4

The Three String Network

This chapter is devoted to study the control problem for the simplest non trivial network of strings that cannot be reduced to a single string: *the three string network*. Most of the results presented here will be generalized later in Chapter 5 to the case of general networks supported by tree-shaped graphs. However, the generality of the problem in that case involves complex notations. It is therefore convenient to first address the simple case of the three string network, for which the main ideas involved in our analysis, that will allow us to address the case of general networks, can be described more transparently.

We first consider the case when two of the three external nodes of the network are controlled. In this case, standard methods based on the d'Alembert formula and energy arguments allow showing that the observability and controllability properties hold in the optimal energy space. We then address the case when a single control acts on one exterior node. In this case the problem is much more complex since the space in which observability and controllability hold depend on the irrationallity properties of the ratios of the lengths of the strings entering in the network. The methods to analyze it also vary significantly and are based on results from Number Theory.

4.1 The Three String Network with Two Controlled Nodes

4.1.1 Equations of Motion of the Network

Let T, ℓ_0, ℓ_1, ℓ_2 be positive numbers. We consider the following non-homogeneous system

$$\begin{cases} u^i_{tt} - u^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_i],\ i=0,1,2, \\ u^0(t,0) = u^1(t,0) = u^2(t,0) & t\in\mathbb{R}, \\ u^0_x(t,0) + u^1_x(t,0) + u^2_x(t,0) = 0 & t\in\mathbb{R}, \\ u^i(t,\ell_i) = v^i(t), \quad u^2(t,\ell_2) = 0 & t\in\mathbb{R}, \qquad i=0,1, \\ u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x) & x\in[0,\ell_i], \quad i=0,1,2 \end{cases} \tag{4.1}$$

which models the vibrations of a network formed by three elastic strings $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ with lengths ℓ_0, ℓ_1, ℓ_2 coupled at one of their extremes. The functions $u^i = u^i(t,x) : [0,\ell_i] \to \mathbb{R}$, $i = 0,1,2$, represent the transversal displacements of the strings. On the free nodes of the strings $\mathbf{e}_0$ and $\mathbf{e}_1$ some external controls v^0 and v^1 act regulating their motion.

Let us observe that in (4.1), the parametrization of the strings has been chosen so that $x = 0$ corresponds to the common node, while $x = \ell_i$ correspond to the exterior nodes of the strings $\mathbf{e}_i$, $i = 1,2$.

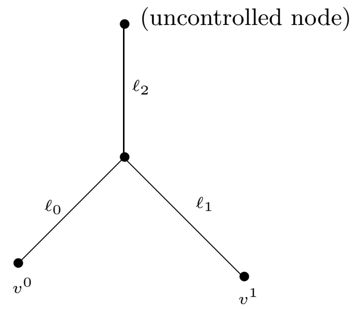

Fig. 4.1. The three string network with two controlled nodes

Let $T > 0$. According to the general results described in Chapter 2, the homogeneous system resulting in the absence of control in (4.1) ($v^0 = v^1 = 0$)

$$\begin{cases} \phi^i_{tt} - \phi^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_i],\ i=0,1,2, \\ \phi^0(t,0) = \phi^1(t,0) = \phi^2(t,0) & t\in\mathbb{R}, \\ \phi^0_x(t,0) + \phi^1_x(t,0) + \phi^2_x(t,0) = 0 & t\in\mathbb{R}, \\ \phi^i(t,\ell_i) = 0 & t\in\mathbb{R}, \qquad i=0,1,2, \\ \phi^i(0,x) = \phi^i_0(x), \quad \phi^i_t(0,x) = \phi^i_1(x) & x\in[0,\ell_i], \quad i=0,1,2, \end{cases} \tag{4.2}$$

has a unique solution $\bar{\phi}$ with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$ satisfying

$$\bar{\phi} \in C([0,T] : V) \bigcap C^1([0,T] : H). \tag{4.3}$$

Recall that the spaces V and H are those defined in Chapter 2, Section 2.2, that is

$$V = \left\{ \bar{\phi} \in \prod_{i=0}^{2} H^1(0,\ell_i) : \phi_0(0) = \phi_1(0) = \phi_2(0),\ \phi_i(\ell_i) = 0,\ i = 0,1,2 \right\},$$

$$H = \prod_{i=0}^{2} L^2(0,\ell_i).$$

The solution $\bar{\phi}$ of (4.2) is expressed in terms of the Fourier coefficients $(\phi_{0,n})$, $(\phi_{1,n})$ of the initial data in the orthonormal basis $(\bar{\theta}_n)$ formed by the eigenfunctions of the elliptic operator $-\Delta_G$ associated to the star-like network under consideration, by the formula

$$\bar{\phi}(t,x) = \sum_{n\in\mathbb{N}} (\phi_{0,n} \cos \lambda_n t + \frac{\phi_{1,n}}{\lambda_n} \sin \lambda_n t) \bar{\theta}_n(x). \tag{4.4}$$

The energy of $\bar{\phi}$, defined as the sum of the energies of the solutions on the strings, is constant in time and, according to the relation (2.25), it may be expressed as

$$\mathbf{E}_{\bar{\phi}} = \sum_{n\in\mathbb{N}} (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2). \tag{4.5}$$

On the other hand, for the non-homogeneous system (4.1), for every $v^0, v^1 \in L^2(0,T)$, there exists a unique solution, defined by transposition that satisfies

$$\bar{u} \in C([0,T]:H) \bigcap C^1([0,T]:V').$$

Here and in the sequel V' denotes the dual of V.

4.1.2 The Control Problem

The control problem in time T for system (4.1) consists in *characterizing the initial states* $(\bar{u}_0, \bar{u}_1) \in H \times V'$ *of the network for which there exist controls* $v^0, v^1 \in L^2(0,T)$ *such that the corresponding solution of* (4.1) *satisfies*

$$\bar{u}(T) = \bar{u}_t(T) = \bar{0}.$$

The control of a three string network from two exterior nodes satisfies the hypotheses of Theorem 2.7. In this case it holds

Theorem 4.1. *System (4.1) is exactly controllable in time*

$$T^* = 2(\ell_2 + \max\{\ell_0, \ell_1\}).$$

Proof. Let us assume that $\ell_0 \geq \ell_1$, such that $T^* = 2(\ell_0 + \ell_2)$. In view of Proposition 2.5, the initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T with controls $v^0, v^1 \in L^2(0,T)$ if, and only if,

$$\int_0^{T^*} \phi_x^0(t,\ell_0)v^0(t)dt + \int_0^{T^*} \phi_x^1(t,\ell_1)v^1(t)dt = -\langle \bar{u}_0, \bar{\phi}_1\rangle_H + \langle \bar{u}_1, \bar{\phi}_0\rangle_{V'\times V},$$

for every solution $\bar{\phi}$ of system (4.2) with initial data $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$. Corollary 3.16 of Theorem 3.4 allows us to ensure that system (4.1) is exactly controllable in time T if, and only if, there exists a constant $C > 0$ such that

$$\int_0^{T^*} |\phi_x^0(t,\ell_0)|^2 dt + \int_0^{T^*} |\phi_x^1(t,\ell_1)|^2 dt \geq C\mathbf{E}_{\bar{\phi}}, \tag{4.6}$$

for every solution $\bar{\phi}$ of the homogeneous system (4.2) with initial state in $Z \times Z$.

In order to prove inequality (4.6), in view of the property of conservation of energy, it suffices to find $\hat{t} \in \mathbb{R}$ such that

$$\int_0^{2(\ell_0+\ell_2)} \left(\left|\phi_x^0(t,\ell_0)\right|^2 + \left|\phi_x^1(t,\ell_1)\right|^2 \right) dt \geq C\mathbf{E}_{\phi^i}(\hat{t}), \qquad i = 0, 1, 2. \tag{4.7}$$

Thanks to Proposition 3.1 estimate (4.7) holds immediately for $i = 0, 1$ (that is, for the components of the solution corresponding to the controlled strings) if $\hat{t} \in [\ell_0, 2\ell_2 + \ell_0]$.

It remains to recover an estimate of the energy of the string corresponding to $i = 2$. The idea is very simple: the d'Alembert formula allows proving the identities

$$\phi_x^0(t,0) = \ell_0^+ \phi_x^0(t,\ell_0), \qquad \phi_t^0(t,0) = \ell_0^- \phi_x^0(t,\ell_0), \tag{4.8}$$

$$\phi_x^1(t,0) = \ell_1^+ \phi_x^1(t,\ell_1), \qquad \phi_t^1(t,0) = \ell_1^- \phi_x^1(t,\ell_1). \tag{4.9}$$

Moreover, in account of the transmission conditions in the common node, we have

$$\phi_t^2(t,0) = \phi_t^1(t,0) = \ell_1^- \phi_x^1(t,\ell_1),$$

$$\phi_x^2(t,0) = -\left(\phi_x^0(t,0) + \phi_x^1(t,0)\right) = \left(\ell_0^+ \phi_x^0(t,\ell_0) + \ell_1^+ \phi_x^1(t,\ell_1)\right).$$

Then, according to Proposition 3.1,

$$\begin{aligned}\mathbf{E}_{\phi^2}(\hat{t}) &\leq \int_{\hat{t}-\ell_2}^{\hat{t}+\ell_2} \left(\left|\phi_t^2(t,0)\right|^2 + \left|\phi_x^2(t,0)\right|^2 \right) dt \\ &= \int_{\hat{t}-\ell_2}^{\hat{t}+\ell_2} \left(\left|\ell_1^- \phi_x^1(t,\ell_1)\right|^2 + \left|\ell_0^+ \phi_x^0(t,\ell_0) + \ell_1^+ \phi_x^1(t,\ell_1)\right|^2 \right) dt.\end{aligned}$$

From this inequality and applying Proposition 3.3 we obtain

$$C\mathbf{E}_{\phi^2}(\hat{t}) \leq \int_{\hat{t}-\ell_2-\ell_1}^{\hat{t}+\ell_2+\ell_1} \left|\phi_x^1(t,\ell_1)\right|^2 dt + \int_{\hat{t}-\ell_0-\ell_2}^{\hat{t}+\ell_0+\ell_2} \left|\phi_x^0(t,\ell_0)\right|^2 dt,$$

and thus, choosing $\hat{t} = \ell_0 + \ell_2$, all the inequalities (4.7) are verified.

Remark 4.2. It is clear that the same procedure would work in the case of a general tree-shaped network controlled from all of its exterior nodes, except one: it suffices to apply an induction argument.

The application of the d'Alembert formula and Proposition 3.1 allow to estimate the norms

$$\int_{\beta+\ell}^{\alpha-\ell} |\phi_x(t,\ell)|^2 \, dt, \qquad \int_{\beta+\ell}^{\alpha-\ell} |\phi_t(t,\ell)|^2 \, dt$$

of the traces ϕ_x and ϕ_t in the extreme $x = \ell$ from the norms

$$\int_{\beta}^{\alpha} |\phi_x(t,0)|^2 \, dt, \qquad \int_{\beta}^{\alpha} |\phi_t(t,0)|^2 \, dt$$

of the traces ϕ_x and ϕ_t in the extreme $x = 0$, provided ϕ solves the wave equation in the space interval $0 < x < \ell$. Thus, in the case of a general tree-shaped network, we start from the controlled nodes (for which $\phi = 0$ and consequently $\phi_t = 0$ and $\phi_x \in L^2(0,T)$) and apply this argument up to the first interior node. Because of the tree-like structure, this provides estimates of the traces ϕ_x and ϕ_t of all the components that are coupled in that node, except for one of them. The two coupling conditions on that node allow then to obtain estimates of the traces of ϕ_t and ϕ_x for the remaining string corresponding to that node. This argument allows getting estimates on all the interior nodes that can be joined to the controlled exterior ones by one single string: the first layer of interior nodes. One can iterate this argument to get control of all the strings of the network. This yields the result by G. Schmidt [108] in Section 2.4.

4.2 A Simpler Problem: Simultaneous Control of Two Strings

The simultaneous control problem of two strings $\mathbf{e}_1$ and $\mathbf{e}_2$ of lengths ℓ_1 and ℓ_2 is similar to the previous one. It was implicitly studied in [57]. Later, in [110] and [14] an essentially complete solution was obtained. The results of [110] are based on a generalization of the Ingham inequality proved in [57]. This technique, however, allowed only to guarantee the controllability of the system in a time larger than the optimal one. In [14] the method of moments was used; this method provided the optimal control time. Here we describe a different method, based on elementary arguments, which in addition provides more information than the previously mentioned techniques.

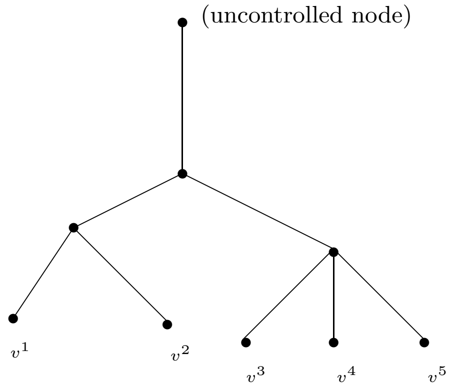

Fig. 4.2. Tree-shaped network with one uncontrolled node

The system corresponding to the simultaneous control of two strings is

$$\begin{cases} u^i_{tt} - u^i_{xx} = 0 & (t,x) \in \mathbb{R}\times[0,\ell_i], \\ u^i(t,\ell_i) = 0, \quad u^i(t,0) = v(t) & t \in \mathbb{R}, \\ u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x) & x \in [0,\ell_i], \end{cases} \tag{4.10}$$

for $i = 1, 2$. In this case the system is constituted by two uncoupled wave equations and the term *simultaneous* refers to the fact that the control v applied to both strings is the same. Chapter 5 of [78] is devoted to this sort of problems, considered for the first time by Russell in [107].

For every $T > 0$ system (4.10) is well posed for initial states $(u^i_0, u^i_1) \in L^2(0,\ell_i) \times H^{-1}(0,\ell_i)$, $i = 1, 2$ and control $v \in L^2(0,T)$: there exists a unique solution satisfying

$$u^i \in C([0,T] : L^2(0,\ell_i)) \cap C^1([0,T] : H^{-1}(0,\ell_i)), \quad i = 1, 2.$$

When $v \equiv 0$ system (4.10) becomes

$$\begin{cases} \phi^i_{tt} - \phi^i_{xx} = 0 & (t,x) \in \mathbb{R}\times[0,\ell_i], \\ \phi^i(t,\ell_i) = \phi^i(t,0) = 0 & t \in \mathbb{R}, \\ \phi^i(0,x) = \phi^i_0(x), \quad \phi^i_t(0,x) = \phi^i_1(x) & x \in [0,\ell_i], \end{cases} \tag{4.11}$$

with $i = 1, 2$. It is constituted by two wave equations with homogeneous Dirichlet boundary conditions, which are *uncoupled*. Both equations are well posed for $(\phi^i_0, \phi^i_1) \in H^1_0(0,\ell_i) \times L^2(0,\ell_i)$ and the corresponding solutions are expressed by the formula

$$\phi^i(t,x) = \sum_{n\in\mathbb{N}} (\phi^i_{0,n} \cos\sigma^i_n t + \frac{\phi^1_{1,n}}{\sigma^i_n} \sin\sigma^i_n t) \sin\sigma^i_n x, \quad i = 1, 2, \tag{4.12}$$

where (σ_n^i) is the sequence of the square roots of the eigenvalues of the string $\mathbf{e}_i$:

$$\sigma_n^i = \frac{n\pi}{\ell_i}, \quad n \in \mathbb{N},$$

and $(\phi_{0,n}^i)$, $(\phi_{1,n}^i)$ are the sequences of the Fourier coefficients of ϕ_0^i, ϕ_1^i, respectively, in the orthogonal basis $(\sin \sigma_n^i x)$ of $L^2(0, \ell_i)$:

$$\phi_0^i(x) = \sum_{n\in\mathbb{N}} \phi_{0,n}^i \sin \sigma_n^i x, \qquad \phi_1^i(x) = \sum_{n\in\mathbb{N}} \phi_{1,n}^i \sin \sigma_n^i x, \qquad i = 1, 2.$$

The control problem in time T consists in *characterizing the initial states* (u_0^i, u_1^i)*,* $i = 1, 2$*, of system* (4.10) *such that there exists* $v \in L^2(0,T)$ *with the property that the solutions* u^1, u^2 *of* (4.10) *satisfy*

$$u^i(T,x) = u_t^i(T,x) = 0, \qquad i = 1, 2,$$

for $x \in [0, \ell_i]$.

Let us observe that, though system (4.11) is constituted by two *uncoupled* equations, the fact that the same control is used generates coupling conditions, similar to those arising in the three string network. In fact, if we apply HUM, it turns out that the simultaneous control problem is equivalent to proving the following observability inequality for (4.11)

$$\int_0^T |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq \sum_{i=1,2} \sum_{n\in\mathbb{N}} (c_n^i)^2 ((\sigma_n^i \phi_{0,n}^i)^2 + (\phi_{1,n}^i)^2). \tag{4.13}$$

If there exist sequences of positive numbers (c_n^i), $i = 1, 2$ such that (4.13) is verified by all the solutions ϕ^1, ϕ^2 of (4.11) with initial states $(\phi_0^1, \phi_1^1) \in Z^1 \times Z^1$, $(\phi_0^2, \phi_1^2) \in Z^2 \times Z^2$, respectively, then, all the initial states (u_0^i, u_1^i), $i = 1, 2$, satisfying

$$\sum_{n\in\mathbb{N}} \frac{1}{(c_n^i)^2} (u_{0,n}^i)^2 + \sum_{n\in\mathbb{N}} \frac{1}{(\sigma_n^i c_n^i)^2} (u_{1,n}^i)^2 < \infty$$

are controllable in time T.

Note that in inequality (4.13) the observed quantity is a combination of the derivatives ϕ_x^i at $x = 0$.

