Γ of $\mathbb{F}_2$, which is isomorphic to $\mathbb{F}_3$, having as free generators, e.g. u^2, v^2 and uv. Let $X_1^{(k)} = \mathrm{Re}\, u^2$, $X_2^{(k)} = \mathrm{Im}\, u^2$, $Y_1^{(k)} = \mathrm{Re}\, v^2$, $Y_2^{(k)} = \mathrm{Im}\, v^2$, $Z_1^{(k)} = \mathrm{Re}\, uv$, $Z_2^{(k)} = \mathrm{Im}\, uv$, $W_1^{(k)} = \frac{1}{k}\mathrm{Re}\, u$, $W_2^{(k)} = \frac{1}{k}\mathrm{Im}\, v$.

Thus if $X_1 = \mathrm{Re}\, u^2$, $X_2 = \mathrm{Im}\, u^2$, $Y_1 = \mathrm{Re}\, v^2$, $Y_2 = \mathrm{Im}\, v^2$, $Z_1 = \mathrm{Re}\, uv$, $Z_2 = \mathrm{Im}\, uv$, $W_1 = 0$, $W_2 = 0$, then $X_j^{(k)} \to X_j$, $Y_j^{(k)} \to Y_j$, $Z_j^{(k)} \to Z_j$ and $W_j^{(k)} \to W_j$ (in norm, hence strongly).

Note finally that for k finite, $X_1^{(k)}, X_2^{(k)}, Y_1^{(k)}, Y_2^{(k)}, Z_1^{(k)}, Z_2^{(k)}, W_1^{(k,)}, W_2^{(k)}$ generate the same algebra as $u^2, v^2, uv, \frac{1}{k}v$, which is the same as the algebra generated by u and v, i.e., the entire group algebra of $\mathbb{F}_2$. Hence $\delta(X_1^{(k)}, X_2^{(k)}, Y_1^{(k)}, Y_2^{(k)}, Z_1^{(k)}, Z_2^{(k)}, W_1^{(k,)}, W_2^{(k)}) = 2$ by Lemma 6. Hence

$$\liminf_k \delta(X_1^{(k)}, X_2^{(k)}, Y_1^{(k)}, Y_2^{(k)}, Z_1^{(k)}, Z_2^{(k)}, W_1^{(k,)}, W_2^{(k)}) = 2$$

On the other hand, $X_1, X_2, Y_1, Y_2, Z_1, Z_2, W_1, W_2$ generate the same algebra as $u^2, v^2, uv, 0$, i.e., the group algebra of $\Gamma \cong \mathbb{F}_3$. Hence

$$\delta(X_1, X_2, Y_1, Y_2, Z_1, Z_2, W_1, W_2) = 3,$$

which is the desired counterexample.

The same example (in view of Lemma 6) also works for δ_0, δ^* and $\delta^\star$.

The following two versions of the question are not ruled out by the counterexample. If either version were to have a positive answer, it would still be sufficient to prove non-isomorphism of free group factors:

Question 8. (a) Let $X_j^{(k)}, X_j \in (M,\tau)$ be self-adjoint variables, $j = 1, \ldots, n$, $k = 1, 2, \ldots$, and assume that $X_j^{(k)} \to X_j$ strongly, $\sup_k \|X_j^{(k)}\| < \infty$. Assume that $X_1, \ldots, X_n$ generate M and that for each k, $X_1^{(k)}, \ldots, X_n^{(k)}$ also generate M. Then is it true that

$$\liminf_k \delta(X_1^{(k)}, \ldots, X_n^{(k)}) \geq \delta(X_1, \ldots, X_n)?$$

(b) A weaker form of the question is the following. Let $X_j^{(k)}, X_j, Y_j \in (M,\tau)$ be self-adjoint variables, $j = 1, \ldots, n$, $k = 1, 2, \ldots$, and assume that $X_j^{(k)} \to X_j$ strongly, $\sup_k \|X_j^{(k)}\| < \infty$. Assume that $Y_1, \ldots, Y_m$ generate M. Then is it true that

$$\liminf_k \delta(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1, \ldots, Y_m) \geq \delta(X_1, \ldots, X_n, Y_1, \ldots, Y_m)?$$

We point out that in the case of δ_0, these questions are actually equivalent to each other and to the statement that $\delta_0(Z_1, \ldots, Z_n)$ only depends on the von Neumann algebra generated by $Z_1, \ldots, Z_n$.

Indeed, it is clear that (a) implies (b).

On the other hand, if we assume that (b) holds, then we can choose $X_j^{(k)}$ to be polynomials in $Y_1, \ldots, Y_m$, so that $\delta_0(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1, \ldots, Y_m) = \delta_0(Y_1, \ldots, Y_m)$ by [10]. Hence $\delta_0(Y_1, \ldots, Y_m) \geq \delta_0(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \geq \delta_0(Y_1, \ldots, Y_m)$, where the first inequality is by (b) and the second inequality is proved in [8]. Hence if $W^*(X_1, \ldots, X_n) = W^*(Y_1, \ldots, Y_m)$, then one has $\delta_0(X_1, \ldots, X_n) = \delta_0(X_1, \ldots, X_n, Y_1, \ldots, Y_m) = \delta_0(Y_1, \ldots, Y_m)$. Hence (b) implies that δ_0 is the same on any generators of M.

Lastrly, if we assume that δ_0 is an invariant of the von Neumann algebra, then (a) clearly holds, since the value of $\delta_0(X_1^{(k)}, \ldots, X_n^{(k)})$ is then independent of k and is equal to $\delta_0(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$.

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Operator Algebras
The Abel Symposium 2004
Bratteli, O.; Neshveyev, S.; Skau, C. (Eds.)
2006, X, 279 p., Hardcover
ISBN: 978-3-540-34196-3