

1. Historical Introduction

If we wish to foresee the future of mathematics our proper course is to study the history and present condition of the science.

Henri Poincaré (1854–1912)

It is not the knowledge but the learning, not the possessing, but the earning, not the being there but the getting there, which gives us the greatest pleasure.

Carl Friedrich Gauss (1777–1855)
to his Hungarian friend Janos Bolyai

For me, as a young man, Hilbert (1858–1943) became the kind of mathematician which I admired, a man with an enormous power of abstract thought, combined with a fully developed sense for the physical reality.

Norbert Wiener (1894–1964)

In the fall 1926, the young John von Neumann (1903–1957) arrived in Göttingen to take up his duties as Hilbert's assistant. These were the hectic years during which quantum mechanics was developing with breakneck speed, with a new idea popping up every few weeks from all over the horizon. The theoretical physicists Born, Dirac, Heisenberg, Jordan, Pauli, and Schrödinger who were developing the new theory were groping for adequate mathematical tools. It finally dawned upon them that their 'observables' had properties which made them look like Hermitean operators in Hilbert space, and that by an extraordinary coincidence, the 'spectrum' of Hilbert (which he had chosen around 1900 from a superficial analogy) was to be the central conception in the explanation of the 'spectra' of atoms. It was therefore natural that they should enlist Hilbert's help to put some mathematical sense in their formal computations. With the assistance of Nordheim and von Neumann, Hilbert first tried integral operators in the space L_2 , but that needed the use of the Dirac delta function δ , a concept which was for the mathematicians of that time self-contradictory. John von Neumann therefore resolved to try another approach.

Jean Dieudonné (1906–1992)
*History of Functional Analysis*¹

Stimulated by an interest in quantum mechanics, John von Neumann began the work in operator theory which he was to continue as long as he lived. Most of the ideas essential for an abstract theory had already been

¹ North-Holland, Amsterdam, 1981 (reprinted with permission).

developed by the Hungarian mathematician Fryges Riesz, who had established the spectral theory for bounded Hermitean operators in a form very much like as regarded now standard. Von Neumann saw the need to extend Riesz's treatment to *unbounded* operators and found a clue to doing this in Carleman's highly original work on integral operators with singular kernels. . .

The result was a paper von Neumann submitted for publication to the *Mathematische Zeitschrift* but later withdrew. The reason for this withdrawal was that in 1928 Erhard Schmidt and myself, independently, saw the role which could be played in the theory by the concept of the adjoint operator, and the importance which should be attached to *self-adjoint operators*. When von Neumann learned from Professor Schmidt of this observation, he was able to rewrite his paper in a much more satisfactory and complete form. . . Incidentally, for permission to withdraw the paper, the publisher exacted from Professor von Neumann a promise to write a book on quantum mechanics. The book soon appeared and has become one of the classics of modern physics.²

Marshall Harvey Stone (1903–1989)

1.1 The Revolution of Physics

At the beginning of the 20th century, Max Planck (1858–1947) and Albert Einstein (1879–1955) completely revolutionized physics. In 1900, Max Planck derived the universal radiation law for stars by postulating that

The action in our world is quantized.

Let us discuss this fundamental physical principle. The action is the most important physical quantity in nature. For any process, the action is the product of energy \times time for a small time interval. The total action during a fixed time interval is then given by an integral summing over small time intervals. The fundamental principle of least (or more precisely, critical) action tells us the following:

A process in nature proceeds in such a way that the action becomes minimal under appropriate boundary conditions.

More precisely, the action is critical. This means that the first variation of the action S vanishes, $\delta S = 0$. In 1918 Emmy Noether (1882–1935) proved a fundamental mathematical theorem. The famous Noether theorem tells us that

Conservation laws in physics are caused by symmetries of physical systems.

² J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955 (first German edition: Springer, Berlin, 1932). This quotation is taken from F. Browder (Ed.), *Functional Analysis and Related Fields*, Springer, Berlin, 1970 (reprinted with permission).

To explain this basic principle for describing nature in terms of mathematics, consider our solar system. The motion of the sun and the planets only depends on the initial positions and initial velocities. Obviously, the motion of the solar system is invariant under time translations, spatial translations, and rotations. This is responsible for conservation of energy, momentum, and angular momentum, respectively. For example, invariance under time translations means the following. If a process of the physical system is possible, $\mathbf{x} = \mathbf{x}(t)$, then each process is also possible which is obtained by time translation, $\mathbf{x} = \mathbf{x}(t + \text{const})$. According to Planck, the smallest amount of action in nature is equal to

$$\boxed{h = 6.260\,0755 \cdot 10^{-34} \text{ Js}} \quad (1.1)$$

where $1 \text{ Joule} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$. We also introduce $\hbar := h/2\pi$. The universal constant h is the famous Planck quantum of action (or the Planck constant). Observe that the action of typical processes in daily life has the magnitude of 1 Js. Therefore, the Planck constant is tiny. Nevertheless, the quantization of action has enormous consequences. For example, consider a mass point on the real line which moves periodically,

$$q(t) = \text{const} \cdot \sin(\omega t)$$

where t denotes time, and ω is called the angular frequency of the harmonic oscillator. Since the sine function has period 2π , the harmonic oscillator has the time period $T = 2\pi/\omega$. By definition, the frequency ν is the number of oscillations per second. Hence $T = 1/\nu$, and $\omega = 2\pi\nu$. If E denotes the energy of the harmonic oscillator, then the product ET is a typical action value for the oscillations of the harmonic oscillator. Therefore, according to Planck's quantization of action, it seems to be quite natural to postulate that $ET = nh$ for $n = 0, 1, 2, \dots$. This yields Planck's quantization rule for the energy of the harmonic oscillator,

$$E = n\hbar\omega, \quad n = 0, 1, 2, \dots$$

from the year 1900. About 25 years later, the young physicist Werner Heisenberg (1901–1976) invented the full quantization procedure of classical mechanics. Using implicitly the commutation rule

$$\boxed{qp - pq = i\hbar} \quad (1.2)$$

for the position q and the momentum p of a quantum particle, Heisenberg obtained the precise formula

$$\boxed{E = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \dots} \quad (1.3)$$

for the quantized energy levels of a harmonic oscillator. Heisenberg's quantum mechanics changed completely the paradigm of physics. In classical physics,

observables are real numbers. In Heisenberg's approach, observables are abstract quantities which obey certain commutation rules. More than fifty years before Heisenberg's discovery, the great Norwegian mathematician Sophus Lie (1842–1899) found out that commutation rules of type (1.2) play a fundamental role when trying to study continuous symmetry groups by means of linearization. In 1934, for this kind of algebraic structure, the term “Lie algebra” was coined by Hermann Weyl (1885–1955). Lie algebras and their generalizations to infinite dimensions, like the Virasoro algebra and supersymmetric algebras in string theory and conformal quantum field theory, are crucial for modern quantum physics. The Heisenberg formula (1.3) tells us that the ground state of each harmonic oscillator has the non-vanishing energy

$$E = \frac{\hbar\omega}{2}. \quad (1.4)$$

This fact causes tremendous difficulties in quantum field theories. Since a quantum field has an infinite number of degrees of freedom, the ground state has an infinite energy. There are tricks to cure the situation a little bit, but the infinite ground state energy is the deeper reason for the appearance of nasty divergent quantities in quantum field theory. Physicists have developed the ingenious method of renormalization in order to extract finite quantities that can be measured in physical experiments. Surprisingly enough, in quantum electrodynamics there is an extremely precise coincidence with the renormalized theoretical values and the values measured in particle accelerator experiments. No one understands this. Nowadays many physicists are convinced that this approach is not the final word. There must be a deeper theory behind. One promising candidate is string theory.

At the end of his life, Albert Einstein wrote the following about his first years.

Between the ages of 12–16, I familiarized myself with the elements of mathematics. In doing so I had the good fortune of discovering books which were not too particular in their logical rigor.

In 1896, at the age of 17, I entered the Swiss Institute of Technology (ETH) in Zurich. There I had excellent teachers, for example, Hurwitz (1859–1919) and Minkowski (1864–1909), so that I really could get a sound mathematical education. However, most of the time, I worked in the physical laboratory, fascinated by the direct contact with experience. The rest of the time I used, in the main, to study at home the works of Kirchhoff (1824–1887), Helmholtz (1821–1894), Hertz (1857–1894), and so on. The fact that I neglected mathematics to a certain extent had its cause not merely in my stronger interest in the natural sciences than in mathematics, but also in the following strange experience. I saw that mathematics was split up into numerous specialities, each of which could easily absorb the short life granted to us. Consequently, I saw myself in the position of Buridan's ass which was unable to decide upon any specific bundle of hay. This was obviously due to the fact that my intuition was not strong enough in the field of mathematics in order to differentiate clearly that

which was fundamentally important, and that which is really basic, from the rest of the more or less dispensable erudition, and it was not clear to me as a student that the approach to a more profound knowledge of the basic principles of physics is tied up with the most intricate mathematical methods. This only dawned upon me gradually after years of independent scientific work. True enough, physics was also divided into separate fields. In this field, however, I soon learned to scent out that which was able to lead to fundamentals.³

After his studies, Einstein got a position at the Swiss patent office in Bern. In 1905 Einstein published four fundamental papers on the theory of special relativity, the equivalence between mass and energy, the Brownian motion, and the light particle hypothesis in Volume 17 of the journal *Annalen der Physik*.

The theory of special relativity completely changed our philosophy about space and time.

According to Einstein, there is no absolute time, but time changes from observer to observer. This follows from the surprising fact that the velocity of light has the same value in each inertial system, which was established experimentally by Albert Michelson (1852–1931) in 1887. From his principle of relativity, Einstein deduced that a point particle of rest mass m_0 and momentum vector \mathbf{p} has a positive energy E given by

$$\boxed{E^2 = m_0^2 c^4 + c^2 \mathbf{p}^2} \quad (1.5)$$

where c denotes the velocity of light in a vacuum. If the particle moves with sub-velocity of light, $\mathbf{x} = \mathbf{x}(t)$, than it has the mass

$$m = \frac{m_0}{\sqrt{1 - \dot{\mathbf{x}}(t)^2/c^2}}. \quad (1.6)$$

If the particle rests, then we get

$$E = m_0 c^2. \quad (1.7)$$

This magic energy formula governs the energy production in our sun by helium synthesis. Thus, our life depends crucially on this formula. Unfortunately, the atomic bomb is based on this formula, too.

Let us now discuss the historical background of Einstein's light particle hypothesis. Maxwell (1831–1879) conjectured in 1862 that light is an electromagnetic wave. In 1886 Heinrich Hertz established the existence of electromagnetic waves by a famous experiment carried out at Kiel University (Germany). When electromagnetic radiation is incident on the surface

³ This is the English translation of Einstein's handwritten letter copied in the following book: Albert Einstein als Philosoph und Naturforscher (Albert Einstein as philosopher and scientist). Edited by P. Schilpp, Kohlhammer Verlag, Stuttgart (printed with permission).

of a metal, it is observed that electrons may be ejected. This phenomenon is called the photoelectric effect. This effect was first observed by Heinrich Hertz in 1887. Fifteen years later, Philipp Lenard (1862–1947) observed that the maximum kinetic energy of the electrons does not depend on the intensity of light. In order to explain the photoelectric effect, Einstein postulated in 1905 that electromagnetic waves are quantized. That is, light consists of light particles (or light quanta) which were coined photons in 1926 by the physical chemist Gilbert Lewis. According to Einstein, a light particle (photon) has the energy E given by Planck's quantum hypothesis,

$$E = h\nu. \quad (1.8)$$

Here, ν is the frequency of light, which is related to the wave length λ by the dispersion relation $\lambda\nu = c$. Hence $E = hc/\lambda$. This means that a blue photon has more energy than a red one. Since a photon moves with light speed, its rest mass must be zero. Thus, from (1.5) we obtain $|\mathbf{p}| = E/c$. If we introduce the angular frequency $\omega = 2\pi\nu$, then we obtain the final expression

$$\boxed{E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}, \quad |\mathbf{k}| = \frac{\omega}{c}} \quad (1.9)$$

for the energy E and the momentum vector \mathbf{p} of a photon. Here, the wave vector \mathbf{k} of length $k = \omega/c$ is parallel to the vector \mathbf{p} . Nowadays we know that light particles are quanta, and that quantum particles are physical objects which possess a strange structure. Quanta combine features of both waves and particles. In the photoelectric effect, a photon hits an electron such that the electron leaves the metal. The energy of the electron is given by

$$E = \hbar\omega - W$$

where the so-called work function W depends on the binding energy of the electrons in the atoms of the metal. This energy formula suggests that for small angular frequencies ω no electrons can leave the metal, since there would be $E < 0$, a contradiction. In fact, this has been observed in experiments. Careful experiments were performed by Millikan (1868–1953) in 1916. He found out that a typical constant in his experiments coincided with the Planck constant, as predicted by Einstein. In 1921 Einstein was awarded the Nobel prize in physics for his services to theoretical physics, and especially for his discovery of the law of the photoelectric effect. As a curiosity let us mention, that Max Planck, while recommending Einstein enthusiastically for a membership in the Prussian Academy in Berlin, wrote the following:

That sometimes, in his speculations, he went too far, such as, for example, in his hypothesis of the light quanta, should not be held too much against him.

1.2 Quantization in a Nutshell

In 1926 Born discovered the fundamental fact that quantum physics is intrinsically connected with random processes. Hence the mathematical theory of probability plays a crucial role in quantum physics. Already Maxwell (1831–1879) had emphasized:

The true logic of this world lies in probability theory.

Before discussing the randomness of quantum processes and the challenge of quantization, let us mention that Maxwell strongly influenced the physics of the 20th century. As we will show later on, Einstein's theory of special relativity follows from the invariance of the Maxwell equations in electromagnetism under Lorentz transformations. Moreover, the generalization of the Maxwell equations from the commutative gauge group $U(1)$ to the non-commutative gauge groups $SU(2)$ and $SU(3)$ leads to the Standard Model in particle physics. Finally, statistical physics can be traced back to Maxwell's statistical velocity distribution of molecules.

From the physical point of view, quantum mechanics and quantum field theory are described best by the Feynman approach via Feynman diagrams, transition amplitudes, Feynman propagators (Green's functions), and functional integrals. In order to make the reader familiar with the fascinating story of this approach, let us start with a quotation taken from Freeman Dyson's book *Disturbing the Universe*, Harper & Row, New York, 1979:⁴

Dick Feynman (1918–1988) was a profoundly original scientist. He refused to take anybody's word for anything. This meant that he was forced to rediscover or reinvent for himself almost the whole physics. It took him five years of concentrated work to reinvent quantum mechanics. He said that he couldn't understand the official version of quantum mechanics that was taught in the textbooks and so he had to begin afresh from the beginning. This was a heroic enterprise. He worked harder during those years than anybody else I ever knew. At the end he had his version of quantum mechanics that he could understand. . .

The calculations that I did for Hans Bethe,⁵ using the orthodox method, took me several months of work and several hundred sheets of paper.

Dick Feynman could get the same answer, calculating on a black-board, in half an hour. . .

In orthodox physics, it can be said: Suppose an electron is in this state at a certain time, then you calculate what it will do next by solving the Schrödinger equation introduced by Schrödinger in 1926. Instead of this, Dick simply said:

⁴ Reprinted by permission of Basic Books, a member of Perseus Books, L.L.C.

⁵ Hans Bethe (1906–2005) was awarded the 1967 Nobel prize in physics for his contributions to nuclear reactions, especially his discoveries concerning the energy production in stars. See H. Bethe, R. Bacher, and M. Livingstone, *Basic Bethe: Seminal Articles on Nuclear Physics 1936–37*, American Institute of Physics, 1986.

The electron does whatever it likes.

A history of the electron is any possible path in space and time. The behavior of the electron is just the result of adding together all the histories according to some simple rules that Dick worked out. I had the enormous luck to be at Cornell in 1948 when the idea was newborn, and to be for a short time Dick's sounding board. . .

Dick distrusted my mathematics and I distrusted his intuition.

Dick fought against my scepticism, arguing that Einstein had failed because he stopped thinking in concrete physical images and became a manipulator of equations. I had to admit that was true. The discoveries of Einstein's earlier years were all based on direct physical intuition. Einstein's later unified theories failed because they were only sets of equations without physical meaning. . .

Nobody but Dick could use his theory. Without success I tried to understand him. . . At the beginning of September after vacations it was time to go back East. I got onto a Greyhound bus and travelled nonstop for three days and nights as far as Chicago. This time I had nobody to talk to. The roads were too bumpy for me to read, and so I sat and looked out of the window and gradually fell into a comfortable stupor. As we were droning across Nebraska on the third day, something suddenly happened. For two weeks I had not thought about physics, and now it came bursting into my consciousness like an explosion. Feynman's pictures and Schwinger's equations began sorting themselves out in my head with a clarity they had never had before. I had no pencil or paper, but everything was so clear I did not need to write it down.

Feynman and Schwinger were just looking at the same set of ideas from two different sides.

Putting their methods together, you would have a theory of quantum electrodynamics that combined the mathematical precision of Schwinger with the practical flexibility of Feynman. . .

During the rest of the day as we watched the sun go down over the prairie, I was mapping out in my head the shape of the paper I would write when I got to Princeton. The title of the paper would be *The radiation theories of Tomonaga, Schwinger, and Feynman*.⁶

For the convenience of the reader, in what follows let us summarize the prototypes of basic formulas for the passage from classical physics to quantum physics. These formulas are special cases of more general approaches due to

- Newton in 1666 (equation of motion),
- Euler in 1744 and Lagrange in 1762 (calculus of variations),
- Fourier in 1807 (Fourier method in the theory of partial differential equations, Fourier series, and Fourier integral),
- Poisson in 1811 (Poisson brackets and conservation laws),

⁶ F. Dyson, Phys. Rev. **75** (1949), 486–502. Freeman Dyson (born 1923) is a member of the Institute for Advanced Study in Princeton (New Jersey, U.S.A.). He made fundamental contributions to quantum field theory, statistical physics, stability of matter, and number theory. This can be found in F. Dyson, *Selected Papers*, Amer. Math. Soc., Providence, Rhode Island, 1996.

