Prologue

We begin with some quotations which exemplify the philosophical underpinnings of this work.

Theoria cum praxi.
Gottfried Wilhelm Leibniz (1646–1716)

> It is very difficult to write mathematics books today. If one does not take pains with the fine points of theorems, explanations, proofs and corollaries, then it won't be a mathematics book; but if one does these things, then the reading of it will be extremely boring.

Johannes Kepler (1571–1630)
Astronomia Nova

> The interaction between physics and mathematics has always played an important role. The physicist who does not have the latest mathematical knowledge available to him is at a distinct disadvantage. The mathematician who shies away from physical applications will most likely miss important insights and motivations.

Marvin Schechter
Operator Methods in Quantum Mechanics[5]

> In 1967 Lenard and I found a proof of the stability of matter. Our proof was so complicated and so unilluminating that it stimulated Lieb and Thirring to find the first decent proof. Why was our proof so bad and why was theirs so good? The reason is simple. Lenard and I began with mathematical tricks and hacked our way through a forest of inequalities without any physical understanding. Lieb and Thirring began with physical understanding and went on to find the appropriate mathematical language to make their understanding rigorous. Our proof was a dead end. Theirs was a gateway to the new world of ideas collected in this book.

Freeman Dyson
From the Preface to *Elliott Lieb's Selecta*[6]

The state of the art in quantum field theory. One of the intellectual fathers of quantum electrodynamics is Freeman Dyson (born in 1923) who

[5] North-Holland, Amsterdam, 1982.
[6] Stability of Matter: From Atoms to Stars, Springer, New York, 2002.

works at the Institute for Advanced Study in Princeton.[7] He characterizes the state of the art in quantum field theory in the following way:

> All through its history, quantum field theory has had two faces, one looking outward, the other looking inward. The outward face looks at nature and gives us numbers that we can calculate and compare with experiments. The inward face looks at mathematical concepts and searches for a consistent foundation on which to build the theory. The outward face shows us brilliantly successful theory, bringing order to the chaos of particle interactions, predicting experimental results with astonishing precision. The inward face shows us a deep mystery. After seventy years of searching, we have found no consistent mathematical basis for the theory. When we try to impose the rigorous standards of pure mathematics, the theory becomes undefined or inconsistent. From the point of view of a pure mathematician, the theory does not exist. This is the *great unsolved paradox of quantum field theory.*
>
> To resolve the paradox, during the last twenty years, quantum field theorists have become string-theorists. String theory is a new version of quantum field theory, exploring the mathematical foundations more deeply and entering a new world of multidimensional geometry. String theory also brings gravitation into the picture, and thereby unifies quantum field theory with general relativity. String theory has already led to important advances in pure mathematics. It has not led to any physical predictions that can be tested by experiment. We do not know whether string theory is a true description of nature. All we know is that it is a rich treasure of new mathematics, with an enticing promise of new physics. During the coming century, string theory will be intensively developed, and, if we are lucky, tested by experiment.[8]

Five golden rules. When writing the latex file of this book on my computer, I had in mind the following five quotations. Let me start with the mathematician Hermann Weyl (1885–1955) who became a successor of Hilbert in Göttingen in 1930 and who left Germany in 1933 when the Nazi regime came to power. Together with Albert Einstein (1879–1955) and John von Neumann (1903–1957), Weyl became a member of the newly founded Institute for Advanced Study in Princeton, New Jersey, U.S.A. in 1933. Hermann Weyl wrote in 1938:[9]

> The stringent precision attainable for mathematical thought has led many authors to a mode of writing which must give the reader an impression of being shut up in a brightly illuminated cell where every detail sticks out with the same dazzling clarity, but without relief. I prefer the open landscape under a clear sky with its depth of perspective, where the wealth of sharply defined nearby details gradually fades away towards the horizon.

[7] F. Dyson, Selected Papers of Freeman Dyson with Commentaries, Amer. Math. Soc., Providence, Rhode Island, 1996. We recommend reading this fascinating volume.

[8] In: Quantum Field Theory, A 20th Century Profile. Edited by A. Mitra, Indian National Science Academy and Hindustan Book Agency, 2000 (reprinted with permission).

[9] H. Weyl, The Classical Groups, Princeton University Press, 1938 (reprinted with permission).

For his fundamental contributions to electroweak interaction inside the Standard Model in particle physics, the physicist Steven Weinberg (born 1933) was awarded the Nobel prize in physics in 1979 together with Sheldon Glashow (born 1932) and Abdus Salam (1926–1996). On the occasion of a conference on the interrelations between mathematics and physics in 1986, Weinberg pointed out the following:[10]

> I am not able to learn any mathematics unless I can see some problem I am going to solve with mathematics, and I don't understand how anyone can teach mathematics without having a battery of problems that the student is going to be inspired to want to solve and then see that he or she can use the tools for solving them.

For his theoretical investigations on parity violation under weak interaction, the physicist Cheng Ning Yang (born 1922) was awarded the Nobel prize in physics in 1957 together with Tsung Dao Lee (born 1926). In an interview, Yang remarked:[11]

> In 1983 I gave a talk on physics in Seoul, South Korea. I joked "There exist only two kinds of modern mathematics books: one which you cannot read beyond the first page and one which you cannot read beyond the first sentence. The *Mathematical Intelligencer* later reprinted this joke of mine. But I suspect many mathematicians themselves agree with me."

