

Contents

Preface	xiii
Introduction	xiii
Organization	xiv
Prerequisites	xv
Bibliography	xv
Acknowledgements	xv
About the Author	xvi
Notation	xvii
Number sets	xvii
Classical matrix groups	xvii
Vector calculus	xviii
Function spaces and multi-index notation	xix
Combinatorial notation	xx

Part I: Symplectic Geometry

1 Symplectic Spaces and Lagrangian Planes	
1.1 Symplectic Vector Spaces	3
1.1.1 Generalities	3
1.1.2 Symplectic bases	7
1.1.3 Differential interpretation of σ	9
1.2 Skew-Orthogonality	11
1.2.1 Isotropic and Lagrangian subspaces	11
1.2.2 The symplectic Gram–Schmidt theorem	12
1.3 The Lagrangian Grassmannian	15
1.3.1 Lagrangian planes	15
1.3.2 The action of $\mathrm{Sp}(n)$ on $\mathrm{Lag}(n)$	18

1.4	The Signature of a Triple of Lagrangian Planes	19
1.4.1	First properties	20
1.4.2	The cocycle property of τ	23
1.4.3	Topological properties of τ	24
2	The Symplectic Group	
2.1	The Standard Symplectic Group	27
2.1.1	Symplectic matrices	29
2.1.2	The unitary group $U(n)$	33
2.1.3	The symplectic algebra	36
2.2	Factorization Results in $Sp(n)$	38
2.2.1	Polar and Cartan decomposition in $Sp(n)$	38
2.2.2	The “pre-Iwasawa” factorization	42
2.2.3	Free symplectic matrices	45
2.3	Hamiltonian Mechanics	50
2.3.1	Hamiltonian flows	51
2.3.2	The variational equation	55
2.3.3	The group $Ham(n)$	58
2.3.4	Hamiltonian periodic orbits	61
3	Multi-Oriented Symplectic Geometry	
3.1	Souriau Mapping and Maslov Index	66
3.1.1	The Souriau mapping	66
3.1.2	Definition of the Maslov index	70
3.1.3	Properties of the Maslov index	72
3.1.4	The Maslov index on $Sp(n)$	73
3.2	The Arnol’d–Leray–Maslov Index	74
3.2.1	The problem	75
3.2.2	The Maslov bundle	79
3.2.3	Explicit construction of the ALM index	80
3.3	q -Symplectic Geometry	84
3.3.1	The identification $Lag_{\infty}(n) = Lag(n) \times \mathbb{Z}$	85
3.3.2	The universal covering $Sp_{\infty}(n)$	87
3.3.3	The action of $Sp_q(n)$ on $Lag_{2q}(n)$	91
4	Intersection Indices in $Lag(n)$ and $Sp(n)$	
4.1	Lagrangian Paths	95
4.1.1	The strata of $Lag(n)$	95
4.1.2	The Lagrangian intersection index	96
4.1.3	Explicit construction of a Lagrangian intersection index	98
4.2	Symplectic Intersection Indices	100

4.2.1	The strata of $\mathrm{Sp}(n)$	100
4.2.2	Construction of a symplectic intersection index	101
4.2.3	Example: spectral flows	102
4.3	The Conley–Zehnder Index	104
4.3.1	Definition of the Conley–Zehnder index	104
4.3.2	The symplectic Cayley transform	106
4.3.3	Definition and properties of $\nu(S_\infty)$	108
4.3.4	Relation between ν and μ_{ℓ_P}	112

Part II: Heisenberg Group, Weyl Calculus, and Metaplectic Representation

5 Lagrangian Manifolds and Quantization

5.1	Lagrangian Manifolds and Phase	123
5.1.1	Definition and examples	124
5.1.2	The phase of a Lagrangian manifold	125
5.1.3	The local expression of a phase	129
5.2	Hamiltonian Motions and Phase	130
5.2.1	The Poincaré–Cartan Invariant	130
5.2.2	Hamilton–Jacobi theory	133
5.2.3	The Hamiltonian phase	136
5.3	Integrable Systems and Lagrangian Tori	139
5.3.1	Poisson brackets	139
5.3.2	Angle-action variables	141
5.3.3	Lagrangian tori	143
5.4	Quantization of Lagrangian Manifolds	145
5.4.1	The Keller–Maslov quantization conditions	145
5.4.2	The case of q -oriented Lagrangian manifolds	147
5.4.3	Waveforms on a Lagrangian Manifold	149
5.5	Heisenberg–Weyl and Grossmann–Royer Operators	152
5.5.1	Definition of the Heisenberg–Weyl operators	152
5.5.2	First properties of the operators $\hat{T}(z)$	154
5.5.3	The Grossmann–Royer operators	156

6 Heisenberg Group and Weyl Operators

6.1	Heisenberg Group and Schrödinger Representation	160
6.1.1	The Heisenberg algebra and group	160
6.1.2	The Schrödinger representation of \mathbf{H}_n	163
6.2	Weyl Operators	166
6.2.1	Basic definitions and properties	167