- Hamilton in 1827 (Hamiltonian and canonical equations),
- Green in 1828 (the method of Green's function in electromagnetism),
- Lie in 1870 (continuous transformation groups (Lie groups) and infinitesimal transformation groups (Lie algebras)),
- Poincaré in 1892 (small divisors in celestial mechanics, and the renormalization of the quasi-periodic motion of planets by adding regularizing terms (also called counterterms) to the Poincaré–Lindsted series),
- Fredholm in 1900 (integral equations),
- Hilbert in 1904 (integral equations, and spectral theory for infinite-dimensional symmetric matrices),
- Emmy Noether in 1918 (symmetry, Lie groups, and conservation laws),
- Wiener in 1923 (Wiener integral for stochastic processes including Brownian motion for diffusion processes),
- von Neumann in 1928 (spectral theory for unbounded self-adjoint operators in Hilbert spaces, and calculus for operators),
- Stone in 1930 (unitary one-parameter groups and the general dynamics of quantum systems).

From the physical point of view, the following formulas are special cases of more general formulas due to Heisenberg in 1925, Born and Jordan in 1926, Schrödinger in 1926, Dirac in 1927, Feynman in 1942, Heisenberg in 1943, Dyson in 1949, Lippmann and Schwinger in 1950. In fact, we will study the following approaches to quantum mechanics:

- the 1925 Heisenberg *particle picture* via time-dependent operators as observables, and the Poisson–Lie operator equation of motion,
- the 1926 Schrödinger *wave picture* via time-dependent quantum states, and the Schrödinger partial differential equation of motion,
- the 1927 Dirac *interaction picture* which describes the motion under an interacting force as a perturbation of the interaction-free dynamics, and
- the 1942 Feynman picture based on a *statistics for possible classical motions* via the *Feynman path integral*, which generalizes the 1923 Wiener integral for the mathematical description of Einstein's Brownian motion in diffusion processes from the year 1905.

The fact that it is possible to describe quantum particles in an equivalent way by either Heisenberg's particle picture or Schrödinger's wave picture reflects a general duality principle in quantum physics:

Quantum particles are more general objects than classical particles and classical waves.

This has been discovered in the history of physics step by step. Note that, for didactic reasons, in this section we will not follow the historical route, but we will present the material in a manner which is most convenient from the modern point of view.⁷ Nowadays most physicists prefer the Feynman

⁷ Remarks on the historical route of quantum mechanics can be found in Sect. 1.3 on page 60.

approach to quantum physics. In what follows we restrict ourselves to formal considerations.

Hints for quick reading. After reading Sects. 1.2.1 through 1.2.4, the reader may pass to Sects. 15.1 through 15.5 on the operator approach to quantum field theory. A rigorous approach to the basic ideas in quantum field theory in terms of a finite-dimensional Hilbert space setting can be found in Chap. 7. The true mathematical difficulties in quantum field theory are related to the infinite-dimensional setting. However, rigorous finite-dimensional models help to understand the mathematical substance of the magic formulas used by physicists in quantum field theory in a formal way. These magic formulas are due to Dyson, Feynman, Schwinger, Gell-Mann and Low, Faddeev and Popov. They can be found in Chaps. 14 through 16. The reader who wants to become familiar, as quickly as possible, with applications of quantum field theory to concrete physical processes in quantum electrodynamics should pass to Volume II.

1.2.1 Basic Formulas

The classical principle of critical action. For the mathematical description of physics, it is crucial that the fundamental processes in nature are governed by an optimality principle called the principle of least action. In fact, the action is not always minimal in nature, but sometimes the action is only critical (also called stationary). Therefore, we have to speak about the principle of critical action. In the history of physics, the role of variational principles was underlined by Fermat (1601–1665), Maupertius (1698–1759), Euler (1707–1783), Lagrange (1736–1813), Gauss (1777–1855), Hamilton (1788–1856), and Jacobi (1804–1851). As the simplest example for the principle of critical action, let us start with the following variational problem

$$\boxed{\int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt = \text{critical!}} \quad (1.10)$$

for the motion, $q = q(t)$, of a classical particle with mass m on the real line (Fig. 1.1). This is called the principle of critical action. Here, $q(t)$ denotes the position of the classical particle at time t . Following Newton, the dot, \dot{q} , denotes the derivative with respect to time. We have to add the boundary condition

$$q(t_0) = x_0, \quad q(t_1) = x_1 \quad (1.11)$$

for given initial point x_0 at the initial time t_0 , and given final point x_1 at the final time t_1 . The function $L = L(q, \dot{q})$ is called Lagrangian; it has the physical dimension of energy. More important than energy is the action S of the classical motion, $q = q(t)$, during the time interval $[t_0, t_1]$. Explicitly,

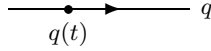


Fig. 1.1. Motion on the real line

$$S[q] := \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt.$$

Here, the action S has the physical dimension of *energy* \times *time*. Each smooth solution of the variational problem (1.10) satisfies the following Euler–Lagrange equation⁸

$$\boxed{\frac{d}{dt} L_{\dot{q}}(q(t), \dot{q}(t)) = L_q(q(t), \dot{q}(t))} \quad (1.12)$$

for all times $t \in [t_0, t_1]$. Since the Lagrangian $L = L(q, \dot{q})$ does not depend explicitly on time t , it is invariant under time translations $t \mapsto t + \text{const}$. By the Noether theorem, each solution, $q = q(t)$, of the Euler–Lagrange equation satisfies the conservation law

$$\frac{dE(t)}{dt} = 0 \quad \text{for all } t \in [t_0, t_1] \quad (1.13)$$

with the momentum function $p(t) := L_{\dot{q}}(q(t), \dot{q}(t))$, and the energy function

$$E(t) := p(t)\dot{q}(t) - L(q(t), \dot{q}(t)).$$

The relation (1.13) tells us that the energy $E(t)$ does not depend on time along the classical trajectory. This is called conservation of energy.⁹ Generally, from the mathematical point of view, the fundamental notion of energy is intimately related to symmetry properties of physical systems, namely, the invariance under time translations.

The Feynman picture of quantum mechanics and the Feynman path integral. We want to consider the motion of a quantum particle on the real line. This motion is described by a complex-valued function $\psi = \psi(x, t)$ whose physical meaning will be explained below. From the physical point of view, the best interpretation of the passage from the classical motion of the particle to the corresponding quantum motion can be obtained by the famous Feynman formula

⁸ The symbol $L_{\dot{q}}$ denotes the partial derivative of the function $L = L(q, \dot{q})$ with respect to the variable \dot{q} .

⁹ Without referring to the general Noether theorem, energy conservation follows directly from the Euler–Lagrange equation, $\dot{p} = L_q$. In fact, differentiation with respect to time t yields

$$\dot{E} = \dot{p}\dot{q} + p\ddot{q} - L_{\dot{q}}\ddot{q} - L_q\dot{q} = (\dot{p} - L_q)\dot{q} = 0.$$

$$\boxed{\mathcal{P}(x_1, t_1; x_0, t_0) = \int e^{iS[q]/\hbar} \mathcal{D}q.} \quad (1.14)$$

Formally, we sum over all possible classical trajectories $q = q(t)$ which satisfy the boundary condition (1.11). The integral from (1.14) is called a Feynman path integral (or functional integral). It represents a statistic over the possible trajectories of the classical particle. The statistical weight $e^{iS[q]/\hbar}$ for each trajectory depends on the classical action $S[q]$ of the trajectory. Since the quantity $S[q]$ has the physical dimension of *energy* \times *time*, we have to divide the action $S[q]$ by a constant \hbar of the same dimension in order to get a dimensionless argument of the exponential function.

This way, Planck's constant of action, \hbar , appears in a natural way.

It was noticed by Feynman that, because of the smallness of the Planck constant \hbar , a formal application of Kelvin's method of stationary phase in optics tells us that, naturally enough, the main contribution to the integral (1.14) comes from that classical trajectory which corresponds to the solution of the principle of critical action (1.10).

Quantum motion is obtained from classical motion by adding random quantum fluctuations.

In particular, quantum mechanics passes over to classical mechanics if

$$\hbar \rightarrow 0.$$

This limit corresponds to the passage from wave optics to geometric optics if the wave length of light goes to zero, $\lambda \rightarrow 0$.

In terms of mathematics, quantization of a classical theory corresponds to a deformation of the classical theory which depends on the Planck parameter \hbar .

It remains to discuss the physical meaning of the Feynman propagator kernel \mathcal{P} and of the function ψ . According to Feynman's 1942 dissertation at Princeton University, the dynamics of the function ψ is governed by the crucial formula

$$\boxed{\psi(x_1, t_1) = \int_{-\infty}^{\infty} \mathcal{P}(x_1, t_1; x_0, t_0) \psi(x_0, t_0) dx_0} \quad (1.15)$$

for all positions x_1 and all times $t_1 \geq t_0$. We write this briefly as

$$\psi(t_1) = P(t_1, t_0) \psi(t_0). \quad (1.16)$$

The operator $P(t_1, t_0)$ is called the Feynman propagator.

The simplest example corresponds to the motion of a free particle on the real line. In this case, the Lagrangian reads as

$$L(q, \dot{q}) := \frac{m\dot{q}^2}{2}.$$

The corresponding Euler–Lagrange equation, $m\ddot{q} = 0$, has the general solution

$$q(t) = x_0 + v(t - t_0), \quad t \in \mathbb{R}.$$

This describes the motion of a point particle on the real line with the constant velocity v . The corresponding Feynman propagator kernel is given by

$$\mathcal{P}(x_1, t_1; x_0, t_0) = \sqrt{\frac{m}{2\pi i \hbar (t_1 - t_0)}} \cdot e^{-\frac{m(x_1 - x_0)^2}{2i\hbar(t_1 - t_0)}}.$$

Here, we choose the value $\sqrt{i} := e^{i\pi/4}$ for the square root of the imaginary unit. Using the replacement $it \mapsto t$, the Feynman kernel, \mathcal{P} , passes over to Fourier’s heat kernel for both the propagation of heat and the diffusion of particles (Brownian motion) on the real line. This will be studied in Sect. 11.1.3 on page 589.

Born’s interpretation of the Schrödinger ψ -function. In 1926 Schrödinger formulated his famous partial differential equation for some wave function ψ which will be considered below. Surprisingly enough, Schrödinger was very successful in computing the quantized energy levels of the hydrogen atom, but he did not know the physical meaning of the function ψ . This was discovered by Born a few months later by studying scattering processes for electrons.

According to Born, the value $|\psi(x, t)|^2$ plays the role of a probability density.

This changes physics dramatically. In contrast to classical physics, quantum processes are random processes. More precisely, we have to distinguish between the following two cases.

- (i) Case 1: One single particle. Suppose that $\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx < \infty$. Then the function ψ describes a single particle on the real line. The quotient

$$\frac{\int_a^b |\psi(x, t)|^2 dx}{\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx}$$

is equal to the probability of finding the particle in the interval $[a, b]$ of the real line at time t . In addition, let us introduce the momentum operator P and the position operator Q ,

$$(P\psi)(x) := -i\hbar \frac{d}{dx} \psi(x), \quad (Q\psi)(x) := x\psi(x), \quad x \in \mathbb{R},$$

along with the inner product

$$\langle \chi | \psi \rangle := \int_{-\infty}^{\infty} \chi(x)^\dagger \psi(x) dx.$$

For a classical particle on the real line, position q and momentum $p = mv$ (mass \times velocity) can be measured with arbitrarily high precision at the same time. This is not true anymore for a quantum particle on the real line. In fact, in the quantum state $\psi = \psi(x)$ it is only possible to measure the mean position \bar{q} and the fluctuation Δq of the position. Explicitly,

$$\bar{q} = \frac{\langle \psi | Q \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int_{-\infty}^{\infty} \psi(x)^{\dagger} x \psi(x) dx}{\int_{-\infty}^{\infty} \psi(x)^{\dagger} \psi(x) dx}$$

and

$$(\Delta q)^2 = \frac{\langle \psi | (Q - \bar{q})^2 \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int_{-\infty}^{\infty} \psi(x)^{\dagger} (x - \bar{q})^2 \psi(x) dx}{\int_{-\infty}^{\infty} \psi(x)^{\dagger} \psi(x) dx}.$$

Similarly, for the measured mean momentum \bar{p} and its fluctuation Δp , we get

$$\bar{p} = \frac{\langle \psi | P \psi \rangle}{\langle \psi | \psi \rangle}, \quad (\Delta p)^2 = \frac{\langle \psi | (P - \bar{p})^2 \psi \rangle}{\langle \psi | \psi \rangle}.$$

If the function $\psi = \psi(x, t)$ also depends on time t , then so do the measured values. In 1927 Heisenberg showed that there holds the following fundamental inequality for the fluctuations:

$$\boxed{\Delta p \Delta q \geq \frac{\hbar}{2}} \quad (1.17)$$

This famous uncertainty inequality tells us that, in contrast to classical particles, it is not possible to measure precisely position and momentum of a quantum particle at the same time. If the measurement of the position of a quantum particle is fairly precise (i.e., Δq is small), then the velocity of the particle is quite uncertain (i.e., Δp is large). Conversely, if the velocity is known fairly precisely (i.e., Δp is small), then the position of the quantum particle is quite uncertain (i.e., Δq is large).

(ii) Case 2: Homogeneous stream of particles. Suppose that

$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = \infty.$$

Then the function ψ describes a flow of identical particles on the real line which has the current density vector

$$\mathbf{J}(x, t) := \varrho(x, t) \mathbf{v}(x, t)$$

with the particle density $\varrho(x, t) := |\psi(x, t)|^2$ and the velocity vector $\mathbf{v}(x, t)$ at the point x at time t . Explicitly,

$$\mathbf{J}(x, t) = \Re \left(\psi(x, t)^{\dagger} \cdot \frac{P}{m} \psi(x, t) \right) \mathbf{e}.$$

The unit vector \mathbf{e} points from left to right on the real line. For example, choose the function

$$\psi(x, t) := \sqrt{\varrho_0} \cdot e^{i(px - E_p t)/\hbar}$$

with fixed real number p and $E_p := p^2/2m$. This function describes a homogeneous stream of particles with momentum p , energy E_p , and particle density $\varrho(x, t) = |\psi(x, t)|^2 = \varrho_0$. The particle momentum p is an eigenvalue of the momentum operator P ,

$$P\psi = p\psi,$$

and the particle energy E_p is an eigenvalue of the energy operator,

$$i\hbar \frac{\partial}{\partial t} \psi = E_p \psi.$$

Moreover, we have $\mathbf{J}(x, t) = \varrho_0 \mathbf{v}$ along with the velocity vector $\mathbf{v} = p\mathbf{e}/m$.

Observe that the measured values of a single particle are based on the inner product $\langle \psi | \varphi \rangle$. This is the key to John von Neumann's Hilbert space approach to quantum mechanics from the late 1920s. However, the Hilbert space setting is not sufficient, since states ψ with $\langle \psi | \psi \rangle = \infty$ appear which do not lie in the Hilbert space under consideration. To include such states, one has to use Gelfand's theory of rigged Hilbert spaces from the 1950s which is based on the notion of generalized functions (distributions). This will be studied in Sect. 12.2 on page 677.

The Schrödinger wave picture for quantum mechanics on the real line. Consider the special case where the Lagrangian is given by

$$L(q, \dot{q}) := \frac{m\dot{q}^2}{2} - \kappa U(q).$$

Here, the Euler–Lagrange equation of motion reads as

$$m\ddot{q} = -\kappa U'(q).$$

This is the Newtonian equation of motion with the force $F(q) := -\kappa U'(q)$. For the momentum, $p(t) = m\dot{q}(t)$. The function U is called the potential, and the real number κ is called coupling constant.¹⁰ For the motion $q = q(t)$ of the classical particle on the real line, we have conservation of energy, i.e., there exists a constant E such that

$$\boxed{E = \frac{p(t)^2}{2m} + \kappa U(q(t))} \quad (1.18)$$

¹⁰ In many cases, the coupling constant κ is small. Then, it is possible to apply the methods of perturbation theory. This is of fundamental importance for quantum field theory.

for all times t . Here, E is called the energy of the motion. Using now the elegant Schrödinger quantization rule

$$E \Rightarrow i\hbar \frac{\partial}{\partial t}, \quad p \Rightarrow P,$$

the classical energy equation (1.18) passes over to the following famous Schrödinger equation

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi = \left(\frac{P^2}{2m} + \kappa U \right) \psi.} \quad (1.19)$$

Explicitly,

$$i\hbar \dot{\psi}(x, t) = -\frac{\hbar^2}{2m} \psi_{xx}(x, t) + \kappa U(x) \psi(x, t). \quad (1.20)$$

Recall that the dot, $\dot{\psi}$, denotes the partial derivative with respect to time. In the history of mathematics and physics, it was gradually discovered that the solutions of partial differential equations can be represented by integral formulas with appropriate kernels which are called Green's functions. In particular, the corresponding solution formula for the initial-value problem to the Schrödinger equation (1.20) reads as

$$\boxed{\psi(x, t) = \int_{-\infty}^{\infty} \mathcal{G}(x, t; y, t_0) \psi(y, t_0) dy} \quad (1.21)$$

for all positions $x \in \mathbb{R}$ and all times $t \geq t_0$. Comparing this with Feynman's formula (1.15), we see that

$$\mathcal{G}(x, t; y, t_0) \equiv \mathcal{P}(x, t; y, t_0).$$

Consequently, the Feynman propagator kernel is nothing than the Green's function to the initial-value problem for the Schrödinger equation (1.20). In fact, if we know the initial state $\psi(x, t_0)$ of the quantum particle at the initial time t_0 , then formula (1.21) determines the state $\psi = \psi(x, t)$ for all times $t \geq t_0$ in the future. This tells us that

The Feynman propagator kernel knows all about the motion of the quantum particle on the real line.

By physical considerations, Feynman discovered that the Feynman kernel can be represented by a path integral of the form (1.14). This discovery was crucial for the further development of quantum field theory.

Schrödinger's method for computing quantized energies by solving eigenvalue problems. Motivated by the classical Fourier method, Schrödinger made the separation ansatz

$$\psi(x, t) = \varphi(x) e^{-iEt/\hbar}.$$

Then, the time-dependent Schrödinger equation (1.19) passes over to the stationary Schrödinger equation

$$E\varphi(x) = -\frac{\hbar^2}{2m} \varphi''(x) + \kappa U(x)\varphi(x), \quad x \in \mathbb{R}.$$

This is an eigenvalue problem for the unknown energy E . Using this method, Schrödinger computed the quantized energies of quantum particles. Interestingly enough, Schrödinger did not know the precise physical meaning of the complex wave function ψ . This problem was solved by Born in 1926; Born discovered the statistical interpretation of $|\psi(x, t)|^2$ discussed above. Schrödinger and Born were awarded the Nobel prize in physics in 1933 and 1954, respectively.