The interrelations between mathematics and modern physics have been promoted by Sir Michael Atiyah (born 1929) on a very deep level. In 1966, the young Atiyah was awarded the Fields medal. In an interview, Atiyah emphasized the following:[12]

> The more I have learned about physics, the more convinced I am that physics provides, in a sense, the deepest applications of mathematics. The mathematical problems that have been solved, or techniques that have arisen out of physics in the past, have been the lifeblood of mathematics... The really deep questions are still in the physical sciences. For the health of mathematics at its research level, I think it is very important to maintain that link as much as possible.

The development of modern quantum field theory has been strongly influenced by the pioneering ideas of the physicist Richard Feynman (1918–1988). In 1965, for his contributions to the foundation of quantum electrodynamics, Feynman was awarded the Nobel prize in physics together with Julian Schwinger (1918–1994) and Sin-Itiro Tomonaga (1906–1979). In the beginning of the 1960s, Feynman held his famous *Feynman lectures* at the California Institute of Technology in Pasadena. In the preface to the printed version of the lectures, Feynman told his students the following:

> Finally, may I add that the main purpose of my teaching has not been to prepare you for some examination – it was not even to prepare you to

[10] Notices Amer. Math. Soc. **33** (1986), 716–733 (reprinted with permission).
[11] Mathematical Intelligencer **15** (1993), 13–21 (reprinted with permission).
[12] Mathematical Intelligencer **6** (1984), 9–19 (reprinted with permission).

serve industry or military. I wanted most to give you some appreciation of the wonderful world and the physicist's way of looking at it, which, I believe, is a major part of the true culture of modern times.[13]

The fascination of quantum field theory. As a typical example, let us consider the anomalous magnetic moment of the electron. This is given by the following formula

$$\boxed{\mathbf{M}_e = -\frac{e}{2m_e}\, g_e \mathbf{S}}$$

with the so-called gyromagnetic factor

$$g_e = 2(1+a)$$

of the electron. Here, m_e is the mass of the electron, $-e$ is the negative electric charge of the electron. The spin vector $\mathbf{S}$ has the length $\hbar/2$, where h denotes Planck's quantum of action, and $\hbar := h/2\pi$. High-precision experiments yield the value

$$a_{\text{exp}} = 0.001\ 159\ 652\ \mathit{188\ 4} \pm 0.000\ 000\ 000\ \mathit{004\ 3}.$$

Quantum electrodynamics is able to predict this result with high accuracy. The theory yields the following value

$$\begin{aligned} a = \frac{\alpha}{2\pi} - 0.328\ 478\ 965\left(\frac{\alpha}{\pi}\right)^2 &+ (1.175\ 62 \pm 0.000\ 56)\left(\frac{\alpha}{\pi}\right)^3 \\ &-(1.472 \pm 0.152)\left(\frac{\alpha}{\pi}\right)^4 \end{aligned} \tag{0.1}$$

with the electromagnetic fine structure constant

$$\alpha = \frac{1}{137.035\ 989\ \mathit{500} \pm 0.000\ 000\ \mathit{061}}.$$

Explicitly,

$$a = 0.001\ 159\ 652\ \mathit{164} \pm 0.000\ 000\ 000\ \mathit{108}.$$

The error is due to the uncertainty of the electromagnetic fine structure constant α. Observe that 9 digits coincide between the experimental value a_{exp} and the theoretical value a.

The theoretical result (0.1) represents a highlight in modern theoretical physics. The single terms with respect to powers of the fine structure constant α have been obtained by using the method of perturbation theory. In order to represent graphically the single terms appearing in perturbation theory, Richard Feynman (1918–1988) invented the language of Feynman diagrams in about 1945.[14] For example, Fig. 0.1 shows some simple Feynman diagrams

[13] R. Feynman, R. Leighton, and M. Sands, The Feynman Lectures in Physics, Addison-Wesley, Reading, Massachusetts, 1963.

[14] For the history of this approach, see the quotation on page 27.

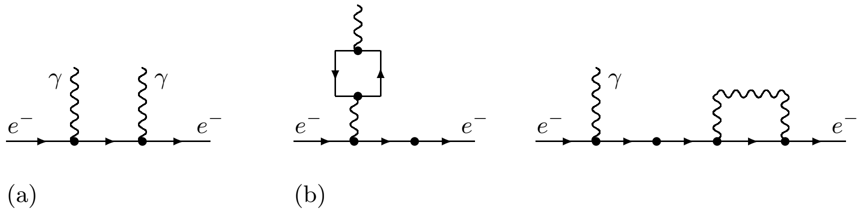

Fig. 0.1. Feynman diagrams

for the Compton scattering between electrons and photons. In higher order of perturbation theory, the Feynman diagrams become more and more complex. In particular, in order to get the α^3-term of (0.1), one has to use 72 Feynman diagrams. The computation of the α^3-term has taken 20 years. The α^4-term from (0.1) is based on 891 Feynman diagrams. The computation has been done mainly by numerical approximation methods. This needed years of supercomputer time.[15] The mathematical situation becomes horrible because of the following fact.

> *Many of the Feynman diagrams correspond to divergent higher-dimensional integrals called algebraic Feynman integrals.*

Physicists invented the ingenious method of renormalization in order to give the apparently meaningless integrals a precise interpretation. Renormalization plays a fundamental role in quantum field theory. Physicists do not expect that the perturbation series (0.1) is part of a convergent power series expansion with respect to the variable α at the origin. Suppose that there would exist such a convergent power series expansion

$$a = \sum_{n=1}^{\infty} a_n \alpha^n, \qquad |\alpha| \le \alpha_0$$

near the origin $\alpha = 0$. This series would then converge for small negative values of α. However, such a negative coupling constant would correspond to a repelling force which destroys the system. This argument is due to Dyson.[16]

> *Therefore, we do not expect that the series* (0.1) *is convergent.*

In Sect. 15.5.2, we will show that each formal power series expansion is indeed the asymptotic expansion of some analytic function in an angular domain, by the famous 1916 Ritt theorem in mathematics.