Our main observability result for the solutions of (4.11) is as follows:

Theorem 4.3. *Let* $T^* = 2(\ell_1 + \ell_2)$*. The following inequalities take place*

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq \ell_1 \sum_{n\in\mathbb{N}} \left(\sin \sigma_n^1 \ell_2\right)^2 \left((\sigma_n^1 \phi_{0,n}^1)^2 + (\phi_{1,n}^1)^2\right),$$

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq \ell_2 \sum_{n\in\mathbb{N}} \left(\sin \sigma_n^2 \ell_1\right)^2 \left((\sigma_n^2 \phi_{0,n}^2)^2 + (\phi_{1,n}^2)^2\right),$$

for any solution of (4.11) with initial states $(\phi_0^i, \phi_1^i) \in Z^i \times Z^i$*,* $i = 1, 2$*.*

Proof. We prove the second inequality; the first one can be proved in a similar way.

Let us observe that, due to the $2\ell_1$-periodicity in time of the component ϕ^1 of the solution of (4.11), it follows that $\ell_1^- \phi_x^1(t,0) = 0$, where ℓ_1^- is the operator defined by (3.7) corresponding to ℓ_1. Then, if we apply Proposition 3.3 we obtain

$$\begin{aligned}\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt &\geq \int_{\ell_1}^{T^*-\ell_1} |\ell_1^- \phi_x^1(t,0) + \ell_1^- \phi_x^2(t,0)|^2 dt \\ &= \int_{\ell_1}^{T^*-\ell_1} |\ell_1^- \phi_x^2(t,0)|^2 dt. \end{aligned} \tag{4.14}$$

On the other hand, $\psi = \ell_1^- \phi^2$ is a solution of the equation

$$\psi_{tt} - \psi_{xx} = 0$$

in $\mathbb{R} \times [0, \ell_2]$ and thus, from Proposition 3.1, it follows

$$\int_{\ell_2}^{T^*-\ell_2} |\psi_x(t,0)|^2 dt \geq 4\mathbf{E}_\psi. \tag{4.15}$$

Taking into account that $\psi_x(t,0) = \ell_1^- \phi_x^2(t,0)$, from (4.14) and (4.15) we obtain

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq 4\mathbf{E}_{\ell_1^- \phi^2}. \tag{4.16}$$

It just remains to compute the energy $\mathbf{E}_{\ell_1^- \phi^2}$. From (4.12) we obtain that

$$\ell_1^- \phi^2(t,x) = \sum_{n\in\mathbb{N}} (\phi_{0,n}^2 \ell_1^- \cos \sigma_n^2 t + \frac{\phi_{1,n}^2}{\sigma_n^2} \ell_1^- \sin \sigma_n^2 t) \sin \sigma_n^2 x. \tag{4.17}$$

In view of the relations

$$\begin{aligned}\ell_1^- \cos \sigma_n^2 t &= \frac{1}{2}\left(\cos \sigma_n^2 (t+\ell_1) - \cos \sigma_n^2 (t-\ell_1)\right) = -\sin \sigma_n^2 \ell_1 \sin \sigma_n^2 t, \\ \ell_1^- \sin \sigma_n^2 t &= \frac{1}{2}\left(\sin \sigma_n^2 (t+\ell_1) - \sin \sigma_n^2 (t-\ell_1)\right) = \sin \sigma_n^2 \ell_1 \cos \sigma_n^2 t,\end{aligned}$$

equality (4.17) becomes

$$\ell_1^- \phi^2(t,x) = \sum_{n\in\mathbb{N}} \sin \sigma_n^2 \ell_1 \left(\frac{\phi_{1,n}^2}{\sigma_n^2} \cos \sigma_n^2 t - \phi_{0,n}^2 \sin \sigma_n^2 t \right) \sin \sigma_n^2 x.$$

If we apply formula (2.25) for the energy it follows

$$\mathbf{E}_{\ell_1^- \phi^2} = \frac{\ell_2}{4} \sum_{n\in\mathbb{N}} \left(\sin \sigma_n^2 \ell_1\right)^2 \left((\sigma_n^2 \phi_{0,n}^1)^2 + (\phi_{1,n}^2)^2 \right).$$

It suffices to replace the latter expression in (4.16) to obtain the observability inequality of the theorem.

Remark 4.4. It is important to remark that the results of Theorem 4.3 are not enhanced if we take a larger observation time. Indeed, due to the ℓ_1-periodicity of ϕ^1 and the ℓ_2-periodicity of ϕ^2 in time we have

$$\ell_1^- \ell_2^- \left(\phi_x^1(t,0) + \phi_x^2(t,0)\right) = 0$$

for every $t \in \mathbb{R}$. This implies that, for every $T \geq T^*$ there exists a constant $C_T > 0$ such that

$$\int_0^T \left|\phi_x^1(t,0) + \phi_x^2(t,0)\right|^2 dt \leq C_T \int_0^{T^*} \left|\phi_x^1(t,0) + \phi_x^2(t,0)\right|^2 dt.$$

Consequently, if

$$\int_0^T \left|\phi_x^1(t,0) + \phi_x^2(t,0)\right|^2 dt$$

defines a norm in the space of initial states for system (4.11) for some $T \geq T^*$, so does

$$\int_0^{T^*} \left|\phi_x^1(t,0) + \phi_x^2(t,0)\right|^2 dt$$

and both norms are equivalent.

4.2.1 Identification of Controllable Subspaces

The aim of this subsection is to identify subspaces of controllable initial data of system (4.10) in time $T \geq 2(\ell_1 + \ell_2)$ with the aid of Theorem 4.3.

An easily identifiable subspace is that of the finite linear combinations of the eigenfunctions. The following holds

Proposition 4.5. *System (4.10) is spectrally controllable in some time $T \geq 2(\ell_1 + \ell_2)$ if, and only if, the quotient ℓ_1/ℓ_2 is an irrational number.*

Proof. If ℓ_1/ℓ_2 is irrational then the coefficients $\sin \sigma_n^1 \ell_2$, $\sin \sigma_n^2 \ell_1$, $n \in \mathbb{N}$, appearing in the inequalities in Proposition 4.3 are all different from zero. Indeed, if $\sin \sigma_n^1 \ell_2 = 0$ for some n, then there would exist $k \in \mathbb{N}$ such that

$$\frac{n\pi}{\ell_1} \ell_2 = k\pi,$$

that is, $\ell_1/\ell_2 = n/k \in \mathbb{Q}$. Then, the initial states (u_0^i, u_1^i), $i = 1, 2$, satisfying

$$\sum_{n \in \mathbb{N}} \frac{1}{(\sin \sigma_n^1 \ell_2)^2} (u_{0,n}^1)^2 + \sum_{n \in \mathbb{N}} \frac{1}{(\sigma_n^1 \sin \sigma_n^1 \ell_2)^2} (u_{1,n}^1)^2 < \infty, \qquad (4.18)$$

$$\sum_{n \in \mathbb{N}} \frac{1}{(\sin \sigma_n^2 \ell_1)^2} (u_{0,n}^2)^2 + \sum_{n \in \mathbb{N}} \frac{1}{(\sigma_n^2 \sin \sigma_n^2 \ell_1)^2} (u_{1,n}^2)^2 < \infty, \qquad (4.19)$$

are controllable in time $T \geq 2(\ell_1 + \ell_2)$. In particular, the initial states $(u_0^1, u_1^1) \in Z^1 \times Z^1$, $(u_0^2, u_1^2) \in Z^2 \times Z^2$ are controllable.

Let us now see that the condition $\ell_1/\ell_2 \notin \mathbb{Q}$ is also necessary for approximate controllability and consequently for spectral controllability. If $\ell_1/\ell_2 = n/k$ with $n, k \in \mathbb{N}$ then, for every $p \in \mathbb{N}$ the functions

$$\phi^1(t,x) = \sin\frac{pn\pi t}{\ell_1}\sin\frac{pn\pi x}{\ell_1}, \qquad \phi^2(t,x) = -\sin\frac{pk\pi t}{\ell_2}\sin\frac{pk\pi x}{\ell_2},$$

are solutions of (4.11) and satisfy

$$\phi_x^1(t,0) + \phi_x^2(t,0) \equiv 0.$$

Consequently, system (4.10) is not approximately controllable and, in particular, is not spectrally controllable.

To further pursue in the identification of controllable initial states of system (4.10) with the aid of Theorem 4.3 we need some definitions from Number Theory. For $\eta \in \mathbb{R}$ we denote by $|||\eta|||$ the distance from η to the set $\mathbb{Z}$:

$$|||\eta||| = |\min\{x \in \mathbb{R} : \ \eta - x \in \mathbb{Z}\}| \, .$$

Proposition 4.6. *If ℓ_1/ℓ_2 is irrational, then all the initial states (u_0^1, u_1^1), (u_0^2, u_1^2) satisfying*

$$\sum_{n\in\mathbb{N}} \frac{1}{|||n\frac{\ell_2}{\ell_1}|||^2}(u_{0,n}^1)^2 + \sum_{n\in\mathbb{N}} \frac{1}{n^2|||n\frac{\ell_2}{\ell_1}|||^2}(u_{1,n}^1)^2 < \infty, \tag{4.20}$$

$$\sum_{n\in\mathbb{N}} \frac{1}{|||n\frac{\ell_1}{\ell_2}|||^2}(u_{0,n}^2)^2 + \sum_{n\in\mathbb{N}} \frac{1}{n^2|||n\frac{\ell_1}{\ell_2}|||^2}(u_{1,n}^2)^2 < \infty, \tag{4.21}$$

are controllable in time $T \geq 2(\ell_1 + \ell_2)$.

Proof. Let us observe that for each $x \in \mathbb{R}$

$$2|||\frac{x}{\pi}||| \leq |\sin x| \leq \pi|||\frac{x}{\pi}||| \tag{4.22}$$

(the proof of this fact may be found in Proposition A.1 in Appendix A).

Then,

$$2|||n\frac{\ell_2}{\ell_1}||| \leq \left|\sin\sigma_n^1\ell_2\right| \leq \pi|||n\frac{\ell_2}{\ell_1}|||, \qquad 2|||n\frac{\ell_1}{\ell_2}||| \leq \left|\sin\sigma_n^2\ell_1\right| \leq \pi|||n\frac{\ell_1}{\ell_2}|||.$$

Thus, relations (4.20)-(4.21) are equivalent to (4.18)-(4.19).

Therefore, in order to characterize subspaces of controllable initial states for (4.10) it suffices to estimate the norms of the sequences $|||n\frac{\ell_2}{\ell_1}|||$, $|||n\frac{\ell_1}{\ell_2}|||$, $n \in \mathbb{N}$.

A natural way of getting additional information is the following: let $\rho : \mathbb{R} \to \mathbb{R}_+$ be an increasing function and define

$$\mathbf{\Psi}_\rho = \left\{ x \in \mathbb{R}_+ : \liminf_{n\to\infty} |||nx|||\rho(n) > 0 \right\}.$$

Then, if $\ell_1/\ell_2, \ell_2/\ell_1 \in \Psi_\rho$ the inequalities

$$\sum_{n\in\mathbb{N}} \rho^2(n)(u^1_{0,n})^2 + \sum_{n\in\mathbb{N}} \frac{\rho^2(n)}{n^2}(u^1_{1,n})^2 < \infty, \tag{4.23}$$

$$\sum_{n\in\mathbb{N}} \rho^2(n)(u^2_{0,n})^2 + \sum_{n\in\mathbb{N}} \frac{\rho^2(n)}{n^2}(u^2_{1,n})^2 < \infty, \tag{4.24}$$

guarantee the controllability of the initial state (u^1_0, u^1_1), (u^2_0, u^2_1).

In what follows we restrict ourselves to the case $\rho(x) = x^\alpha$ with $\alpha > 0$. This choice is motivated by two reasons. The first one is that it leads to the identification of subspaces of controllable initial states of the form

$$(u^i_0, u^i_1) \in \hat{H}^\alpha(0, \ell_i) \times \hat{H}^{\alpha-1}(0, \ell_i),$$

where

$$\hat{H}^s(0, \ell_i) = \left\{ u(x) = \sum_{n\in\mathbb{N}} u_n \sin \sigma^i_n x : \sum_{n\in\mathbb{N}} n^{2s}(u_n)^2 < \infty \right\}.$$

Let us note that $\hat{H}^s(0, \ell_i)$ is the domain of the $s/2$-power of the laplacian and it is a closed subspace of the Sobolev space $H^s(0, \ell_i)$ with certain additional boundary conditions. In particular, $\hat{H}^1(0, \ell_i) = H^1_0(0, \ell_i)$ and $\hat{H}^0(0, \ell_i) = L^2(0, \ell_i)$.

The second reason for this choice of the function ρ is that the problem of describing the sets

$$\Psi_\alpha := \mathbf{\Psi}_{x^\alpha} = \left\{ x \in \mathbb{R}_+ : \liminf_{n\to\infty} |||nx|||n^\alpha > 0 \right\},$$

is a classical and difficult one in Number Theory. In [69] and [26] the reader may find information on this topic. We refer also to Appendix A where we have gathered the most relevant facts, which will be used in what follows.

We now summarize the main consequences of our analysis distinguishing positive and negative results:

Positive results

The following results are known

1) For every $\alpha > 0$ the sets Ψ_α have the following property: *if* $\xi \in \Psi_\alpha$ *then* $1/\xi \in \Psi_\alpha$.

2) Ψ_1 coincides with the set of irrational numbers $\eta \in \mathbb{R}$ having a continuous fraction expansion $[a_0, a_1, ..., a_n, ...]$ (see, e.g., [69], p. 6) with bounded (a_n). The set Ψ_1 is not countable but has zero Lebesgue measure.

3) For every $\varepsilon > 0$ the complement of the set $\Psi_{1+\varepsilon}$ is of measure zero (see Proposition A.5 in Appendix A). This set is usually denoted in the literature as $\mathbf{B}_\varepsilon \subset \mathbb{R}$. As a consequence of Roth's theorem (Theorem A.4), the set $\mathbf{B}_\varepsilon$ contains all the algebraic irrational numbers, that is, all the roots of polynomials of degree greater than one with integer coefficients.

As a consequence we obtain

Corollary 4.7. *a) If $\ell_1/\ell_2 \in \mathbf{B}_\varepsilon$ then the subspace of initial states*

$$(u_0^i, u_1^i) \in \hat{H}^{1+\varepsilon}(0, \ell_i) \times \hat{H}^{\varepsilon}(0, \ell_i),$$

is controllable in any time $T \geq 2(\ell_1+\ell_2)$. In particular, if ℓ_1/ℓ_2 is an algebraic irrational number, this subspace is controllable for any $\varepsilon > 0$.

b) If ℓ_1/ℓ_2 admits a bounded expansion in continuous fraction then, the subspace of initial states $(u_0^i, u_1^i) \in H_0^1(0, \ell_i) \times L^2(0, \ell_i)$, is controllable in any time $T \geq 2(\ell_1 + \ell_2)$.

Remark 4.8. Note that, in both cases, the space of controllable data is at most $H_0^1(0, \ell_i) \times L^2(0, \ell_i)$. It is important to remark that this space is smaller by one derivative than the controllable space for each individual string $L^2(0, \ell_i) \times H^{-1}(0, \ell_i)$. Therefore we see that, even though simultaneous controllability is possible under suitable assumptions on the legths ℓ_1, ℓ_2, it necessarily holds in a weaker space, with a loss of one derivative.

Negative results

We now describe some results on the lack of controllability that may be obtained as a consequence of the characterization in Proposition 4.6.

Proposition 4.9. *If there exists a sequence $(n_k) \subset \mathbb{N}$ such that*

$$|||n_k \frac{\ell_1}{\ell_2}|||\rho(n_k) \to 0 \quad or \quad |||n_k \frac{\ell_2}{\ell_1}|||\rho(n_k) \to 0, \qquad k \to \infty,$$

then, there exist initial states (u_0^1, u_1^1), (u_0^2, u_1^2) satisfying (4.23)-(4.24) which are not controllable in any finite time T.

Proof. Recall that the fact that all the initial states satisfying (4.23)-(4.24) are controllable in time T is equivalent to the inequalities:

$$\int_0^T |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq C_1 \sum_{n\in\mathbb{N}} \frac{1}{\rho^2(n)} \left((\frac{n\pi}{\ell_1}\phi_{0,n}^1)^2 + (\phi_{1,n}^1)^2 \right), \tag{4.25}$$

$$\int_0^T |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq C_2 \sum_{n\in\mathbb{N}} \frac{1}{\rho^2(n)} \left((\frac{n\pi}{\ell_2}\phi_{0,n}^2)^2 + (\phi_{1,n}^2)^2 \right), \tag{4.26}$$

for any solution of (4.11) with initial states $(\phi_0^i, \phi_1^i) \in Z^i \times Z^i$, $i = 1, 2$.

When

$$|||n_k \frac{\ell_1}{\ell_2}|||\rho(n_k) \to 0 \tag{4.27}$$

inequality (4.26) is impossible. Indeed, from (4.27) it holds that, for every $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that

$$\left| n_k \frac{\ell_1}{\ell_2} - m_k \right| \rho(n_k) \to 0.$$

Then,

$$\left| \sigma_{n_k}^2 - \sigma_{m_k}^1 \right| \rho(n_k) = \left| \frac{\pi n_k}{\ell_2} - \frac{\pi m_k}{\ell_1} \right| \rho(n_k) \to 0. \tag{4.28}$$

On the other hand, after replacing in (4.26) the solutions

$$\phi_k^1(t,x) = \cos \sigma_{m_k}^1 t \sin \sigma_{m_k}^1 x, \qquad \phi_k^2(t,x) = -\cos \sigma_{n_k}^2 t \sin \sigma_{n_k}^2 x,$$

it holds

$$\int_0^T |\sigma_{m_k}^1 \cos \sigma_{m_k}^1 t - \sigma_{n_k}^2 \cos \sigma_{n_k}^2 t|^2 dt \geq C_2 \rho^{-2}(n_k)(\sigma_{n_k}^2)^2$$

and then

$$\left| \sigma_{n_k}^2 - \sigma_{m_k}^1 \right|^2 \geq C \rho^{-2}(n_k). \tag{4.29}$$

Here we have used the inequality

$$\int_0^T |x \cos xt - y \cos yt|^2 dt \leq 4|x-y|^2 x^2 T,$$

for $y \geq x \geq 1$, which is easily obtained using, for example, the mean value theorem.

Thus, from (4.29) we obtain

$$\left| \sigma_{n_k}^2 - \sigma_{m_k}^1 \right| \rho(n_k) \geq C,$$

what contradicts (4.28).