Von Neumann's solution of the Schrödinger equation. Let us introduce the free Hamiltonian

$$H_0 := \frac{P^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

along with the Hamiltonian

$$H := H_0 + \kappa U.$$

Then, the Schrödinger equation (1.19) can be written as an operator equation of the form

$$\boxed{i\hbar \dot{\psi}(t) = H\psi(t)} \tag{1.22}$$

for all times $t \in \mathbb{R}$. If we consider equation (1.22) as a classical ordinary differential equation, then the solution reads as

$$\boxed{\psi(t) = e^{-i(t-t_0)H/\hbar} \psi(t_0) \quad \text{for all } t \in \mathbb{R}.} \tag{1.23}$$

Let us now discuss the mathematical meaning of the operator $e^{-itH/\hbar}$.

Von Neumann's operator calculus. In the late 1920s, von Neumann developed a calculus for unbounded self-adjoint operators on Hilbert spaces which gives the exponential function for operators a precise meaning under appropriate assumptions on the Hamiltonian H .¹¹ In the sense of von Neumann's operator calculus, formula (1.23) solves the Schrödinger equation (1.19), and it describes completely the dynamics of the quantum particle on the real line. Comparing this with equation (1.16), $\psi(t) = P(t, t_0)\psi(t_0)$, we see that the Feynman propagator is given by

¹¹ The operator H has to be a self-adjoint operator on the Hilbert space $L_2(\mathbb{R})$. See Zeidler (1995), Vol. 1.

$$P(t, t_0) = e^{-i(t-t_0)H/\hbar} \quad \text{for all } t, t_0 \in \mathbb{R}.$$

Generalizing Hilbert's spectral theory from 1904 for bounded symmetric operators to unbounded self-adjoint operators in 1928, von Neumann justified the formula

$$H = \int_{\lambda=-\infty}^{\infty} \lambda dE_{\lambda}$$

where the operators $E_{\lambda} : X \rightarrow X$ are orthogonal projection operators on the Hilbert space $X = L_2(\mathbb{R})$. Moreover, $f(H) = \int_{\lambda=-\infty}^{\infty} f(\lambda) dE_{\lambda}$. In particular,

$$e^{-itH/\hbar} = \int_{\lambda=-\infty}^{\infty} e^{-it\lambda/\hbar} dE_{\lambda}.$$

The family $\{E_{\lambda}\}$ of operators E_{λ} with $\lambda \in \mathbb{R}$ is called the spectral family of the Hamiltonian H . In order to discuss the physical meaning of the spectral family, choose a fixed particle state on the real line, $\psi \in X$, normalized by

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1.$$

Suppose we are measuring the energy E of the particle in the state ψ . Then, the probability P of finding the measured value E in the interval J is given by the integral

$$P := \int_J dF(\lambda).$$

Here, the function $F(\lambda_0) := \int_{]-\infty, \lambda_0[} d\sigma(\lambda)$ with $\sigma(\lambda) := \langle \psi | E_{\lambda} \psi \rangle$ represents the probability distribution of the energy of the quantum particle on the real line in the given state ψ . By the theory of probability, for the mean value \overline{E} and the fluctuation ΔE of the measured energy, we get

$$\overline{E} = \int_{-\infty}^{\infty} \lambda dF(\lambda), \quad (\Delta E)^2 = \int_{-\infty}^{\infty} (\lambda - \overline{E})^2 dF(\lambda).$$

In particular, if the function F is smooth, then $dF(\lambda)$ can be replaced by $F'(\lambda)d\lambda$ where $F'(\lambda) = \sigma(\lambda)$.

According to von Neumann, the spectral family can be constructed for all self-adjoint operators on Hilbert spaces. Such operators represent observables in quantum mechanics. Therefore, the theory of spectral families allows us to describe the random measurements of observables in quantum mechanics.¹²

Heisenberg's S -matrix for scattering processes. The most important processes for elementary particles are scattering processes in particle

¹² As an introduction to functional analysis and its applications to mathematical physics, we recommend Zeidler (1995), Vols. 1, 2.

accelerators. Therefore, physicists are mainly interested in computing scattering processes. For this, the main tool is the S -matrix which was introduced by Wheeler in 1937 and Heisenberg in 1943. The further development was strongly influenced by Heisenberg's paper from 1943.

The S -matrix is closely related to physical quantities which can be measured in experiments.

This underlines the importance of the S -matrix. Let us discuss this in terms of scattering processes on the real line. To this end, set

$$\psi_{\text{in}}(x, t) := \sqrt{\varrho_{\text{in}}} \cdot e^{i(p_{\text{in}}x - E_{\text{in}}t)/\hbar}$$

and

$$\psi_{\text{out}}(x, t) := \sqrt{\varrho_{\text{out}}} \cdot e^{i(p_{\text{out}}x - E_{\text{out}}t)/\hbar}.$$

We regard ψ_{in} as a stream of incoming free particles with

- momentum vector $\mathbf{p}_{\text{in}} = p_{\text{in}}\mathbf{e}$ of the incoming particles,
- velocity vector $\mathbf{v}_{\text{in}} = \mathbf{p}_{\text{in}}/m$ of the incoming particles,
- energy $E_{\text{in}} = p_{\text{in}}^2/2m$ of the incoming particles, and
- particle density ϱ_{in} of the incoming particles.

The corresponding quantities of the outgoing particles are defined similarly. We assume that the potential U is concentrated in a neighborhood of the origin. The incoming free particles are then scattered at the potential κU . This means that some particles are reflected at the potential wall and some of them penetrate the potential wall. We are interested in those particles which penetrate the potential wall. This yields a stream of outgoing particles described by the function ψ_{out} . Let us introduce the Feynman transition amplitude

$$\boxed{S(q, t_1; p, t_0) := \langle \psi_{\text{out}} | P(t_1, t_0) \psi_{\text{in}} \rangle}.$$

In this connection, observe that the Feynman propagator $P(t_1, t_0)$ sends the incoming state ψ_{in} at time t_0 to the state

$$\psi(t_1) = P(t_1, t_0)\psi(t_0)$$

at time t_1 , and we compare the actual state $\psi(t_1)$ with the possible outgoing state $\psi_{\text{out}}(t_1)$ at time t_1 by computing the inner product $\langle \psi_{\text{out}}(t_1) | \psi(t_1) \rangle$. In terms of the Feynman propagator kernel \mathcal{P} , the transition amplitude $S(q, t_1; p, t_0)$ is equal to

$$\int_{-\infty}^{\infty} \psi_{\text{out}}(x_1, t_1)^\dagger \left(\int_{-\infty}^{\infty} \mathcal{P}(x_1, t_1; x_0, t_0) \psi_{\text{in}}(x_0, t_0) dx_0 \right) dx_1.$$

Carrying out a physical experiment, we are interested in the transition probability from the incoming particle stream at time t_0 to the outgoing stream

$\psi_{\text{out}}(t_1)$ at time t_1 . It turns out that this transition probability¹³ is given by the expression

$$|\mathbf{S}(q, t_1; p, t_0)|^2.$$

To free ourselves from the arbitrary choice of the initial time t_0 and the final time t_1 , we pass over to the formal limit $t_0 \rightarrow -\infty$ and $t_1 \rightarrow +\infty$. This way, we get

$$\mathbf{S}(q, +\infty; p, -\infty) := \lim_{t_1 \rightarrow +\infty} \lim_{t_0 \rightarrow -\infty} \mathbf{S}(q, t_1; p, t_0).$$

Physicists call this the S -matrix element for particles with incoming momentum p and outgoing momentum q . The set of all these S -matrix elements forms the S -matrix \mathbf{S} . The corresponding transition probability for the particle stream is then given by the key formula

$$|\mathbf{S}(q, +\infty; p, -\infty)|^2.$$

It turns out that these S -matrix elements vanish if $q \neq p$, by energy conservation.

The Lippmann–Schwinger integral equation. It is possible to compute the S -matrix elements by solving the following Lippmann–Schwinger integral equation

$$\varphi(x) = \varphi_0(x) - \kappa \int_{-\infty}^{\infty} \mathcal{G}(x, y) U(y) \varphi(y) dy \quad (1.24)$$

with $\varphi_0(x) := e^{ipx/\hbar}$ and the kernel

$$\mathcal{G}(x, y) := im \frac{e^{ip|x-y|/\hbar}}{\hbar p}.$$

The function

$$\psi_{\text{in}}(x, t) := e^{-iE_p(t-t_0)/\hbar} \varphi_0(x)$$

describes the incoming particle stream with momentum p . If the function $\varphi = \varphi(x)$ is a solution of (1.24), then

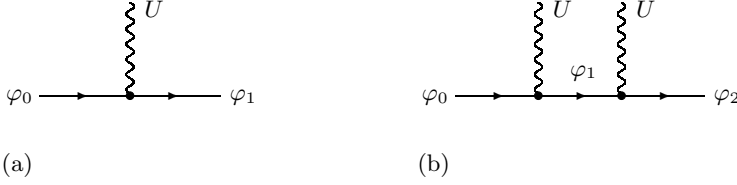
$$\psi(x, t) := e^{-iE_p(t-t_0)/\hbar} \varphi(x)$$

with $E_p := p^2/2m$ is a solution of the Schrödinger equation (1.19) which describes the scattering of the incoming particle stream ψ_{in} . Finally, we compare the scattered particle stream with the outgoing particle stream

$$\psi_{\text{out}}(x, t) := e^{-iE_p(t-t_1)/\hbar} \varphi_0(x)$$

of momentum p . This implies

¹³ In mathematics, transition probability is called conditional probability.

**Fig. 1.2.** Feynman diagrams

$$\begin{aligned}
 \langle \psi_{\text{out}} | \psi \rangle &= \int_{-\infty}^{\infty} \psi_{\text{out}}(x, t)^{\dagger} \psi(x, t) dx \\
 &= e^{-iE_p(t_1 - t_0)} \int_{-\infty}^{\infty} \varphi_0(x)^{\dagger} \varphi(x) dx = e^{-iE_p(t_1 - t_0)} \langle \varphi_0 | \varphi \rangle.
 \end{aligned}$$

Letting $t_0 \rightarrow -\infty$ and $t_1 \rightarrow +\infty$, we obtain the formal limit

$$|\mathcal{S}(p, +\infty; p, -\infty)|^2 = |\langle \varphi_0 | \varphi \rangle|^2$$

which depends on the solution φ of the time-independent Lippmann–Schwinger integral equation.¹⁴ The point is that the solution φ can be computed approximately by using the following iterative method

$$\varphi_{n+1}(x) = \varphi_0(x) - \kappa \int_{-\infty}^{\infty} \mathcal{G}(x, y) U(y) \varphi_n(y) dy, \quad n = 0, 1, 2, \dots$$

The method of Feynman diagrams. The basic idea of Feynman diagrams is to represent graphically the approximations $\varphi_1, \varphi_2, \dots$. This technique is widely used in elementary particle physics; it helps to simplify the computation of scattering processes, and gives physical insight. In the present case, for small coupling constant κ , the first approximation

$$\varphi_1(x) = \varphi_0(x) - \kappa \int_{-\infty}^{\infty} \mathcal{G}(x, y) U(y) \varphi_0(y) dy$$

is called the Born approximation; it was used by Born in 1926 in order to compute scattering processes for electrons. The Feynman diagram for φ_1 is pictured in Fig. 1.2(a). Intuitively, the interaction between the “particle” φ_0 and the potential U yields the “particle” φ_1 . The second approximation φ_2 is given by

$$\varphi_2(x) = \varphi_0(x) - \kappa \int_{-\infty}^{\infty} \mathcal{G}(x, y) U(y) \varphi_1(y) dy.$$

The corresponding Feynman diagram is pictured in Fig. 1.2(b). Intuitively, the interaction between the “particle” φ_1 and the potential U yields the “particle” φ_2 . Observe that

¹⁴ This approach is called stationary scattering theory. See Zeidler (1995), Vol. 1, Sect. 5.24.5.

As a rule, each iterative method in mathematics and physics can be represented graphically by Feynman diagrams.

The Heisenberg–Born–Jordan commutation relation. For the momentum operator P and the position operator Q ,

$$QP\psi - PQ\psi = -i\hbar(x\psi'(x) - (x\psi(x))') = i\hbar\psi(x).$$

Letting $[Q, P]_- := QP - PQ$, we obtain

$$[Q, P]_- = i\hbar I$$

where I denotes the identity operator. This is the commutation rule (1.2) which appeared at the birth of modern quantum mechanics in 1925. The interesting history of the commutation relation will be discussed in Sect. 1.3 on page 60.

The Heisenberg particle picture for quantum mechanics on the real line. It was discovered in the 1920s that completely different approaches to quantum mechanics are in fact equivalent. Let us discuss the equivalence between the Schrödinger picture and the Heisenberg picture which was invented before the Schrödinger equation. In the Schrödinger picture, the dynamics of the time-dependent wave function $\psi = \psi(t)$ is governed by the equation

$$\psi(t) = e^{-itH/\hbar}\psi(0).$$

In the Heisenberg picture, we introduce the time-dependent momentum operator

$$P(t) := e^{itH/\hbar}Pe^{-itH/\hbar},$$

the time-dependent position operator

$$Q(t) := e^{itH/\hbar}Qe^{-itH/\hbar},$$

and the time-independent state $\psi(0)$. For the measured mean-value of momentum in the Schrödinger picture, we get

$$\bar{p} = \frac{\langle \psi(t) | P\psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}.$$

Moreover, we get

$$\bar{p} = \frac{\langle \psi(0) | P(t)\psi(0) \rangle}{\langle \psi(0) | \psi(0) \rangle}.$$

in the Heisenberg picture.¹⁵ Differentiation with respect to time t yields

¹⁵ Note that $\langle A\chi | \psi \rangle = \langle \chi | A^\dagger \psi \rangle$. Since $H^\dagger = H$ and $(e^{-itH/\hbar})^\dagger = e^{itH/\hbar}$, we obtain that the inner product $\langle \psi(t) | P\psi(t) \rangle$ is equal to

$$\langle e^{-itH/\hbar}\psi(0) | Pe^{-itH/\hbar}\psi(0) \rangle = \langle \psi(0) | e^{itH/\hbar}Pe^{-itH/\hbar}\psi(0) \rangle = \langle \psi(0) | P(t)\psi(0) \rangle.$$

As an introduction to the theory of Hilbert spaces, we recommend Zeidler (1995), Vol. 1.

$$i\hbar\dot{P}(t) = P(t)H - HP(t).$$

This way, we obtain the equations of motion in the Heisenberg picture,

$$\boxed{i\hbar\dot{P}(t) = [P(t), H]_-, \quad i\hbar\dot{Q}(t) = [Q(t), H]_-,}$$

along with the commutation relations $[Q(t), P(t)]_- = i\hbar I$ for all times $t \in \mathbb{R}$.

The Dirac interaction picture for quantum mechanics on the real line. In order to study the dynamics of perturbed systems, Dirac introduced the so-called interaction picture which is crucial for quantum field theory. Let us discuss the basic idea of this approach. We start with the Schrödinger equation

$$\boxed{i\hbar\dot{\psi}(t) = (H_0 + \kappa U(t))\psi(t).} \quad (1.25)$$

Here, the potential $U = U(t)$ is allowed to depend on time t . The following arguments are well-known for classical ordinary differential equations. The point is that we will apply formally these arguments to operator differential equations, too. The first trick is to introduce the new function

$$\Psi(t) := e^{i(t-t_0)H_0/\hbar} \psi(t).$$

Then $\Psi(t_0) = \psi(t_0)$. From the Schrödinger equation (1.25), we get the new differential equation

$$\boxed{i\hbar\dot{\Psi}(t) = \kappa U(t)\Psi(t), \quad t \in \mathbb{R}} \quad (1.26)$$

with the transformed potential¹⁶

$$U(t) := e^{i(t-t_0)H_0/\hbar} U(t) e^{-i(t-t_0)H_0/\hbar}.$$

Differentiation of the function

$$F(t) = F(t_0) + \int_{t_0}^t f(\tau) d\tau$$

with respect to time t yields $\dot{F}(t) = f(t)$. Therefore, the differential equation (1.26) is equivalent to the Volterra integral equation

$$\Psi(t) = \Psi_0 + \frac{\kappa}{i\hbar} \int_{t_0}^t U(\tau)\Psi(\tau) d\tau \quad (1.27)$$

¹⁶ In fact, for simplifying notation, we set $\hbar := 1$ and $t_0 := 0$. Then

$$i\dot{\Psi}(t) = -H_0 e^{itH_0} \psi(t) + e^{itH_0} i\dot{\psi}(t).$$

Using $i\dot{\psi} = (H_0 + \kappa U)\psi$, we obtain the claim (1.26).

with $\Psi_0 := \Psi(t_0) = \psi(t_0)$. This integral equation can be solved by means of the following iterative method

$$\Psi_{n+1}(t) = \Psi_0 + \frac{\kappa}{i\hbar} \int_{t_0}^t U(\tau) \Psi_n(\tau) d\tau, \quad n = 0, 1, 2, \dots$$

This yields the solution

$$\Psi(t) = \Psi_0 + \sum_{n=1}^{\infty} \frac{\kappa^n}{(i\hbar)^n} \int U(t_1) \cdots U(t_n) \Psi_0 \quad (1.28)$$

where we use the convention $\int := \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n$.