[15] See M. Veltman, Facts and Mysteries in Elementary Particle Physics, World Scientific, Singapore, 2003; this is a beautiful history of modern elementary particle physics.

[16] F. Dyson, Divergence of perturbation theory in quantum electrodynamics, Phys. Rev. **85** (1952), 631–632.

From the mathematical point of view, the best approach to renormalization was created by Epstein and Glaser in 1973. The Epstein–Glaser theory avoids the use of divergent integrals and their regularization, but relies on the power of the modern theory of distributions (generalized functions).

Physicists have also computed the magnetic moment of the myon. As for the electron, the coincidence between theory and experiment is of fantastic accuracy. Here, the theory takes all of the contributions coming from electromagnetic, weak, strong, and gravitative interaction into account.[17]

It is a challenge for the mathematics of the future to completely understand formula (0.1)*.*

Let us now briefly discuss the content of Volumes I through VI of this monograph.

Volume I. The first volume entitled *Basics in Mathematics and Physics* is structured in the following way.

Part I: Introduction
- Chapter 1: Historical Introduction
- Chapter 2: Phenomenology of the Standard Model in Particle Physics
- Chapter 3: The Challenge of Different Scales in Nature.

Part II: Basic Techniques in Mathematics
- Chapter 4: Analyticity
- Chapter 5: A Glance at Topology
- Chapter 6: Many-Particle Systems
- Chapter 7: Rigorous Finite-Dimensional Magic Formulas of Quantum Field Theory
- Chapter 8: Rigorous Finite-Dimensional Perturbation Theory
- Chapter 9: Calculus for Grassmann Variables
- Chapter 10: Infinite-Dimensional Hilbert Spaces
- Chapter 11: Distributions and Green's Functions
- Chapter 12: Distributions and Quantum Physics.

Part III: Heuristic Magic Formulas of Quantum Field Theory
- Chapter 13: Basic Strategies in Quantum Field Theory
- Chapter 14: The Response Approach
- Chapter 15: The Operator Approach
- Chapter 16: Peculiarities of Gauge Theories
- Chapter 17: A Panorama of the Literature.

Describing the content of Volume I by a parable, we will first enter a mountain railway in order to reach easily and quickly the top of the desired mountain and to admire the beautiful mountain ranges. Later on we will try to climb to the top along the rocks.

[17] See M. Böhm, A. Denner, and H. Joos, Gauge Theories of the Strong and Electroweak Interaction, Teubner, Stuttgart, 2001, p. 80.

In particular, the heuristic magic formulas from Part III should help the reader to understand quickly the language of physicists in quantum field theory. These magic formulas are non-rigorous from the mathematical point of view, but they are extremely useful for computing physical effects.

Modern elementary particle physics is based on the Standard Model in particle physics introduced in the late 1960s and the early 1970s. Before studying thoroughly the Standard Model in the next volumes, we will discuss the phenomenology of this model in the present volume. It is the goal of quantum field theory to compute

- the cross sections of scattering processes in particle accelerators which characterize the behavior of the scattered particles,
- the masses of stable elementary particles (e.g., the proton mass as a bound state of three quarks), and
- the lifetime of unstable elementary particles in particle accelerators.

To this end, physicists use the methods of perturbation theory. Fortunately enough, the computations can be based on only a few basic formulas which we call magic formulas. The magic formulas of quantum theory are extremely useful for describing the experimental data observed in particle accelerators, but they are only valid on a quite formal level.

This difficulty is typical for present quantum field theory.

To help the reader in understanding the formal approach used in physics, we consider the finite-dimensional situation in the key Chapter 7.

In the finite-dimensional case, we will rigorously prove all of the magic formulas used by physicists in quantum field theory.

Furthermore, we relate physics to the following fields of mathematics:

- causality and the analyticity of complex-valued functions,
- many-particle systems, the Casimir effect in quantum field theory, and number theory,
- propagation of physical effects, distributions (generalized functions), and the Green's function,
- rigorous justification of the elegant Dirac calculus,
- duality in physics (time and energy, time and frequency, position and momentum) and harmonic analysis (Fourier series, Fourier transformation, Laplace transformation, Mellin transformation, von Neumann's general operator calculus for self-adjoint operators, Gelfand triplets and generalized eigenfunctions),
- the relation between renormalization, resonances, and bifurcation,
- dynamical systems, Lie groups, and the renormalization group,
- fundamental limits in physics,
- topology in physics (Chern numbers and topological quantum numbers),
- probability, Brownian motion, and the Wiener integral,

- the Feynman path integral,
- Hadamard's integrals and algebraic Feynman integrals.

In fact, this covers a broad range of physical and mathematical subjects.

Volume II. The second volume entitled *Quantum Electrodynamics* consists of the following parts.[18]

Part I: Introduction
- Chapter 1: Mathematical Principles of Modern Natural Philosophy
- Chapter 2: The Basic Strategy of Extracting Finite Information from Infinities – Ariadne's Thread in Renormalization Theory
- Chapter 3: The Power of Combinatorics and Hopf Algebras
- Chapter 4: The Strategy of Equivalence Classes in Mathematics.

Part II: Basic Ideas in Classical Mechanics
- Chapter 5: Geometrical Optics
- Chapter 6: The Principle of Critical Action and the Harmonic Oscillator – Ariadne's Thread in Classical Mechanics.