6.2.2	Relation with ordinary pseudo-differential calculus	170
6.3	Continuity and Composition	174
6.3.1	Continuity properties of Weyl operators	174
6.3.2	Composition of Weyl operators	179
6.3.3	Quantization versus dequantization	183
6.4	The Wigner–Moyal Transform	185
6.4.1	Definition and first properties	186
6.4.2	Wigner transform and probability	189
6.4.3	On the range of the Wigner transform	192
7	The Metaplectic Group	
7.1	Definition and Properties of $\text{Mp}(n)$	196
7.1.1	Quadratic Fourier transforms	196
7.1.2	The projection $\pi^{\text{Mp}} : \text{Mp}(n) \longrightarrow \text{Sp}(n)$	199
7.1.3	Metaplectic covariance of Weyl calculus	204
7.2	The Metaplectic Algebra	208
7.2.1	Quadratic Hamiltonians	208
7.2.2	The Schrödinger equation	209
7.2.3	The action of $\text{Mp}(n)$ on Gaussians: dynamical approach	212
7.3	Maslov Indices on $\text{Mp}(n)$	214
7.3.1	The Maslov index $\hat{\mu}(\hat{S})$	215
7.3.2	The Maslov indices $\hat{\mu}_\ell(\hat{S})$	220
7.4	The Weyl Symbol of a Metaplectic Operator	222
7.4.1	The operators $\hat{R}_\nu(S)$	223
7.4.2	Relation with the Conley–Zehnder index	227

Part III: Quantum Mechanics in Phase Space

8	The Uncertainty Principle	
8.1	States and Observables	238
8.1.1	Classical mechanics	238
8.1.2	Quantum mechanics	239
8.2	The Quantum Mechanical Covariance Matrix	239
8.2.1	Covariance matrices	240
8.2.2	The uncertainty principle	240
8.3	Symplectic Spectrum and Williamson’s Theorem	244
8.3.1	Williamson normal form	244
8.3.2	The symplectic spectrum	246
8.3.3	The notion of symplectic capacity	248

8.3.4	Admissible covariance matrices	252
8.4	Wigner Ellipsoids	253
8.4.1	Phase space ellipsoids	253
8.4.2	Wigner ellipsoids and quantum blobs	255
8.4.3	Wigner ellipsoids of subsystems	258
8.4.4	Uncertainty and symplectic capacity	261
8.5	Gaussian States	262
8.5.1	The Wigner transform of a Gaussian	263
8.5.2	Gaussians and quantum blobs	265
8.5.3	Averaging over quantum blobs	266
9	The Density Operator	
9.1	Trace-Class and Hilbert–Schmidt Operators	272
9.1.1	Trace-class operators	272
9.1.2	Hilbert–Schmidt operators	279
9.2	Integral Operators	282
9.2.1	Operators with L^2 kernels	282
9.2.2	Integral trace-class operators	285
9.2.3	Integral Hilbert–Schmidt operators	288
9.3	The Density Operator of a Quantum State	291
9.3.1	Pure and mixed quantum states	291
9.3.2	Time-evolution of the density operator	296
9.3.3	Gaussian mixed states	298
10	A Phase Space Weyl Calculus	
10.1	Introduction and Discussion	304
10.1.1	Discussion of Schrödinger’s argument	304
10.1.2	The Heisenberg group revisited	307
10.1.3	The Stone–von Neumann theorem	309
10.2	The Wigner Wave-Packet Transform	310
10.2.1	Definition of U_ϕ	310
10.2.2	The range of U_ϕ	314
10.3	Phase-Space Weyl Operators	317
10.3.1	Useful intertwining formulae	317
10.3.2	Properties of phase-space Weyl operators	319
10.3.3	Metaplectic covariance	321
10.4	Schrödinger Equation in Phase Space	324
10.4.1	Derivation of the equation (10.39)	324
10.4.2	The case of quadratic Hamiltonians	325
10.4.3	Probabilistic interpretation	327
10.5	Conclusion	331

A	Classical Lie Groups	
A.1	General Properties	333
A.2	The Baker–Campbell–Hausdorff Formula	335
A.3	One-parameter Subgroups of $\mathrm{GL}(m, \mathbb{R})$	335
B	Covering Spaces and Groups	
C	Pseudo-Differential Operators	
C.1	The Classes $S_{\rho,\delta}^m, L_{\rho,\delta}^m$	342
C.2	Composition and Adjoint	342
D	Basics of Probability Theory	
D.1	Elementary Concepts	345
D.2	Gaussian Densities	347
	Solutions to Selected Exercises	349
	Bibliography	355
	Index	365

Symplectic Geometry and Quantum Mechanics

de Gosson, M.A.

2006, XX, 368 p., Hardcover

ISBN: 978-3-7643-7574-4

A product of Birkhäuser Basel