The first important consequence of Proposition 4.9 is based on the Dirichlet theorem: *for all $\alpha < 1$, $\xi \in \mathbb{R}$ and $\varepsilon > 0$ there exists an infinite number of values of n such that $|||n\xi|||n^\alpha < \varepsilon$* (see [26], Section I.5).

Corollary 4.10. *For all values ℓ_1, ℓ_2 of the lengths of the strings and every $\alpha < 1$ there exist initial states*

$$(u_0^i, u_1^i) \in \hat{H}^\alpha(0, \ell_i) \times \hat{H}^{\alpha-1}(0, \ell_i), \qquad i = 1, 2,$$

which are not controllable in any finite time T. In particular, there exist non-controllable initial states in $L^2(0, \ell_i) \times H^{-1}(0, \ell_i)$, and, consequently, system (4.10) is not exactly controllable in any finite time.

Remark 4.11. According to this corollary the positive result in Corollary 4.7 is sharp since, whatever ℓ_1/ℓ_2 is, the space of controllable data is at most $\hat{H}^1 \times \hat{L}^2$ and, therefore, there is a loss of, at least, one derivative.

However, there are irrational values of ℓ_1/ℓ_2 for which the space of controllable data can be as small as one wishes. The following result of negative character is based on a construction due to Liouville.

Let us consider the series

$$\xi = \sum_{k \in \mathbb{N}} 10^{-a_k}, \tag{4.30}$$

where (a_k) is an increasing sequence of natural numbers. Then, for each $p \in \mathbb{N}$,

$$\min_{m \in \mathbb{Z}} |\xi 10^{a_p} - m| = 10^{a_p} \sum_{k>p} 10^{-a_k} < 10^{a_p - a_{p+1}} \sum_{k \geq 0} 10^{-k} < 10^{1+a_p-a_{p+1}}. \tag{4.31}$$

Let us assume that $\rho : \mathbb{R} \to \mathbb{R}_+$ is an increasing function. Fix $\varepsilon > 0$ and choose a sequence (a_k) that verifies, for every $k \in \mathbb{N}$,

$$10^{1+a_k-a_{k+1}} < \frac{\varepsilon}{\rho(10^{a_k})}, \tag{4.32}$$

or equivalently,

$$a_{k+1} > 1 + a_k + \lg \frac{\rho(10^{a_k})}{\varepsilon}.$$

Then, in view of (4.31) and (4.32) the number ξ defined by (4.30) satisfies

$$\min_{m \in \mathbb{Z}} |\xi 10^{a_p} - m| < \frac{\varepsilon}{\rho(10^{a_p})}.$$

Thus, for the sequence of natural numbers $n_p = 10^{a_p}$, $p \in \mathbb{N}$, it holds

$$|||n_p \xi|||\rho(n_p) < \varepsilon.$$

Summarizing, it is possible to construct real numbers ξ, which are approximated by rational ones faster than any given order ρ.

From Proposition 4.9 it follows:

Corollary 4.12. *For any increasing function $\rho : \mathbb{R} \to \mathbb{R}_+$, there exist values of the lengths ℓ_1, ℓ_2 of the strings and initial data in the subspace defined by (4.23)-(4.24), which are not controllable in any finite time T. In other words, the subspace of controllable initial states may be arbitrarily small.*

Remark 4.13. Numbers of the form (4.30) are the so-called Liouville's numbers. The discovery of such numbers had a transcendental importance in the history of Mathematics: Liouville proved that, if ξ is an algebraic irrational number of order p (that is, ξ is a root of a polynomial of degree p with rational coefficients and there is no polynomial of smaller degree having that property) then, the inequality

$$|\xi n - m| < \frac{1}{n^{p-1}}$$

has no solutions $m, n \in \mathbb{N}$. Therefore, the numbers defined by (4.30) are not algebraic. This fact allowed to show for the first time the existence of non algebraic numbers.

Remark 4.14. It is important to remark the elementary character of the proof of Theorem 4.3 if we compare it with the previously mentioned approaches based on generalized Ingham inequalities. We do not need to analyze the spectral gap and to use Ingham type inequalities. Moreover, our technique provides the optimal observation time.

4.3 The Three String Network with One Controlled Node

The rest of this chapter is devoted to the study of the control problem for the three string network with a single exterior controlled node. It can be reduced to the previously studied problem on the simultaneous control of two strings (see also Section 4.8). We prefer however to develop another method to directly address the three string network since it also works for general tree-shaped networks, a topic that we shall develop in Chapter 5.

The motion of the network is described by the system

$$\begin{cases} u^i_{tt} - u^i_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_i],\ i = 0,1,2, \\ u^0(t,0) = u^1(t,0) = u^2(t,0) & t \in \mathbb{R} \\ u^0_x(t,0) + u^1_x(t,0) + u^2_x(t,0) = 0 & t \in \mathbb{R} \\ u^0(t,\ell_0) = v(t), \quad u^i(t,\ell_i) = 0 & t \in \mathbb{R} \qquad i = 1,2, \\ u^i(0,x) = u^i_0(x), \quad u^i_t(0,x) = u^i_1(x) & x \in [0,\ell_i], \quad i = 0,1,2, \end{cases} \tag{4.33}$$

which coincides with (4.1), except by the fact that now $v^1 = 0$. In other words, one single control is now entering on the system.

Let us observe that the homogeneous version of system (4.33), that is, when $v = 0$, coincides with (4.2).

From the general results of Chapter 2 we know that the observability inequality

$$\int_0^T |\phi^0_x(t,\ell_0)|^2 dt \geq \sum_{n\in\mathbb{N}} c_n^2 (\mu_n \phi^2_{0,n} + \phi^2_{1,n}), \tag{4.34}$$

for every solution $\bar{\phi}$ of (4.2) with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$, and a sequence (c_n) of positive weights, is equivalent to the fact that the space of initial states $(\bar{u}_0, \bar{u}_1)$ verifying

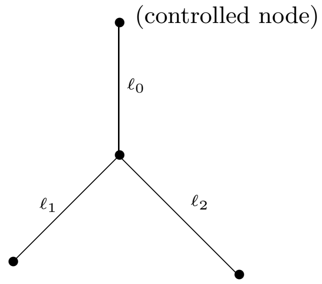

Fig. 4.3. The three string network with a single controlled node

$$\sum_{n\in\mathbb{N}} \frac{u_{0,n}^2}{c_n^2} < \infty, \qquad \sum_{n\in\mathbb{N}} \frac{u_{1,n}^2}{c_n^2 \mu_n} < \infty, \tag{4.35}$$

is controllable in time T.

Then the exact controllability property of system (4.33) in $H \times V^{'}$ is equivalent to the existence of a subsequence (c_n) satisfying (4.34) and having a positive lower bound:

$$c_n \geq c > 0, \qquad n \in \mathbb{N}. \tag{4.36}$$

Unfortunately, that is impossible for system (4.2), whatever the values of the lengths ℓ_0, ℓ_1, ℓ_2 of the strings are. Indeed, as we shall see in Proposition 4.23, for all lengths ℓ_0, ℓ_1, ℓ_2, there exists a subsequence $(n_k) \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} (\lambda_{n_k+1} - \lambda_{n_k}) = 0. \tag{4.37}$$

Consequently, the infimum of the gap of consecutive eigenfrequencies entering in the Fourier development of solutions vanishes, and this for all values of ℓ_0, ℓ_1, and ℓ_2. We shall refer to this situation by saying simply that the spectral gap vanishes.

The spectral gap and the values of the weights (c_n) entering in (4.34) are closely related. Indeed, consider the solution

$$\bar{\phi}^k(t,x) = \frac{1}{\varkappa_{n_k+1}} \cos \lambda_{n_k+1} t \; \bar{\theta}_{n_k+1}(x) - \frac{1}{\varkappa_{n_k}} \cos \lambda_{n_k} t \; \bar{\theta}_{n_k}(x),$$

of (4.2), where

$$\varkappa_n = \theta^0_{n,x}(\ell_0).$$

Then

$$\phi^0_x(t,\ell_0) = \cos \lambda_{n_k+1} t - \cos \lambda_{n_k} t.$$

We have clearly,

$$\int_0^T |\cos\lambda_{n_k+1}t - \cos\lambda_{n_k}t|^2 dt \le \frac{T^3}{3}|\lambda_{n_k+1} - \lambda_{n_k}|^2. \tag{4.38}$$

On the other hand, if (4.34) were true we would also have

$$\begin{aligned}\min(c_{n_k}, c_{n_k+1})\left(\frac{\lambda^2_{n_k+1}}{\varkappa^2_{n_k+1}} + \frac{\lambda^2_{n_k}}{\varkappa^2_{n_k}}\right) &\le \int_0^T |\cos\lambda_{n_k+1}t - \cos\lambda_{n_k}t|^2 dt \\ &= \int_0^T |\phi^0_x(t,\ell_0)|^2 dt. \end{aligned} \tag{4.39}$$

But $|\varkappa_n| \le C\lambda_n$ (see formula (4.88) in Remark 4.31).

Then, as a consequence of (4.38) and (4.39) we have

$$\min(c_{n_k}, c_{n_k+1}) \le C|\lambda_{n_k+1} - \lambda_{n_k}|^2,$$

for every k.

As a consequence, if the spectral gap condition is not satisfied and (4.37) holds, a sequence (c_n) such that the observaility property (4.34) holds is necessarily such that

$$\liminf_{n\to\infty} c_n = 0. \tag{4.40}$$

As we said above, whatever the values of the lengths ℓ_0, ℓ_1 and ℓ_2 are, the spectral gap of the three-string network vanishes. Thus, system (4.33) is not exactly controllable in the optimal space $H \times V'$ for any choice of the values of ℓ_0, ℓ_1, ℓ_2 and T. In this section we will prove inequalities of the form (4.34) for sequences of non-trivial weights (c_n) satisfying weakened positivity conditions. This will lead to controllability results in smaller spaces of initial data.

4.4 An Observability Inequality

In this section we prove the following property for the solutions of (4.2) which holds for all values of the lengths ℓ_i:

Theorem 4.15. *There exists a positive constant C such that every solution $\bar\phi$ of (4.2) with initial data $(\bar\phi_0, \bar\phi_1) \in V \times H$ satisfies the inequalities*

$$\int_0^{T^*} |\phi^0_x(t,\ell_0)|^2 dt \ge C\mathbf{E}_{\ell_j^-\bar\phi}, \quad j = 1, 2, \tag{4.41}$$

where $T^* = 2(\ell_0 + \ell_1 + \ell_2)$.

In Theorem 4.15, $\ell_j^-\bar\phi$ stands for the function obtained by applying the operator ℓ^- defined in (3.7) for $\ell = \ell_j$ to the solution $\bar\phi$ of (4.2). Due to the linearity of ℓ_j^-, the function $\ell_j^- \bar{\phi}$ is also a solution of (4.2). In particular, its energy $\mathbf{E}_{\ell_j^- \bar{\phi}}(t)$ is conserved in time.

It is natural to try to proceed as in Section 4.1, that is, to apply Proposition 3.1 to estimate the energy of each string. This will allow us to show that, for every $\hat{t} \in \mathbb{R}$ and $i = 1, 2$,

$$\mathbf{E}_{\phi^i}(\hat{t}) \le C \left(\int |\phi_x^i(t,0)|^2 dt + \int |\phi_t^i(t,0)|^2 dt \right),$$

$$\mathbf{E}_{\phi^0}(\hat{t}) \le C \left(\int |\phi_x^0(t,\ell_0)|^2 dt + \int |\phi_t^0(t,\ell_0)|^2 dt \right) = \int |\phi_x^0(t,\ell_0)|^2 dt.$$

Note that we have not written the limits in the integrals explicitly. We will come back later to that issue in detail.

Thus, if we were able to prove that there exists $C > 0$ such that, for $i = 1, 2$,

$$\int |\phi_x^i(t,0)|^2 dt \le C \int_0^T |\phi_x^0(t,\ell_0)|^2 dt, \tag{4.42}$$

$$\int |\phi_t^i(t,0)|^2 dt \le C \int_0^T |\phi_x^0(t,\ell_0)|^2 dt, \tag{4.43}$$

we would obtain inequality (4.41).

Inequality (4.43) can be proved without difficulty for $i = 1, 2$: in view of the coupling conditions,

$$\phi_t^1(t,0) = \phi_t^2(t,0) = \phi_t^0(t,0) = \ell_0^- \phi_x^0(t,\ell_0), \tag{4.44}$$

and then

$$\int |\phi_t^i(t,0)|^2 dt = \int |\ell_0^- \phi_x^0(t,\ell_0)|^2 dt \le \int |\phi_x^0(t,\ell_0)|^2 dt, \qquad i = 1, 2.$$

However, the inequalities (4.42) fail to hold[1]. In spite of what happens with $\phi_t^1(.,0)$, $\phi_t^2(.,0)$, the traces $\phi_x^1(.,0)$, $\phi_x^2(.,0)$ can not be expressed in a direct way as a function of $\phi_x^0(.,\ell_0)$; for them the coupling condition in this node allows simply deducing that

$$\phi_x^1(t,0) + \phi_x^2(t,0) = -\ell_0^+ \phi_x^0(t,\ell_0).$$

In other words, we get complete information on the linear combination $\phi_x^1(t,0) + \phi_x^2(t,0)$ but not on each individual string, i. e. on $\phi_x^1(t,0)$ and $\phi_x^2(t,0)$ separately.

Nevertheless, the boundary conditions $\phi_t^1(t,\ell_1) = \phi_t^2(t,\ell_2) = 0$ provide additional information that may be explicitly written as follows:

[1] If these inequalities were true, the whole energy space $H \times V'$ would be controllable and, as we have seen above, this is not true because of the lack of spectral gap.

$$0 = \phi_x^i(t, \ell_1) = \ell_i^+ \phi_t^i(t, 0) + \ell_i^- \phi_x^i(t, 0), \qquad i = 1, 2,$$

from where it holds

$$\ell_i^- \phi_x^i(t, 0) = -\ell_i^+ \phi_t^i(t, 0) = \ell_i^+ \ell_0^- \phi_x^0(t, \ell_0), \qquad i = 1, 2. \tag{4.45}$$

In this way, we arrive to the system of equations

$$\begin{cases} \phi_x^1(t, 0) + \phi_x^2(t, 0) = f(t), \\ \ell_1^- \phi_x^1(t, 0) = g_1(t), \qquad \ell_2^- \phi_x^2(t, 0) = g_2(t), \end{cases} \tag{4.46}$$

which is satisfied by the traces $\phi_x^1(., 0)$, $\phi_x^2(., 0)$, where

$$f(t) = -\ell_0^+ \phi_x^0(t, \ell_0), \qquad g_i(t) = \ell_i^+ \ell_0^- \phi_x^0(t, \ell_0), \qquad i = 1, 2.$$

Let us observe that f, g_1, g_2 are functions such that their norms in L^2 are bounded above in terms of the L^2-norm of $\phi_x^0(., \ell_0)$ with the help of Proposition 3.3.

As we shall see, the information (4.46) is sufficient to prove inequality (4.41). The idea of the proof is the following: if we apply, for example, the operator ℓ_1^- to the first equation in (4.46) we obtain the uncoupled conditions

$$\ell_1^- \phi_x^1(., 0) = g_1, \qquad \ell_1^- \phi_x^2(., 0) = \ell_1^- f - g_1. \tag{4.47}$$

Due to the linearity of system (4.2) and of the operators ℓ_1^- and ℓ_2^-, if $\bar{\phi}$ is a solution of (4.2) the functions $\ell_1^- \bar{\phi}$ and $\ell_2^- \bar{\phi}$ (the operators act in this case over the variable t) are also solutions of (4.2). Besides, the following identities take place

$$(\ell_j^- \phi^i)_x = \ell_j^- \phi_x^i, \quad (\ell_j^- \phi^i)_t = \ell_1^- \phi_t^i\ , \quad \text{for} \quad i = 0, 1, 2, \ \text{ and } \ j = 1, 2.$$

Thus the solution $\bar{w} = \ell_1^- \bar{\phi}$ of (4.2) satisfies

$$(w^i)_x = \ell_1^- \phi_x^i, \quad (w^i)_t = \ell_1^- \phi_t^i\ , \quad \text{for} \quad i = 0, 1, 2,$$

and the relations (4.47) become

$$w_x^1(., 0) = g_1, \qquad w_x^2(., 0) = \ell_1^- f - g_1.$$

This implies that the L^2-norms of $w_x^1(., 0)$, $w_x^2(., 0)$, may be fully estimated in terms of the L^2-norm of $\phi_x^0(., \ell_0)$. Of course, the same happens with the traces $w_t^1(., 0)$, $w_t^2(., 0)$ and $w_x^0(., \ell_0)$ due to the continuity of ℓ_1^- (Proposition 3.3). With this it may be proved that

$$\int |\phi_x^0(t, \ell_0)|^2 dt \geq C \mathbf{E}_{\bar{w}}.$$

Following this simple argument we get:

Proposition 4.16. *There exists a positive constant C such that every solution $\bar\phi$ of (4.2) with initial data $(\bar\phi_0, \bar\phi_1) \in V \times H$ satisfies the inequalities*

$$\int_0^{T_j} |\phi_x^0(t,\ell_0)|^2 dt \geq C\mathbf{E}_{\ell_j^-\bar\phi}, \quad j = 1, 2,$$

where $T_j = 2(\ell_0 + \ell_j + \max\{\ell_1, \ell_2\})$.

Remark 4.17. This result is very close to Theorem 4.15 but in a larger observation time. If $\ell_1 < \ell_2$ the inequality of the theorem is immediately obtained for $j = 1$, since $T_1 = T^*$. But the time needed for $j = 2$ is larger in Proposition 4.16.

Proof. The proof of this Proposition is almost complete; we just need to follow carefully the integration intervals to obtain the indicated observation time. We will prove the assertion for $i = 1$; for $i = 2$ the proof is, obviously, similar.