Part III: Basic Ideas in Quantum Mechanics
- Chapter 7: Quantization of the Harmonic Oscillator – Ariadne's Thread in Quantization
- Chapter 8: Quantum Particles on the Real Line – Ariadne's Thread in Scattering Theory
- Chapter 9: A Glance at General Scattering Theory.

Part IV: Quantum Electrodynamics (QED)
- Chapter 10: Creation and Annihilation Operators
- Chapter 11: The Basic Equations in Quantum Electrodynamics
- Chapter 12: The Free Quantum Fields of Electrons, Positrons, and Photons
- Chapter 13: The Interacting Quantum Field, and the Magic Dyson Series for the S-Matrix
- Chapter 14: The Beauty of Feynman Diagrams in QED
- Chapter 15: Applications to Physical Effects.

Part V: Renormalization
- Chapter 16: The Continuum Limit
- Chapter 17: Radiative Corrections of Lowest Order
- Chapter 18: A Glance at Renormalization to all Orders of Perturbation Theory
- Chapter 19: Perspectives.

The final goal of quantum field theory is the foundation of a rigorous mathematical theory which contains the Standard Model as a special low-energy approximation. At present we are far away from reaching this final goal. From

[18] This volume appeared in 2008.

the physical point of view, the most successful quantum field theory is quantum electrodynamics. This will be studied in Volume II along with some applications to important physical processes like Compton scattering between electrons and photons, the spontaneous emission of light by molecules, Cherenkov radiation of fast electrons, the Lamb shift in the hydrogen spectrum, and the anomalous magnetic moment of the electron. Generally, we try to include both interesting mathematics and interesting physics. In particular, we will discuss the relation of renormalization in physics to the following mathematical subjects: Euler's gamma function, the Riemann–Liouville integral, and dimensional regularization; Borel summation of divergent series; pseudo-convergence of iterative methods for ill-posed problems, Hopf algebras and Rota–Baxter algebras; theory of categories; wave front sets and the theory of distributions, Euler's and Feynman's mathemagics. Some important, but lengthy computations of physical effects in quantum electrodynamics based on renormalization theory are postponed to Volume III.

Volume III. The fundamental forces in the universe are described by gauge field theories which generalize both Gauss' surface theory and Maxwell's theory of electromagnetism. The third volume entitled *Gauge Theory* is divided into the following parts.

Part I: The Euclidean Manifold as a Paradigm
- Chapter 1: The Euclidean Space E_3
- Chapter 2: Algebras and Supersymmetry
- Chapter 3: The Euclidean Manifold $\mathbb{E}^3$
- Chapter 4: The Lie Group of Rotations and the Lie Algebra of Infinitesimal Rotations
- Chapter 5: Temperature Fields on the Euclidean Manifold and the Lie Derivative
- Chapter 6: The Lie Algebra of Velocity Vector Fields on the Euclidean Manifold
- Chapter 7: The Beauty of Differential Forms.

Part II: The Sphere as the Paradigm of a Curved Surface
- Chapter 8: The Gauss Method of Quadratic Forms
- Chapter 9: The Cartan Method of Moving Frames and Fiber Bundles.

Part III: Observers and Invariants
- Chapter 10: Change of Local Coordinates for Two Observers
- Chapter 11: Families of Observers and Cocycles
- Chapter 12: Linear Connections.

Part IV: Einstein's Theory of Special Relativity
- Chapter 13: The Importance of Inertial Systems
- Chapter 14: The Mathematical Structure of the Minkowski Space
- Chapter 15: The Poincaré Group and the Electromagnetic Field
- Chapter 16: Spinor Calculus.

Part V: Ariadne's Thread in Gauge Theory
- Chapter 17: The $SU(N)$-Gauge Theory as a Paradigm

- Chapter 18: Models in Solid State Physics.

Part VI: The Fundamental Forces in Nature

- Chapter 19: The Clifford Algebra of the Minkowski Space and the Relativistic Electron
- Chapter 20: The Hydrogen Atom as a Paradigm
- Chapter 21: The Stability of Matter
- Chapter 22: The Standard Model in Particle Physics
- Chapter 23: Einstein's Theory of General Relativity and Cosmology.

Part VII: Radiative Corrections in Quantum Electrodynamics (QED)

- Chapter 24: Dimensional Regularization of Critical Feynman Diagrams
- Chapter 25: The Electron in an External Electromagnetic Field
- Chapter 26: The Lamb Shift.

Part VIII: A Glance at String Theory

- Chapter 27: Minimal Surfaces
- Chapter 28: Strings and the Graviton.

Interestingly enough, it turns out that the Standard Model in particle physics is related to many deep questions in both mathematics and physics. We will see that the question about the structure of the fundamental forces in nature has influenced implicitly or explicitly the development of a large part of mathematics. One of our heros will be Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time. We will encounter his highly influential work again and again. In the German Museum in Munich, one can read the following inscription under Gauss' impressive portrait:

> His spirit lifted the deepest secrets of numbers, space, and nature; he measured the orbits of the planets, the form and the forces of the earth; in his mind he carried the mathematical science of a coming century.

On the occasion of Gauss' death, Sartorius von Waltershausen wrote the following in 1855:

> From time to time in the past, certain brilliant, unusually gifted personalities have arisen from their environment, who by virtue of the creative power of their thoughts and the energy of their actions have had such an overall positive influence on the intellectual development of mankind, that they at the same time stand tall as markers between the centuries... Such epoch-making mental giants in the history of mathematics and the natural sciences are Archimedes of Syracuse in ancient times, Newton toward the end of the dark ages and Gauss in our present day, whose shining, glorious career has come to an end after the cold hand of death touched his at one time deeply-thinking head on February 23 of this year.

