Preface

Introduction

We have been experiencing since the 1970s a process of "symplectization" of Science especially since it has been realized that symplectic geometry is the natural language of both classical mechanics in its Hamiltonian formulation, and of its refinement, *quantum mechanics*. The purpose of this book is to provide core material in the symplectic treatment of quantum mechanics, in both its semi-classical and in its "full-blown" operator-theoretical formulation, with a special emphasis on so-called phase-space techniques. It is also intended to be a work of reference for the reading of more advanced texts in the rapidly expanding areas of symplectic geometry and topology, where the prerequisites are too often assumed to be "well-known" by the reader. This book will therefore be useful for both pure mathematicians and mathematical physicists. My dearest wish is that the somewhat novel presentation of some well-established topics (for example the uncertainty principle and Schrödinger's equation) will perhaps shed some new light on the fascinating subject of quantization and may open new perspectives for future interdisciplinary research.

I have tried to present a balanced account of topics playing a central role in the "symplectization of quantum mechanics" but of course this book in great part represents my own tastes. Some important topics are lacking (or are only alluded to): for instance Kirillov theory, coadjoint orbits, or spectral theory. We will moreover almost exclusively be working in flat symplectic space: the slight loss in generality is, from my point of view, compensated by the fact that simple things are not hidden behind complicated "intrinsic" notation.

The reader will find the style in which this book has been written very traditional: I have been following the classical pattern "Definition–Lemma–Theorem–Corollary". Some readers will inevitably find this way of writing medieval practice; it is still, in my opinion, the best way to make a mathematical text easily accessible. Since this book is intended to be used in graduate courses as well as for reference, we have included in the text carefully chosen exercises to enhance the understanding of the concepts that are introduced. Some of these exercises should be viewed as useful complements: the reader is encouraged to spend some time on them (solutions of selected exercises are given at the end of the book).

Organization

This book consists of three parts which can to a large extent be read independently of each other:

- The first part (partly based on my monograph [61] *Maslov Classes, Metaplectic Representation and Lagrangian Quantization*) is joint work with Serge de Gosson. It is intended to be a rigorous presentation of the basics of symplectic geometry (Chapters I and II) and of its multiply-oriented extension "q-symplectic geometry" (Chapter III); complete proofs are given, and some new results are presented. The basic tool for the understanding and study of q-symplectic geometry is the Arnold–Leray–Maslov (For short: *ALM*) index and its topological and combinatorial properties. In Chapter IV we study and extend to the degenerate case diverse Lagrangian and symplectic intersection indices with a special emphasis on the Conley–Zehnder index; the latter not only plays an important role in the modern study of periodic Hamiltonian orbits, but is also essential in the theory of the metaplectic group and its applications to the study of quantum systems with chaotic classical counterpart. A remarkable fact is that all these intersection indices are easily reduced to one mathematical object, the ALM index.
- In the second part we begin by studying thoroughly the notion of phase of a Lagrangian manifold (Chapter V). That notion, together with the properties of the ALM index defined in Chapter III, allows us to view quantized Lagrangian manifolds as those on which one can define a generalized notion of wave function. Another attractive feature of the phase of a Lagrangian manifold is that it allows a geometric definition of the Heisenberg–Weyl operators, and hence of the Heisenberg group and algebra; these are studied in detail in Chapter VI, together with the related notions of Weyl operator and Wigner–Moyal transform, which are the keys to quantum mechanics in phase space. In Chapter VII we study the metaplectic group and the associated Maslov indices, which are, surprisingly enough, related to the ALM index in a crucial way.
- In the third and last part we begin by giving a rigorous geometrical treatment of the uncertainty principle of quantum mechanics Chapter VIII). We show that this principle can be expressed in terms of the notion of symplectic capacity, which is closely related to Williamson's diagonalization theorem in the linear case, and to Gromov's non-squeezing theorem in the general case. We thereafter (Chapter IX) expose in detail the machinery of Hilbert–Schmidt and trace-class operators, which allows us a rigorous mathematical treatment of the fundamental notion of density matrix. Finally, in Chapter X (and this is definitely one of the novelties compared to traditional texts) we extend the Weyl pseudo-differential calculus to phase space, using Stone and von Neumann's theorem on the irreducible representations of the Heisenberg group. This allows us to derive by a rigorous method a "Schrödinger equation

in phase space" whose solutions are related to those of the usual Schrödinger equation by a "wave-packet transform" generalizing the physicist's Bargmann transform.