Let us observe first that, since $w_t^0(t,\ell_0) = 0$, $w_x^0(t,\ell_0) = \ell_1^- \phi_x^0(t,\ell_0)$, then, in view of Proposition 3.1, for any $\hat t \in \mathbb{R}$, the energy $\mathbf{E}_{w^0}$ of the solution $\bar w$ on the string $\mathbf{e}_0$ satisfies

$$\mathbf{E}_{w^0}(\hat t) \leq C \int_{\hat t - \ell_0}^{\hat t + \ell_0} |w_x^0(t,\ell_0)|^2 dt = C \int_{\hat t - \ell_0}^{\hat t + \ell_0} |\ell_1^- \phi_x^0(t,\ell_0)|^2 dt,$$

and from Proposition 3.3 it follows that, for $\hat t \in [\ell_0 + \ell_1, T_1 - \ell_0 - \ell_1]$,

$$\mathbf{E}_{w^0}(\hat t) \leq C \int_{\hat t - \ell_0 - \ell_1}^{\hat t + \ell_0 + \ell_1} |\phi_x^0(t,\ell_0)|^2 dt \leq C \int_0^{T_1} |\phi_x^0(t,\ell_0)|^2 dt. \tag{4.48}$$

We claim that it is sufficient to prove the existence of $C > 0$ and $\hat t \in \mathbb{R}$ such that

$$\int_{\hat t - \ell_i}^{\hat t - \ell_i} |w_x^i(t,0)|^2 dt \leq C \int_0^{T_1} |\phi_x^0(t,\ell_0)|^2 dt, \tag{4.49}$$

$$\int_{\hat t - \ell_i}^{\hat t - \ell_i} |w_t^i(t,0)|^2 dt \leq C \int_0^{T_1} |\phi_x^0(t,\ell_0)|^2 dt, \tag{4.50}$$

$i = 1, 2$, for every solution of (4.2). Indeed, if this were true, then, based on Proposition 3.1 we would obtain

$$\mathbf{E}_{w^i}(\hat t) \leq C \int_0^{T_1} |\phi_x^0(t,\ell_0)|^2 dt, \quad i = 0, 1, 2$$

and then, in account of (4.48),

$$\mathbf{E}_{\bar w}(\hat t) = \mathbf{E}_{w^0}(\hat t) + \mathbf{E}_{w^1}(\hat t) + \mathbf{E}_{w^2}(\hat t) \leq C \int_0^{T_1} |\phi_x^0(t,\ell_0)|^2 dt.$$

So, we concentrate ourselves in proving inequalities (4.49)-(4.50). As it has been pointed out above in the formulas (4.44)-(4.45), we have the equalities

$$w_t^i(t,0) = \ell_i^- \phi_t^i(t,0) = \ell_i^- \ell_0^- \phi_x^0(t,\ell_0), \qquad i = 1,2, \tag{4.51}$$

$$w_x^1(t,0) = \ell_1^- \phi_x^1(t,0) = -\ell_1^+ \phi_t^1(t,0),\ w_t^2(t,0) = w_t^1(t,0) = \ell_1^- \ell_0^- \phi_x^0(t,\ell_0). \tag{4.52}$$

Then, combining (4.51) with Proposition 3.3, we can ensure that, for any $\hat{t} \in \mathbb{R}$,

$$\int_{\hat{t}-\ell_1}^{\hat{t}+\ell_1} |w_t^1(t,0)|^2 dt = \int_{\hat{t}-\ell_1}^{\hat{t}+\ell_1} |\ell_1^- \ell_0^- \phi_x^0(t,0)|^2 dt \le C \int_{\hat{t}-\ell_0-2\ell_1}^{\hat{t}+\ell_0+2\ell_1} |\phi_x^0(t,\ell_0)|^2 dt.$$

In a similar way, the following inequalities hold

$$\begin{aligned}
&\int_{\hat{t}-\ell_1}^{\hat{t}+\ell_1} |w_x^1(t,0)|^2 dt \le C \int_{\hat{t}-\ell_0-2\ell_1}^{\hat{t}+\ell_0+2\ell_1} |\phi_x^0(t,\ell_0)|^2 dt, \\
&\int_{\hat{t}-\ell_2}^{\hat{t}+\ell_2} |w_t^2(t,0)|^2 dt \le C \int_{\hat{t}-\ell_0-\ell_1-\ell_2}^{\hat{t}+\ell_0+\ell_1+\ell_2} |\phi_x^0(t,\ell_0)|^2 dt, \\
&\int_{\hat{t}-\ell_2}^{\hat{t}+\ell_2} |w_x^2(t,0)|^2 dt \le C \int_{\hat{t}-\ell_0-\ell_1-\ell_2}^{\hat{t}+\ell_0+\ell_1+\ell_2} |\phi_x^0(t,\ell_0)|^2 dt.
\end{aligned} \tag{4.53}$$

Now it is easy to see that if we choose $\hat{t} = \ell_0 + \ell_1 + \max\{\ell_1, \ell_2\}$ in (4.53) we obtain[2] the inequalities (4.49)-(4.50). With this the proposition is proved.

Let us assume now that $\ell_1 \le \ell_2$. Then $T_1 = 2(\ell_0 + \ell_1 + \ell_2)$ and $T_2 = 2(\ell_0 + \ell_1 + \ell_2) \ge T_1$. But, in fact, the value of the observation time T_2 may be reduced.

The possibility of choosing an observation time smaller than T_2 (T_1 already coincides with T^*), which allows to obtain the assertion of the theorem from Proposition 4.16, is based on a property of *generalized periodicity* in time of the solutions of the homogeneous system (4.2) (see Proposition 4.18), which guarantees that all the L^2-information on the traces $\phi_x^0(t,\ell_0)$ of these solutions may be obtained in a time interval of length T^*. This makes observation in a larger time superfluous.

We define the operator

$$\mathcal{Q} := \ell_0^+ \ell_1^- \ell_2^- + \ell_0^- \ell_1^+ \ell_2^- + \ell_0^- \ell_1^- \ell_2^+. \tag{4.54}$$

Then,

[2] We have that $\hat{t} \in [\ell_0 + \ell_1, T_1 - \ell_0 - \ell_1]$, what is necessary for inequality (4.48). This value of $\hat{t}$ has been chosen so that the numbers $\hat{t} - \ell_0 - 2\ell_1$, $\hat{t} + \ell_0 + 2\ell_1$, $\hat{t} - \ell_0 - \ell_1 - \ell_2$, $\hat{t} + \ell_0 + \ell_1 + \ell_2$ belong to the interval $[0, T_1]$.

Proposition 4.18. *For every solution* $\bar{\phi}$ *of (4.2) with initial data* $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$ *it holds*

$$\mathcal{Q}\phi_x^0(t, \ell_0) = 0.$$

Proof. From the relations $\phi_t^0(t, 0) = -\ell_0^- \phi_x^0(t, \ell_0)$, $\phi_x^0(t, 0) = \ell_0^+ \phi_x^0(t, \ell_0)$, we have that

$$\begin{aligned}\mathcal{Q}\phi_x^0(t, \ell_0) &= \ell_1^- \ell_2^- (\ell_0^+ \phi_x^0(t, \ell_0)) + (\ell_1^+ \ell_2^- + \ell_1^- \ell_2^+) \ell_0^- \phi_x^0(t, \ell_0) \\ &= \ell_1^- \ell_2^- \phi_x^0(t, 0) - (\ell_1^+ \ell_2^- + \ell_1^- \ell_2^+) \phi_t^0(t, 0).\end{aligned}$$

Recalling that $\phi_t^0(t, 0) = \phi_t^1(t, 0) = \phi_t^2(t, 0)$ we obtain

$$\mathcal{Q}\phi_x^0(t, \ell_0) = \ell_1^- \ell_2^- \phi_x^0(t, 0) + \ell_2^- (-\ell_1^+ \phi_t^1(t, 0)) + \ell_1^- (-\ell_2^+ \phi_t^2(t, 0)). \qquad (4.55)$$

Applying the equalities $\ell_1^+ \phi_t^1(t, 0) + \ell_1^- \phi_x^1(t, 0) = 0$, $\ell_2^+ \phi_t^2(t, 0) + \ell_2^- \phi_x^2(t, 0) = 0$ (obtained as in the proof of Proposition 4.16 from the boundary conditions) in (4.55) it holds

$$\begin{aligned}\mathcal{Q}\phi_x^0(t, \ell_0) &= \ell_1^- \ell_2^- \phi_x^0(t, 0) + \ell_2^- \ell_1^- \phi_t^1(t, 0) + \ell_1^- \ell_2^- \phi_t^2(t, 0) \\ &= \ell_1^- \ell_2^- (\phi_x^0(t, 0) + \phi_t^1(t, 0) + \phi_t^2(t, 0)) = 0.\end{aligned}$$

The usefulness of Proposition 4.18 in our context relies on the following property:

Lemma 4.19. *For every* $T > 0$ *there exists a constant* $C_T > 0$ *such that every continuous function* ψ*, which is a solution of* $Q\psi \equiv 0$*, satisfies the inequality*

$$\int_0^T |\psi(t)|^2 dt \le C_T \int_0^{T^*} |\psi(t)|^2 dt, \qquad (4.56)$$

where, as before, $T^* = 2(\ell_0 + \ell_1 + \ell_2)$*.*

This fact, when applied to $\phi_x^0(t, \ell_0)$, yields the assertion of Theorem 4.15 from Proposition 4.16.

The proof of this property will be given in Chapter 5 for a larger class of operators $\mathcal{Q}$. Now we restrict ourselves to the particular version corresponding to the operator $\mathcal{Q}$ defined by (4.54), which allows to illustrate clearly the idea of the proof in the general case.

Let us denote $\ell_* = \min\{\ell_0, \ell_1, \ell_2\}$. We will prove that, for all $T > 0$ and ψ satisfying $Q\psi \equiv 0$,

$$\int_0^T |\psi(t)|^2 dt \le \int_0^{T^*} |\psi(t)|^2 dt + C \int_0^{T - 2\ell_*} |\psi(t)|^2 dt. \qquad (4.57)$$

From this inequality we can get (4.56), by iterating it as many times as necessary to obtain $T - 2n\ell_* \le T^*$. Indeed, in such case

$$\begin{aligned}\int_0^T |\psi(t)|^2 dt &\leq (1 + C + ... + C^{n-1}) \int_0^{T^*} |\psi(t)|^2 dt \\ &\quad + C^n \int_0^{T-2n\ell_*} |\psi(t)|^2 dt \\ &\leq (1 + ... + C^n) \int_0^{T^*} |\psi(t)|^2 dt,\end{aligned} \tag{4.58}$$

which is the claimed inequality (4.56).

Let us observe that, according to the definition of $\mathcal{Q}$, the equality $\mathcal{Q}\psi \equiv 0$ may be written as

$$(\ell_0^+ \ell_1^- \ell_2^- + \ell_0^- \ell_1^+ \ell_2^- + \ell_0^- \ell_1^- \ell_2^+)\psi \equiv 0$$

and then, from the definition of the operators $\ell_i^{\pm}$, it follows that[3]

$$\begin{aligned}&3\psi(t + \ell_0 + \ell_1 + \ell_2) - \psi(t + \ell_0 + \ell_1 - \ell_2) - \psi(t + \ell_0 - \ell_1 + \ell_2) \\ &-\psi(t - \ell_0 - \ell_1 - \ell_2) - -\psi(t - \ell_0 + \ell_1 + \ell_2) \\ &-\psi(t - \ell_0 + \ell_1 - \ell_2) - \psi(t - \ell_0 - \ell_1 + \ell_2) + 3\psi(t - \ell_0 - \ell_1 - \ell_2) = 0.\end{aligned}$$

Replacing the variable t by $t - (\ell_0 + \ell_1 + \ell_2)$ the previous identity may be written as

$$\psi(t) = \sum_{\tau \in \Gamma} c_\tau \psi(t - \tau),$$

where

$$\Gamma := \{2\ell_0, 2\ell_1, 2\ell_2, 2(\ell_0 + \ell_1), 2(\ell_0 + \ell_2), 2(\ell_1 + \ell_2), 2(\ell_0 + \ell_1 + \ell_2)\}$$

with coefficients c_τ taking the values 1 or $-1/3$. From this, using the Cauchy-Schwarz inequality, we get

$$\int_{T^*}^T |\psi(t)|^2 dt \leq C \sum_{\tau \in \Gamma} \int_{T^*}^T |\psi(t - \tau)|^2 dt = C \sum_{\tau \in \Gamma} \int_{T^* - \tau}^{T - \tau} |\psi(t)|^2 dt, \tag{4.59}$$

for some constant C independent of ψ (in fact, we may take $C = 55/9$).

But every $\tau \in \Gamma$ satisfies $2\ell^* \leq \tau \leq T^*$, so we have $T^* - \tau \geq 0$ and $T - \tau \leq T - 2\ell_*$; and then the following inequality holds

$$\int_{T^* - \tau}^{T - \tau} |\psi(t)|^2 dt \leq \int_0^{T - 2\ell_*} |\psi(t)|^2 dt.$$

Using the latter inequality in (4.59) we obtain

$$\int_{T^*}^T |\psi(t)|^2 dt \leq C \sum_{\tau \in \Gamma} \int_0^{T - 2\ell_*} |\psi(t)|^2 dt = 7C \int_0^{T - 2\ell_*} |\psi(t)|^2 dt.$$

[3] It is a lengthly, but completely elementary computation.

Finally

$$\int_0^T |\psi(t)|^2 dt = \int_0^{T^*} |\psi(t)|^2 dt + \int_{T^*}^T |\psi(t)|^2 dt$$
$$\leq \int_0^{T^*} |\psi(t)|^2 dt + C \int_0^{T-2\ell_*} |\psi(t)|^2 dt, \tag{4.60}$$

which is the inequality (4.57).

Thus, we have proved inequality (4.56) and with this, Theorem 4.15 is also proved.

4.5 Properties of the Sequence of Eigenvalues

The relation of the operator $\mathcal{Q}$ with system (4.2) is not purely technical. This operator is closely related to the boundary conditions (4.2), as Proposition 4.18 shows. Besides, the operator $\mathcal{Q}$ is linked in a direct way to the spectrum of $-\Delta_G$. Indeed, if we apply $\mathcal{Q}$ to the function $e^{i\lambda t}$, where λ is an arbitrary real number, since $\ell^+ e^{i\lambda t} = \cos\lambda\ell \; e^{i\lambda t}$ and $\ell^- e^{i\lambda t} = i\sin\lambda\ell \; e^{i\lambda t}$, we obtain

$$\begin{aligned}\mathcal{Q}e^{i\lambda t} &= \left(\ell_0^+\ell_1^-\ell_2^- + \ell_0^-\ell_1^+\ell_2^- + \ell_0^-\ell_1^-\ell_2^+\right) e^{i\lambda t}\\ &= -\left(\cos\lambda\ell_0 \sin\lambda\ell_1 \sin\lambda\ell_2 + \sin\lambda\ell_0 \cos\lambda\ell_1 \sin\lambda\ell_2\right.\\ &\qquad \left.+ \sin\lambda\ell_0 \sin\lambda\ell_1 \cos\lambda\ell_2\right) e^{i\lambda t}.\end{aligned}$$

Thus, we have

$$\mathcal{Q}e^{i\lambda t} = q(\lambda)e^{i\lambda t}$$

with

$$q(\lambda) := -(\cos\lambda\ell_0 \sin\lambda\ell_1 \sin\lambda\ell_2 + \sin\lambda\ell_0 \cos\lambda\ell_1 \sin\lambda\ell_2 + \sin\lambda\ell_0 \sin\lambda\ell_1 \cos\lambda\ell_2). \tag{4.61}$$

The following holds

Proposition 4.20. *Let $\lambda \neq 0$. Then λ^2 is an eigenvalue of $-\Delta_G$ if, and only if, $q(\lambda) = 0$.*

Proof. The necessity of this condition is immediate: if λ^2 is an eigenvalue with associated eigenfunction $\bar{\theta}$ then the function $\bar{\phi}(t,x) = e^{i\lambda t}\bar{\theta}(x)$ is a solution of (4.2). According to Proposition 4.18 we have

$$0 = \mathcal{Q}\phi_x^0(t,\ell_0) = \mathcal{Q}e^{i\lambda t}\theta_x^0(\ell_0) = q(\lambda)e^{i\lambda t}\theta_x^0(\ell_0).$$

From this inequality it holds $q(\lambda) = 0$ if $\theta_x^0(\ell_0) \neq 0$. On the other hand, if $\theta_x^0(\ell_0) = 0$ then the function θ^0, which is a solution of a second order ordinary differential equation satisfies $\theta^0(\ell_0) = \theta_x^0(\ell_0) = 0$, and this implies $\theta^0 \equiv 0$; in particular, $\theta^0(0) = 0$. From the continuity conditions at $x = 0$ we have that

$\theta^1(0) = \theta^2(0) = 0$. This means that λ^2 is also an eigenvalue of the strings $\mathbf{e}_1$ and $\mathbf{e}_2$ and therefore,

$$\sin \lambda \ell_1 = \sin \lambda \ell_2 = 0.$$

If we replace these equalities in (4.61), we obtain $q(\lambda) = 0$.

Now we will see that the condition $q(\lambda) = 0$ is also sufficient for λ^2 to be an eigenvalue. To do this we construct a non-zero eigenfunction $\bar{\theta}$ associated to λ^2. We look for the components of $\bar{\theta}$ in the form:

$$\theta^i(x) = a_i \sin \lambda(x - \ell_i), \quad i = 0, 1, 2. \tag{4.62}$$

This guarantees that the boundary conditions at the external nodes ($\theta^i(\ell_i) = 0$) are satisfied. The remaining boundary conditions at $x = 0$ lead to the linear system

$$a_0 \sin \lambda \ell_0 = a_1 \sin \lambda \ell_1 = a_2 \sin \lambda \ell_2, \tag{4.63}$$

$$a_0 \lambda \cos \lambda \ell_0 + a_1 \lambda \cos \lambda \ell_1 + a_2 \lambda \cos \lambda \ell_2 = 0, \tag{4.64}$$

whose determinant coincides with $\lambda q(\lambda)$. Thus, if $q(\lambda) = 0$ we can find numbers a_0, a_1, a_2, not all of them equal to zero, such that the function $\bar{\theta}$ defined by (4.62) is an eigenfunction.