The magic Dyson series. It is our goal to simplify the solution formula (1.28) by introducing the chronological operator \mathcal{T} . We will obtain the elegant formula

$$\boxed{\Psi(t) = \mathcal{T} \exp \left(\frac{\kappa}{i\hbar} \int_{t_0}^t U(\tau) d\tau \right) \Psi_0.} \quad (1.29)$$

Explicitly, we get the following Dyson series

$$\Psi(t) = \Psi_0 + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!(i\hbar)^n} \int_{t_0}^t \cdots \int_{t_0}^t \mathcal{T}(U(t_1) \cdots U(t_n)) \Psi_0 dt_1 \cdots dt_n.$$

Here, the chronological operator \mathcal{T} organizes the factors in such a way that time is increasing from right to left.¹⁷ For example,

$$\mathcal{T}(U(t_1)U(t_2)) := \begin{cases} U(t_1)U(t_2) & \text{if } t_1 \geq t_2, \\ U(t_2)U(t_1) & \text{if } t_2 > t_1. \end{cases}$$

More generally,

$$\mathcal{T}(U(t_1)U(t_2) \cdots U(t_n)) := U(t_{1'})U(t_{2'}) \cdots U(t_{n'})$$

where $t_{1'}, \dots, t_{n'}$ is a permutation of t_1, \dots, t_n such that $t_{1'} \geq t_{2'} \geq \dots \geq t_{n'}$. Consider now the integral

$$J := \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 U(t_1)U(t_2) = \int_{t_0}^t \int_{t_0}^t U(t_1)U(t_2) \theta(t_1 - t_2) dt_1 dt_2$$

where $\theta(t) := 1$ if $t \geq 0$, and $\theta(t) := 0$ if $t < 0$. Using a permutation of indices,

¹⁷ In the present case, the use of the chronological operator \mathcal{T} is trivial, since $U(t_1)$ commutes with $U(t_2)$. However, the chronological operator is crucial if one wants a straightforward generalization of the argument above to finite-dimensional systems of classical ordinary differential equations and infinite-dimensional operator equations appearing in quantum field theory.

$$J = \int_{t_0}^t \int_{t_0}^t \frac{1}{2} \left(\mathcal{U}(t_1) \mathcal{U}(t_2) \theta(t_1 - t_2) + \mathcal{U}(t_2) \mathcal{U}(t_1) \theta(t_2 - t_1) \right) dt_1 dt_2.$$

Hence $J = \int_{t_0}^t \int_{t_0}^t \frac{1}{2} \mathcal{T}(\mathcal{U}(t_1) \mathcal{U}(t_2)) dt_1 dt_2$. Similarly, we proceed for $n = 3, 4, \dots$ in order to get the desired relation (1.29). \square

Summarizing, in the Dirac interaction picture we pass from the Schrödinger state function $\psi = \psi(t)$ and the potential $U = U(t)$ to the Dirac state function Ψ and the Dirac potential \mathcal{U} , respectively. Explicitly,

$$\Psi(t) := e^{i(t-t_0)H_0/\hbar} \psi(t), \quad \mathcal{U}(t) := e^{i(t-t_0)H_0/\hbar} U(t) e^{-i(t-t_0)H_0/\hbar}$$

for all times $t \in \mathbb{R}$. Here, the Hamiltonian H_0 describes the free dynamics in the Schrödinger picture. The Dirac state function Ψ satisfies the modified Schrödinger equation

$$i\hbar \dot{\Psi}(t) = \kappa \mathcal{U}(t) \Psi(t), \quad t \in \mathbb{R}$$

and the Dirac potential \mathcal{U} satisfies the equation of motion

$$i\hbar \dot{\mathcal{U}}(t) = [\mathcal{U}(t), H_0]_- + V(t), \quad t \in \mathbb{R}$$

with $V(t) := e^{i(t-t_0)H_0/\hbar} \dot{U}(t) e^{-i(t-t_0)H_0/\hbar}$. If the potential $U(t)$ is time-independent, then

$$i\hbar \dot{\mathcal{U}}(t) = [\mathcal{U}(t), H_0]_-, \quad t \in \mathbb{R}.$$

This is the Heisenberg equation of motion with respect to the unperturbed Hamiltonian H_0 .

In Chap. 15 we will use this type of argument in order to reduce the investigation of interacting quantum fields to free quantum fields.

Perturbation theory. Suppose that the coupling constant κ is small. We then obtain the first-order (or Born) approximation

$$\Psi(t) = \Psi_0 + \frac{\kappa}{i\hbar} \int_{t_0}^t \mathcal{U}(\tau) \Psi_0 d\tau.$$

Thus, for the solution $\psi(t) = e^{-i(t-t_0)H_0/\hbar} \Psi(t)$ of the Schrödinger equation (1.25), we obtain the first-order approximation

$$\psi(t) = \psi_0(t) + \frac{\kappa}{i\hbar} \int_{t_0}^t e^{-i(t-\tau)H_0/\hbar} U(\tau) \psi_0(\tau) d\tau, \quad t \in \mathbb{R}$$

which represents the first-order perturbation of the free dynamics

$$\psi_0(t) := e^{-i(t-t_0)H_0/\hbar} \psi_0(t_0).$$

1.2.2 The Fundamental Role of the Harmonic Oscillator in Quantum Field Theory

The present paper seeks to establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which in principle are observable... We shall restrict ourselves to problems involving one degree of freedom.¹⁸

Werner Heisenberg, 1925

Since the 1920s, the experience of physicists has shown that

Quantum fields can be treated as nonlinear perturbations of an infinite number of uncoupled quantized harmonic oscillators.

All the computations of physical effects in quantum field theory done by physicists have been based on this general principle. It is the long-term desire of physicists to replace this local approach by a more powerful global approach. The harmonic oscillator and its relations to quantum field theory will be thoroughly studied in Volume II. At this point, let us only sketch the basic ideas.

The classical harmonic oscillator and Poisson brackets. The Newtonian equation of motion for a harmonic oscillator of mass m and coupling constant $\kappa > 0$ on the real line reads as

$$m\ddot{q}(t) = -\kappa q(t), \quad t \in \mathbb{R}. \quad (1.30)$$

This is the simplest oscillating system in physics. The equation of motion (1.30) possesses the following general solution

$$q(t) = \sqrt{\frac{\hbar}{2m\omega}} (ae^{-i\omega t} + a^\dagger e^{i\omega t}), \quad t \in \mathbb{R} \quad (1.31)$$

along with the angular frequency $\omega := \sqrt{\kappa/m}$. The Fourier coefficient a is a complex number. Introducing the momentum $p(t) := m\dot{q}(t)$ at time t , the relations between the Fourier coefficient a , the conjugate complex value a^\dagger , and the initial values of the harmonic oscillator are given by

$$q(0) = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \quad p(0) = i\sqrt{\frac{\hbar m\omega}{2}} (a^\dagger - a). \quad (1.32)$$

The following three equivalent formulations were studied in the history of classical mechanics.¹⁹

¹⁸ W. Heisenberg, Quantum-theoretical re-interpretation of kinematic and mechanical relations, *Zeitschrift für Physik* **33** (1925), 879–893. This paper founded quantum mechanics. Heisenberg was awarded the Nobel prize in physics in 1932.

¹⁹ A detailed investigation of the harmonic oscillator can be found in Volume II.

- (i) The Lagrangian approach: The function

$$L(q, \dot{q}) := \frac{1}{2}m\dot{q}^2 - \frac{1}{2}\kappa q^2$$

represents the Lagrangian of the harmonic oscillator. The Euler–Lagrange equation $\dot{p} = L_q$ is equivalent to the Newtonian equation (1.30). Along each trajectory $q = q(t)$ of the harmonic oscillator, we have energy conservation

$$E = \frac{p(t)^2}{2m} + \frac{\kappa q(t)^2}{2} \quad \text{for all } t \in \mathbb{R}.$$

- (ii) The Hamiltonian approach: Introducing the Hamiltonian

$$H(q, p) := \frac{p^2}{2m} + \frac{\kappa q^2}{2},$$

the Newtonian equation (1.30) is equivalent to the following Hamiltonian equations of motion

$$\dot{p} = -H_q, \quad \dot{q} = H_p. \quad (1.33)$$

This is also called canonical equation. Explicitly, $\dot{p}(t) = -\kappa q(t)$ and $\dot{q}(t) = p(t)/m$. We will show in Volume II that there is a symplectic structure behind the Hamiltonian approach.

- (iii) The Poissonian approach: Let us introduce the Poisson bracket

$$\{A(q, p), B(q, p)\} := A_q B_p - B_q A_p$$

where A_q denotes the partial derivative with respect to the variable q . The Hamiltonian equations (1.33) of motion can then be written as

$$\boxed{\dot{p} = \{p, H\}, \quad \dot{q} = \{q, H\}.} \quad (1.34)$$

This reveals the Poissonian structure behind classical mechanics. Moreover, we have

$$\boxed{\{q, p\} = 1.} \quad (1.35)$$

We will show below that the equations (1.34) and (1.35) are the key to the quantization of the harmonic oscillator. This was discovered gradually by Heisenberg, Born, Jordan, and Dirac in 1925/1926. Let us discuss this.

Heisenberg’s philosophical principle. In 1925 Heisenberg wanted to understand atomic spectra. As a mathematical model, he considered the infinite scheme

$$q(t) = (q_{nm} e^{i\omega_{nm}t}), \quad n, m = 1, 2, \dots$$

of angular frequencies ω_{nm} and complex-valued amplitudes q_{nm} . Following Einstein and Bohr, Heisenberg postulated that the angular frequencies are related to the possible energies of the system by the equation

$$\omega_{nm} = \frac{E_n - E_m}{\hbar}.$$

It was his goal to compute the energy levels E_1, E_2, \dots and the intensities of the spectral lines which are proportional to the squares $|q_{nm}|^2$. To this end, Heisenberg developed some simple rules for the scheme. Finally, he got the crucial energy relation

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega, \quad n = 0, 1, 2, \dots$$

This was the birth of modern quantum mechanics. From the philosophical point of view, Heisenberg did only use quantities which are directly related to physical experiments in the spectroscopy of atoms and molecules. In particular, he did not use the notion of trajectory or velocity of a quantum particle. In the same philosophical spirit, Heisenberg introduced the S -matrix in 1943; this approach has been very successful in elementary particle physics.

Heisenberg did not know the mathematical notion of matrix. In fact, in his 1925 paper he invented matrix multiplication by using physical arguments. When reading Heisenberg's manuscript, Born remembered some course in matrix calculus from the time of his studies; he conjectured that there should hold the following commutation relation

$$q(t)p(t) - p(t)q(t) = i\hbar I \quad \text{for all } t \in \mathbb{R} \quad (1.36)$$

with $p(t) := m\dot{q}(t) = (im\omega_{nm}q_{nm}e^{i\omega_{nm}t})$. Recall that the symbol I denotes the identity operator. Born, himself, could prove (1.36) only for the diagonal elements. The general proof was then obtained with the help of his young assistant Pascual Jordan in Göttingen. For this historical reason, the commutation relation (1.36) will be called the Heisenberg–Born–Jordan commutation relation in what follows.

The Heisenberg picture of the quantum harmonic oscillator and Lie brackets. Let us now formulate Heisenberg's approach to quantum mechanics in the manner polished by Born, Jordan, and Dirac. For the quantum harmonic oscillator, the classical motion

$$q = q(t), \quad t \in \mathbb{R}$$

is replaced by the operator-valued function $q = q(t)$. Moreover, in order to obtain the equation of motion, we use the Poissonian approach, and we replace the classical Poisson bracket by the Lie bracket. Explicitly,

$$\{A, B\} \Rightarrow \frac{1}{i\hbar} [A, B]_-$$

where $[A, B]_- := AB - BA$.²⁰ From (1.34) and (1.35), we get the equations of motion

$$i\hbar\dot{p}(t) = [p(t), H(t)]_-, \quad i\hbar\dot{q}(t) = [q(t), H(t)]_-, \quad (1.37)$$

and the commutation relation (1.36) along with the Hamiltonian

$$H(t) = \frac{p(t)^2}{2m} + \frac{\kappa q(t)^2}{2}.$$

It turns out that

This problem can be solved easily by using the classical solution (1.31) and by replacing the Fourier coefficient a by an operator. Here, we have to assume that the operator a and its adjoint a^\dagger satisfy the following commutation relation

$$[a, a^\dagger]_- = I. \quad (1.38)$$

This method is called Fourier quantization. In Volume II, we will use this method in order to obtain quantum electrodynamics as a quantum field theory which generalizes classical Maxwell's theory of electromagnetism.

The Schrödinger picture. The Schrödinger equation for the harmonic oscillator reads as

$$i\hbar\dot{\psi}(x, t) = H\psi(x, t), \quad x, t \in \mathbb{R}$$

with the momentum operator $(P\psi)(x, t) := -i\hbar\psi_x(x, t)$, the position operator $(Q\psi)(x, t) := x\psi(x, t)$, and the Hamiltonian

$$H := \frac{P^2}{2m} + \frac{\kappa Q^2}{2}.$$

Explicitly, the Schrödinger equation reads as

$$i\hbar\dot{\psi}(x, t) = -\frac{\hbar^2}{2m} \psi_{xx}(x, t) + \frac{\kappa}{2} x^2 \psi(x, t).$$

The ansatz $\psi(x, t) = \varphi(x)e^{-iEt/\hbar}$ yields the stationary Schrödinger equation

$$E\varphi = H\varphi. \quad (1.39)$$

²⁰ This general rule is due to Dirac. In 1928, Jordan and Wigner discovered that one has to replace the commutator $[A, B]_-$ for bosons (e.g., photons) by the anticommutator $[A, B]_+ := AB + BA$ in the case of fermions (e.g., electrons).

Explicitly, $E\varphi(x) = -\frac{\hbar^2}{2m}\varphi''(x) + \frac{\kappa}{2}x^2\varphi(x)$. This is an eigenvalue problem for computing the unknown energy E . From classical analysis it is known that the Hermite functions $\varphi_0, \varphi_1, \dots$ are eigenfunctions of (1.39). Let us use the language of physicists in order to obtain these eigenfunctions in a very elegant manner. Motivated by (1.32), we set $p(0) := P, q(0) := Q$, and hence

$$Q = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \quad P = i\sqrt{\frac{\hbar m\omega}{2}} (a^\dagger - a).$$

To simplify notation, let $m = \omega = \hbar = 1$. This implies $\kappa = 1$. Then,

$$a = \frac{1}{\sqrt{2}} (Q + iP), \quad a^\dagger = \frac{1}{\sqrt{2}} (Q - iP).$$

It follows from the commutation relation $[Q, P]_- = iI$ that this choice of the operator a satisfies the commutation relation (1.38). We will show in Volume II, using only the commutation relations, that the functions

$$\varphi_n := \frac{1}{\sqrt{n!}} (a^\dagger)^n \varphi_0, \quad n = 0, 1, 2, \dots$$

with $\varphi_0(x) := c_0 e^{-x^2/2}$ are eigensolutions of the equation

$$E_n \varphi_n = H \varphi_n, \quad n = 0, 1, 2, \dots$$

with the eigenvalues $E_n := (n + \frac{1}{2})$.²¹ If we choose the constant $c_0 := \pi^{-1/4}$, then

$$\langle \varphi_n | \varphi_m \rangle := \int_{-\infty}^{\infty} \varphi_n(x)^\dagger \varphi_m(x) dx = \delta_{nm}, \quad n, m = 0, 1, 2, \dots$$

In other words, the eigenfunctions $\varphi_0, \varphi_1, \dots$ form an orthonormal system in the Hilbert space $L_2(\mathbb{R})$.²² For the original Schrödinger equation, we get the solutions

$$\psi_n(x, t) = \varphi_n(x) e^{-iE_n t/\hbar}, \quad n = 0, 1, 2, \dots$$

which describe quantum oscillations of the quantum particle on the real line with energy $E_n = (n + \frac{1}{2})\hbar\omega$.

The Feynman picture. Using the eigenfunctions ψ_0, ψ_1, \dots , we can construct the Feynman propagator kernel

$$\mathcal{P}(x, t; y, t_0) = \sum_{n=0}^{\infty} \psi_n(x, t) \psi_n(y, t_0)^\dagger.$$

²¹ This corresponds to $E_n = (n + \frac{1}{2}) \hbar\omega$ when our simplification $\hbar = \omega = m = 1$ drops out.

²² In fact, it is shown in Zeidler (1995), Vol. 1, Sect. 3.4 that this orthonormal system is complete.

This kernel knows all about the dynamics of the quantum harmonic oscillator. In fact, suppose that we are given the wave function $\psi(x, t_0) := \varphi(x)$ at the initial time t_0 . For arbitrary points x on the real line and arbitrary real time t , the wave function is then given by the formula

$$\boxed{\psi(x, t) = \int_{-\infty}^{\infty} \mathcal{P}(x, t; y, t_0) \varphi(y) dy.} \quad (1.40)$$

The fundamental role of Green's functions in mathematics and physics. In terms of physics, the Feynman propagator kernel \mathcal{P} allows the following intuitive interpretation. Choose the initial state

$$\varphi(x) := \varphi_0 \delta(x - x_0)$$

where φ_0 is a fixed complex number.²³ Formally, this corresponds to an initial state which is sharply concentrated at the point x_0 at the initial time t_0 . By (1.40), we get the solution

$$\psi(x, t) = \varphi_0 \mathcal{P}(x, t; x_0, t_0)$$

for all positions $x \in \mathbb{R}$ and all times $t \geq t_0$. Thus, the Feynman propagator describes the propagation of a sharply concentrated initial state. Formula (1.40) tells us then that the general dynamics is the superposition of sharply concentrated initial states $\varphi(x_0) \delta(x - x_0)$. This is the special case of a general strategy in mathematics and physics called the strategy of the Green's function:

- Study first the response of a given physical system under the action of a sharply concentrated external force. This response corresponds to the Green's function of the system.
- The total response of the system under the action of a general external force can then be described by the superposition of sharply concentrated forces.

The response approach to quantum field theory will be studied in Chap. 14.

The importance of Fock states in quantum field theory. In the example above, the states

$$\varphi_n := \frac{1}{\sqrt{n!}} (a^\dagger)^n \varphi_0, \quad n = 0, 1, 2, \dots$$

span the Hilbert space $L_2(\mathbb{R})$. These states are called Fock states, and $L_2(\mathbb{R})$ is called the corresponding Fock space. The state φ_0 represents the ground state, and we have

$$a\varphi_0 = 0.$$

²³ The meaning of the Dirac delta function δ can be found on page 592.

Furthermore, for the operator $N := a^\dagger a$, we get

$$N\varphi_n = n\varphi_n, \quad n = 0, 1, 2, \dots$$

In Chap. 15 we will generalize this model to quantum field theory. Then, the following will happen:

- The state φ_0 passes over to the vacuum state $|0\rangle$ of a free quantum field.
- The operator a^\dagger is called creation operator.
- The Fock state φ_n corresponds to a state which consists of n particles.
- Because $a\varphi_0 = 0$, the operator a is called annihilation operator.
- The Fock state φ_n is a common eigenstate of the energy operator H and the particle number operator N with the eigenvalue n which counts the number of particles of φ_n .