Another hero will be Bernhard Riemann (1826–1866) – a pupil of Gauss. Riemann's legacy influenced strongly mathematics and physics of the 20th century, as we will show in this treatise.[19]

[19] We also recommend the beautiful monograph written by Krzysztof Maurin, The Riemann Legacy, Kluwer, Dordrecht, 1997.

The two Standard Models in modern physics concerning cosmology and elementary particles are closely related to modern differential geometry. This will be thoroughly studied in Volume III. We will show that both Einstein's general theory of relativity and the Standard Model in particle physics are gauge theories. From the mathematical point of view, the fundamental forces in nature are curvatures of appropriate fiber bundles. Historically, mathematicians have tried to understand the curvature of geometric objects. At the very beginning, there was Gauss' *theorema egregium*[20] telling us the crucial fact that curvature is an intrinsic property of a surface. On the other side, in the history of physics, physicists have tried to understand the forces in nature. Nowadays we know that both mathematicians and physicists have approached the same goal coming from different sides. We can summarize this by saying briefly that

$$\textit{force} = \textit{curvature}.$$

From the physical point of view, the parallel transport of physical information plays the fundamental role in gauge theory. For the convenience of the reader, we will also discuss in Volume III that many of the mathematical concepts arising in quantum field theory are rooted in the geometry of the Euclidean space (e.g., Lie groups and Lie algebras, operator algebras, Grassmann algebras, Clifford algebras, differential forms and cohomology, Hodge duality, projective structures, symplectic structures, contact structures, conformal structures, Riemann surfaces, and supersymmetry).

Volume IV. Quantum physics differs from classical relativistic field theories by adding the process of quantization. From the physical point of view, there appear additional quantum effects based on random quantum fluctuations. From the mathematical point of view, one has to deform classical theories in an appropriate way. Volume IV is devoted to the mathematical methods of quantization. For this, we coin the term *Quantum Mathematics*. This is a branch of mathematics. Volume IV represents the first systematic textbook on *Quantum Mathematics*. This volume will be divided into the following parts.

Part I: Quantization
Part II: Quantum Information
Part III: Symmetry, Groups, and Hopf Algebras
Part IV: Observables and Operator Algebras
Part V: Cohomology and Homology
Part VI: Physical Fields, Fiber Bundles, and Sheaves.

Typically, quantum fields are interacting physical systems with an infinite number of degrees of freedom and very strong singularities. In mathematics,

- interactions lead to nonlinear terms, and
- infinite-dimensional systems are described in terms of functional analysis.

[20] The Latin expression *theorema egregium* means *the beautiful theorem*.

Therefore, the right mathematical setting for quantum field theory is nonlinear functional analysis. This branch of mathematics has been very successful in the rigorous treatment of nonlinear partial differential equations concerning elasticity and plasticity theory, hydrodynamics, and the theory of general relativity. But the actual state of the art of nonlinear functional analysis does not yet allow for the rigorous investigation of realistic models in quantum field theory, like the Standard Model in particle physics. Physicists say, we cannot wait until mathematics is ready. Therefore, we have to develop our own non-rigorous methods, and we have to check the success of our methods by comparing them with experimental data. In order to help mathematicians to enter the world of physicists, we will proceed as follows.

(i) Rigorous methods: We first develop quantum mathematics in finite-dimensional spaces. In this case, we can use rigorous methods based on the theory of Hilbert spaces, operator algebras, and discrete functional integrals.
(ii) Formal methods. The formulas from (i) can be generalized in a straightforward, but formal way to infinite-dimensional systems.

This way, the mathematician should learn where the formulas of the physicists come from and how to handle these formulas in order to compute physical effects. What remains is to solve the open problem of rigorous justification.

> *The point will be the investigation of limits and pseudo-limits if the number of particles goes to infinity.*

By a pseudo-limit, we understand the extraction of maximal information from an ill-defined object, as in the method of renormalization. The experience of physicists and mathematicians shows that we cannot expect the limits or pseudo-limits to exist for all possible quantities. The rule of thumb is as follows: concentrate on quantities which can be measured in physical experiments. This seriously complicates the subject. We will frequently encounter the Feynman functional integral. From the mnemonic point of view, this is a marvellous tool. But it lacks mathematical rigor. We will follow the advise given by Evariste Galois (1811–1832):

> Unfortunately what is little recognized is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties.

Volume V. The mathematician should notice that it is the ultimate goal of a physicist to compute real numbers which can be measured in physical experiments. For reaching this goal, the physicist mixes rigorous arguments with heuristic ones in an ingenious way. In order to make mathematicians familiar with this method of doing science, in Volume V we will study the physics of the Standard Model in particle physics. In particular, we will show how to compute a number of physical effects. In this respect, symmetries will play an important role. For example, this will concern the representation theory of compact Lie groups (e.g., gauge groups in gauge theory), noncompact

Lie groups (the Poincaré group and its universal covering group in relativistic physics), infinite-dimensional Lie algebras (e.g., the Virasoro algebra in string theory), and supersymmetric generalizations.