Remark 4.21. The proof of the fact that the condition given in Proposition 4.20 is necessary may be done in a simpler way without using the operator $\mathcal{Q}$. Indeed, in view of the boundary conditions, an eigenfunction is necessarily of the form (4.62). If this eigenfunction does not vanish identically, the determinant of the liner system (4.63) is equal to zero. Thus, $q(\lambda) = 0$. However, we have used the operator $\mathcal{Q}$ because this is the natural technique we will use in Chapter 5 to address more general networks.

An important consequence of Proposition 4.20 is the following. Let us denote by (σ_n) the increasing sequence formed by the elements of the set

$$\Sigma = \frac{\pi}{\ell_0}\mathbb{N} \cup \frac{\pi}{\ell_1}\mathbb{N} \cup \frac{\pi}{\ell_2}\mathbb{N}. \tag{4.65}$$

The elements of Σ are the positive square roots of the eigenvalues of the decoupled strings with homogeneous Dirichlet boundary conditions. Let (λ_n) be the increasing sequence formed by the positive square roots of the eigenvalues of the network. Then, for every $n \in \mathbb{N}$,

$$\lambda_n < \sigma_n < \lambda_{n+1} < \sigma_{n+1}. \tag{4.66}$$

Indeed, in every interval (σ_n, σ_{n+1}) the function $q(\lambda)$ may be expressed as

$$q(\lambda) = h_1(\lambda) h_2(\lambda),$$

where

$$h_1(\lambda) = \sin\lambda\ell_0 \sin\lambda\ell_1 \sin\lambda\ell_2, \qquad h_2(\lambda) = \cot\lambda\ell_0 + \cot\lambda\ell_1 + \cot\lambda\ell_2.$$

Observe that under the hypothesis that all the ratios ℓ_i/ℓ_j $(i \neq j)$ are irrational, the numbers λ_n are the positive zeros of $h_2(\lambda)$, while σ_n are the points where $h_2(\lambda) \to \pm\infty$. It suffices now to note that on every interval (σ_n, σ_{n+1}) the function $h_2(\lambda)$ is strictly increasing to conclude that, necessarily, the numbers σ_n and λ_n alternate, that is, (4.66) is verified.

Inequalities (4.66) allow to obtain information on the numbers λ_n from the properties of the sequence (σ_n). Indeed, observe that from (4.66) we obtain that, for every $n \in \mathbb{N}$,

$$\lambda_{n+4} - \lambda_n > \sigma_{n+3} - \sigma_n. \tag{4.67}$$

But, for every $n \in \mathbb{N}$, among the four numbers $\sigma_n, \sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3}$ there are at least two corresponding to the same string. Consequently, for every $n \in \mathbb{N}$, there exists $i \in \{0, 1, 2\}$ such that

$$\sigma_{n+3} - \sigma_n > \frac{\pi}{\ell_i}.$$

Therefore, for every $n \in \mathbb{N}$,

$$\sigma_{n+3} - \sigma_n > \pi \min\left(\frac{1}{\ell_0}, \frac{1}{\ell_1}, \frac{1}{\ell_2}\right).$$

In other words, the following generalized separation property holds:

Proposition 4.22. *For every $n \in \mathbb{N}$ it holds*

$$\lambda_{n+4} - \lambda_n > \pi \min\left(\frac{1}{\ell_0}, \frac{1}{\ell_1}, \frac{1}{\ell_2}\right). \tag{4.68}$$

This generalized separation property (4.68) allows to apply the technique derived from Theorem 3.29 developed in [11] and [17] in the proof of observability inequalities of the type (4.34).

On the other hand, if n_λ and n_σ denote respectively, the counting functions[4] of the sequences (λ_n) and (σ_n) then

$$n_\sigma(r) \leq n_\lambda(r) \leq n_\sigma(r) + 1.$$

The function $n_\sigma(r)$ coincides with the sum of the counting functions of the sequences $(n\pi/\ell_0)$, $(n\pi/\ell_1)$ and $(n\pi/\ell_2)$. It holds

$$n_\sigma(r) = [\frac{r\ell_0}{\pi}] + [\frac{r\ell_1}{\pi}] + [\frac{r\ell_2}{\pi}],$$

where $[\eta]$ denotes the integer part of the number η. Therefore, we obtain

[4] The counting function $n(r)$ of the sequence of positive numbers (λ_n) is the number of elements of the sequence contained on the interval $(0, r]$.

$$r\frac{\ell_0+\ell_1+\ell_2}{\pi}-3\leq n_\lambda(r)\leq 1+r\frac{\ell_0+\ell_1+\ell_2}{\pi}. \tag{4.69}$$

Then, the sequence (λ_n) has density

$$D(\lambda_n)=\lim_{r\to\infty}\frac{n_\lambda(r)}{r}=\frac{\ell_0+\ell_1+\ell_2}{\pi},$$

which coincides with the density of the sequence (σ_n). It is essentially due to this reason that the time of observation $T^*=2(\ell_0+\ell_1+\ell_2)=2\pi D(\lambda_n)$ appearing in Theorem 4.15 is optimal. Indeed, the Beurling-Malliavin theorem ensures that for a sequence (λ_n) with density D the sequence $\left(e^{it\lambda_n}\right)$ is complete in $L^2(0,T)$ if $T<2\pi D$. This implies that a non-trivial inequality of the type (4.34) cannot be proved for $T<2\pi D$.

Let us also observe that inequality (4.69) implies that

$$\lim_{n\to\infty}\frac{\lambda_n}{n}=\frac{\pi}{\ell_0+\ell_1+\ell_2}. \tag{4.70}$$

This shows that the eigenvalues of the network behave asymptotically as the eigenvalues of a single string of length $\ell_0+\ell_1+\ell_2$.

Another important consequence of the inequalities (4.66) is the following:

Proposition 4.23. *For any values ℓ_0, ℓ_1, ℓ_2 of the lengths of the strings there exists a subsequence $(n_k)\subset\mathbb{N}$ such that*

$$\lim_{k\to\infty}(\lambda_{n_k+1}-\lambda_{n_k})=0.$$

Proof. According to Dirichlet's theorem on the simultaneous approximation of real numbers by rational ones (see [26], Section I.5), for every $\varepsilon>0$ there exist infinitely many values of $k\in\mathbb{N}$ for which there exist natural numbers p_k, q_k such that

$$\left|k\frac{\ell_1}{\ell_0}-p_k\right|<\varepsilon,\qquad \left|k\frac{\ell_2}{\ell_0}-q_k\right|<\varepsilon.$$

Then

$$\left|\frac{\pi k}{\ell_0}-\frac{\pi p_k}{\ell_1}\right|<\varepsilon',\qquad \left|\frac{\pi k}{\ell_0}-\frac{\pi q_k}{\ell_2}\right|<\varepsilon', \tag{4.71}$$

where

$$\varepsilon'=\max\left\{\frac{\pi\varepsilon}{\ell_1},\frac{\pi\varepsilon}{\ell_2}\right\}.$$

Let $n_k\in\mathbb{N}$ be such that

$$\sigma_{n_k}=\min\left\{\frac{\pi k}{\ell_0},\frac{\pi p_k}{\ell_1},\frac{\pi q_k}{\ell_2}\right\}.$$

Then, from (4.71) we obtain the inequalities

$$\sigma_{n_k+2}-\sigma_{n_k}<\varepsilon'.$$

But, in view of (4.66), the latter inequality implies

$$\lambda_{n_k+1} - \lambda_{n_k} < \varepsilon',$$

for infinitely many values of $k \in \mathbb{N}$.

Taking into account that ε' may be chosen arbitrarily small, the assertion of the proposition is obtained.

Remark 4.24. In a similar way we can prove that for any values ℓ_0, ℓ_1, ℓ_2 of the lengths of the strings there exists a subsequence $(n_k) \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} (\lambda_{n_k+3} - \lambda_{n_k}) = 0.$$

Thus, four is the minimal spectral step to ensure the generalized separation property of Proposition 4.22 to hold.

4.6 Fourier Representation of the Observability Inequality

Our aim in this section is to express the inequalities of Theorem 4.15 in terms of the Fourier coefficients of the initial data of the solution $\bar{\phi}$ of (4.2). To do this, we have to characterize $\mathbf{E}_{\ell_j^- \bar{\phi}}$, $j = 1, 2$, in terms of those coefficients.

If $\bar{\phi}_0, \bar{\phi}_1 \in Z$, that is, if the sequences $(\phi_{0,n})$ and $(\phi_{1,n})$ are finite, then from the formula (4.4) it follows

$$\ell_j^- \bar{\phi}(x) = \sum_{n\in\mathbb{N}} (\phi_{0,n} \ell_j^- \cos \lambda_n t + \frac{\phi_{1,n}}{\lambda_n} \ell_j^- \sin \lambda_n t) \bar{\theta}_n(x). \tag{4.72}$$

But

$$\ell_j^- \cos \lambda_n t = \frac{1}{2} (\cos \lambda_n (t + \ell_j) - \cos \lambda_n (t - \ell_j)) = -\sin \lambda_n \ell_j \sin \lambda_n t,$$
$$\ell_j^- \sin \lambda_n t = \frac{1}{2} (\sin \lambda_n (t + \ell_j) - \sin \lambda_n (t - \ell_j)) = \sin \lambda_n \ell_j \cos \lambda_n t.$$

Replacing these relations in (4.72) we obtain

$$\ell_j^- \bar{\phi}(x) = \sum_{n\in\mathbb{N}} \sin \lambda_n \ell_j (\frac{\phi_{1,n}}{\lambda_n} \cos \lambda_n t - \phi_{0,n} \sin \lambda_n t) \bar{\theta}_n(x). \tag{4.73}$$

Using the formula (4.5) for the energy of $\ell_j^- \bar{\phi}$ we arrive to

$$\mathbf{E}_{\ell_j^- \bar{\phi}} = \sum_{n\in\mathbb{N}} \sin^2 \lambda_n \ell_j (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2) \tag{4.74}$$

and therefore, Theorem 4.15 allows us to ensure that there exists a constant $C > 0$ such that the inequalities

$$\int_0^{T^*} |\phi_x^0(t,\ell_0)|^2 dt \geq C \sum_{n\in\mathbb{N}} \sin^2 \lambda_n \ell_j (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2), \quad j=1,2,$$

are verified for every $\bar{\phi}_0, \bar{\phi}_1 \in Z$. Since $Z \times Z$ is dense in $V \times H$, this inequality is still valid for all $\bar{\phi}_0 \in V$, $\bar{\phi}_1 \in H$.

If we denote

$$c_n := \max\{|\sin \lambda_n \ell_1|, |\sin \lambda_n \ell_2|\} \tag{4.75}$$

we have:

Theorem 4.25. *There exists a positive constant C such that every solution $\bar{\phi}$ of (4.2) with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$ satisfies*

$$\int_0^{T^*} |\phi_x^0(t,\ell_0)|^2 dt \geq C \sum_{n\in\mathbb{N}} c_n^2 (\mu_n \phi_{0,n}^2 + \phi_{1,n}^2). \tag{4.76}$$

4.7 Study of the Weights c_n

Theorem 4.25 provides a satisfactory result as it allows to ensure the controllability of the subspace of initial data defined by (4.35). However, that subspace depends on the coefficients c_n.

Let us observe first that when the ratio ℓ_1/ℓ_2 is a rational number there exist infinitely many linearly independent eigenfunctions which vanish identically on the controlled string. They are constructed as follows. If $\ell_1/\ell_2 = p/q$ with $p, q \in \mathbb{Z}$ then, for $k \in \mathbb{Z}$ we define the functions $\bar{\psi}_k = (\psi_k^0, \psi_k^1, \psi_k^2)$ by

$$\psi_k^0 \equiv 0, \quad \psi_k^1 = \sin \frac{kp\pi x}{\ell_1}, \quad \psi_k^2 = -\sin \frac{kq\pi x}{\ell_2}.$$

This fact implies that when ℓ_1/ℓ_2 is rational we cannot obtain an inequality like the one given in Theorem 4.25, with other non-vanishing coefficients c_n, not necessarily those defined by (4.75). Indeed, the solutions of (4.2) defined by

$$\bar{\phi}_k = \sin \frac{kp\pi t}{\ell_1} \bar{\psi}_k$$

satisfy (see figure below)

$$\phi_{k,x}^0(t,\ell_0) \equiv 0.$$

Thus, the condition $\ell_1/\ell_2 \notin \mathbb{Q}$ is necessary for an inequality like (4.76) to hold. In fact this condition is also sufficient:

Proposition 4.26. *If the ratio ℓ_1/ℓ_2 is an irrational number, then all the coefficients c_n, $n \in \mathbb{N}$, defined by (4.75), are different from zero.*

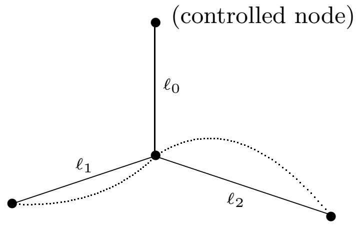

Fig. 4.4. A localized vibration when $\ell_2/\ell_1 \in \mathbb{Q}$.

Proof. It suffices to observe that $c_n = 0$ implies $|\sin \lambda_n \ell_1| = |\sin \lambda_n \ell_2| = 0$, and then

$$\lambda_n \ell_1 = p\pi, \qquad \lambda_n \ell_2 = q\pi.$$

An that is

$$\frac{\ell_1}{\ell_2} = \frac{p}{q} \in \mathbb{Q},$$

for some integers p and q. This contradicts the assumption $\ell_1/\ell_2 \notin \mathbb{Q}$.

Summarizing we have obtained the following characterization of the lengths of the strings for which the system is approximately or spectrally controllable in some finite time.

Corollary 4.27. *The following properties of the system of the three string network are equivalent:*

1) The system is spectrally controllable in time $T \geq T^*$;
2) The system is approximately controllable in time $T \geq T^*$;
3) The ratio ℓ_1/ℓ_2 is an irrational number.

The analysis of the previous section shows that for $T \geq T^*$ the space of observable and/or controllable states may be expressed in Fourier series by means of the weights c_n. In what follows we give sufficient conditions over the values of ℓ_0, ℓ_1, ℓ_2 allowing to ensure that for some $\alpha \in \mathbb{R}$ all the initial states $(\bar{u}_0, \bar{u}_1) \in \mathcal{W}^\alpha$ are controllable in time $T^* = 2(\ell_0 + \ell_1 + \ell_2)$. More precisely, if we define

$$\Phi_\alpha = \left\{ (\ell_0, \ell_1, \ell_2) \in \mathbb{R}_+^3 : \mathcal{W}^\alpha \subset \mathcal{W}_{T^*} \right\}$$

($\mathcal{W}_{T^*}$ is the subspace of controllable states in time T^*), our aim is to give explicit conditions guaranteeing that $(\ell_0, \ell_1, \ell_2) \in \Phi_\alpha$.

First, observe that if the weights c_n are such that for some $C > 0$

$$c_n \geq C \lambda_n^{-\alpha}, \tag{4.77}$$

$n \in \mathbb{N}$, then, as it has been pointed out in Section 4.3, $(\ell_0, \ell_1, \ell_2) \in \Phi_\alpha$.

Let us consider the function

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda) := (|\sin \lambda\ell_1| + |\sin \lambda\ell_2|)\lambda^\alpha.$$

It is also clear from (4.75) that, if for some values ℓ_1, ℓ_2 the function $\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda)$ has a positive lower bound:

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda) \geq a > 0 \text{ for all } \lambda \in \mathbb{R}_+,$$

then, for every $\ell_0 \in \mathbb{R}_+$, inequality (4.77) holds. Thus, we will be concerned with the study of the function $\mathbf{a}$.

The following holds:

Proposition 4.28. *If there exists a constant $C > 0$ such that*

$$|||n\frac{\ell_1}{\ell_2}||| \geq Cn^{-\alpha}$$

for every $n \in \mathbb{N}$, then

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda) \geq a > 0 \text{ for every } \lambda \geq 1. \tag{4.78}$$

Proof. Let us assume that the inequality (4.78) is false. Then there exists a sequence (λ_k) such that

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda_k) = |\sin \lambda_k\ell_1|\lambda_k^\alpha + |\sin \lambda_k\ell_2|\lambda_k^\alpha \to 0 \ (k \to \infty). \tag{4.79}$$

On the other hand, necessarily, $\lambda_k \to \infty$. Indeed, if (λ_k) has a finite limit point $\lambda_* \geq 1$ then, the continuity of $\mathbf{a}^\alpha$ yields

$$\mathbf{a}^\alpha(\ell_1, \ell_2, \lambda_*) = 0.$$

Consequently

$$\sin \lambda_*\ell_1 = \sin \lambda_*\ell_2 = 0,$$

and this may only happen if ℓ_1/ℓ_2 is a rational number. But in such case there exist values of $n \in \mathbb{N}$ such that

$$|||n\frac{\ell_1}{\ell_2}||| = 0,$$

and this contradicts the hypothesis of the proposition. Thus we can assume that $\lambda_k \to \infty$.

From (4.79) it holds

$$|\sin \lambda_k\ell_1|\lambda_k^\alpha \to 0, \qquad |\sin \lambda_k\ell_2|\lambda_k^\alpha \to 0. \tag{4.80}$$

Let us denote for every $k \in \mathbb{N}$,

$$\varepsilon_k := |||\frac{\ell_1\lambda_k}{\pi}|||, \qquad m_k := \frac{\ell_1\lambda_k}{\pi} - \varepsilon_k \in \mathbb{N},$$

$$\delta_k := |||\frac{\ell_2 \lambda_k}{\pi}|||, \qquad n_k := \frac{\ell_2 \lambda_k}{\pi} - \delta_k \in \mathbb{N}.$$

Since $0 \le \varepsilon_k, \delta_k \le 1/2$,

$$\lim_{k\to\infty} \frac{m_k}{\lambda_k} = \frac{1}{\pi}, \qquad \lim_{k\to\infty} \frac{n_k}{\lambda_k} = \frac{1}{\pi}.$$

In particular, as $\lambda_k \to \infty$, the same happens with the sequences m_k and n_k. Besides,

$$\pi\varepsilon_k \le 2|\sin \varepsilon_k \pi| = |\sin(m_k + \varepsilon_k)\pi| = \sin \lambda_k \ell_1,$$

and thus, from (4.80) we obtain $\varepsilon_k \lambda_k^\alpha \to 0$. Analogously, $\delta_k \lambda_k^\alpha \to 0$.