1.2.3 Quantum Fields and Second Quantization

Quantum field theory was founded by Heisenberg and Pauli in 1929.²⁴ From the physical point of view the following is crucial:

A quantum field can be treated as a system of an infinite number of quantum particles where creation and annihilation of particles are possible.

In particular, for studying the radiation of atoms and molecules, one has to consider the quantum field of photons. In quantum electrodynamics, one investigates the quantum field of electrons, positrons, and photons.

The second quantization of the Schrödinger equation. As a prototype, let us consider the quantum field corresponding to the Schrödinger equation. We will proceed in several steps.

- Step 1: Classical mechanics. We start with a classical particle on the real line. The principle of critical action reads as

$$\int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt = \text{critical!}$$

along with the boundary condition “ $q(t) = \text{given}$ ” for $t = t_0, t_1$. This leads to the Euler–Lagrange equation

$$\frac{d}{dt} L_{\dot{q}}(q(t), \dot{q}(t)) = L_q(q(t), \dot{q}(t))$$

which describes the motion, $q = q(t)$, of the classical particle on the real line.²⁵ Let us consider the special case where

²⁴ W. Heisenberg and W. Pauli, On quantum field theory (in German), *Zeitschrift für Physik* **56** (1929), 1–61; **59** (1930), 108–190.

²⁵ The derivation of the Euler–Lagrange equation in classical mechanics along with symplectic and Poissonian geometry will be studied in Chaps. 4 and 5 of Vol. II.

$$L(q, \dot{q}) := \frac{m\dot{q}^2}{2} - \kappa U(q).$$

We define the momentum $p := L_{\dot{q}}$ and the Hamiltonian $H := p\dot{q} - L$. Hence $p = m\dot{q}$, and

$$H = \frac{p^2}{2m} + \kappa U(q).$$

Set $H(t) := \frac{p(t)^2}{2m} + \kappa U(q(t))$. By energy conservation, we have

$$H(t) = H(0) \quad \text{for all } t \in \mathbb{R},$$

for each smooth solution $q = q(t)$ of the Euler–Lagrange equation.

- Step 2: First quantization by using Heisenberg’s particle picture. We want to describe a quantum particle on the real line.

To this end, we replace the classical trajectory $q = q(t)$ by an operator-valued function.

This implies the operators $p(t) := m\dot{q}(t)$ and $H(t)$ as given above. More precisely, for each time t , we have the commutation relation

$$[q(t), p(t)]_- = i\hbar I$$

and the following equations of motion²⁶

$$\boxed{i\hbar\dot{q}(t) = [q(t), H(t)]_-, \quad i\hbar\dot{p}(t) = [p(t), H(t)]_-}.$$

We will show in Volume II that this implies the Newtonian equation of motion $m\ddot{q}(t) = -\kappa U'(q(t))$. Furthermore, the energy operator $H(t)$ does not depend on time t . To simplify notation, this operator is denoted by the symbol H .

- Step 3: First quantization by using Schrödinger’s wave picture. Here, the quantum particle on the real line is described by the complex-valued wave function $\psi = \psi(x, t)$ which satisfies the Schrödinger equation

$$i\hbar\dot{\psi}(x, t) = -\frac{\hbar^2}{2m} \psi_{xx}(x, t) + \kappa U(x)\psi(x, t). \quad (1.41)$$

First of all we want to derive the Schrödinger equation by the principle of critical action of the form

$$\int_{t_0}^{t_1} \left(\int_{x_0}^{x_1} \mathcal{L} \, dx \right) dt = \text{critical!} \quad (1.42)$$

along with the boundary condition “ $\psi(x, t) = \text{given}$ ” on the boundary $\partial\Omega$ of the rectangle $\Omega := \{(x, t) \in \mathbb{R}^2 : x_0 \leq x \leq x_1, t_0 \leq t \leq t_1\}$. Explicitly, for the Lagrangian density,

²⁶ Recall that $[A, B]_- := AB - BA$.

$$\mathcal{L}(\psi, \dot{\psi}, \psi_x; \psi^\dagger, \dot{\psi}^\dagger, \psi_x^\dagger) := i\hbar\psi^\dagger\dot{\psi} - \frac{\hbar^2}{2m}\psi_x^\dagger\psi_x - \kappa U\psi^\dagger\psi$$

with the real potential $U = U(x)$. Recall that $\dot{\psi}$ denotes the partial derivative with respect to time t . Each classical solution $\psi = \psi(x, t)$ of (1.42) satisfies the two Euler–Lagrange equations

$$\frac{\partial}{\partial x}\mathcal{L}_{\psi_x} + \frac{\partial}{\partial t}\mathcal{L}_{\dot{\psi}} = \mathcal{L}_\psi \quad (1.43)$$

and

$$\frac{\partial}{\partial x}\mathcal{L}_{\psi_x^\dagger} + \frac{\partial}{\partial t}\mathcal{L}_{\dot{\psi}^\dagger} = \mathcal{L}_{\psi^\dagger}. \quad (1.44)$$

Equation (1.43) is precisely the Schrödinger equation (1.41), whereas equation (1.44) is obtained from the Schrödinger equation by applying the operation of complex conjugation, that is,

$$-i\hbar\dot{\psi}^\dagger = -\frac{\hbar^2}{2m}\psi_{xx}^\dagger + \kappa U\psi^\dagger.$$

Thus, equation (1.44) does not provide us any new information. Introduce the momentum

$$\pi := \mathcal{L}_{\dot{\psi}}.$$

Explicitly, $\pi(x, t) = i\hbar\psi^\dagger(x, t)$. Moreover, we introduce the Hamiltonian density

$$\mathcal{H} := \pi\dot{\psi} - \mathcal{L}$$

and the Hamiltonian $H := \int_{-\infty}^{\infty} \mathcal{H} dx$. Explicitly,

$$\mathcal{H} = \frac{\hbar^2}{2m} \psi_x^\dagger\psi_x + U\psi^\dagger\psi.$$

Here, H represents the energy of the classical field ψ .

- Step 4: Second quantization of the Schrödinger equation and the quantum field. We now want to describe an infinite number of quantum particles on the real line including the creation and annihilation of particles.

To this end, we replace the classical wave function $\psi = \psi(x, t)$ by an operator-valued function.

More precisely, $\psi(x, t)$ is an operator which, for all positions $x, y \in \mathbb{R}$ and all times $t \in \mathbb{R}$, satisfies the so-called canonical commutation relations

$$\begin{aligned} [\psi(x, t), \pi(y, t)]_- &= i\hbar\delta(x - y), \\ [\psi(x, t), \psi(y, t)]_- &= [\pi(x, t), \pi(y, t)]_- = 0 \end{aligned}$$

along with the equations of motion

$$\boxed{i\hbar\dot{\psi} = [\psi, H]_-, \quad i\hbar\dot{\pi} = [\pi, H]_-}$$

It turns out that this implies the Schrödinger equation for the quantum field $\psi = \psi(x, t)$.²⁷

The prototype of a quantum field and the method of Fourier quantization. Suppose that we know a system $\varphi_0, \varphi_1, \dots$ of eigensolutions of the stationary Schrödinger equation,

$$E\varphi_n = -\frac{\hbar^2}{2m}(\varphi_n)_{xx} + \kappa U\varphi_n, \quad n = 0, 1, 2, \dots$$

where $\varphi_0, \varphi_1, \dots$ represents a complete orthonormal system in the Hilbert space $L_2(\mathbb{R})$. The Fourier series

$$\psi(x, t) = \sum_{n=0}^{\infty} \varphi_n(x) e^{-iE_n t/\hbar} a_n \quad (1.45)$$

with complex numbers a_0, a_1, \dots is then a solution of the Schrödinger equation. Replace now the classical Fourier coefficients by operators a_0, a_1, \dots which, for all $n, m = 0, 1, \dots$ satisfy the commutation relations

$$[a_n, a_m^\dagger]_- = \delta_{nm} I, \quad [a_n, a_m]_- = [a_n^\dagger, a_m^\dagger]_- = 0.$$

The classical field ψ from (1.45) passes then over to a quantum field which consists of an infinite number of particles having the energies E_0, E_1, \dots . We assume that there exists a state $|0\rangle$ which is free of particles. This state of lowest energy E_0 is called ground state (or vacuum). The symbol

$$a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_N}^\dagger |0\rangle$$

represents then a state of the quantum field which consists of precisely N free particles possessing the energies E_{i_1}, \dots, E_{i_N} . Moreover, the symbol

$$\psi_{\text{free}}^\dagger(x_1, t) \cdots \psi_{\text{free}}^\dagger(x_N, t) |0\rangle$$

represents a state at time t which is related to N free particles. Here, it is important to distinguish between

- the ground state $|0\rangle$ of the free quantum field ψ_{free} without any interactions,
- and the ground state $|0_{\text{int}}\rangle$ of the interacting quantum field ψ .

The main trouble of quantum field theories concerns the investigation of interacting quantum fields in rigorous mathematical terms.

²⁷ The Dirac delta function δ represents a generalization of the Kronecker symbol δ_{ij} to infinite degrees of freedom. In particular, $\delta(x - y) = 0$ if $x \neq y$. For the heuristic and rigorous definition of δ , see pages 593 and 612, respectively.

Commutation relations for creation and annihilation operators.

In elementary particle physics, we have to distinguish between bosons (particles of integer spin, e.g., photons) and fermions (particles of half-integer spin, e.g., electrons). The prototype of commutation relations for annihilation operators $a(\mathbf{p})$ and creation operators $a^\dagger(\mathbf{p})$ of bosonic particles of a momentum vector \mathbf{p} reads as²⁸

$$\boxed{[a(\mathbf{p}), a^\dagger(\mathbf{q})]_- = \delta_{\mathbf{p}\mathbf{q}} I, \quad [a(\mathbf{p}), a(\mathbf{q})]_- = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})]_- = 0}$$

for all momentum vectors \mathbf{p}, \mathbf{q} which lie on a fixed lattice of width Δp in 3-dimensional momentum space. Here, we use the 3-dimensional Kronecker symbol defined by $\delta_{\mathbf{p}\mathbf{p}} := 1$ and $\delta_{\mathbf{p}\mathbf{q}} = 0$ if $\mathbf{p} \neq \mathbf{q}$. Physicists pass to the formal continuum limit. To consider this, let us rescale the annihilation and creation operators by setting

$$\mathbf{a}(\mathbf{p}) := \frac{a(\mathbf{p})}{\sqrt{(\Delta p)^3}}, \quad \mathbf{a}^\dagger(\mathbf{p}) := \frac{a^\dagger(\mathbf{p})}{\sqrt{(\Delta p)^3}}.$$

Hence

$$[\mathbf{a}(\mathbf{p}), \mathbf{a}^\dagger(\mathbf{q})]_- = \frac{\delta_{\mathbf{p}\mathbf{q}}}{(\Delta p)^3} I, \quad [\mathbf{a}(\mathbf{p}), \mathbf{a}(\mathbf{q})]_- = [\mathbf{a}^\dagger(\mathbf{p}), \mathbf{a}^\dagger(\mathbf{q})]_- = 0.$$

The formal continuum limit $\Delta p \rightarrow 0$ yields then

$$[\mathbf{a}(\mathbf{p}), \mathbf{a}^\dagger(\mathbf{q})]_- = \delta^3(\mathbf{p} - \mathbf{q}) I, \quad [\mathbf{a}(\mathbf{p}), \mathbf{a}(\mathbf{q})]_- = [\mathbf{a}^\dagger(\mathbf{p}), \mathbf{a}^\dagger(\mathbf{q})]_- = 0$$

for all 3-dimensional momentum vectors \mathbf{p} and \mathbf{q} . The relation between the discrete Dirac delta function and its continuum limit is studied on page 675. The rigorous mathematical approach to creation and annihilation operators for free quantum particles in terms of the so-called Fock space can be found in Volume II.

The fundamental role of correlations of a quantum field. The experience of physicists in quantum physics shows that one should prefer the study of quantities which are related to measurements in physical experiments. From the physical point of view, we can measure

- cross sections of scattering processes for elementary particles, and
- masses of bound particles (like the proton as a bound state of three quarks).

It turns out that these quantities are related to correlations between different space-time points of the quantum field. According to Feynman, the basic quantity is the correlation function

$$\boxed{G_2(x_1, t_1; x_2, t_2) := \langle 0_{\text{int}} | \mathcal{T} \psi(x_1, t_1) \psi^\dagger(x_2, t_2) | 0_{\text{int}} \rangle}$$

²⁸ Recall that $[A, B]_- := AB - BA$ and $[A, B]_+ := AB + BA$. For fermions, one has to replace the Lie bracket $[\cdot, \cdot]_-$ by the Jordan–Wigner bracket $[\cdot, \cdot]_+$.

which is also called the 2-point Green's function of the interacting quantum field ψ . This function describes the correlation between the quantum field at position x_1 at time t_1 and the quantum field at position x_2 and time t_2 . Here, the symbol \mathcal{T} denotes the chronological operator. Explicitly,

$$\mathcal{T}(\psi(x_1, t_1)\psi^\dagger(x_2, t_2)) := \begin{cases} \psi(x_1, t_1)\psi^\dagger(x_2, t_2) & \text{if } t_1 \geq t_2, \\ \psi^\dagger(x_2, t_2)\psi(x_1, t_1) & \text{if } t_2 > t_1. \end{cases}$$

It turns out that

The 2-point Green's function G_2 of a quantum field is a highly singular mathematical object.

This fact causes serious mathematical difficulties. Similarly, the $2n$ -point Green's function is obtained by replacing the product $\psi(x_1, t_1)\psi^\dagger(x_2, t_2)$ by a product of $2n$ field operators. For example, the 4-point Green's function G_4 is given by

$$\langle 0_{\text{int}} | \mathcal{T} \psi(x_1, t_1) \psi(x_2, t_2) \psi^\dagger(x_3, t_3) \psi^\dagger(x_4, t_4) | 0_{\text{int}} \rangle.$$

The Green's functions G_2, G_4, G_6, \dots of a quantum field are closely related to the moments of the quantum field which contain the information on the probability structure of the quantum field.

1.2.4 The Importance of Functional Integrals

The Feynman picture of quantum field theory and the method of moments. For quantum field theory, it is crucial that the $2n$ -point Green's functions can be expressed by Feynman functional integrals (also called path integrals). For example,

$$G_2(x, t; y, s) = \frac{\int \psi(x, t) \psi^\dagger(y, s) e^{iS[\psi, \psi^\dagger]/\hbar} \mathcal{D}\psi \mathcal{D}\psi^\dagger}{\int e^{iS[\psi, \psi^\dagger]/\hbar} \mathcal{D}\psi \mathcal{D}\psi^\dagger}$$

where we integrate over all possible classical fields ψ, ψ^\dagger . In this connection, we use the classical action

$$S[\psi, \psi^\dagger] := \int_{\mathbb{R}^2} \mathcal{L} dx dt$$

where the Lagrangian density \mathcal{L} depends on ψ, ψ^\dagger , and their first-order partial derivatives. The crucial point is that the formula for G_2 makes also sense if the Lagrangian \mathcal{L} contains nonlinear terms in ψ and ψ^\dagger which describe self-interactions of the quantum field. For example, we may replace the potential U by the field product $\psi^\dagger\psi$. Then

$$\mathcal{L} := i\hbar\psi^\dagger\dot{\psi} - \frac{\hbar^2}{2m}\psi_x^\dagger\psi_x - \kappa(\psi^\dagger)^2\psi^2.$$

The relation to the Gaussian distribution in the theory of probability. For $k = 0, 1, 2, \dots$, the quantity

$$M_k := \frac{\int_{\mathbb{R}} x^k e^{-x^2/2\sigma^2} dx}{\int_{\mathbb{R}} e^{-x^2/2\sigma^2} dx}$$

is called the k th moment of the Gaussian distribution in the theory of probability. By the classical moment theorem, a probability distribution is uniquely determined by its infinite series M_0, M_1, M_2, \dots of moments.²⁹ There exists the following simple trick for computing the moments. We introduce the function

$$Z(J) := \mathcal{C} \int_{\mathbb{R}} e^{-x^2/2\sigma^2} e^{Jx} dx$$

of the real variable J where the normalization constant \mathcal{C} is chosen in such a way that $Z(0) = 1$. Then, for $k = 0, 1, 2, \dots$,

$$M_k = \frac{d^k Z(0)}{dJ^k}.$$

Naturally enough, the function $Z = Z(J)$ is called the generating function of the moments. Physicists call this the Wick moment trick. It is quite remarkable that the investigation of general interacting quantum fields can be based on an infinite-dimensional version of the Wick trick. The point is that classical integrals have to be replaced by Feynman functional integrals. Here, physicists start with the so-called generating functional integral

$$Z(J, J^\dagger) = \mathcal{C} \int e^{iS[\psi, \psi^\dagger]/\hbar} e^{\int_{\mathbb{R}^2} (\psi J + \psi^\dagger J^\dagger) dx dt} \mathcal{D}\psi \mathcal{D}\psi^\dagger$$

where the normalization \mathcal{C} is chosen in such a way that $Z(0, 0) = 1$. Then

$$G_2(x, t; y, s) = \left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z(0, 0)}{\delta J(x, t) \delta J^\dagger(y, s)}.$$

Analogously, one obtains the higher-order Green's functions by applying functional derivatives of higher order. The precise definition of functional derivatives can be found in Sect. 7.20.1 on page 398. Note that

Functional derivatives and functional integrals are natural generalizations of classical partial derivatives and classical multidimensional integrals to infinite dimensions, respectively.

They have been used systematically by physicists in the 20th century in order to generalize the classical calculus due to Newton and Leibniz to an infinite number of degrees of freedom which appear typically in quantum field theory.

²⁹ See Zeidler (1995), Vol. 2, Sect. 1.4.