Volume VI. The last volume will be devoted to combining the Standard Model in particle physics with gravitation. We will study several possible approaches to this fascinating, but still completely open problem. The leading candidate is string theory. In connection with the string theory of physicists, a completely new way of thinking has emerged which we will call *physical mathematics*, a term already used in Kishore Marathe's nice survey article on the role of knot theory in modern mathematics, physics, and biology.[21] Distinguish the following:

- By mathematical physics, we traditionally understand a branch of mathematics which answers questions coming from physics by applying rigorous mathematical methods. The heart of mathematical physics are mathematical proofs (e.g., existence proofs for solutions of partial differential equations or operator equations).
- By physical mathematics, we understand a branch of physics which is motivated by the question about the fundamental forces in nature. Using physical pictures, physicists are able to conjecture deep mathematical results (e.g., the existence and the properties of new topological invariants for manifolds and knots). The heart of physical mathematics is physical intuition, but not the mathematical proof.

The hero of physical mathematics is the physicist Edward Witten (born 1951) from the Institute for Advanced Study in Princeton. At the International Congress of Mathematicians in Kyoto (Japan) in 1990, Witten was awarded the Fields medal. In the last 15 years, physical mathematics was very successful in feeding fascinating new ideas into mathematics. The main method of physical mathematics goes like this:

- start with a model in quantum field theory based on an appropriate Lagrangian;
- quantize this model by means of the corresponding Feynman functional integral;
- extract essential information from the functional integral by using the method of stationary phase.

The point is that this method yields beautiful mathematical conjectures, but it is not able to give rigorous proofs. Unfortunately, for getting proofs, mathematicians have to follow quite different sophisticated routes. It is a challenge to mathematicians to understand better the magic weapon of physical mathematics.

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[21] K. Marathe, A chapter in physical mathematics: theory of knots in the sciences, pp. 873–888. In: Mathematics Unlimited – 2001 and Beyond edited by B. Engquist and W. Schmid, Springer, Berlin, 2001.

> *The magic weapon of physical mathematics will be called the Witten functor. This functor translates physical structures into mathematical structures.*

With respect to the Witten functor, one observes the following general evolution principle in mathematics.

(i) From quantity to quality: In the 1920s, modern algebra was founded by passing from concrete mathematical objects like numbers to abstract mathematical structures like groups, rings, fields, and algebras. Here, one only considers the relations between the objects, but not the individual structure of the objects. For example, Emmy Noether emphasized in the 1920s that, in the setting of algebraic topology created by Poincaré at the end of the 19th century, it is very useful to pass from Betti numbers to homology groups. In turn, it was discovered in the 1930s that cohomology groups are in fact richer in structure than homology groups. The point is that cohomology groups possess a natural multiplicative structure which generates the cohomology ring of topological spaces. For example, the product $\mathbb{S}^2 \times \mathbb{S}^4$ of a 2-dimensional sphere with a 4-dimensional sphere has the same homology and cohomology groups as the 3-dimensional complex projective space $\mathbb{P}^3_{\mathbb{C}}$. However, these two manifolds are not topologically equivalent, since their cohomology rings are different.

(ii) Combining abstract structures with each other: For example, Lie groups are obtained by combining the notion of manifold with the notion of group. In turn, fiber bundles occur by combining manifolds with Lie groups.

(iii) Functors between abstract structures: In the late 1940s, the theory of categories emerged in the context of algebraic topology. For example, the Galois functor simplifies the study of field extensions by mapping fields to groups. The Lie functor simplifies the investigation of Lie groups by mapping Lie groups to Lie algebras. Moreover, the homology functor simplifies the structural analysis of topological spaces (geometric objects) by mapping topological spaces to groups called homology groups. Combining the homology functor with the general concept of duality, we arrive at the cohomology functor which maps topological spaces to cohomology groups. Cohomology plays a fundamental role in modern physics.

(iv) Statistics of abstract structures: In physical mathematics, one considers the statistics of physical states in terms of functional integrals. The point is that the states are equivalence classes of mathematical structures. In the language of mathematics, the physical state spaces are moduli spaces. For example, in string theory the states of strings are Riemann surfaces modulo conformal equivalence. Thus, the state space of all those strings which possess a fixed genus g is nothing other than Riemann's famous moduli space $\mathcal{M}_g$ which can be described by a universal covering space of $\mathcal{M}_g$ called the Teichmüller space $\mathcal{T}_g$. Mathematicians know that the theory of moduli spaces is a challenge in algebraic geometry, since such

> objects carry singularities, as a rule. Physicists expect that those singularities are responsible for essential physical effects.

Another typical feature of physical mathematics is the description of many-particle systems by *partition functions* which encode essential information. As we will show, the Feynman functional integral is nothing other than a partition function which encodes the essential properties of quantum fields. From the physical point of view, the Riemann zeta function is a partition function for the infinite system of prime numbers. The notion of partition function unifies

- statistical physics,
- quantum mechanics,
- quantum field theory, and
- number theory.

Summarizing, I dare say that

> *The most important notion of modern physics is the Feynman functional integral as a partition function for the states of many-particle systems.*

It is a challenge of mathematics to understand this notion in a better way than known today.

A panorama of mathematics. For the investigation of problems in quantum field theory, we need a broad spectrum of mathematical branches. This concerns

(a) algebra, algebraic geometry, and number theory,
(b) analysis and functional analysis,
(c) geometry and topology,
(d) information theory, theory of probability, and stochastic processes,
(e) scientific computing.