Then we have

$$\begin{aligned} \lim_{k\to\infty} \left(n_k \frac{\ell_1}{\ell_2} - m_k \right) n_k^\alpha &= \frac{\pi}{\ell_2} \lim_{k\to\infty} \left(\varepsilon_k \frac{n_k}{\lambda_k} - \delta_k \frac{m_k}{\lambda_k} \right) n_k^\alpha \\ &= \frac{1}{\ell_2} \lim_{k\to\infty} (\varepsilon_k n_k^\alpha - \delta_k n_k^\alpha) = 0. \end{aligned} \tag{4.81}$$

From this

$$|||n_k \frac{\ell_1}{\ell_2}||| n_k^\alpha \le \left(n_k \frac{\ell_1}{\ell_2} - m_k \right) n_k^\alpha \to 0,$$

what contradicts the fact $|||n\ell_1/\ell_2||| \ge Cn^{-\alpha}$ for all $n \in \mathbb{N}$.

The condition provided by Proposition 4.28, which implies the inequality (4.77), is sufficient for $\mathcal{W}^\alpha \subset \mathcal{W}_T$ too. Now we will see that this condition is also necessary in the following sense

Proposition 4.29. *If there exists a sequence of natural numbers $n_k \to \infty$, for which*

$$|||n_k \frac{\ell_1}{\ell_2}||| n_k^\alpha \to 0, \ \text{as } k \to \infty,$$

then there exist values of $\ell_0 \in \mathbb{R}$ such that for every $T > 0$, the space $\mathcal{W}^\alpha$ is not contained in $\mathcal{W}_T$. That is, there exist initial states in $\mathcal{W}^\alpha$, which are not controllable.

Proof. It suffices to choose ℓ_0 such that

$$|||n_k \frac{\ell_0}{\ell_2}||| n_k^\alpha \to 0, \ \text{as } k \to \infty.$$

In fact, let m and $\tilde{m}$ be the closest natural numbers to $n_k \ell_1/\ell_2$ and $n_k \ell_0/\ell_2$, respectively. Let (σ_p) be the sequence defined by (4.65) and denote by p_k the index for which

$$\sigma_{p_k} = \min \left(n_k \frac{\pi}{\ell_2}, m \frac{\pi}{\ell_1}, \tilde{m} \frac{\pi}{\ell_0} \right).$$

Then we have

$$|\sigma_{p_k+2} - \sigma_{p_k}|\,\sigma_{p_k}^{\alpha} \to 0,$$

and this implies, in view of the inequalities (4.66),

$$|\lambda_{p_k+1} - \lambda_{p_k}|\,\lambda_{p_k}^{\alpha} \to 0. \tag{4.82}$$

On the other hand, the fact that all the initial states $(\bar{u}_0, \bar{u}_1) \in \mathcal{W}^{\alpha}$ are controllable in time T is equivalent to the following inequality

$$\int_0^T |\phi_x^0(t,\ell_0)|^2 dt \geq C \sum_{n\in\mathbb{N}} \lambda_n^{2\alpha}(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2),$$

for all solution of (4.2) with $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$. However, proceeding as in Section 4.3, it can be easily proved that, due to (4.82), the latter inequality is impossible.

As in the case of the simultaneous control of two strings, Proposition 4.28 reduces the problem of identifying subspaces of controllable initial states to the following diophantine approximation problem: given $\alpha > 0$, to determine the values of ℓ_1/ℓ_2 for which there exists a constant $C > 0$ such that the inequality

$$|||n\ell_1/\ell_2||| \geq Cn^{-\alpha}, \tag{4.83}$$

is true for each $n \in \mathbb{N}$.

In view of the results described in Section 4.2.1 we obtain:

Corollary 4.30. *a) If $\ell_1/\ell_2 \in B_\varepsilon$ then, the space $\mathcal{W}^{1+\varepsilon}$ is controllable in any time $T \geq T^*$. In particular, if ℓ_1/ℓ_2 is an algebraic irrational number then, $\mathcal{W}^{1+\varepsilon}$ is controllable for any $\varepsilon > 0$.*

b) If ℓ_1/ℓ_2 admits a bounded expansion in continuous fraction then the subspace $\mathcal{W}^1$ is controllable in any time $T \geq T^*$.

c) There exist values of the lengths ℓ_0, ℓ_1, ℓ_2 such that no subspace of the form $\mathcal{W}^{\alpha}$ is controllable in finite time T.

Remark 4.31. As we will see later, the numbers $\varkappa_n = \theta_{n,x}^0(\ell_0)$, where $\bar{\theta}_n$ are the eigenfunctions of the elliptic problem associated to (4.2), are relevant for the control problem when we attempt to prove the observability inequalities in a direct way using Fourier series.

The eigenfunctions $\bar{\theta}_n$ may be explicitly expressed in terms of the eigenvalues λ_n,

$$\bar{\theta}_n = \begin{pmatrix} \theta_n^0 \\ \theta_n^1 \\ \theta_n^2 \end{pmatrix} = \gamma_n \begin{pmatrix} \dfrac{\sin\lambda_n(\ell_0 - x)}{\sin\lambda_n\ell_0} \\ \dfrac{\sin\lambda_n(\ell_1 - x)}{\sin\lambda_n\ell_1} \\ \dfrac{\sin\lambda_n(\ell_2 - x)}{\sin\lambda_n\ell_2} \end{pmatrix},$$

where

$$\gamma_n = \sqrt{2}\left\{\frac{\ell_0}{\sin^2\lambda_n\ell_0} + \frac{\ell_1}{\sin^2\lambda_n\ell_1} + \frac{\ell_2}{\sin^2\lambda_n\ell_2}\right\}^{-\frac{1}{2}}.$$

Then

$$\varkappa_n = -\lambda_n\sqrt{2}\left\{\ell_0 + \ell_1\frac{\sin^2\lambda_n\ell_0}{\sin^2\lambda_n\ell_1} + \ell_2\frac{\sin^2\lambda_n\ell_0}{\sin^2\lambda_n\ell_2}\right\}^{-\frac{1}{2}}.$$

The following rough estimate is true

$$|\varkappa_n| \geq \lambda_n\sqrt{\frac{2}{\ell_0+\ell_1+\ell_2}}\,|\sin\lambda_n\ell_1\sin\lambda_n\ell_2|\,. \tag{4.84}$$

If the lengths ℓ_0, ℓ_1, ℓ_2 are linearly independent over $\mathbb{Q}$ and the ratios ℓ_i/ℓ_j are algebraic numbers (see condition (S) in Appendix A for more details) then, according to Proposition A.11 from Appendix A, for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|\sin\lambda_n\ell_1| \geq \frac{C_\varepsilon}{\lambda_n^{1+\varepsilon}}, \qquad |\sin\lambda_n\ell_2| \geq \frac{C_\varepsilon}{\lambda_n^{1+\varepsilon}}, \quad n \in \mathbb{N}, \tag{4.85}$$

and with this, from (4.84) it follows

$$|\varkappa_n| \geq \frac{C_\varepsilon}{\lambda_n^{1+\varepsilon}}. \tag{4.86}$$

However, with the aid of Theorem 4.25 it is possible to establish more precise estimates under weaker conditions on the lengths. Indeed, it suffices to apply the inequality of this theorem to the solutions $\sin\lambda_n t\,\bar{\theta}_n$ and $\cos\lambda_n t\,\bar{\theta}_n$ to get

$$|\varkappa_n|^2\int_0^T|\cos\lambda_n t|^2\,dt \geq Cc_n^2\lambda_n^2, \qquad |\varkappa_n|^2\int_0^T|\sin\lambda_n t|^2\,dt \geq Cc_n^2\lambda_n^2.$$

From this we obtain

$$|\varkappa_n| \geq C\lambda_n\max\{|\sin\lambda_n\ell_1|, |\sin\lambda_n\ell_2|\}.$$

This inequality is obviously sharper than (4.84). Consequently, if the ratio ℓ_1/ℓ_2 belongs to $\mathbf{B}_\varepsilon$ then,

$$|\varkappa_n| \geq \frac{C}{\lambda_n^\varepsilon}. \tag{4.87}$$

In spite of estimate (4.84), the latter does not impose any restriction over ℓ_0.

Proceeding in a similar way, from the inequality (2.29) we obtain

$$C\lambda_n \geq |\varkappa_n|\,, \tag{4.88}$$

independently of the values of the lengths.

Note that these two inequalities (4.87) and (4.88) reflect correctly the defect of at least one derivative on the boundary observation of the energy of individual eigenfunctions from the boundary measurement made on one single external node.

4.8 Relation Between the Simultaneous Control of Two Strings and the Control of the Three String Network from One Exterior Node

As the reader has noticed already, the conditions on the lengths of the strings that allow identifying subspaces of controllable initial states are the same for the simultaneous control of two strings and for the control of the three string network from one exterior node. Besides, when these conditions are satisfied, the corresponding subspaces of controllable initial states coincide, up to the boundary conditions, on the uncontrolled strings. There is indeed a very close connection between these two problems that explains this analogy:

Theorem 4.32. *If $\mathcal{V}$ is a subspace of controllable initial states in time T for the simultaneous control of two strings (4.10) then the subspace $(L^2(0,\ell_0) \times H^{-1}(0,\ell_0)) \times \mathcal{V}$ of initial states for the system of the three string network (4.33) is controllable in time $T + 2\ell_0$.*

For the proof of this fact we need some preliminary elements. We consider the spaces

$$\begin{aligned} \mathcal{W}_0 &= \left\{ (\bar{\phi}_0, \bar{\phi}_1) \in V \times H : \quad \phi_0^0(0) = \phi_0^1(0) = \phi_0^2(0) = 0 \right\}, \\ \mathcal{V}_0 &= \left(H_0^1(0,\ell_1) \times L^2(0,\ell_1) \right) \times \left(H_0^1(0,\ell_2) \times L^2(0,\ell_2) \right). \end{aligned}$$

For $(\bar{\phi}_0, \bar{\phi}_1) \in \mathcal{W}_0$ we denote by $\bar{\phi}$ the solution of the homogeneous system for the three string network (4.2) with initial state $(\bar{\phi}_0, \bar{\phi}_1)$. We also consider $\bar{\psi} = (0, \psi^1, \psi^2)$, where (ψ^1, ψ^2) is the solution of the homogeneous system (4.11) with initial states (ϕ_0^1, ϕ_1^1) and (ϕ_0^2, ϕ_1^2), respectively.

Let us choose $T \geq 2(\ell_0 + \ell_1 + \ell_2)$ and denote

$$||(\bar{\phi}_0, \bar{\phi}_1)||_E^2 := \int_0^T |\phi_x^0(t,\ell_0)|^2 dt,$$

$$||(\bar{\phi}_0, \bar{\phi}_1)||_S^2 := \int_{\ell_0}^{T-\ell_0} |\psi_x^1(t,0) + \psi_x^2(t,0)|^2 dt.$$

In view of the results of Proposition 4.5 and Corollary 4.27, the functions $||.||_E$ and $||.||_S$ define norms in $\mathcal{W}_0$ and $\mathcal{V}_0$ respectively, if, and only if, ℓ_1/ℓ_2 is an irrational number and $T \geq 2(\ell_0 + \ell_1 + \ell_2)$.

Proposition 4.33. *There exists a constant $C > 0$ such that for every $(\bar{\phi}_0, \bar{\phi}_1) \in \mathcal{W}_0$,*

$$C||(\bar{\phi}_0, \bar{\phi}_1)||_E^2 \geq ||(\phi_0^0, \phi_1^0)||^2_{H_0^1(0,\ell_0) \times L^2(0,\ell_0)} + ||(\bar{\phi}_0, \bar{\phi}_1)||_S^2.$$

Proof. Let us observe that, if we apply D'Alembert formulas (3.5) to the component ϕ^0 we have, in account of the fact that $\phi_t^0(t,\ell_0) \equiv 0$,

$$\phi_x^0(t,0) = \ell_0^- \phi_x^0(t,\ell_0), \qquad \phi_t^0(t,0) = -\ell_0^+ \phi_x^0(t,\ell_0).$$

Then, from Proposition 3.3 we obtain the inequality

$$\int_0^T |\phi_x^0(t,\ell_0)|^2 dt \geq \max\{\int_{\ell_0}^{T-\ell_0} |\phi_x^0(t,0)|^2 dt, \int_{\ell_0}^{T-\ell_0} |\phi_t^0(t,0)|^2 dt\}. \quad (4.89)$$

On the other hand, if $\mathbf{E}_{\phi^0}$ is the energy of the component ϕ^0, from Proposition 3.1 it follows,

$$\mathbf{E}_{\phi^0}(0) \leq \int_{-\ell_0}^{\ell_0} |\phi_x^0(t,\ell_0)|^2 dt.$$

And then, from property (4.56) (see Proposition 4.18)

$$||(\phi_0^0, \phi_1^0)||^2_{H_0^1 \times L^2} = 2\mathbf{E}_{\phi^0}(0) \leq C \int_0^T |\phi_x^0(t,\ell_0)|^2 dt. \quad (4.90)$$

Let us observe now that the solution $\bar{\phi}$ may be decomposed as

$$\bar{\phi} = \bar{\psi} + \bar{\omega}, \quad (4.91)$$

were $\bar{\omega} = (\omega^0, \omega^1, \omega^2)$ is the unique solution of the problem

$$\begin{cases} \omega^i_{tt} - \omega^i_{xx} = 0 & & \text{in } \mathbb{R} \times [0,\ell_i], & i = 0,1,2, \\ \omega^i(t,\ell_i) = 0, & \omega^i(t,0) = \phi^i(t,0) & \text{in } \mathbb{R}, & i = 0,1,2, \\ \omega^0(0,x) = \phi^0_0(x), & \omega^0_t(0,x) = \phi^0_1(x) & \text{in } [0,\ell_0], & \\ \omega^i(0,x) = \omega^i_t(0,x) = 0, & & \text{in } [0,\ell_i], & i = 1,2. \end{cases} \quad (4.92)$$

Indeed, for every $i = 0,1,2$, the function $\bar{\eta} = \bar{\phi} - \bar{\psi} - \bar{\omega}$ satisfies

$$\begin{cases} \eta^i_{tt} - \eta^i_{xx} = 0 & \text{in } \mathbb{R} \times [0,\ell_i], \\ \eta^i(t,0) = \eta^i(t,\ell_i) = 0 & \text{in } \mathbb{R}, \\ \eta^i(0,x) = \eta^i_t(0,x) = 0 & \text{in } [0,\ell_i]. \end{cases}$$

Thus, $\bar{\eta} \equiv \bar{0}$, that is, (4.91) is verified. In particular,

$$\phi^i_x(t,0) = \psi^i_x(t,0) + \omega^i_x(t,0), \qquad i = 0,1,2. \quad (4.93)$$

In view of the coupling conditions

$$\phi^0_x(t,0) + \phi^1_x(t,0) + \phi^2_x(t,0) = 0,$$

from (4.93) it follows that

$$-\phi^0_x(t,0) = \psi^1_x(t,0) + \psi^2_x(t,0) + \omega^1_x(t,0) + \omega^2_x(t,0). \quad (4.94)$$

Thus, we have

$$\begin{aligned}\int_{\ell_0}^{T-\ell_0} & |\psi_x^1(t,0)+\psi_x^2(t,0)|^2 dt \\ &\leq \int_{\ell_0}^{T-\ell_0} |\psi_x^1(t,0)+\psi_x^2(t,0)+\omega_x^1(t,0)+\omega_x^2(t,0)|^2 dt \\ &\quad + \int_{\ell_0}^{T-\ell_0} |\omega_x^1(t,0)+\omega_x^2(t,0)|^2 dt \\ &\leq \int_{\ell_0}^{T-\ell_0} |\phi_x^0(t,0)|^2 dt + \int_{\ell_0}^{T-\ell_0} |\omega_x^1(t,0)+\omega_x^2(t,0)|^2 dt.\end{aligned}$$

On the other hand, if we apply Lemma 4.2 from [51] to the system (4.92), we obtain that there exists a constant $C>0$ such that

$$\int_{\ell_0}^{T-\ell_0} |\omega_x^1(t,0)+\omega_x^2(t,0)|^2 dt \leq C\int_{\ell_0}^{T-\ell_0} |\omega_t^0(t,0)|^2 dt = C\int_{\ell_0}^{T-\ell_0} |\phi_t^0(t,0)|^2 dt.$$

So, we arrive to the inequality

$$\int_{\ell_0}^{T-\ell_0} |\psi_x^1(t,0)+\psi_x^2(t,0)|^2 dt \leq \int_{\ell_0}^{T-\ell_0} |\phi_x^0(t,0)|^2 dt + C\int_{\ell_0}^{T-\ell_0} |\phi_t^0(t,0)|^2 dt,$$

and, in view of (4.89),

$$\int_{\ell_0}^{T-\ell_0} |\psi_x^1(t,0)+\psi_x^2(t,0)|^2 dt \leq C\int_0^T |\phi_x^0(t,\ell_0)|^2 dt. \tag{4.95}$$

Finally, combining the inequalities (4.90) and (4.95) the assertion of the proposition is obtained.

Proposition 4.34. *Let $\bar{g}\in H$ be a continuous function such that $g^0(0)\neq 0$. Then, there exists a constant $C>0$ such that for every $(\bar{\phi}_0,\bar{\phi}_1)\in \mathcal{W}_0$ and every $\lambda\in\mathbb{R}$,*

$$C||(\bar{\phi}_0+\lambda\bar{g},\bar{\phi}_1)||_E \geq ||(\phi_0^0,\phi_1^0)||_{H_0^1\times L^2} + ||(\bar{\phi}_0,\bar{\phi}_1)||_S.$$

Proof. Let us denote by $\bar{\varphi}_\lambda$ the solution of system (4.33) with initial state $(\bar{\phi}_0+\lambda\bar{g},\bar{\phi}_1)$. Let us observe that

$$|\lambda|^2 = |\phi_0^0(0)+\lambda g^0(0)|^2 \leq C||\phi_0^0+\lambda g^0||_{H^1}^2 \leq C\mathbf{E}_{\varphi_\lambda^0}(0). \tag{4.96}$$

As in the proof of Proposition 4.33 we may show that

$$\mathbf{E}_{\varphi_\lambda^0}(0) \leq C||(\bar{\phi}_0+\lambda\bar{g},\bar{\phi}_1)||_E^2. \tag{4.97}$$

Then, from the relations (4.96), (4.97) we have

$$||(\bar{\phi}_0,\bar{\phi}_1)||_E \leq ||(\bar{\phi}_0+\lambda,\bar{\phi}_1)||_E + |\lambda|||(\bar{g},\bar{0})||_E \leq C||(\bar{\phi}_0+\lambda\bar{g},\bar{\phi}_1)||_E$$

and the assertion holds from Proposition 4.33.