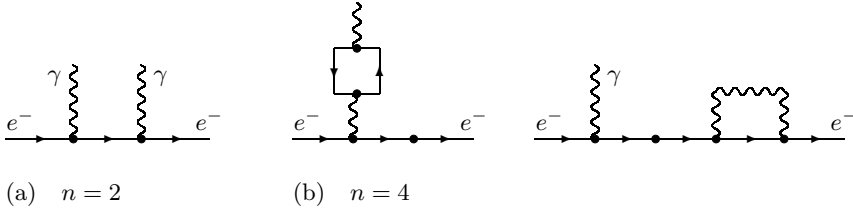


Fig. 1.3. Virtual particles

The Feynman approach to interacting quantum fields can be based on the moments of “infinite-dimensional Gaussian distributions”.

This will be thoroughly studied in the volumes of this treatise.

Virtual particles. It turns out that the Feynman diagrams from Fig. 1.2 on page 41 can be generalized to quantum fields. Fig. 1.3 displays some Feynman diagrams which describe the scattering of one electron with one photon (Compton effect). The diagrams are nothing more than a graphical representation of the analytic expressions of perturbation theory. If one gives the diagrams a physical interpretation, then there occur, for example, photons for which the physical relation

$$E^2 = c^2 \mathbf{p}^2$$

between energy E and momentum vector \mathbf{p} is violated. Such particles are called virtual photons by physicists.³⁰ Without using virtual particles, the computation of scattering processes would give wrong results. This shows that

The interactions between quantum fields are based on both real and virtual particles.

The Bethe–Salpeter equation for bound states. The state

$$\psi^\dagger(x, t) \psi^\dagger(y, s) |0_{\text{int}}\rangle$$

of the quantum field ψ describes the physics of systems consisting of two quantum particles. In particular, we expect that this state also contains bound states between two particles. Let $|p\rangle$ denote a one-particle state of momentum p . In 1951 Bethe and Salpeter used the so-called Bethe amplitudes

$$\chi_p(x, t; y, s) := \langle 0_{\text{int}} | \mathcal{T} \psi(x, t) \psi^\dagger(y, s) | p \rangle$$

in order to derive an integral equation for χ_p which contains the unknown energy E of the bound state as an eigenvalue parameter. This is the famous

³⁰ The Compton effect will be thoroughly studied in Volume II in the context of quantum electrodynamics.

Bethe–Salpeter equation which will be considered in Volume V on the physics of the Standard Model.³¹

1.3 The Role of Göttingen

One cannot comprehend what it is one possesses if one has not understood what one's predecessors possessed.

Johann Wolfgang von Goethe (1749–1832)

I am ill-mannered, for I take a lively interest in a mathematical object only where I see a prospect of a clever connection of ideas or of results recommended by elegance or generality.

Carl Friedrich Gauss (1777–1855)

In 1807, Carl Friedrich Gauss got the position as professor for astronomy and director of the observatory in Göttingen. His successors were Lejeune Dirichlet (1805–1859) and Bernhard Riemann (1826–1866) in Göttingen. In 1871, Felix Klein (1849–1925) finished his habilitation in Göttingen. After that he received professorships in Erlangen, Munich, and Leipzig. In 1881 he founded the Mathematical Institute of Leipzig University. After his move to Göttingen in 1886, Sophus Lie (1842–1899) became Klein's successor in Leipzig. Initiated by Klein, Hilbert received a professorship at Göttingen University in 1895. Under Hilbert, Göttingen became an extremely active place in mathematics. In the 1920s, Emmy Noether (1882–1935) revolutionized algebra in Göttingen. The young mathematician Bartel Leendert van der Waerden (1903–1996) attended the lectures given by Emmy Noether and Emil Artin. The result was his highly influential monograph *Modern Algebra*, Springer, Berlin 1930.

The emergence of quantum mechanics. In the late 1920s, Göttingen was the intellectual center for the development of the new quantum physics by Heisenberg (1901–1976), Born (1882–1970), Jordan (1902–1980), Pauli (1900–1958), and von Neumann (1903–1957). Students and scientists from all over the world came to Göttingen in order to take part in this scientific revolution. Among the visitors were the young physicists Vladimir Fock (1890–1974), Lev Landau (1908–1968), and Robert Oppenheimer (1904–1967). In order to get an impression of the flair of the early days of quantum mechanics, let us first quote Werner Heisenberg (1901–1976) and Paul Dirac (1902–1984) who gave *Evening Lectures* at the International Center for Theoretical Physics in Trieste (Italy) in 1968. They were invited by Abdus Salam (1926–1996).³² Heisenberg pointed out the following:

³¹ See Bethe and Salpeter (1957), Itzykson and Zuber (1981), Sect. 10.2, and Gross (1993), Sect. 12.5.

³² A. Salam (Ed.) (1968). The author would like to thank Professor Armin Uhlmann (Leipzig) for drawing his attention to these beautiful “Evening Lectures”. The history of quantum physics can be found in J. Mehra and H. Rechenberg (2002),

I had the impression from my conversation with Bohr (1885–1962) that one should go away from all these classical concepts, one should not speak of the orbit of an electron. . .

When I came back from Copenhagen to Göttingen I decided that I should again try to do some kind of guess work there, namely, to guess the intensities in the hydrogen spectrum. . . That was early in the summer 1925 and I failed completely. The formulae got too complicated. . . At the same time I also felt, if the mechanical system would be simpler, then it might be possible just to do the same thing as Kramers (1894–1952) and I had done in Copenhagen and to guess the amplitudes. Therefore I turned from the hydrogen atom to the anharmonic oscillator, which was a very simple model. Just then I became ill and went to the island of Heligoland to recover. There I had plenty of time to do my calculations. It turned out that it really was quite simple to translate classical mechanics into quantum mechanics. But I should mention one important point. It was not sufficient simply to say “let us take some frequencies and amplitudes to replace orbit quantities” and use a kind of multiplication which we had already used in Copenhagen and which later turned out to be equivalent to matrix multiplication. . .

It turned out that one could replace the quantum conditions of Bohr’s theory by a formula which was essentially equivalent to the sum-rule by Thomas and Kuhn. . . I was however not able to get a neat mathematical scheme out of it. Very soon afterwards both Born and Jordan in Göttingen and Dirac in Cambridge were able to invent a perfectly closed mathematical scheme; Dirac with very ingenious new methods on q -numbers and Born and Jordan with more conventional methods of matrices³³ . . .

When you try too much for rigorous mathematical methods you fix your attention on those points which are not important from the physics point and thereby you get away from the experimental situation. If you try to solve a problem by rather dirty mathematics, as I have mostly done, then you are forced always to think of the experimental situation; and whatever formulae you write down, you try to compare the formulae with reality and thereby, somehow, you get closer to reality than by looking for the rigorous methods. But this may, of course, be different for different people. . .

In 1926 Niels Bohr and I discussed the question on the physical interpretation of quantum mechanics many, many nights and we were frequently in a state of despair. Bohr tried more in the direction of duality between waves and particles; I preferred to start from the mathematical formalism and to look for a consistent interpretation. Finally Bohr went to Norway to think alone about the problem and I remained in Copenhagen. Then I remembered Einstein’s remark in our discussion. I remembered that Einstein had said that “It is the theory which decides what can be observed.” From there it was easy to turn around our question and not to ask “How can I represent in quantum mechanics this orbit of an electron in a cloud

Vols. 1–6. As an introduction to the development of quantum mechanics in the 1920s, we recommend van der Waerden (1968). For the history of quantum field theory, we refer to Schweber (1994) and Weinberg (1995), Vol. 1, Chap. 1.

³³ Dirac’s q -numbers (quantum numbers) are abstract operators in the sense of modern functional analysis, whereas Born and Jordan used concrete realizations of the operators in the form of infinite-dimensional complex matrices.

chamber?”, but rather to ask “Is it not true that always only such situations occur in nature, even in a cloud chamber, which can be described by the mathematical formalism of quantum mechanics?” By turning around I had to investigate what can be described in this formalism; and then it was very easily seen, especially when one used the new mathematical discoveries of Dirac and Jordan about transformation theory, that one could not describe at the same time the exact position and the exact velocity of an electron; one had these *uncertainty relations*. In this way things became clear. When Bohr returned to Copenhagen, he had found an equivalent interpretation with his concept of complementarity, so finally we all agreed that now we had understood quantum theory. . .

Again we met a difficult situation in 1927 when Einstein and Bohr discussed these matters at the Solvay Conference. Almost every day the sequence of events was the following. We all lived in the same hotel. In the morning for breakfast Einstein would appear and tell Bohr a new fictitious experiment in which he could disprove the uncertainty relations and thereby our interpretation of quantum theory. Then Bohr, Pauli and I would be very worried, we would follow Bohr and Einstein to the meeting and would discuss this problem all day. But at night for dinner usually Bohr had solved the problem and he gave the answer to Einstein, so then we felt that everything was alright and Einstein was a bit sorry about that and said he would think about it. Next morning he would bring a new fictitious experiment, again we had to discuss, and so on. This went on for quite a number of days and at the end of the conference the Copenhagen physicists had the feeling that they had won the battle and that actually Einstein could not make any real objection. . . Einstein never accepted the probabilistic interpretation of quantum mechanics. He said : “God does not play at dice.”

Dirac emphasized the following in his *Evening Lecture* at Trieste:

I have the best of reasons for being an admirer of Werner Heisenberg. He and I were young research students at the same time, about the same age, working on the same problem. Heisenberg succeeded where I failed. There was a large mass of spectroscopic data accumulated at that time and Heisenberg found out the proper way of handling it. In doing so, he started the golden age of theoretical physics. . .

One can distinguish between two main procedures for a theoretical physicist. One of them is to work from the *experimental basis*. For this, one must keep in close touch with the experimental physicists. One reads about all the results they obtain and tries to fit them into a comprehensive and satisfying scheme.

The other procedure is to work from the *mathematical basis*. One examines and criticizes the existing theory. One tries to pin-point the faults in it and then tries to remove them. The difficulty here is to remove the faults without destroying the very great success of the existing theory. . .

This is illustrated by the discovery of quantum mechanics. Two men are involved, Heisenberg and Schrödinger. Heisenberg was working from the experimental basis, using the results of spectroscopy, which by 1925 had accumulated an enormous amount of data. . . It was Heisenberg’s genius that he was able to pick out the important things from the great wealth

of information and arrange them in a natural scheme. He was thus led to matrices. . .

Schrödinger's approach was quite different. He worked from the mathematical basis. He was not well informed about the latest spectroscopic results, like Heisenberg was, but had the idea at the back of his mind that spectral frequencies should be fixed by eigenvalue equations, something like those that fix the frequencies of systems of vibrating strings. He had this idea for a long time, and was eventually able to find the right equation, in an indirect way. . .

Heisenberg and Schrödinger gave us two forms of quantum mechanics, which were soon found to be equivalent. They provided two pictures, with a certain mathematical transformation connecting them. I joined in the early work on quantum mechanics, following the procedure based on mathematics, with a very abstract point of view. I took the noncommutative algebra which was suggested by Heisenberg's matrices as the main feature for a new dynamics. . .

The following quotation is taken from Max Born's fascinating book *Physics in my Generation*, Springer, New York, 1969:

In Göttingen we also took part in the attempts to distill the unknown mechanics of the atom out of the experimental results. The logical difficulty became ever more acute. Investigations on scattering and dispersions of light showed that Einstein's conception of transition probability as a measure of the strength of an oscillation was not adequate. . . The art of guessing correct formulas, which depart from the classical formulas but pass over into them in the sense of Bohr's correspondence principle, was brought to considerable perfection. . .

This period was brought to a sudden end by Heisenberg, who was my assistant at that time. He cut the Gordian knot by a philosophical principle and replaced guesswork by a mathematical rule. The principle asserts that concepts and pictures that do not correspond to physically observable facts should not be used in theoretical description. When Einstein, in setting up his theory of relativity, eliminated his concepts of the absolute velocity of a body and of the absolute simultaneity of two events at different places, he was making use of the same principle. Heisenberg banished the picture of electron orbits with definite radii and periods of rotation, because these quantities are not observable; he demanded that the theory should be built up by means of quadratic arrays. Instead of describing the motion by giving a coordinate as a function of time $x = x(t)$, one ought to determine an array of transition probabilities (x_{ij}) . To me the decisive part in his work is the requirement that one must find a rule whereby from a given array

$$\begin{pmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

the array for the square $(x^2)_{ij}$ may be found (or, in general, the multiplication law of such arrays).

By consideration of known examples discovered by guesswork, Heisenberg found this rule and applied it with success to simple examples such as

the harmonic and anharmonic oscillator. This was in the summer 1925. Heisenberg, suffering from a severe attack of hay fever, took leave of absence for a course of treatment at the seaside and handed over his paper to me for publication, if I thought I could do anything about it.

The significance of the idea was immediately clear to me, and I sent the manuscript to the publisher.³⁴ Heisenberg's rule of multiplication left me no peace, and after a week of intensive thought and trial, I suddenly remembered an algebraic theory that I had learned from my teacher, Rosanes, in Breslau. Such quadratic arrays are quite familiar to mathematicians and are called matrices, in association with a definite rule of multiplication. I applied this rule to Heisenberg's quantum condition and found that it agreed for the diagonal elements. It was easy to guess what the remaining elements must be, namely, null; and immediately there stood before me the strange formula

$$qp - pq = i\hbar. \quad (1.46)$$

This meant that the coordinates q and momenta p are not to be represented by the values of numbers but by symbols whose product depends on the order of multiplication – which do not “commute”, as we say.

My excitement over this result was like that of the mariner who, after long voyaging, sees the desired land from afar, and my only regret was that Heisenberg was not with me. I was convinced from the first that we had stumbled on the truth. Yet again a large part was only guesswork, in particular the vanishing of the non-diagonal elements in the foregoing expression. For this problem, I secured the collaboration of my pupil Pascual Jordan, and in a few days we succeeded in showing that I had guessed correctly. The joint paper written by Jordan and myself³⁵ contains the most important principles of quantum mechanics, including its extension to electrodynamics. . .

There followed a hectic period of collaboration among the three of us, rendered difficult by Heisenberg's absence. There was a lively interchange of letters. . . The result was a three-man paper,³⁶ which brought the formal side of the investigation to a certain degree of completeness. Before this paper appeared, the first dramatic surprise occurred: Paul Dirac's paper on the same subject.³⁷ The stimulus received through a lecture by Heisenberg in Cambridge led him to results similar to ours in Göttingen, with the difference that he did not have recourse to the known matrix theory of the mathematicians but discovered for himself and elaborated the doctrine of such non-commuting symbols.

³⁴ W. Heisenberg, Quantum-theoretical re-interpretation of kinematic and mechanical relations, *Zeitschrift für Physik* **33** (1925), 879–893.

³⁵ M. Born and P. Jordan, On quantum mechanics, *Zeitschrift für Physik* **34** (1925), 883–888.

³⁶ M. Born, W. Heisenberg, and P. Jordan, On quantum mechanics II, *Zeitschrift für Physik* **35** (1926), 557–615.

³⁷ P. Dirac, Quantum mechanics and a preliminary investigation of the hydrogen atom, *Proc. Roy. Soc. A* **110** (1926), 561–569.

The first nontrivial and physically important application of quantum mechanics was made soon afterwards by Wolfgang Pauli,³⁸ who calculated the stationary energy values of the hydrogen atom by the matrix method and found complete agreement with Bohr's 1913 formulas. From this moment there was no longer any doubt about the correctness of the theory among physicists. . .

What the real significance of the formalism might be was, however, by no means clear. Mathematics, as often happens, was wiser than interpretative thought. While we were still discussing the point, there occurred the second dramatic surprise: the appearance of Schrödinger's celebrated paper.³⁹ He followed quite a different line of thought, which derived from Prince Louis de Broglie (1892–1987). The latter had a few years previously made the bold assertion, supported by brilliant theoretical considerations, that wave-corpuscle dualism, familiar to physicists in the case of light, must also be exhibited by electrons; to each freely movable electron there belongs, according to these ideas, a plane wave of perfectly definite wave length, determined by Planck's constant and mass. . . Schrödinger extended de Broglie's wave equation, which applied to free motion, to the case in which forces act. . . and he succeeded in deriving the stationary states of the hydrogen atom as monochromatic solutions of his wave equation not extending to infinity. For a short while, at the beginning of 1926, it looked as if suddenly there were two self-contained but entirely distinct systems of explanation in the field – matrix mechanics and wave mechanics. But Schrödinger himself soon demonstrated their complete equivalence.

Wave mechanics enjoyed much greater popularity than the Göttingen or Cambridge version of quantum mechanics. Wave mechanics operates with a wave function ψ , which – at least in the case of one particle – can be pictured in space, and it employs the mathematical methods of partial differential equations familiar to every physicist.

It appeared to me that it was not possible to arrive at a clear interpretation of the Schrödinger ψ -function by considering bound electrons. I had therefore been at pains, as early as the end of 1925. . . I was at that time the guest of the Massachusetts Institute of Technology in the U.S.A., and there I found in Norbert Wiener (1894–1964) a distinguished collaborator. In a joint 1926 paper we replaced the matrix by the general concept of an operator and, in this way, made possible the description of aperiodic processes. . . Once more an idea of Einstein's gave the lead. He had thought to make the duality of particles (light quanta or photons) and waves comprehensible by interpreting the square of the optical wave amplitudes as probability density for the occurrence of photons. This idea could at once be extended to Schrödinger's ψ -function:

The square of the amplitude, $|\psi|^2$, must represent the probability density for electrons (or other particles).

To assert this was easy; but how could I prove this? For this purpose atomic scattering processes suggested themselves. A shower of electrons coming from an infinite distance, represented by a wave of known intensity (that

³⁸ W. Pauli, On the hydrogen spectrum from the standpoint of the new quantum mechanics, *Zeitschrift für Physik* **36** (1926), 336–363.

³⁹ E. Schrödinger, Quantization as an eigenvalue problem, *Ann. Phys.* **9** (1926), 361–376.

is, $|\psi|^2$) impinge on an obstacle say a heavy atom... In the same way that the water wave caused by a steamer excites secondary circular waves in striking a pile, the incident electron wave is partly transformed by the atom into a secondary spherical wave, whose amplitude of oscillations ψ is different in different directions. The square of the amplitude $|\psi|^2$ of this wave at a great distance from the scattering center then determines the relative probability of scattering in its dependence of direction... Soon Wentzel succeeded in deriving Rutherford's celebrated 1911 formula for the scattering of α -particles from my theory.

But the factor that contributed more than these successes to the speed of acceptance of my statistical interpretation of the ψ -function was a 1927 paper by Heisenberg that contained his celebrated uncertainty relationship, through which the revolutionary character of the new conception was first made clear.