In particular, we will deal with the following subjects:

- Lie groups and symmetry, Lie algebras, Kac–Moody algebras (gauge groups, permutation groups, the Poincaré group in relativistic physics, conformal symmetry),
- graded Lie algebras (supersymmetry between bosons and fermions),
- calculus of variations and partial differential equations (the principle of critical action),
- distributions (also called generalized functions) and partial differential equations (Green's functions, correlation functions, propagator kernels, or resolvent kernels),
- distributions and renormalization (the Epstein–Glaser approach to quantum field theory via the S-matrix),
- geometric optics and Huygens' principle (symplectic geometry, contact transformations, Poisson structures, Finsler geometry),

- Einstein's Brownian motion, diffusion, stochastic processes and the Wiener integral, Feynman's functional integrals, Gaussian integrals in the theory of probability, Fresnel integrals in geometric optics, the method of stationary phase,
- non-Euclidean geometry, covariant derivatives and connections on fiber bundles (Einstein's theory of general relativity for the universe, and the Standard Model in elementary particle physics),
- the geometrization of physics (Minkowski space geometry and Einstein's theory of special relativity, pseudo-Riemannian geometry and Einstein's theory of general relativity, Hilbert space geometry and quantum states, projective geometry and quantum states, Kähler geometry and strings, conformal geometry and strings),
- spectral theory for operators in Hilbert spaces and quantum systems,
- operator algebras and many-particle systems (states and observables),
- quantization of classical systems (method of operator algebras, Feynman's functional integrals, Weyl quantization, geometric quantization, deformation quantization, stochastic quantization, the Riemann–Hilbert problem, Hopf algebras and renormalization),
- combinatorics (Feynman diagrams, Hopf algebras),
- quantum information, quantum computers, and operator algebras,
- conformal quantum field theory and operator algebras,
- noncommutative geometry and operator algebras,
- vertex algebras (sporadic groups, monster and moonshine),
- Grassmann algebras and differential forms (de Rham cohomology),
- cohomology, Hilbert's theory of syzygies, and BRST quantization of gauge field theories,
- number theory and statistical physics,
- topology (mapping degree, Hopf bundle, Morse theory, Lyusternik–Schnirelman theory, homology, cohomology, homotopy, characteristic classes, homological algebra, K-theory),
- topological quantum numbers (e.g., the Gauss–Bonnet theorem, Chern classes, and Chern numbers, Morse numbers, Floer homology),
- the Riemann–Roch–Hirzebruch theorem and the Atiyah–Singer index theorem,
- analytic continuation, functions of several complex variables (sheaf theory),
- string theory, conformal symmetry, moduli spaces of Riemann surfaces, and Kähler manifolds.

The role of proofs. Mathematics relies on proofs based on perfect logic. The reader should note that, in this treatise, the terms

- proposition,
- theorem (important proposition), and
- proof

are used in the rigorous sense of mathematics. In addition, for helping the reader in understanding the basic ideas, we also use 'motivations', 'formal proofs', 'heuristic arguments' and so on, which emphasize intuition, but lack rigor. Because of the rich material to be studied, it is impossible to provide the reader with full proofs for all the different subjects. However, for missing proofs we add references to carefully selected sources. Many of the missing proofs can be found in the following monographs:

- E. Zeidler, Applied Functional Analysis, Vols. 1, 2, Springer, New York. 1995.
- E. Zeidler, Nonlinear Functional Analysis and its Applications, Vols. 1–4, Springer, New York, 1985–88.

For getting an overview, the reader should also consult the following book:[22]

- E. Zeidler (Ed.), Oxford Users' Guide to Mathematics, Oxford University Press, 2004 (1300 pages).

At the end of the *Oxford Users' Guide to Mathematics*, the interested reader may find a chronology of mathematics and physics from ancient to present times embedded in the cultural history of mankind.

Perspectives. At the International Congress of Mathematicians in Paris in 1900, Hilbert formulated 23 open problems for the mathematics of the 20th century. Many of them have been solved.[23] Hilbert said the following to the audience in 1900:

> Each progress in mathematics is based on the discovery of stronger tools and easier methods, which at the same time makes it easier to understand earlier methods. By making these stronger tools and easier methods his own, it is possible for the individual researcher to orientate himself in the different branches of mathematics...
> When the answer to a mathematical problem cannot be found, then the reason is frequently that we have not recognized the general idea from which the given problem only appears as a link in a chain of related problems...
> The organic unity of mathematics is inherent in the nature of this science, for mathematics is the foundation of all exact knowledge of natural phenomena.

For the 21th century, the open problem of quantum field theory represents a great challenge. It is completely unclear how long the solution of this problem will take. In fact, there are long-term problems in mathematics. As an example, let us consider *Fermat's Last Theorem* where the solution needed more than 350 years. In ancient times, Pythagoras (508–500 B.C.) knew that the equation

$$x^2 + y^2 = z^2$$

[22] The German version reads as E. Zeidler, Teubner-Taschenbuch der Mathematik, Vols. 1, 2, Teubner, Wiesbaden, 2003. The English translation of the second volume is in preparation.

[23] See D. Hilbert, Mathematical Problems, Bull. Amer. Math. Soc. **8** (1902), 437–479, and B. Yandell, The Honors Class: Hilbert's Problems and Their Solvers, Natick, Massachusetts, 2001.

has an infinite number of integer solutions (e.g., $x = 3, y = 4, z = 5$). In 1637, Pierre de Fermat (1601–1665), claimed that the equation

$$x^n + y^n = z^n, \qquad n = 3, 4, \ldots$$

has no nontrivial integer solution. In his copy of the *Arithmetica* by Diophantus (250 A.C.), Fermat wrote the following:

> It is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvellous proof of this, which however the margin is not large enough to contain.