Proof (Proof of Theorem 4.32). Let us denote by $\mathcal{F}_S$ and $\mathcal{F}_E$ the completions of H and $\mathcal{V}_0$ with the norms $||.||_E$ and $||.||_S$, respectively. In account of the fact that

$$H = \mathbb{R}\bar{g} + \mathcal{W}_0,$$

Proposition 4.33 allows us to ensure that

$$\left(H_0^1(0,\ell_0) \times L^2(0,\ell_0)\right) \times \mathcal{F}_S \supset \mathcal{F}_E.$$

Then, the spaces of controllable initial states $\mathcal{C}_E = \mathcal{F}'_E$, $\mathcal{C}_S = \mathcal{F}'_S$ of systems (4.33) and (4.10) given by HUM satisfy the relation

$$\mathcal{C}_E \subset (L^2(0,\ell_0) \times H^{-1}(0,\ell_0)) \times \mathcal{C}_S.$$

In particular, if $\mathcal{V} \subset \mathcal{C}_S$ then

$$\mathcal{V} \subset (L^2(0,\ell_0) \times H^{-1}(0,\ell_0)) \times \mathcal{V},$$

and this is the assertion of the theorem.

Remark 4.35. The advantage of this approach that consists in deducing the controllability of the three string network from the simultaneous control of two strings is that:

a) It provides subspaces of controllable initial states of system (4.33) in which no restriction is imposed on the regularity of the components (u_0^0, u_1^0), other than being in $L^2(0,\ell_0) \times H^{-1}(0,\ell_0)$, in spite of what is needed in Corollary 4.30;

b) It is clear that no restriction on the length ℓ_0 is needed provided the control time is large enough ($T \geq 2(\ell_0 + \ell_1 + \ell_2)$). This is in agreement with common sense. Indeed, one expects that the initial data over the string whose external node is being controlled should be controlled correctly and that only further requirements should be needed on the data over the other two strings. Obviously, for that to be the case one has to take into account the extra time ($2\ell_0$) that controlling the third string $(0,\ell_0)$ adds.

4.9 Lack of Observability in Small Time

Due to the finite speed of propagation of waves along the strings of the network (equal to one in this case), it is natural to expect that, when the control time T is small the system is neither controllable nor observable. It turns out that this occurs whenever $T < T^* = 2(\ell_0 + \ell_1 + \ell_2)$. Consequently, the control time $T^* = 2(\ell_0 + \ell_1 + \ell_2)$ obtained in previous sections turns out to be sharp.

For an arbitrary network, the lack of spectral controllability for values of T smaller than twice its lengths may be proved on the basis of results from the Theory on Non Harmonic Fourier Series (more precisely, the Beurling-Malliavin theorem) and the asymptotic properties of the sequence of eigenvalues of the problem (see Chapter 6). However, for the three string network it is possible to give a completely elementary proof of this fact based on the explicit construction (shown in Figure 4.5) of a solution $\bar{\phi}$ of (4.2), whose trace $\phi_x^0(.,\ell_0)$ in the observation point vanishes during a time $T < T^*$. This allows to ensure that the system (4.33) is not even approximately controllable $T < T^*$.

In Figure 4.5 we draw the projection of the support of the solution to the uncontrolled strings $(0,\ell_1)$ and $(0,\ell_2)$. The Figure shows a family of rays crossing each other at the connecting node $x = 0$ so that waves cancel when crossing at $x = 0$. This can be done in any time $\tau < 2(\ell_1 + \ell_2)$. One can then extend this solution by zero to the observed string $(0,\ell_0)$ guaranteeing that the observed trace at $x = \ell_0$ vanishes for a time interval of length $T < 2(\ell_0 + \ell_1 + \ell_2)$.

The construction is thus the same as that of the lack of simultaneous approximate controllability of two strings of lengths a and b from a common end-point in time less than $T < 2(a + b)$. This is precisely the situation we represent in Figure 4.5. When $T \geq 2(a + b)$ this construction can only be performed when a/b is rational (an then this may be done for all time $T > 0$). This is in agreement with our results on the simultaneous controllability of two strings in section 4.2 that guarantee approximate and spectral controllability as soon as a/b is irrational and $T \geq 2(a + b)$.

The following holds

Theorem 4.36. *Let $T < T^*$. Then, there exist non-zero initial states*

$$(\bar{\phi}_0, \bar{\phi}_1) \in \bigcap_{\alpha \in \mathbb{R}} \mathcal{W}^\alpha,$$

for which the solution $\bar{\phi}$ of (4.2) satisfies

$$\phi_x^0(t,\ell_0) = 0 \qquad in \;\; [0,T]. \tag{4.98}$$

In the proof of this theorem we use some technical results. Let $T > 0$ and $0 < \sigma < T$. We define the operator $I_\sigma : L^2(0,T) \to L^2(0,T-\sigma)$ by the formula

$$(I_\sigma f)(t) := \int_t^{t+\sigma} f(\tau) d\tau.$$

For arbitrary values of σ_1, $\sigma_2 \in (0,T)$ the system of functional equations

$$\begin{cases} I_{\sigma_i} f_i = 0 & \text{a. e. in } (0, T - \sigma_i) \quad i = 1,2, \\ f_1 + f_2 = 0 & \text{a. e. in } (0,T), \end{cases} \tag{4.99}$$

admits the trivial solution $f_1 = f_2 = 0$. We need to study for which values of T this is the only solution of (4.99). The answer is given by the following

Lemma 4.37. *Let $T_0 = \sigma_1 + \sigma_2$. Then, if $T < T_0$, system (4.99) admits non-trivial solutions $f_i \in C^\infty([0,T])$, $i = 1,2$.*

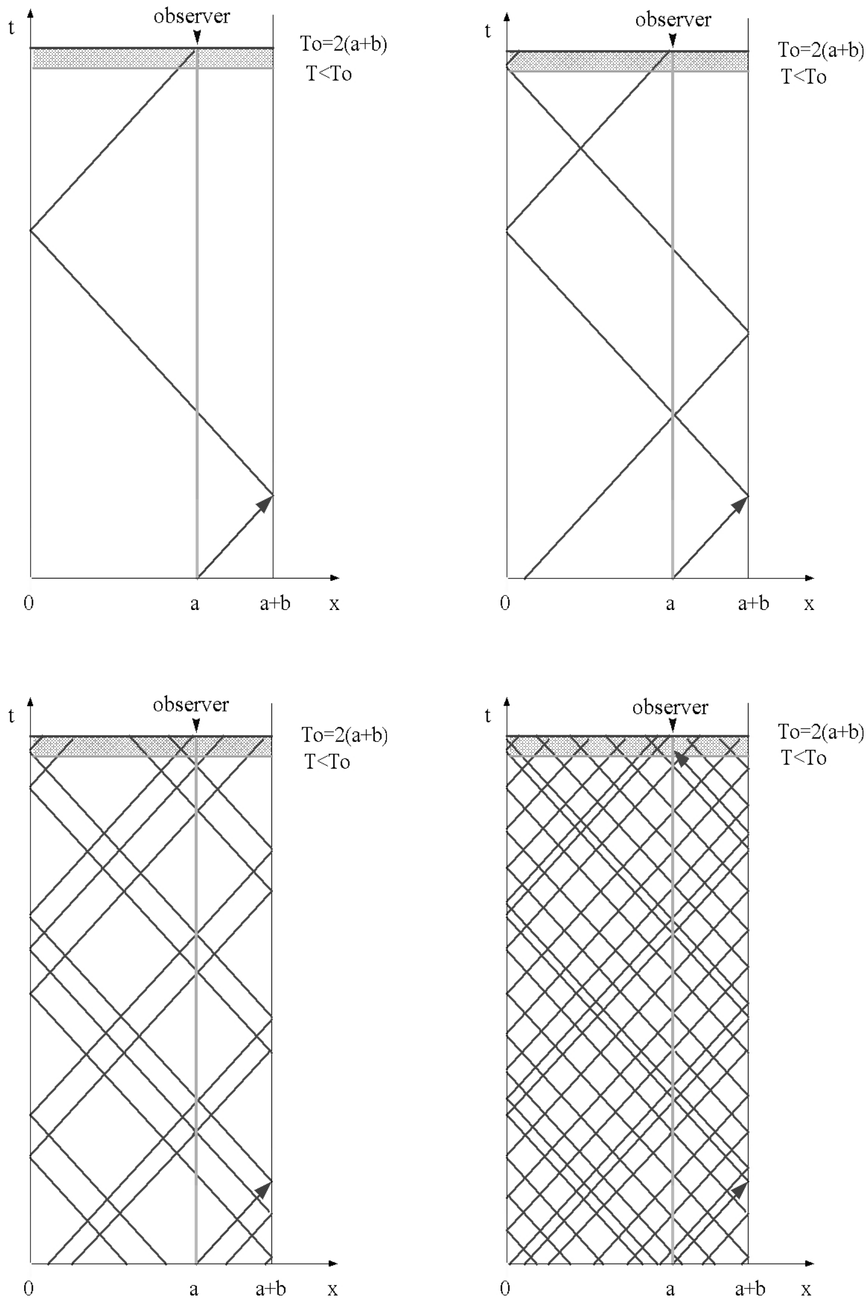

Fig. 4.5. Construction of the support of a non-observable solution

Before proving this lemma let us see how Theorem 4.36 may be obtained from it. It is clear that it is sufficient to prove Theorem 4.36 for large values of T so that we assume, without loss of generality, that $T \geq 2(\ell_0 + \hat{\ell})$, where $\hat{\ell} = \max(\ell_1, \ell_2)$.

Let f_1, f_2 be non-zero solutions of (4.99) for $\sigma_1 = 2\ell_1$, $\sigma_2 = 2\ell_2$ and $\tilde{T} = T - 2\ell_0$. We define the functions

$$\phi^i(t,x) = \frac{1}{2}\int_{t-x}^{t+x} f_i(\tau - \ell_0)d\tau, \qquad i = 1, 2,$$

for $x \in [0, \ell_i]$, $t \in [x + \ell_0, T - \ell_0 - x]$. These functions satisfy

$$(\mathrm{S}_i) \qquad \begin{cases} \phi^i_{tt}(t,x) = \phi^i_{xx}(t,x) \\ \phi^i(t,0) = \phi^i(t,\ell_i) = 0, \end{cases}$$

whenever $x \in [0, \ell_i]$ and $t \in [\ell_0 + x, T - \ell_0 - x]$.

Each of the functions ϕ^i may be extended to a solution of (S_i), which we will denote again by ϕ^i, defined in the region $[\ell_0, T - \ell_0] \times [0, \ell_i]$. Note that these functions have been chosen such that $\phi^i_x(t,0) = f_i(t)$ for $t \in [\ell_0, T - \ell_0]$. Besides,

$$\phi^1_x(t,0) + \phi^2_x(t,0) = f_1(t) + f_2(t) = 0 \quad \text{and} \quad \phi^1(t,0) = \phi^2(t,0) = 0.$$

Then, $\bar{\phi} = (\phi^0 = 0, \phi^1, \phi^2)$ is a solution of (4.2) defined in the time interval $[\ell_0, T - \ell_0]$. Consequently, the unique solution of (4.2) defined on $[0, T]$ that coincides with $\bar{\phi}$ on $[\ell_0, T - \ell_0]$ satisfies the vanishing condition (4.98).

It just remains to prove that the initial data of $\bar{\phi}$ belong to $\mathcal{W}^{\alpha}$ for every real α.

As $\phi^0 \equiv 0$ and $f_1, f_2 \in C^\infty([0,T])$ this is equivalent to proving that for some $T^* \in [\ell_0, T - \ell_0]$ and every $k \in \mathbb{N}$ the following inequalities hold

$$\frac{\partial^{2k}}{\partial x^{2k}}\phi^i(T^*,0) = \frac{\partial^{2k}}{\partial x^{2k}}\phi^i(T^*,\ell_i) = \frac{\partial^{2k+1}}{\partial x^{2k+1}}\phi^1(T^*,0) + \frac{\partial^{2k+1}}{\partial x^{2k+1}}\phi^2(T^*,0) = 0, \tag{4.100}$$

and

$$\frac{\partial^{2k}}{\partial x^{2k}}\phi^i_t(T^*,0) = \frac{\partial^{2k}}{\partial x^{2k}}\phi^i_t(T^*,\ell_i) = \frac{\partial^{2k+1}}{\partial x^{2k+1}}\phi^1_t(T^*,0) + \frac{\partial^{2k+1}}{\partial x^{2k+1}}\phi^2_t(T^*,0) = 0, \tag{4.101}$$

for $i = 1, 2$. These facts guarantee and characterize the property that the data of the constructed solution $\bar{\phi}$ belong to any power of the domain of the generator of the three-string system.

To check these identities we first observe that, if f is a smooth function, then

$$(I_\sigma f)^{(k)}(t) = (I_\sigma f^{(k)})(t).$$

This implies that, if f_1 and f_2 are smooth solutions of (4.99) then so are the functions $f_1^{(k)}$, $f_2^{(k)}$ for all $k \geq 1$.

Note also that, since ϕ^i solve the wave equation, in (4.100) and (4.101) one can replace any derivative of even order in x by the derivative of the same order in t.

Combining these facts and choosing, e.g., $T^* = \ell_0 + \hat{\ell}$ we obtain the equalities (4.100) and (4.101).

The proof of Lemma 4.37 is based on the following facts:

Proposition 4.38. *If $\sigma_1/\sigma_2 \in \mathbb{Q}$ then, for every $T > 0$ there exist non-trivial functions $\varphi \in C^\infty([0,T])$ such that*

$$I_{\sigma_1}\varphi = 0 \quad in \quad [0, T - \sigma_1], \quad I_{\sigma_2}\varphi = 0 \quad in \quad [0, T - \sigma_2]. \tag{4.102}$$

Proof. If $\sigma_1/\sigma_2 \in \mathbb{Q}$ there exist numbers $p, q \in \mathbb{N}$, $\gamma \in \mathbb{R}$ such that

$$\frac{\sigma_1}{p} = \frac{\sigma_2}{q} = \gamma.$$

Let $\varphi \in C^\infty(\mathbb{R})$ be a non trivial, γ-periodic function such that

$$\int_0^\gamma \varphi(\tau)d\tau = 0.$$

Then,

$$I_{\sigma_1}\varphi = \int_t^{t+\sigma_1} \varphi(\tau)d\tau = \int_t^{t+\gamma p} \varphi(\tau)d\tau = p\int_t^{t+\gamma} \varphi(\tau)d\tau = 0.$$

In a similar way it may be proved that $I_{\sigma_2}\varphi = 0$.

Proposition 4.39. *Let $\varepsilon > 0$, $T = \sigma_1 + \sigma_2 - \varepsilon$ and $\sigma_1/\sigma_2 \notin \mathbb{Q}$. Then, there exists a non-zero function $\varphi \in C^\infty([0,T])$, such that (4.102) holds.*

Proof. The real number σ_2 may be expressed as $\sigma_2 = n\sigma_1 + \omega$, $n \in \mathbb{N}$, $\omega \in (0, \sigma_1)$. Since σ_1/σ_2 is irrational, so are ω/σ_1 and ω/σ_2. Let us consider the sequence $\{\omega_k\}$ defined by

$$\omega_k \in (0, \sigma_1), \quad k\omega - \omega_k \in \sigma_1\mathbb{Z}$$

(the values of $k\omega$ modulo σ_1). As a consequence of the irrationality of ω/σ_1, we have $\omega_k \neq \omega_l$ if $k \neq l$ and that the sequence $\{\omega_k\}$ is dense in the interval $[0, \sigma_1]$. Then there exist $k_1 < 0$, $k_2 > 0$ such that $\omega_{k_1}, \omega_{k_2} \in [\sigma_1 - \varepsilon, \sigma_1)$ and $\omega_k \in [0, \sigma_1 - \varepsilon)$ for every k satisfying $k_1 < k < k_2$[5].

Now let us define the subsets of $[0, \sigma_1)$:

$$\Omega_k = (\omega_k, \omega_k + \gamma)$$

[5] In other words, $k_1 < 0$ and $k_2 > 0$ are the values of k with the smallest non-zero absolute value such that ω_k is in the interval $[\sigma_1 - \varepsilon, \sigma_1)$.

for $k_1 < k \le k_2$, where $\gamma > 0$ is sufficiently small so that it holds

$$\overline{\Omega_k} \cap \overline{\Omega_l} = \emptyset \ \text{ if } k \neq l \ \text{ and } \ \Omega_k \subset (0, \sigma_1) \ \text{ for } k_1 < k, l \le k_2.$$

It is not difficult to show that the sets Ω_k have the following properties:
(i) if $t \in \Omega_k$ with $k_1 < k < k_2$ and $t = \omega_k + \tau$ for some $\tau \in (0, \gamma)$ then,

$$\begin{aligned} t + \omega &= \omega_{k+1} + \tau & &\text{if } \omega_k < \omega_{k+1}, \\ t + \omega &= \omega_{k+1} + \tau - \sigma_1 & &\text{if } \omega_{k+1} < \omega_k. \end{aligned}$$

(ii) if $t \in [0, \sigma_1 - \varepsilon) \setminus \cup\Omega_k$, then,

$$\begin{aligned} t + \omega &\notin \cup\Omega_k & &\text{if } t < \sigma_1 - \omega, \\ t + \omega - \sigma_1 &\notin \cup\Omega_k & &\text{if } t \ge \sigma_1 - \omega. \end{aligned}$$

Let us choose now a function $\psi \in C^\infty([0, \sigma_1])$ with support contained in the interval $(0, \gamma)$ and satisfying $\int_0^\gamma \psi(\tau) d\tau = 0$ and define the function φ in $[0, \sigma_1]$ by

$$\varphi(t) = \begin{cases} \varphi(t - \omega_k) & \text{if } t \in \Omega_k, \\ 0 & \text{if } t \in [0, \sigma_1] \setminus \cup\Omega_k. \end{cases}$$

Then it follows $\varphi \in C^\infty([0, \sigma_1])$ and $\operatorname{supp}\varphi \subset \cup\Omega_k \subset (0, \sigma_1)$. In particular, the σ_1-periodic extension of φ to $\mathbb{R}$, which we still denote by φ, verifies $\varphi \in C^\infty(\mathbb{R})$.