In 1927 Heisenberg left Göttingen in order to get a professorship at Leipzig University. Four years later, van der Waerden came to Leipzig as a professor of mathematics. According to his own words, he liked very much to attend Heisenberg's seminars. In 1932 van der Waerden published his nicely written book *Group Theory and Quantum Mechanics* about applications of group theory to the spectra of non-relativistic molecules and Dirac's relativistic electron.

The challenge of quantum electrodynamics. Already in the early days of quantum mechanics, physicists tried to understand the quantization of the electromagnetic field. Heisenberg and Pauli published two fundamental papers in 1929 and 1930.⁴⁰ Quantum electrodynamics was fully developed in the late 1940s. In order to handle meaningless infinite expressions, physicists developed the method of renormalization in the 1930s and 1940s. This ingenious method allows physicists to extract the relevant physical information from mathematically meaningless expressions. We will thoroughly study this important point later on. From the physical point of view, the following is crucial:

- The singularities of the Green's functions reflect both the complicated structure of the ground state of a quantum field and the complex interactions between the unobservable ground state and the real world, by means of quantum fluctuations.
- The renormalization procedure indicates that the present quantum field theory is not a basic theory, but only an effective theory which averages a deeper physical structure, at a fairly low energy scale.

In 1965, Julian Schwinger said in his Nobel prize speech:

The relativistic quantum theory of fields was born some thirty-five years ago through the paternal efforts of Dirac, Heisenberg, Pauli and others. It was a somewhat retarded youngster, however, and first reached adolescence seventeen years later, an event we are gathered here to celebrate.

⁴⁰ W. Heisenberg and W. Pauli, On quantum field theory (in German) *Zeitschrift für Physik* **56** (1929), 1–61; **59** (1930), 168–190.

1.4 The Göttingen Tragedy

Wistfully I recall how, during the Nazi occupation of Poland (1939–1945), Edward Marczewski introduced the arcana of analysis to me. In those dark days these were for me bright moments, for which I am infinitely grateful to him.

Krzysztof Maurin (born 1923)

In 1933 the Nazi regime reached the political power in Germany. Because of racist repression, the best scientists left Germany, among them Emmy Noether (1882–1935), Emil Artin (1898–1972), Paul Bernays (1888–1977), Max Born (1882–1970), Richard Courant (1882–1972), Albert Einstein (1879–1955), Kurt-Otto Friedrichs (1901–1982), Leon Lichtenstein (1878–1933), and Hermann Weyl (1885–1955). Furthermore, Edmund Landau (1877–1938) and the Nobel laureate James Franck (1884–1964) lost their positions at Göttingen University. Hilbert's best friend, Otto Blumenthal (1876–1944), was murdered in the Nazi concentration camp Terežin (Theresienstadt). In 1934 Hilbert was asked by the Nazi minister of education about the flourishing scientific life in Göttingen. Hilbert answered:

There is no mathematics anymore in Göttingen.

The Göttingen tradition moved to the United States of America. Richard Courant founded the famous Courant Institute at New York University (NYU). In this context, we recommend reading the two beautiful biographies about David Hilbert and Richard Courant written by Constance Reid.⁴¹

After working as professor in Zurich and Prague, Einstein was appointed as director of the Kaiser-Wilhelm Institute for Physics in Berlin in 1914. In 1933 Einstein emigrated to the United States of America where he got a professorship at the Institute for Advanced Study in Princeton, New Jersey.⁴² Einstein lived there until his death in 1955.

In 1933 John von Neumann obtained a professorship at the Institute for Advanced Study in Princeton. Von Neumann was one of the greatest mathematicians of the 20th century. His fundamental contributions concern game theory, mathematical economics, mathematical logic, lattice theory, operator theory in Hilbert spaces, operator algebras, mathematical foundations of quantum mechanics, theory of Lie groups, measure theory, statistical physics, ergodic theory, construction of the first computers ENIAC and MANIAC, and foundations of computer science in the 1940s, shock waves, turbulence, mete-

⁴¹ C. Reid, *Hilbert*, Springer 1970, New York, and C. Reid, *Courant in Göttingen and New York: the Life of an Improbable Mathematician*, Springer, New York, 1976.

⁴² This institute was founded in 1930. The history of this famous research institute can be found in E. Regis, *Who Got Einstein's Office? Eccentricity and Genius at the Institute for Advanced Study in Princeton*, Addison-Wesley, Reading, Massachusetts.

orology, and numerical mathematics. In the context with John von Neumann, let us quote Peter Hilton:⁴³

For a mentor of Ph.D. candidates it would be most easy to educate a poor applied mathematician. The next simplest thing would be to educate a poor pure mathematician. Then an entire quantum gap lies between the education of a good pure mathematician, and finally, an enormous quantum gap, the education of a good applied mathematician. For the latter task, especially after the death of John von Neumann, I would consider no one sufficiently qualified.

The knowledge and abilities which are nowadays required of a really successful applied mathematician, presume an extraordinary high intellectual standard, and, even for the career of our present-day students, it is almost impossible to predict which parts of mathematics will prove most suited for applications.

Besides John von Neumann, another hero of the 20th century mathematics is Hermann Weyl. He studied mathematics in Göttingen from 1903 until 1908. He attended lectures given by Carathéodory (1873–1950), Hilbert (1858–1943), Klein (1849–1925), Koebe (1882–1945), and Zermelo (1871–1953). He was Hilbert's most gifted pupil. From 1913 until 1930, Weyl worked at the Swiss Institute of Technology (ETH) in Zurich. In 1930 he became Hilbert's successor at Göttingen University. Three years later, Hermann Weyl left Germany and joined Einstein and von Neumann at the Institute for Advanced Study in Princeton. Hermann Weyl influenced very strongly the relations between mathematics and physics. He wrote a number of classical monographs about Riemann surfaces, theory of general relativity, group theory and quantum mechanics, representation theory of the classical Lie groups, symmetry, and philosophical questions. His books became bibles for both physicists and mathematicians. In particular, the idea of gauge field theory can be traced back to Weyl's monograph *Space, Time, Matter* from 1923 where he presented Einstein's theory of general relativity along with his own ideas about a general theory of matter, based on scaling invariance and the conformal group. In 1944 Hermann Weyl wrote the following in the Bulletin of the American Mathematical Society:

A great master of mathematics passed away when Hilbert died in Göttingen on February 14, 1943, at the age of eighty-one. In retrospect, it seems to us that the era of mathematics upon which he impressed the seal of his spirit, and which is now sinking below the horizon, achieved a more perfect balance than has prevailed before or since, between the mastering of single concrete problems and the formation of general abstract concepts. Hilbert's own work contributed not a little to bringing about this happy equilibrium, and the direction we have since proceeded can in many instances be traced back to this impulse. No mathematician of equal stature has arisen from our generation. . .

Hilbert was singularly free from national and racial prejudice; in all public questions, be they political, social, or spiritual, he stood forever on the side

⁴³ In: M. Otte (Ed.), *Mathematiker über Mathematik*, Springer, Berlin 1973.

of freedom, frequently in isolated opposition against the compact majority of his environment... It was not mere chance, when the Nazis “purged” the universities in 1933 and their hand fell most heavily on the Hilbert school that Hilbert’s most intimate collaborators left Germany voluntarily or under the pressure of Nazi persecution. He himself was too old, and stayed behind; but the years after 1933 became years of ever-deepening tragic loneliness.

1.5 Highlights in the Sciences

1.5.1 The Nobel Prize

The history of quantum physics in the 20th century is reflected best by the Nobel laureates in physics, chemistry, and medicine. For the convenience of the reader, here is a selection of topics.⁴⁴

- (i) **Radioactivity:** Bequerel, and Marie and Pierre Curie 1903 (natural radioactivity), Rutherford 1908 (chemistry of radioactive substances), Marie Curie 1911 (radium), Irène Joliot-Curie and Frédéric Joliot 1935 (artificial radioactivity), Hahn 1944 (uranium fission).
- (ii) **Rays:** Röntgen 1901 (X-rays or Röntgen rays), Lorentz and Zeeman 1902 (influence of magnetism on radiation phenomena), Lenard 1905 (cathode rays), Michelson 1907 (spectroscopic experiments), Wien 1911 (radiation of heat), Sir William Henry Bragg and Sir William Lawrence Bragg 1913 (analysis of crystal structure by means of X-rays), Laue 1914 (diffraction of X-rays by crystals), Stark 1919 (splitting of spectral lines in electric fields), Franck and Hertz 1925 (observation of energy quantization in mercury atoms), Perrin 1926 (measurement of the size of atoms), Compton 1927 (Compton effect), Wilson 1927 (electron tracks in a cloud chamber), Sir Raman 1930 (scattering of light by atoms), Debye 1936 (investigation of molecular structure by diffraction of X-rays and electrons in gases), Hess 1936 (discovery of cosmic rays), Cherenkov 1958 (Cherenkov radiation), Bethe 1967 (energy production in stars), Gabor 1971 (holograph method), Townes, Basov, and Prochorov 1973 (laser), Ryle and Hewish 1974 (radio astronomy and pulsars), Penzias and Wilson 1978 (discovery of cosmic microwave radiation coming from the early universe), Chandrasekhar 1983 (theory of the structure and formation of stars), Fowler 1983 (theory of the formation of chemical elements in the universe), Hulse and Taylor 1993 (detection of gravitational waves coming from binary neutron stars).
- (iii) **Structure of matter:** Sir Joseph John Thomson 1906 (conduction of electricity by gases), Einstein 1921 (photoelectric effect), Millikan 1923 (measurement of the charge of the electron), Chadwick 1935 (discovery of the neutron), Anderson 1936 (discovery of the positron), Fermi 1938 (experimental production of new radioactive elements and the discovery of nuclear reactions brought about by slow neutrons), Lawrence 1939 (construction of the cyclotron as particle accelerator), Stern 1943 (measurement of the magnetic moment of the proton), Yukawa 1949 (theoretical prediction of the existence of π -mesons in 1935), Powell 1950 (experimental discovery of π -mesons), Lamb 1955 (precision test of quantum electrodynamics by measurement of the $2s$ and $2p$ energy difference

⁴⁴ For more details, we refer to the literature about Nobel laureates summarized on page 945.

- in the hydrogen atom), Kusch 1955 (measurement of the magnetic moment of the electron), Chamberlain and Segré 1959 (discovery of the antiproton), Hofstadter 1961 (experimental discovery of the internal structure of the proton), Richter and Ting 1976 (experimental discovery of the charm quark), Fitch and Cronin 1980 (experimental discovery of CP violation in weak interaction), Rubbia and van der Meer 1984 (experimental discovery of the W^\pm -bosons and the Z^0 -boson in electroweak interaction), Lederman, Schwartz, and Steinberger 1988 (experimental discovery of the muon neutrino), Friedman, Kendall, and Taylor 1990 (experimental discovery of the quark structure of the proton by investigating deep inelastic scattering of electrons on protons), Perl 1995 (experimental discovery of the tau lepton), Reines 1995 (detection of the neutrino), Curl, Kroto, and Smalley 1996 (fullerenes).
- (iv) **Quantum mechanics:** Planck 1918 (existence of energy quanta and foundation of quantum physics), Bohr 1922 (semiclassical model of the atom), de Broglie 1929 (wave nature of electrons), Heisenberg 1932 (foundation of quantum mechanics), Dirac and Schrödinger 1933 (new productive forms of atomic theory), Pauli 1945 (exclusion principle), Born 1954 (statistical interpretation of Schrödinger's wave function), Pauling 1954 (chemical bond), Wigner 1963 (symmetry principles), Goeppert-Mayer and Jensen 1963 (nuclear shell), Glauber, Hänsch, and Hall 2005 (quantum optics).
 - (v) **Quantum chemistry:** Fukui and Hoffmann 1981 (course of chemical reactions), Kohn 1998 (density functional theory), Pople 1998 (computational methods in quantum chemistry).
 - (vi) **The emergence of the Standard Model of elementary particle physics:** Yang and Lee 1957 (theory of parity violation in weak interaction), Feynman, Schwinger, and Tomonaga 1965 (theory of quantum electrodynamics), Gell-Mann 1969 (classification of elementary particles and interactions), Glashow, Salam, and Weinberg 1979 (theory of electroweak interaction as a gauge theory), Rubbia and van der Meer 1984 (experimental discovery of the intermediate vector bosons), 't Hooft and Veltman 1999 (renormalization of the gauge field theory for electroweak interaction), Gross, Politzer, and Wilczek 2004 (asymptotic freedom of quarks), Mather and Smoot 2006 (cosmic background radiation), Nambu, Kobayashi, and Masukawa 2008 (symmetry breaking in elementary particle physics).
 - (vii) **Solid states and condensed matter:** Kammerlingh-Onnes 1913 (superconductivity), Bardeen, Brattain, and Shocklee 1956 (transistor), Mößbauer 1961 (recoilless gamma emission from nuclei), Landau 1962 (theory of phase transitions in condensed matter and liquid helium), Alfvén 1970 (magnetohydrodynamics and plasma physics), Néel 1970 (antiferromagnetism), Bardeen, Cooper, and Schrieffer 1972 (theory of Cooper pairs and superconductivity), Esaki and Giaever 1973 (tunnelling effects in semiconductors and superconductors), Josephson 1973 (super-current through a tunnel barrier and the Josephson effect), Wilson 1980 (critical phenomena, phase transitions, and the renormalization group technique for combining different scales in nature with each other), von Klitzing 1985 (experimental discovery of the quantum Hall effect), Müller and Bednorz 1987 (experimental discovery of high-temperature superconductivity in ceramic materials), Ramsey 1989 (hydrogen maser), Dehmelt and Paul 1989 (trapping of ions), de Gennes 1991 (liquid crystals and polymers), Chu, Tannoudji, and Phillips 1998 (cooling and trapping of atoms with laser light), Laughlin, Störmer, and Tsu 1999 (experimental discovery of the fractional quantum Hall effect), Cornell, Ketterle, and Wieman 2001 (experimental realization of Bose-Einstein condensation in dilute gases), Abrikosov, Ginzburg, and Leggett 2003 (superconductivity and superfluidity).

- (viii) **Non-equilibrium thermodynamics:** Onsager 1968 (Onsager's law), Prigogine 1977 (dissipative structures).
- (ix) **Computer technology:** Alferov and Kroemer 2000 (semiconductor heterostructures used in high-speed electronics and opto-electronics), Kirby 2000 (invention of integrated circuits), Fert and Grünberg 2007 (giant magnetoresistance).
- (x) **Molecular biology:** Crick and Watson 1962 (double helix of DNA), Holley, Har Gobind Khorana, and Nirenberg 1968 (genetic code and protein synthesis), Delbrück, Hershey, and Luria 1969 (replication mechanism and the genetic structure of viruses), Diefenbacher, Huber, and Michel 1988 (determination of a 3-dimensional structure of a photosynthetic reaction), Fenn and Tanaka 2002 (development of soft desorption ionization methods for mass spectrometric analysis of biological macromolecules), Wüthrich 2002 (development of nuclear magnetic resonance spectroscopy for determining the three-dimensional structure of biological macromolecules in solution).

1.5.2 The Fields Medal in Mathematics

The International Congress of Mathematicians (ICM) takes place every four years. In 1924, a resolution was adopted that at each ICM, two gold medals should be awarded to recognize outstanding mathematical achievement. Professor J. C. Fields, a Canadian mathematician who was secretary of the 1924 Congress, later donated funds establishing the medals which were named in his honor. Consistent with Field's wish that the awards recognize both existing work and the promise of future achievement, it was agreed to restrict the medal to mathematicians not over forty in the year of the congress. In 1966 it was agreed, because of great extension of mathematical research, up to four medals could be awarded at each Congress. The following list of Fields medallists reflects important progress in mathematics. The Fields medal has a very high reputation.⁴⁵

- 1936 Oslo: Ahlfors (quasi-conformal maps), Douglas (minimal surfaces).
- 1950 Cambridge, Massachusetts: Laurent Schwartz (generalized functions), Selberg (elementary proof of the prime number theorem).
- 1954 Amsterdam: Kodaira (harmonic integrals in algebraic geometry), Serre (homotopy groups of spheres).
- 1958 Edinburgh: Roth (rational approximations to algebraic numbers), Thom (cobordism theory for manifolds).
- 1962 Stockholm: Hörmander (general theory of linear partial differential equations), Milnor (exotic spheres).
- 1966 Moscow: Atiyah (K -theory for vector bundles), Cohen (continuum hypothesis), Smale (proof of the Poincaré conjecture for n -dimensional spheres with $n \geq 5$, general structure of dynamical systems), Grothendieck (nuclear spaces, schemes in algebraic geometry).
- 1970 Nice: Baker (theory of transcendental numbers), Hironaka (blowing-up of singularities of algebraic varieties), Novikov (homology and homotopy theory), Thompson (group theory).

⁴⁵ We refer to M. Atiyah and D. Iagolnitzer (Eds.), *Fields Medallists' Lectures*, World Scientific, Singapore, 2003.