The history of this problem can be found in the bestseller by Simon Singh, Fermat's Last Theorem: The Story of a Riddle that Confounded the World's Greatest Minds for 358 Years, Fourth Estate, London, 1997. The final proof was given by Andrew Wiles (born 1953) in Princeton in 1994.[24] The proof, based on the Galois functor, is of extraordinary complexity, and it uses many sophisticated tools from number theory and algebraic geometry developed in the 19th and 20th century. However, in the sense of Hilbert's philosophy for hard problems quoted above, let us describe the basic idea behind the solution. In this connection, it turns out that there is a beautiful geometric result of general interest behind Fermat's Last Theorem.[25] The fundamental geometric result tells us that[26]

(M) *Each elliptic curve is modular.*

Roughly speaking, the proof of Fermat's last theorem proceeds now like this:

(i) Suppose that Fermat's claim is wrong. Then, there exists a nontrivial triplet x, y, z of integers such that $x^n + y^n = z^n$ for some fixed natural number $n \geq 3$.
(ii) The triplet x, y, z can be used in order to construct a specific elliptic curve (the Frey curve), which is not modular, a contradiction to (M).

It remains to sketch the meaning of the geometric principle (M). To begin with, consider the equation of the complex unit circle

$$x^2 + y^2 = 1$$

where x and y are complex parameters. The unit circle allows a parametrization either by periodic functions,

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24 A. Wiles, Modular elliptic curves and Fermat's Last Theorem, Ann. Math. **142** (1994), 443–551.

25 We refer to the beautiful lecture given by Don Zagier, Leçon inaugurale, Jeudi 17 Mai 2001, Collège de France, Paris. See also H. Darmon, A proof of the full Shimura–Taniyama–Weil conjecture is announced, Notices Amer. Math. Soc. **46** (1999), 1397–1401. Much background material can be found in the fascinating textbook by Y. Hellagouarch, Invitation to the Mathematics of Fermat–Wiles, Academic Press, New York.

26 A comprehensive survey article on modular forms can be found in Zagier (1995).

$$x = \cos\varphi, \qquad y = \sin\varphi, \qquad \varphi \in \mathbb{C},$$

or by rational functions,

$$x = \frac{2}{1+t^2} - 1, \qquad y = \frac{2t}{1+t^2}, \qquad t \in \mathbb{C},$$

provided we set $t := \tan\frac{\varphi}{2}$. Recall that each compact Riemann surface of genus zero is conformally and topologically equivalent to the real two-dimensional sphere called the Riemann sphere. In particular, the complex unit circle considered above is such a Riemann surface of genus zero. Moreover, compact Riemann surfaces of genus one are conformally and topologically equivalent to some real two-dimensional torus. Such Riemann surfaces are also called elliptic curves. For example, given three pairwise different complex numbers e_1, e_2, e_3, the equation

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3)$$

with complex parameters x and y represents an elliptic curve which allows the global parametrization

$$x = \wp(t), \qquad y = \wp'(t), \qquad t \in \mathbb{C}$$

by the Weierstrass $\wp$-function. This is an elliptic (i.e., double-periodic) function whose two complex periods depend on e_1, e_2, e_3. The fundamental geometric result reads now as follows:

(i) Each compact Riemann surface of genus zero (i.e., each complex curve of circle type) allows two global parametrizations by either periodic functions or rational functions.
(ii) Each compact Riemann surface of genus one (i.e., each elliptic curve) allows two global parametrizations by either double-periodic functions or modular functions.
(iii) Each compact Riemann surface of genus $g \geq 2$ can be globally parametrized by automorphic functions.[27]

The global parametrization (i) of elliptic curves by elliptic functions is one of the most famous results of 19th century mathematics due to Jacobi, Riemann, and Weierstrass. The general result (ii) on the global parametrization of elliptic curves by modular functions was only proved in 1999, i.e., it was shown that the full Shimura–Taniyama–Weil conjecture is true. Statement (iii) represents the famous uniformization theorem for compact Riemann surfaces which was proved independently by Koebe and Poincaré in 1907 after strong efforts made by Poincaré and Klein. The existence of double-periodic functions was discovered by Gauss in 1797 while studying the geometric properties

[27] Much material on Riemann surfaces, elliptic curves, zeta functions, Galois theory, and so on, can be found in the volume edited by M. Waldschmidt et al., From Number Theory to Physics, Springer, New York, 1995.

of the lemniscate introduced by Jakob Bernoulli (1654–1705). Therefore, the innocent looking three statements (i), (ii), (iii) above are the result of 200 years of intense mathematical research. Summarizing, in the sense of Hilbert, the famous Fermat conjecture could finally be solved because it could be reduced to the general idea of modular curves. In a fascinating essay on the future of mathematics, Arthur Jaffe (born 1937) from Harvard University wrote the following:[28]

> Mathematical research should be as broad and as original as possible, with very long range-goals. We expect history to repeat itself: we expect that the most profound and useful future applications of mathematics cannot be predicted today, since they will arise from mathematics yet to be discovered.

Studying the physics and mathematics of the fundamental forces in nature, there arises the question about the philosophical background. Concerning this, let me finish with two quotations. Erich Worbs writes in his Gauss biography:

> Sartorius von Waltershausen reports that Gauss once said there were questions of infinitely higher value than the mathematical ones, namely, those about our relation to God, our determination, and our future. Only, he concluded, their solutions lie far beyond our comprehension, and completely outside the field of science.

In the Harnack Building of the Max Planck Society in Berlin, one can read the following words by Johann Wolfgang von Goethe:

> The greatest joy of a thinking man is to have explored the explorable and just to admire the unexplorable.

[28] Ordering the universe: the role of mathematics, Notices Amer. Math. Soc. **236** (1984), 589–608.

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