Now, let us check that φ is in addition one of the functions, whose existence is asserted in the Proposition.

Let $t_1, t_2 \in [0, \sigma_1] \setminus \cup\Omega_k$, then

$$\begin{aligned} \int_{t_1}^{t_2} \varphi(\tau) d\tau &= \sum_{m : \Omega_m \subset (t_1, t_2)} \int_{\Omega_m} \varphi(\tau) d\tau = \sum_m \int_{\omega_m}^{\omega_m + \gamma} \varphi(\tau - \omega_m) d\tau \\ &= \sum_m \int_0^\gamma \varphi(\tau) d\tau = 0. \end{aligned}$$

In particular, if we choose $t_1 = 0$, $t_2 = \sigma_1$ we get $\int_0^{\sigma_1} \varphi(\tau) d\tau = 0$, and therefore, since φ is σ_1-periodic, $I_{\sigma_1}\varphi = 0$ in $\mathbb{R}$.

It remains to calculate $I_{\sigma_2}\varphi$ for the values of t in the interval $[0, \sigma_1 - \varepsilon)$. Two cases are possible:

Case 1. $t \in \Omega_k$ for some k. Then, $t = \omega_k + \tau$ with $\tau \in (0, \gamma)$. We will assume that $\omega_k < \omega_{k+1}$, since when $\omega_{k+1} < \omega_k$ the result is obtained in a similar way.

In view of the property (i) of the sets Ω_k mentioned above, we obtain

$$\begin{aligned} I_{\sigma_2}\varphi(t) &= \int_t^{t+\omega} \varphi(s) ds = \int_{\omega_k + \tau}^{\omega_{k+1} + \tau} \varphi(s) ds \\ &= \int_{\omega_k + \tau}^{\omega_k + \gamma} \varphi(s) ds + \int_{\omega_{k+1}}^{\omega_{k+1} + \tau} \varphi(s) ds + \int_{\omega_k + \gamma}^{\omega_{k+1}} \varphi(s) ds. \end{aligned}$$

But $\omega_k+\gamma$ and ω_{k+1} do not belong to $\cup\Omega_k$, $\int_{\omega_k+\gamma}^{\omega_{k+1}}\varphi(s)ds=0$, and thus

$$\begin{aligned} I_{\sigma_2}\varphi(t) &= \int_{\omega_k+\tau}^{\omega_k+\gamma}\varphi(s-\omega_k)ds+\int_{\omega_{k+1}}^{\omega_{k+1}+\tau}\varphi(s-\omega_{k+1})ds \\ &= \int_\tau^\gamma \psi(s)ds+\int_0^\tau \psi(s)ds=0. \end{aligned}$$

Case 2. $t\notin\cup\Omega_k$. In view of the property (ii), if $t<\sigma_1-\omega$, then $t+\omega$ does not belong to $\cup\Omega_k$ and we have

$$I_{\sigma_2}\varphi(t)=\int_t^{t+\omega}\varphi(s)ds=0.$$

If $t\geq\sigma_1-\omega$, then $t-\sigma_1+\omega\notin\cup\Omega_k$ and it holds

$$I_{\sigma_2}\varphi(t)=\int_t^{t+\omega-\sigma_1}\varphi(s)ds=0.$$

This proves the Proposition.

Proof (Proof of Lemma 4.37). It follows immediately from the previous propositions. It suffices to take $f_1=\varphi$ and $f_2=-\varphi$ according to Propositions 4.38 or 4.39, depending on whether σ_1/σ_2 is rational or irrational.

4.10 Application of the Method of Moments to the Control of the Three String Network

In this section we study the problem of moments (3.40), i. e.

$$\int_0^T \varkappa_{|k|}e^{i\lambda_k t}\ h(t)dt=u_{1,|k|}-i\lambda_k u_{0,|k|}\quad \text{for every } k\in\mathbb{Z}_*, \tag{4.103}$$

for the three string network. Recall that, in view of the results of Section 3.3, the existence of a solution $h\in L^2(0,T)$ of the problem of moments (4.103) is equivalent to the controllability in time T of the initial state $(\bar{u}_0,\bar{u}_1)$ with

$$\bar{u}_0=\sum_{n\in\mathbb{N}}u_{0,n}\bar{\theta}_n,\qquad \bar{u}_1=\sum_{n\in\mathbb{N}}u_{1,n}\bar{\theta}_n.$$

Thus, this is an alternative way to study the control problem for system (4.33).

Performing the change of variable $t\to t-T/2$ in (4.103), we obtain

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\lambda_n t}\ h(t-\frac{T}{2})dt=\frac{1}{\varkappa_n}\left(u_{1,n}-i\lambda_n u_{0,n}\right)e^{i\lambda_n T/2}.$$

Denoting $m_n := \varkappa_n^{-1}(u_{1,n} - i\lambda_n u_{0,n})\, e^{i\lambda_n T/2}$, $A := T/2$, problem (4.103) will be written in the form (3.28). This implies, in account of Proposition 3.18, that, if we can construct a sequence (v_n) biorthogonal to $\left(e^{i\lambda_n t}\right)$ in $L^2(-T/2, T/2)$ then the initial states satisfying

$$\sum_{n\in\mathbb{Z}_*} \left| \frac{1}{\varkappa_n} (u_{1,n} - i\lambda_n u_{0,n})\, e^{i\lambda_n T/2} \right| \|v_n\|_{L^2} < \infty \tag{4.104}$$

are controllable in time T with control

$$v = \sum_{n\in\mathbb{Z}_*} \frac{1}{\varkappa_n} (u_{1,n} - i\lambda_n u_{0,n})\, e^{i\lambda_n T/2} v_n.$$

Inequality (4.104) is equivalent to

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{|\varkappa_n|} \left(|u_{1,n}| + \lambda_n |u_{0,n}|\right) \|v_n\|_{L^2} < \infty.$$

In particular, if the biorthogonal sequence (v_n) has been obtained from a generating function F then all the initial states satisfying

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{|\varkappa_n|\, |F'(\lambda_n)|} \left(|u_{1,n}| + \lambda_n |u_{0,n}|\right) < \infty \tag{4.105}$$

are controllable in time T.

Let us remark that for the three string network it is easy to construct a generating function, since we already know a function that vanishes at the numbers λ_n. Indeed, recall that, as it has been shown in Proposition 4.20, $q(\lambda_n) = 0$, where

$$\begin{aligned} q(z) &= \cos z\ell_0 \sin z\ell_1 \sin z\ell_2 + \sin z\ell_0 \cos z\ell_1 \sin z\ell_2 \\ &\quad + \sin z\ell_0 \cos z\ell_1 \sin z\ell_1 \cos z\ell_2, \end{aligned} \tag{4.106}$$

and this is an entire function bounded on the real axis: $|q(z)| \le 3$.

On the other hand, if we replace in (4.106) $\cos(z\ell_k)$ and $\sin(z\ell_k)$ by their expressions in terms of complex exponentials

$$\cos z\ell_k = \frac{1}{2}\left(e^{iz\ell_k} + e^{-iz\ell_k}\right), \qquad \sin z\ell_k = -\frac{i}{2}\left(e^{iz\ell_k} - e^{-iz\ell_k}\right),$$

we see that q may be written as a linear combination of eight terms of the form e^{izh}, with

$$|h| \le \ell_0 + \ell_1 + \ell_2.$$

Then, there exists a constant $C > 0$ such that, for every $z \in \mathbb{C}$,

$$|q(z)| \le Ce^{|z|(\ell_0+\ell_1+\ell_2)},$$

that is, the function q is of exponential type at most $\ell_0 + \ell_1 + \ell_2$.

Then, based on the results of Subsection 3.3.1, we can assert that there exists a sequence (v_n) biorthogonal to $\left(e^{i\lambda_n t}\right)$ in any interval $(-T/2, T/2)$ with $T \geq 2(\ell_0 + \ell_1 + \ell_2)$ that satisfies

$$||v_n||_{L^2(-\frac{T}{2},\frac{T}{2})} \leq \frac{C}{|q'(\lambda_n)|}, \quad n \in \mathbb{N}, \tag{4.107}$$

where the constant $C > 0$ does not depend on n.

This guarantees immediately that the spaces of sequences for which the problem of moments (4.103) has a solution is dense in l^2. Therefore, the space of controllable initial states in time $T \geq 2(\ell_0 + \ell_1 + \ell_2)$, is dense in $H \times V'$. Moreover, all the initial states from $Z \times Z$ are controllable. That means that spectral controllability holds as well.

Now we estimate $|q'(\lambda_n)|$ in order to identify larger subspaces of controllable initial states. Observe that the function q may be written in the form

$$q(z) = \sin z\ell_0 \sin z\ell_1 \sin z\ell_2 \left(\cot z\ell_0 + \cot z\ell_1 + \cot z\ell_2\right). \tag{4.108}$$

Then

$$|q'(\lambda_n)| = |\sin \lambda_n\ell_0 \sin \lambda_n\ell_1 \sin \lambda_n\ell_2|\, \mathbf{A}_n, \tag{4.109}$$

with

$$\mathbf{A}_n = \left(\frac{\ell_0}{\sin^2 \lambda_n\ell_0} + \frac{\ell_1}{\sin^2 \lambda_n\ell_1} + \frac{\ell_2}{\sin^2 \lambda_n\ell_2}\right).$$

In account of (4.105), we can ensure that the initial states satisfying

$$\sum_{n\in\mathbb{Z}_*} \frac{1}{|\varkappa_n|\,|q'(\lambda_n)|} \left(|u_{1,n}| + \lambda_n |u_{0,n}|\right) < \infty \tag{4.110}$$

are controllable.

To make this condition more precise, we need to estimate the product $|\varkappa_n|\,|q'(\lambda_n)|$. Recall that (see Remark 4.31)

$$|\varkappa_n| = \frac{\sqrt{2}\lambda_n}{|\sin \lambda_n\ell_0|} \mathbf{A}_n^{-\frac{1}{2}},$$

and thus we have

$$|\varkappa_n|\,|q'(\lambda_n)| = \sqrt{2}\lambda_n\, |\sin \lambda_n\ell_1 \sin \lambda_n\ell_2|\, \mathbf{A}_n^{\frac{1}{2}}.$$

Then,

$$\begin{aligned}|\varkappa_n|^2\,|q'(\lambda_n)|^2 &= 2\lambda_n^2\, |\sin \lambda_n\ell_1 \sin \lambda_n\ell_2|^2 \left(\frac{\ell_0}{\sin^2 \lambda_n\ell_0} + \frac{\ell_1}{\sin^2 \lambda_n\ell_1} + \frac{\ell_2}{\sin^2 \lambda_n\ell_2}\right)\\ &\geq 2\lambda_n^2 \left(\ell_1 \sin^2 \lambda_n\ell_2 + \ell_2 \sin^2 \ell_1\right) \geq C\lambda_n^2 c_n^2.\end{aligned}$$

Here $c_n = \max\left(\left|\sin \lambda_n \ell_1\right|, \left|\sin \lambda_n \ell_2\right|\right)$ are the coefficients defined by (4.75) in Section 4.6.

With this we can conclude that a sufficient condition for the initial state $(\bar{u}_0, \bar{u}_1)$ to be controllable is

$$\sum_{n \in \mathbb{Z}_*} \frac{1}{c_n \lambda_n} \left(\left| u_{1,n} \right| + \lambda_n \left| u_{0,n} \right| \right) < \infty. \tag{4.111}$$

Let us observe that this result is weaker than that given in Proposition 4.6, since, if the initial state $(\bar{u}_0, \bar{u}_1)$ satisfies (4.111) then it also satisfies

$$\sum_{n \in \mathbb{Z}_*} \frac{1}{c_n^2 \lambda_n^2} \left(u_{1,n}^2 + \lambda_n u_{0,n}^2 \right) < \infty.$$

Let us choose $\delta > 0$. The series

$$\sum_{n \in \mathbb{Z}_*} \frac{1}{\lambda_n^{1+\delta}}$$

converges for every $\delta > 0$, as

$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{\pi}{\ell_0 + \ell_1 + \ell_2}.$$

Then, with the help of the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} &\sum_{n \in \mathbb{Z}_*} \frac{1}{c_n \lambda_n} \left(\left| u_{1,n} \right| + \lambda_n \left| u_{0,n} \right| \right) \\ &\quad \leq \left(\sum_{n \in \mathbb{Z}_*} \frac{\lambda_n^{\delta - 1}}{c_n^2} \left(u_{1,n}^2 + \lambda_n^2 u_{0,n}^2 \right) \right)^{1/2} \left(\sum_{n \in \mathbb{Z}_*} \frac{1}{\lambda_n^{1+\delta}} \right)^{1/2} \\ &\quad \leq C \left(\sum_{n \in \mathbb{Z}_*} \frac{\lambda_n^{\delta - 1}}{c_n^2} \left(u_{1,n}^2 + \lambda_n^2 u_{0,n}^2 \right) \right)^{1/2}. \end{aligned}$$

Thus, for (4.111) to be verified and consequently for the initial state $(\bar{u}_0, \bar{u}_1)$ to be controllable in time T, it is sufficient that

$$\sum_{n \in \mathbb{Z}_*} \frac{\lambda_n^{\delta - 1}}{c_n^2} \left(u_{1,n}^2 + \lambda_n^2 u_{0,n}^2 \right) < \infty.$$

In particular, if $\ell_1/\ell_2 \in \mathbf{B}_\varepsilon$, in view of (4.85) we have

$$c_n \geq \frac{C}{\lambda_n^{1+\varepsilon}},$$

and, consequently, the controllability condition (4.111) obtained with the method of moments guarantees that all the initial states from

$$(\bar{u}_0, \bar{u}_1) \in \mathcal{W}^{\frac{3}{2}+\varepsilon+\delta} = V^{\frac{3}{2}+\varepsilon+\delta} \times V^{\frac{1}{2}+\varepsilon+\delta},$$

with arbitrarily small $\delta > 0$ are controllable.

Thus, we need roughly $1/2$ more derivatives in L^2 on the initial data for the method of moments than in Corollary 4.30. This difference may be due to the possible inaccuracy in the estimates of the sequence $|\varkappa_n| \, |q'(\lambda_n)|$.

Remark 4.40. According to Proposition 3.25, once we have identified subspaces of controllable initial states in time T of the form $\mathcal{W}^r$, we can construct *a posteriori* a sequence $(\tilde{v}_n)$, biorthogonal to $\left(e^{i\lambda_n t}\right)$ in $L^2(0,T)$, satisfying

$$\|\tilde{v}_n\|_{L^2(0,T)} \le C\lambda_n^{r-1}.$$

Thus, in view of Corollary 4.30, if $\ell_1/\ell_2 \in \mathbf{B}_\varepsilon$, for the system of the three string network, a sequence $(\tilde{v}_n)$ biorthogonal to $\left(e^{i\lambda_n t}\right)$ in $L^2(0,T)$ can be constructed verifying

$$\|\tilde{v}_n\|_{L^2(0,T)} \le C\lambda_n^{\varepsilon}. \tag{4.112}$$

Let us remark that the biorthogonal sequence (v_n) used in this section does not necessarily coincide with $(\tilde{v}_n)$. Recall in addition, that we do not resort to that sequence, since we got information on controllable subspaces without using the information provided by Corollary 4.30.

Let us try a sharper estimate of the sequence (v_n). In view of (4.107) it suffices to estimate $|q'(\lambda_n)|$. From equality (4.108) we obtain

$$\begin{aligned}|q'(\lambda_n)| &\ge \ell_0 \left|\sin\lambda_n\ell_1 \sin\lambda_n\ell_2\right| + \ell_1 \left|\sin\lambda_n\ell_0 \sin\lambda_n\ell_2\right| + \ell_2 \left|\sin\lambda_n\ell_0 \sin\lambda_n\ell_1\right| \\ &\ge Cs(\lambda, \ell_0, \ell_1, \ell_2),\end{aligned}$$

where we have denoted

$$s(\lambda, \ell_0, \ell_1, \ell_2) := \left|\sin\lambda_n\ell_0\right| \left|\sin\lambda_n\ell_1\right| + \left|\sin\lambda_n\ell_0\right| \left|\sin\lambda_n\ell_2\right| + \left|\sin\lambda_n\ell_1\right| \left|\sin\lambda_n\ell_2\right|.$$

To obtain lower bounds of the function s we need to impose additional restrictions on the lengths ℓ_0, ℓ_1, ℓ_2. Let us assume that those lengths satisfy the following rational approximation conditions, which we will call briefly *conditions* (S) (see also Definition A.9 in Appendix A):

- ℓ_0, ℓ_1, ℓ_2 are linearly independent over the field $\mathbb{Q}$ of rational numbers;
- all the ratios ℓ_i/ℓ_j are algebraic numbers, that is, roots of polynomials with rational coefficients.

Under these hypotheses in Proposition A.11 it is proved that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for every $n = 1, 2, ...$, the following inequality holds

$$s(\lambda_n, \ell_0, \ell_1, \ell_2) \ge C_\varepsilon \left(\lambda_n\right)^{-1-e}.$$

This guarantees that

$$\|v_n\|_{L^2(-T/2,T/2)} \leq C\lambda_n^{1+\varepsilon}.$$

Unfortunately, we have imposed restrictive conditions on ℓ_0 and we have been able to prove an estimate weaker than (4.112). This could be caused by two reasons: that the norms of the elements of the sequence (v_n) are actually larger than those of the elements of the sequences $(\tilde{v}_n)$ or that the technique we have used to estimate $|q'(\lambda_n)|$ is not sharp.

The obtention of the optimal controllability results for the three string network by the method of moments is therefore an open problem.

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