- 1974 Vancouver: Bombieri (analytic number theory and geometry of numbers), Mumford (Abelian varieties).
- 1978 Helsinki: Deligne (proof of the modified Riemann conjecture due to Weil for algebraic varieties over finite fields), Fefferman (singular integral operators, analytic functions of several variables), Margulis (structure of discrete Lie subgroups with fixed volume), Quillen (proof of the Serre conjecture on projective modules, cohomology of groups).
- 1982 Warsaw: Connes (structure of von Neumann algebras of type III), Thurston (hyperbolic structure of 3-dimensional manifolds), Shing-Tung Yau (positive mass theorem in general relativity, proof of the Calabi conjecture for Kähler manifolds).
- 1986 Berkeley: Donaldson (Yang–Mills equations and the differential topology of 4-dimensional manifolds), Faltings (proof of the Mordell conjecture for Diophantine equations), Freedman (proof of the Poincaré conjecture for 4-dimensional spheres).
- 1990 Kyoto: Drinfeld (quantum groups and Galois groups), Jones (von Neumann algebras and Jones polynomials in knot theory), Mori (classification of 3-dimensional algebraic varieties), Witten (supersymmetry and Morse theory, global anomalies, supersymmetric index theory, rigidity theorems for representations of Lie groups in string theory, spin structure, and a new approach to the positive mass theorem).
- 1994 Zurich: Bourgain (nonlinear Schrödinger equation, geometry of Banach spaces, ergodic theory, and analytic number theory), Pierre-Louis Lions (nonlinear partial differential equations, Boltzmann equation, viscosity method for the Hamilton–Jacobi equation, compressible fluids, Hartree–Fock equation, anisotropic diffusion and image processing), Yoccoz (stability of dynamical systems), Zelmanov (Lie algebras and Jordan algebras).
- 1998 Berlin: Borchers (representation of the monster group, modular forms), Gowers (geometry of Banach spaces, combinatorics), Kontsevich (Poisson structures and quantum deformations, equivalence of two models in quantum gravitation, effective knot invariants in topology), McMullen (complex dynamics and hyperbolic geometry), Wiles (special tribute for proving Fermat’s last theorem).
- 2002 Beijing: Lafforgue (Langlands program for function fields, deep connections between number theory, analysis, and group representation theory), Voevodsky (proof of the Milnor conjecture in algebraic K -theory, motivic cohomology theory).
- 2006 Madrid: Okunkov (theory of probability, representation theory and algebraic geometry), Perelman (proof of the Poincaré conjecture via the Ricci flow – Perelman turned down the Fields medal), Tao (partial differential equations, combinatorics, harmonic analysis and additive number theory), Werner (stochastic Loewner evolution, geometry of two-dimensional Brownian motion, conformal field theory).

This list shows convincingly that the great achievements of mathematics in the 20th century are related to the efforts made by the great masters of mathematics and physics in the 18th and 19th century.

1.5.3 The Nevanlinna Prize in Computer Sciences

Since 1982, parallel to the Fields medal, the Nevanlinna prize has been awarded for outstanding contributions to computer sciences.

- 1983 Warsaw: Tarjan (construction of highly effective algorithms).

- 1986 Berkeley: Valiant (complexity theory, random algorithms).
- 1990 Kyoto: Razborov (complexity of networks).
- 1994 Avi Wigderson (verification of proofs).
- 1998 Berlin: Shore (theory of quantum computation).
- 2002 Beijing: Sudan (probabilistic algorithms for checking the correctness of proofs).
- 2006 Madrid: Kleinberg (effective algorithm for ranking Web pages – nodes in a directed graph – by assigning an authority value and a hub value to each page).

1.5.4 The Gauss Prize in Mathematics

The Gauss prize was founded in 2006 by the International Mathematical Union (IMU) and the German Mathematical Society (DMV). It will be awarded parallel to the Fields medal.

- 2006 Itô (solution of stochastic differential equations).

1.5.5 The Wolf Prize in Physics

Every year the Israelitic Parliament (the Knesseth) confers the Wolf prize to outstanding scientists for their life-work in the fields of agriculture, arts, chemistry, mathematics, medicine, and physics. Here is the list of physicists who were awarded the Wolf prize.

- 1978 Wu (experimental discovery of parity violation in weak interaction in 1957).
- 1979 Uhlenbeck (experimental discovery of the electron spin in 1922, together with the late Goudsmith), Occhialini (experimental discovery of electron pair production and the charged pion).
- 1980 Fisher, Kadanoff, and Wilson (critical phenomena and phase transitions).
- 1981 Dyson, 't Hooft, and Weisskopf (quantum theory of fields).
- 1982 Lederman and Perl (discovery of the bottom quark).
- 1983/84 Hahn (spin echo), Hirsh (transmission electron microscope), Maiman (first operating laser).
- 1984/85 Herring and Nozieres (electrons in metals).
- 1986 Feigenbaum and Libchhaber (universal laws in turbulence; theory and experiments).
- 1987 Friedman, Rossi, and Giacconi (solar X-rays).
- 1988 Penrose and Hawking (necessity of cosmic singularities).
- 1989 not awarded.
- 1990 de Gennes and Thouless (complex condensed matter, liquid crystals, and disordered 2-dimensional systems).
- 1991 Goldhaber and Telegdi (weak interaction).
- 1992 Taylor (radio pulsar).
- 1993 Mandelbrot (fractals).
- 1994/95 Ginzburg (superconductivity and Ginzburg–Landau equation), Nambu (superconductivity, colored quarks).
- 1996 not awarded.
- 1996/97 Wheeler (black holes, quantum gravity, and nuclear fission).
- 1998 Aharonov and Berry (global quantum effects, Aharonov–Bohm effect, and Berry phase).
- 1999 Shechtman (experimental discovery of quasi-crystals).

- 2000 Davis and Koshiba (neutrino astronomy).
- 2001 not awarded.
- 2002/03 Leggett (superfluidity of the light helium isotope and macroscopic quantum phenomena), Halperin (two-dimensional melting, disordered systems, and strongly interacting electrons).
- 2004 Brout and Higgs (mass generation by local gauge symmetry in the world of subatomic particles).
- 2005 Kleppner (hydrogen maser, Rydberg atoms, and Bose–Einstein condensation).
- 2006/2007 Fert and Grünberg (giant magnetoresistance).
- 2008 not awarded.

1.5.6 The Wolf Prize in Mathematics

The following mathematicians were awarded the Wolf prize.⁴⁶

- 1978 Gelfand (functional analysis, group representations, and seminal contributions to many parts of mathematics), Siegel (number theory, analytic functions of several variables, celestial mechanics).
- 1979 Leray (application of topological methods to differential equations), Weil (algebraic-geometric methods in number theory).
- 1980 Henri Cartan (algebraic topology, homological algebra, sheaf theory, and analytic functions of several variables), Kolmogorov (foundation of probability theory, stochastic processes, ergodic theory, Fourier analysis, and celestial mechanics).
- 1981 Ahlfors (geometric function theory), Zariski (commutative algebra and algebraic geometry).
- 1982 Whitney (algebraic topology and differential topology), Krein (functional analysis and its applications).
- 1983/84 Chern (global differential geometry), Erdős (discrete mathematics: number theory, combinatorics, graph theory, probability).
- 1984/85 Kodaira (complex manifolds and algebraic varieties), Hans Lewy (partial differential equations).
- 1986 Eilenberg (algebraic topology and homological algebra), Selberg (number theory, discrete groups, and automorphic functions).
- 1987 Itô (stochastic differential equations), Lax (linear and nonlinear partial differential equations, direct and inverse scattering theory, and shock waves).
- 1988 Hirzebruch (topology, algebraic geometry, and index theory), Hörmander (partial differential equations, pseudo-differential operators, Fourier integral operators).
- 1989 Calderon (singular integral operators and partial differential equations), Milnor (geometry, algebraic and differential topology).
- 1990 de Giorgi (calculus of variations and partial differential equations).
- 1991 not awarded.
- 1992 Carleson (Fourier analysis, complex analysis, and quasi-conformal mappings), Thompson (finite groups).
- 1993 Gromov (global Riemannian geometry and symplectic geometry), Tits (algebraic groups, Tits buildings).

⁴⁶ We refer to S. Chern and F. Hirzebruch (Eds.), *Wolf Prize in Mathematics*, Vols. 1, 2, World Scientific, Singapore, 2001.

- 1994 Moser (stability in Hamiltonian mechanics, celestial mechanics, and non-linear partial differential equations).
- 1995/96 Langlands (number theory, automorphic forms, group representations, and the Langlands program on noncommutative class field theory), Wiles (proof of Fermat's last theorem).
- 1997 Joseph Keller (electromagnetic, acoustic, and optical wave propagation; fluid, solid, and quantum mechanics, statistical physics), Sinai (ergodic theory and statistical mechanics, dynamical systems).
- 1998 not awarded.
- 1999 Lovasz (combinatorics, theoretical computer sciences, combinatorial optimization), Stein (Fourier analysis and harmonic analysis).
- 2000 Bott (topology, differential geometry, Lie groups), Serre (topology, algebra, and algebraic geometry).
- 2001 Arnold (dynamical systems and singularity theory).
- 2002/03 Sato (hyperfunctions and microfunction theory, holonomic quantum field theory), Tate (algebraic number theory).
- 2004 not awarded.
- 2005 Margulis (theory of lattices in semi-simple Lie groups and striking applications of this to ergodic theory, representation theory, number theory, combinatorics, and measure theory), Novikov (algebraic and differential topology, algebraic-geometric methods in mathematical physics).
- 2006/07 Smale (differential geometry, dynamical systems, mathematical economics, numerical analysis), Furstenberg (ergodic theory, topological dynamics, analysis on symmetric spaces and homogeneous flows).
- 2008 Deligne (arithmetic, proof of the Weil conjecture, mixed Hodge theory, Riemann–Hilbert correspondence), Griffiths (variations of Hodge structures, periods of Abelian integrals, complex differential geometry), Mumford (algebraic surfaces, geometric invariant theory, foundations of the modern theory of moduli of curves and theta functions).

1.5.7 The Abel Prize in Mathematics

The Abel prize was founded in 2003 by the Norwegian government. This new prize is intended to play the role of the Nobel prize in mathematics.

- 2003 Serre (algebra, number theory, and topology).
- 2004 Atiyah and Singer (analysis, differential geometry, topology, and the Atiyah–Singer index theorem).
- 2005 Lax (linear and nonlinear partial differential equations, solitons and Lax pairs, scattering theory, shock waves).
- 2006 Carleson (harmonic analysis and smooth dynamical systems).
- 2007 Varadhan (theory of probability, unified theory of large deviations).
- 2008 Thompson and Tits (group theory).
- 2009 Gromov (differential geometry).

1.6 The Emergence of Physical Mathematics – a New Dimension of Mathematics

At the International Congress of Mathematicians in Kyoto in 1990, the young physicist Edward Witten (Institute for Advanced Study in Princeton) was

awarded the Fields medal in mathematics. In his laudation for Edward Witten, Sir Michael Atiyah emphasized the following:⁴⁷

The past decade has seen a remarkable renaissance in the interaction between mathematics and physics. This has been mainly due to the increasingly sophisticated mathematical models employed by elementary particle physicists, and the consequent need to use the appropriate mathematical machinery. In particular, because of the strongly non-linear nature of the theories involved, topological ideas and methods have played a prominent part.

The mathematical community has benefited from this interaction in two ways. First, and more conventionally, mathematicians have been spurred into learning some of the relevant physics and collaborating with colleagues in theoretical physics. Second, and more surprisingly, many of the ideas emanating from physics have led to significant new insights in purely mathematical problems, and remarkable discoveries have been made in consequence. The main input from physics has come from quantum field theory. While the analytic foundations of quantum field theory have been intensively studied by mathematicians for many years, the new stimulus has involved the more formal (algebraic, geometric, topological) aspects.

In all this large and exciting field, which involves many of the leading physicists and mathematicians in the world, Edward Witten stands clearly as the most influential and dominating figure. Although he is definitely a physicist his command of mathematics is rivalled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by a brilliant application of physical insight leading to new and deep mathematical theorems.

In 1986, the American Mathematical Society invited mathematicians and physicists to a joined symposium devoted to *Mathematics: the Unifying Thread in Science*. In his quite remarkable speech, the physicist Steven Weinberg pointed out the following.⁴⁸

String theory is right now the hot topic of theoretical physics. According to this picture, the fundamental constituents of nature are not, in fact, particles, or even fields, but are instead little strings, little elementary rubber bands that go zipping around, each in its own state of vibration. In these theories what we call a particle is just a string in a particular state of vibration, and what we call a reaction among particles, is just the collision of two or more strings, each in its own state of vibration, forming a single joined string which then later breaks up, forming several independent strings, each again in its own mode of vibration.

⁴⁷ M. Atiyah and D. Iagolnitzer (Eds.), *Fields Medallists' Lectures*, 1997. Reprinted by permission of World Scientific Publishing Co. Pte. Ltd, Singapore.

⁴⁸ *Notices Amer. Math. Soc.* **33** (1986), 716–733 (reprinted with permission). For his fundamental contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, Steven Weinberg was awarded the Nobel prize in physics in 1979. He wrote the standard textbook of modern quantum field theory, S. Weinberg, *Quantum Field Theory*, Vols. 1–3, Cambridge University Press.

It seems like a strange notion for physicists to have come to after all these years of talking about particles and fields, and it would take too long to explain why we think this is not an unreasonable picture of nature, but perhaps I can summarize it in one sentence:

String theories incorporate gravitation.

In fact, not only do they incorporate it, you cannot have a string theory without gravitation. The graviton, the quantum of gravitational radiation, the particle which is transmitted when a gravitational force is exerted between two masses, is just the lowest mode of vibration of a fundamental closed string (closed meaning that it is a loop). Not only do they include and necessitate gravitation, but these string theories for the first time allow a description of gravitation on a microscopic quantum level which is free of mathematical inconsistencies.

All other descriptions of gravity broke down mathematically, gave nonsensical results when carried to very small distances or very high energies. String theory is our first chance at a reasonable theory of gravity which extends from the very large down to very small and as such, it is natural that we are all agog over it. String theory itself has focused the attention of physicists on branches of mathematics that most of us weren't fortunate enough to have learned when we were students. You can easily see that a string (just think of a little bit of cord) travelling through space, sweeps out a two-dimensional surface.

A very convenient description of string theory is to say that it is the theory of these two-dimensional Riemann surfaces.

The theory of two-dimensional surfaces is remarkably beautiful. There are ways of classifying all possible two-dimensional surfaces according to their handles on them and the number of boundaries, which simply don't exist in any higher dimension. The theory of two-dimensional surfaces is a branch of mathematics that when you get into it is one of the loveliest things you can learn. It was developed in the 19th century, starting with Riemann, and further developed by mathematicians working in the late 19th century motivated by problems in complex analysis, and then continuing into the 20th century.

There are mathematicians who have spent their whole lives working on this theory of two-dimensional surfaces, who have never heard of string theory (or at least not until very recently). Yet when the physicists started to figure out how to solve the dynamical problems of strings, and they realized what they had to do was to perform sums⁴⁹ over all possible two-dimensional surfaces in order to add up all the ways that reactions could occur, they found the mathematics just ready for their use, developed over the past 100 years.

String theory involves another branch of mathematics which goes back to group theory.

The equations which govern these surfaces have a very large group of symmetries, known as the conformal group.

One description of these symmetries is in terms of an algebraic structure representing all the possible group transformations, which is actually infinite dimensional. Mathematicians have been doing a lot of work developing

⁴⁹ These sums correspond to Feynman functional integrals.

the theory of these *infinite dimensional Lie algebras* which underlie symmetry groups, again without a clear motivation in terms of physics, and certainly without knowing anything about string theory. Yet when the physicists started to work on it, there it was.

Speaking quite personally, I have found it exhilarating at my stage of life to have to go back to school and learn all this wonderful mathematics. Some of us physicists have enjoyed our conversations with mathematicians, in which

We beg them to explain things to us in terms we can understand.

At the same time the mathematicians are pleased and somewhat bemused that we are paying attention to them after all these years. The mathematics department of the University of Texas at Austin now allows the physicists to use one of their lounges – which would have been unlikely in previous years.

Unfortunately, I must admit that there is no experimental evidence yet for string theory, and so, if theoretical physicists are spending more time talking to the mathematicians, they are spending less time talking to the experimentalists, which is not good.

1.7 The Seven Millennium Prize Problems of the Clay Mathematics Institute

At the Second World Congress of Mathematicians in Paris in 1900, in a seminal lecture, Hilbert formulated his famous 23 open problems.⁵⁰ The hundredth anniversary of Hilbert's lecture was celebrated in Paris, in the "Amphithéâtre" of the Collège de France, on May 24, 2000. The Scientific Advisory Board of the newly founded Clay Mathematics Institute (CMI) in Cambridge, Massachusetts, U.S.A., selected seven Millennium prize problems. The Scientific Advisory Board consists of Arthur Jaffe (director of the CMI, Harvard University, U.S.A.), Alain Connes (Institut des Hautes Études Scientifiques (IHÉS) and Collège de France), Andrew Wiles (Princeton University, U.S.A.), and Edward Witten (Institute for Advanced Study, Princeton, U.S.A.). The CMI explains its intention as follows:

Mathematics occupies a privileged place among the sciences. It embodies the quintessence of human knowledge, reaching into every field of human endeavor. The frontiers of mathematical understanding evolve today in deep and unfathomable ways. Fundamental advances go hand in hand with discoveries in all fields of science. Technological applications of mathematics underpin our daily life, including our ability to communicate thanks to cryptography and coding theory, our ability to navigate and to travel, our health and well-being, our security, and they also play a central role in our economy. The evolution of mathematics will remain a central tool to shaping civilization. To appreciate the scope of mathematical truth challenges the capabilities of the human mind.

⁵⁰ See F. E. Browder (Ed.) *Mathematical Developments Arising from Hilbert's Problems*, Amer. Math. Soc., New York, 1976, and B. Yandell, *The Honors Class: Hilbert's Problems and Their Solvers*, Peters, Natick, Massachusetts, 2001.

In order to celebrate mathematics in the new millennium, the CMI has named seven “Millennium prize problems”. The Scientific Advisory Board of the CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a \$ 7 million prize fund to these problems, with \$ 1 million allocated to each.

The seven Millennium prize problems read as follows:

- (i) The Riemann conjecture in number theory on the zeros of the Riemann zeta function and the asymptotics of prime numbers.
- (ii) The Birch and Swinnerton–Dyer conjecture in number theory on the relation between the size of the solution set of a Diophantine equation and the behavior of an associated zeta function near the critical point $s = 1$.
- (iii) The Poincaré conjecture in topology on the exceptional topological structure of the 3-dimensional sphere.
- (iv) The Hodge conjecture in algebraic geometry on the nice structure of projective algebraic varieties.
- (v) The Cook problem in computer sciences of deciding whether an answer that can be quickly checked with inside knowledge, may without such help require much longer to solve, no matter how clever a program we write.
- (vi) The solution of the turbulence problem for viscous fluids modelled by the Navier–Stokes partial differential equations.
- (vii) The rigorous mathematical foundation of a unified quantum field theory for elementary particles.

A detailed description of the problems can be found on the following Internet address:

http://www.claymath.org/prize_problems/

For a detailed discussion of the seven prize problems, we refer to K. Devlin, *The Millennium Problems: the Seven Greatest Unsolved Mathematical Puzzles of Our Time*, Basic Books, New York, 2002.

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Zeidler, E.

2006, XXIV, 1052 p., Hardcover

ISBN: 978-3-540-34762-0