For the reader's convenience I have reviewed some classical topics in a series of Appendices at the end of the book (Classical Lie Groups, Covering Spaces, Pseudo-Differential Operators, Elementary Probability Theory). I hope that this arrangement will help the beginner concentrate on the main text with a minimum of distraction and without being sidetracked by technicalities.

Prerequisites

The mathematical prerequisites for reading with profit most of this book are relatively modest: solid undergraduate courses in linear algebra and advanced calculus, as well as the most basic notions of topology and functional analysis (Hilbert spaces, distribution theory) in principle suffice. Since we will be dealing with problems having their origin in some parts of modern physics, some familiarity with the basics of classical and quantum mechanics is of course helpful.

Bibliography

A few words about the bibliography: I have done my very best to give an accurate and comprehensive list of references. Inevitably, there are omissions; I apologize in advance for these. Some of these omissions are due to sheer ignorance; on the other hand this book exposes techniques and results from diverse fields of mathematics (and mathematical physics); to give a *complete* account of *all* contributions is an impossible task!

Enough said. The book – and the work! – is now yours.

Acknowledgements

While the main part of this book was written during my exile in Sweden, in the lovely little city of Karlskrona, it has benefitted from many visits to various institutions in Europe, Japan, USA, and Brazil. It is my duty – and great pleasure – to thank Professor B.-W. Schulze (Potsdam) for extremely valuable comments and constructive criticisms, and for his kind hospitality: the first part of this book originates in research having been done in the magnificent environment of the *Neues Palais* in the *Sans-Souci park* where the Department of Mathematics of the University of Potsdam is located.

I would like to express very special thanks to Ernst Binz (Mannheim) and Kenro Furutani (Tokyo) for having read and commented upon a preliminary version of this book. Their advice and encouragements have been instrumental. (I am, needless to say, soley responsible for remaining errors or misconceptions!)

I also wish to acknowledge debt and to express my warmest thanks for useful discussions, criticisms, and encouragements to (in alphabetical order): Bernhelm Booss-Bavnbek (Roskilde), Matthias Brack (Regensburg), Serge de Gosson (Växjö), Jürgen Eichhorn (Greifswald), Karl Gustafson (Boulder), Katarina Habermann (Göttingen), Basil Hiley (London), Ronnie Lee (Yale), Robert Littlejohn (Berkeley), James Meiss (Boulder), Walter Schempp (Siegen), Gijs Tuynman (Lille).

The third part of this book (the one devoted to quantum mechanics in phase space) has been deeply influenced by the work of Robert Littlejohn on semiclassical mechanics: may he find here my gratitude for having opened my eyes on the use of Weyl calculus and symplectic methods in quantum mechanics!

I had the opportunity to expose parts of this book during both a graduate course I gave as an Ulam Visiting Professor at the University of Colorado at Boulder during the fall term 2001, and during a course as "First Faculty in Residence" at the same University during the summer session 2003 (I take the opportunity to thank J. Meiss for his kind hospitality and for having provided me with extremely congenial environment).

This work has been partially supported by the State of São Paulo Research Foundation (Fapesp) grant 2005/51766–7; a great thanks to P. Piccione (University of São Paulo) for his hospitality and fascinating conversations about Morse theory and the Maslov index.

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http://www.springer.com/978-3-7643-7574-4

Symplectic Geometry and Quantum Mechanics
de Gosson, M.A.
2006, XX, 368 p., Hardcover
ISBN: 978-3-7643-7574-4
A product of Birkhäuser Basel