Introduction

The n-dimensional metaplectic group $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ is the twofold cover of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$, which is the group of linear transformations of $\mathcal{X} = \mathbb{R}^n \times \mathbb{R}^n$ that preserve the bilinear (alternate) form

$$[\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right), \left(\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right)] = -\langle x,\, \eta\rangle + \langle y,\, \xi\rangle. \tag{0.1}$$

There is a unitary representation of $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ in the Hilbert space $L^2(\mathbb{R}^n)$, called the metaplectic representation, the image of which is the group of transformations generated by the following ones: the linear changes of variables, the operators of multiplication by exponentials with pure imaginary quadratic forms in the exponent, and the Fourier transformation; some normalization factor enters the definition of the operators of the first and third species. The metaplectic representation was introduced in a great generality in [28] – special cases had been considered before, mostly in papers of mathematical physics – and it is of such fundamental importance that the two concepts (the group and the representation) have become virtually indistinguishable. This is not going to be our point of view: indeed, the main point of this work is to show that a certain finite covering of the symplectic group (generally of degree n) has another interesting representation, which enjoys analogues of most of the nicer properties of the metaplectic representation. We shall call it the anaplectic representation – other coinages that may come to your mind sound too medical – and shall consider first the one-dimensional case, the main features of which can be described in quite elementary terms.

It may not be an exaggeration to claim that among the foundational objects of classical analysis, the one-dimensional Gaussian function $e^{-\pi x^2}$ occupies one of the foremost positions: it is central in Fourier analysis and special function theory, everywhere in probability and, through its appearance in theta functions, it is basic in modular form theory as well. With the help of some of its satellites – the Heisenberg representation and Bargmann–Fock transform, the metaplectic representation, the Weyl calculus – it lies again at the core of fundamental methods of harmonic analysis or partial differential equations; it is also the basis of some mathematical techniques used in quantum field theory.

A starting point of the present work might be the fact that there is an alternative to this function, leading to a different kind of analysis but with a possibly wide range of influence too: this is the Bessel function $|x|^{\frac{1}{2}}\, I_{-\frac{1}{4}}(\pi\, x^2)$, which lies in the null space of the (formal) harmonic oscillator. It has at infinity the considerable growth of the more obvious function $|x|^{-\frac{1}{2}}\, e^{\pi x^2}$: therefore, it cannot, in general, occur in integrals on the real line of the usual type. Actually, the development of the present analysis requires that we stray away from the usual one in several aspects. Possibly the only mathematical object which will remain as it stands, at least formally, is the Heisenberg representation: but a new notion of integral – not destroying the invariance under translations – will be needed, and the Fourier transformation and associated Weyl calculus of operators will be replaced by some different, quite parallel objects; finally, the usual L^2 scalar product will have to be changed to an indefinite pseudoscalar product.

Turning to the n-dimensional case, let us first recall that the role of the homogeneous space $\mathrm{Sp}(n,\mathbb{R})/U(n)$ in analysis is well documented. On one hand, it is the set of *complex* polarizations of $\mathcal{X}$, *i.e.,* the set of complex structures on this space such that the symplectic form appears as the imaginary part of some (Hilbert) scalar product on $\mathcal{X}$; on the other hand, it is a Hermitian domain (Siegel's domain), a natural place for analysis in Bergman's style. What is more important here is that one may realize the space $L^2(\mathbb{R}^n)$ as a space of vector-valued functions on Siegel's domain, in a way that makes the metaplectic representation appear as quite natural. To introduce the anaplectic representation, we substitute for Siegel's domain a finite covering $\Sigma^{(n)}$ of the space $U(n)/O(n)$ of *real* polarizations of $\mathcal{X}$, *i.e.,* the space of Lagrangian subspaces of $\mathcal{X}$. Again, we consider a certain space of vector-valued functions on $\Sigma^{(n)}$, getting in a natural way a new representation of some covering of the symplectic group as a result. These functions can in turn be identified with scalar functions on $\mathbb{R}^n$: however, in contradiction to the metaplectic case, the class of functions on $\mathbb{R}^n$ which enter the new analysis consists only of functions which extend as entire functions on $\mathbb{C}^n$. The one–dimensional case of this analysis coincides with the one hinted at above. A common point of the metaplectic and anaplectic representations is that each of the two groups of operators normalizes the group of operators arising from the Heisenberg representation: the latter one is formally the same in both cases. The anaplectic representation (only) can be enriched by a rotation of ninety degrees in the complex coordinates on $\mathbb{C}^n$, an operation that corresponds to the matrix $\left(\begin{smallmatrix} -i\,I & 0 \\ 0 & i\,I \end{smallmatrix}\right)$.

The development of anaplectic analysis calls for mathematical techniques rather different from the usual ones, as it depends as much on elementary real algebraic geometry as on Hilbert space methods. Some of the main questions that have to be tackled concern the analytic continuation of functions, and depend on a careful examination of the singularities of certain fractional-linear transformations; homotopy considerations often play a role too.

