

Preface

In this book, we intend to give an extensive treatment of the basic theory of general near polygons. The subject of near polygons has been around for about 25 years now. Excellent handbooks have appeared on certain important subclasses of near polygons like generalized quadrangles ([82]) and generalized polygons ([100]), but no book has ever occurred dealing with the topic of general near polygons. Although generalized polygons and especially generalized quadrangles are indispensable to the study of near polygons, we do not aim at giving a profound study of these incidence structures. In fact, this book can be seen as complementary to the two above-mentioned books.

Although generalized quadrangles and generalized polygons were intensively studied since they were introduced by Tits in his celebrated paper on triality ([96]), the terminology near polygon first occurred in a paper in 1980. In [91], Shult and Yanushka showed the connection between the so-called tetrahedrally closed line-systems in Euclidean spaces and a class of point-line geometries which they called *near polygons*. In [91], also some very fundamental results regarding the geometric structure of near polygons were obtained, like the existence of quads, a result which was later generalized by Brouwer and Wilbrink [16] who showed that any dense near polygon has convex subpolygons of any feasible diameter. The paper [16] gives for the first time a profound study of dense near polygons. Other important papers on near polygons from the 1980s and the beginning of the 1990s deal with dual polar spaces, the classification of regular near polygons in terms of their parameters and the classification of the slim dense near hexagons. The subject of near polygons has regained interest in the last years. Important new contributions to the theory were the theory of glued near polygons, the theory of valuations and important breakthrough results regarding the classification of dense near polygons with three and four points on every line. These new contributions will be discussed extensively in this book.

This book essentially consists of two main parts. In the first part of the book, which consists of the first five chapters, we develop the basic theory of near polygons. In Chapters 2, 3 and 4, we study three classes of near polygons: the dense, the regular and the glued near polygons. Our treatment of the dense and glued near polygons is rather complete. The treatment of the regular near

polygons is concise and results are not always accompanied with proofs. More detailed information on regular near polygons can be found in the book *Distance-regular graphs* [13] by Brouwer, Cohen and Neumaier. In that book regular near polygons are considered as one of the main classes of distance-regular graphs. In Chapter 5, we discuss the notion of *valuation of a near polygon* which is a very important tool for classifying near polygons.

The second main part of this book, which consists of Chapters 6–9, discusses the problem of classifying all slim dense near polygons. These are dense near polygons with three points on every line. There is a conjecture which states that all these near polygons can be obtained by applying the direct product and the gluing construction to a list of near polygons which consists of five infinite classes and three exceptional near hexagons. In Chapter 6, we will discuss all the near polygons of that list, thereby providing information on their convex subpolygons, spreads of symmetry and valuations. The first main result with respect to the above-mentioned classification was obtained by Brouwer, Cohen, Hall and Wilbrink [12] who succeeded in classifying all slim dense near hexagons. Their proof relies on Fisher’s theory of groups generated by 3-transpositions [72] and Buekenhout’s geometric interpretation of that theory [18]. In Chapter 7, we will give a purely combinatorial proof of the classification of the slim dense near hexagons. In Chapter 8, we classify all slim dense near polygons with a so-called nice chain of convex subpolygons. All known slim dense near polygons, except for the ones with a so-called \mathbb{E}_1 - or \mathbb{E}_2 -hex, have such a chain of subpolygons. In Chapter 9, we will give a complete classification of all slim dense near octagons. In order to obtain the classification results mentioned in Chapters 7, 8 and 9, we must invoke almost the whole theory of near polygons which we developed in the earlier chapters. It is our sincere hope that the techniques provided in Chapters 1–9 will ultimately lead to a complete classification of all slim dense near polygons.

In the final chapter we will discuss nondense slim near hexagons and in the appendix we will give an overview of what is known on the classification of dense near polygons with four points on each line.

The study of near polygons is rather interesting and some of the tools which we developed for dealing with general near polygons seem to have nice applications to some specific classes of near polygons. We give two examples.

Admissible triples were originally introduced for the study of glued near hexagons. One of the advantages of these structures was that they allowed a unified construction for several classes of generalized quadrangles ([29]). Recently, these structures have also been used to obtain some interesting characterization results regarding the symplectic generalized quadrangle $W(q)$ ([49], [55]).

Several of the near polygon techniques which we will discuss throughout this book turn out to be useful for the study of certain important substructures of dual polar spaces. E.g., the study of valuations of dual polar spaces has led to two new classes of hyperplanes in dual polar spaces ([61]). Hyperplanes of dual

polar spaces are often near polygons themselves and/or contain near polygons as substructures, see e.g. [57].

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