Except in the one-dimensional case, it seems unlikely that one could define a space of functions on $\mathbb{R}^n$, invariant under the full anaplectic representation, and on which an invariant pseudoscalar product could be defined. However, anaplectic analysis is not concerned solely with representation by the same name. In anaplectic analysis, the spectrum of the harmonic oscillator L is $\mathbb{Z}$ rather than $\frac{n}{2} + \mathbb{N}$, and the usual creation and annihilation operators become raising and lowering operators; also, unless $n = 1$, all the eigenspaces of L are infinite-dimensional. Provided that $n \not\equiv 0 \mod 4$, one can build, in a way unique up to normalization, a pseudoscalar product on the space generated by the eigenfunctions of L just alluded to, with respect to which the infinitesimal generators of the Heisenberg representation are self-adjoint.

Despite its many similarities with the usual analysis, anaplectic analysis differs from it in two major respects. First, there is no natural embedding of, say, the group of one-dimensional anaplectic transformations into the group of two-dimensional ones, that would generalize what is obtained, in the usual analysis, by regarding one of a pair of variables as a parameter. On the other hand, there is in the usual analysis a class of quite simple functions, to wit the exponentials with a second-order polynomial (the real part of which has a positive-definite top-order part) in the exponent, which resists all operations taken from the Heisenberg representation or the metaplectic representation. No comparable class can be described in such simple terms in anaplectic analysis. This is why non-trivial identities can sometimes be obtained by calculations the analogues of which, in the usual analysis, would not produce anything interesting: examples will occur in Section 10.

In the last chapter, we imbed the one-dimensional anaplectic analysis into a one-parameter family of analyses. There is one such analysis for every complex number $\nu \mod 2$, $\nu \notin \mathbb{Z}$: the case when ν is an integer should be regarded as leading to the usual analysis, the case when $\nu = -\frac{1}{2} \mod 2$ is that considered in Section 1. In each case, there is a translation-invariant concept of integral, an associated Fourier transformation and ν-anaplectic representation. When ν is real, $\nu \notin \mathbb{Z}$, there is on the basic relevant space $\mathfrak{A}_\nu$ a pseudoscalar product, invariant both under the Heisenberg representation and under the ν-anaplectic representation: besides, this latter representation, when restricted to the space of even, or odd, functions on $\mathfrak{A}_\nu$ (this depends on whether $\nu \in]-1,\, 0[+2\,\mathbb{Z}$ or $\nu \in]0,\, 1[+2\,\mathbb{Z}$), is unitarily equivalent to one of the representations of the universal cover of $SL(2,\mathbb{R})$ as made explicit in [18]; not surprisingly, the series that occurs here is one which does not occur in the Plancherel theorem for the group under consideration.

It is our hope, and belief, that anaplectic analysis will prove useful in several domains: in quantum mechanics (especially in relativistic quantum mechanics), in partial differential equations, in special function theory. Let us only observe to start with that a mathematical analysis based on a harmonic oscillator unbounded from below cannot fail to help in questions in which we would like to have time circulate just as well in two directions. Also, the pseudoscalar product which occurs in the one-dimensional anaplectic analysis has a striking similarity to that which plays a role in the covariant formulation [5, p. 384] or [3, p. 68] of quantum electrodynamics. Concerning the possibility of using anaplectic analysis in partial differential equations, this only has, as yet, the status of wishful thinking. We have, however, initiated the study of the anaplectic Weyl calculus: though we have mostly dealt, up to now, with its more formal aspects only, one may expect that some kind of new pseudodifferential analysis will eventually emerge. Under the name of "Krein spaces", the subject of linear spaces with an indefinite metric is currently under much scrutiny, in particular in connection with spectral problems of an unusual type (*cf.* for instance [19]); such a kind of problems has also been considered by several authors [1, 2] for reasons having to do with PT-symmetry. Anaplectic analysis certainly provides a special domain of research related to this question, with a rich harmonic analysis of its own. Also, when it is completed, the anaplectic pseudodifferential analysis might be a useful tool for this kind of problems in general. Some possible connection between the one-dimensional anaplectic pseudodifferential analysis and a variant of the Lax–Phillips scattering theory for the automorphic wave equation has been briefly hinted at at the end of Section 10. Finally, but this goes beyond our current projects, there is the question whether some version of the anaplectic representation could be developed in the case of local fields such as the fields of p-adic numbers or their quadratic extensions, thus following in the steps of Weil's celebrated paper [28] on the metaplectic representation.

Let me apologize to M. Gell–Mann and Y. Ne'eman [8] for my choice of a title: I simply could not resist its poetic appeal. On the other hand, the first section of this volume will show that no other choice was possible.

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