

## Preface

In 1928 the austrian author Egon Friedell wrote in the introduction to his opus magnum *Kulturgeschichte der Neuzeit* [92]:

*Alle Dinge haben ihre Philosophie, ja noch mehr: alle Dinge sind Philosophie. Alle Menschen, Gegenstände und Ereignisse sind Verkörperungen eines bestimmten Naturgedankens, einer eigentümlichen Weltabsicht. Der menschliche Geist hat nach der Idee zu forschen, die in jedem Faktum verborgen liegt, nach dem Gedanken, dessen bloße Form es ist. Die Dinge pflegen oft erst spät, ihren wahren Sinn zu offenbaren.*<sup>1</sup>

And, a few lines after:

*Daß die Dinge geschehen, ist nichts. Daß sie gewusst werden, ist alles.*<sup>2</sup>

Although spoken in the context of cultural history these words may also be applied towards the interpretation of mathematical thought. Friedell seems to say that nothing is just a ‘brute fact’ but the form of an idea which is hidden and has to be discovered in order to be shared by human beings. What really matters is not the mere fact (which in mathematics would be: the truth of a theorem) but is the form of our knowledge of it, the way (*how*) we know things. This means that in order to obtain substantial understanding (‘revealing its true meaning’) it is not enough to just state and prove theorems.

This conviction is at the heart of my efforts in writing this book. It came out of my attempt to deepen (or to establish in the first place) my *own* understanding of its subject. But I hope of course that it will also prove useful to others, and eventually will have its share in the advance of our understanding in general of the mathematical world.

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<sup>1</sup> All things have their philosophy, even more: all things actually *are* philosophy. All men, objects, and events embody a certain thought of nature, a proper intention of the world. The human mind has to inquire the idea which is hidden in each fact, the thought its mere form it is. Things tend to reveal their true meaning only after a long time. (Translation by the author)

<sup>2</sup> That things happen, is nothing. That they are known, is everything. (Translation by the author)

## Topic of the book

The main theme, as the title indicates, is functional calculus. Shortly phrased it is about ‘inserting operators into functions’, in order to render meaningful such expressions as

$$A^\alpha, \quad e^{-tA}, \quad \log A,$$

where  $A$  is an (in general unbounded) operator on a Banach space. The basic objective is quite old, and in fact the Fourier transform provides an early example of a method to define  $f(A)$ , where  $A = \Delta$  is the Laplacian,  $X = \mathbf{L}^2(\mathbb{R})$  and  $f$  is an arbitrary measurable function on  $\mathbb{R}$ . A straightforward generalisation involving self-adjoint (or normal) operators on Hilbert spaces is provided by the Spectral Theorem, but to leave the Hilbert space setting requires a different approach.

Suppose that a class of functions on some set  $\Omega$  has a reproducing kernel, i.e.,

$$f(z) = \int_{\Omega} f(w) K(z, w) \mu(dw) \quad (z \in \Omega)$$

for some measure  $\mu$ , and — for whatever reason — one already ‘knows’ what operator the expression  $K(A, w)$  should yield; then one may try to define

$$f(A) := \int_{\Omega} f(w) K(A, w) \mu(dw).$$

The simplest reproducing kernel is given by the Cauchy integral formula, so that  $K(z, w) = (w - z)^{-1}$ , and  $K(A, w) = R(w, A)$  is just the resolvent of  $A$ . This leads to the ‘ansatz’

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Omega} f(w) R(w, A) dw,$$

an idea which goes back already to RIESZ and DUNFORD, with a more recent extension towards functions which are singular at some points of the boundary of the spectrum. The latter extension is indeed needed, e.g. to treat fractional powers  $A^\alpha$ , and is one of the reasons why functional calculus methods nowadays can be found in very different contexts, from abstract operator theory to evolution equations and numerical analysis of partial differential equations. We invite the reader to have a look at Chapter 9 in order to obtain some impressions of the possible applications of functional calculus.

## Overview

The Cauchy formula encompasses a great flexibility in that its application requires only a spectral condition on the operator  $A$ . Although we mainly treat sectorial operators the approach itself is generic, and since we shall need to use it also for so-called strip-type operators, it seemed reasonable to ask for a more *axiomatic treatment*. This is provided in Chapter 1. *Sectorial operators* are introduced in

Chapter 2 and we give a full account of the basic functional calculus theory of these operators. As an application of this theory and as evidence for its elegance, in Chapter 3 we treat *fractional powers* and *holomorphic semigroups*. Chapter 4 is devoted to the interplay between a sectorial operator  $A$  and its *logarithm*  $\log A$ . One of the main aspects in the theory, subject to extensive research during the last two decades, is the *boundedness* of the  $H^\infty$ -calculus. Chapter 5 provides the necessary background knowledge including *perturbation theory*, Chapter 6 investigates the relation to real *interpolation spaces*. Here we encounter the surprising fact that an operator improves its functional calculus properties in certain interpolation spaces; this is due to the ‘flexible’ descriptions of these spaces in terms of the functional calculus.

Hilbert spaces play a special role in analysis in general and in functional calculus in particular. On the one hand, boundedness of the functional calculus can be deduced directly from *numerical range* conditions. On the other hand, there is an intimate connection with *similarity* problems. Both aspects are extensively studied in Chapter 7.

Chapters 8 and 9 account for *applications* of the theory. We study elliptic operators with constant coefficients and the relation of the functional calculus to Fourier multiplier theory. Then we apply functional calculus methods to a problem from numerical analysis regarding time-discretisation schemes of parabolic equations. Finally, we discuss the so-called maximal regularity problem and the functional calculus approach to its solution.

To make the book as self-contained as possible, we have provided an ample appendix, often also listing the more elementary results, since we thought the reader might be grateful for a comprehensive and nevertheless surveyable account. The appendix consists of six parts. Appendix A deals with operators, in particular their basic *spectral theory*. Our opinion is that a slight increase of generality, namely towards *multi-valued* operators, renders the whole account much easier. (Multi-valued operators will appear in the main text occasionally, but not indispensably.) Appendix B provides basics on *interpolation spaces*. Two more appendices (Appendix C and D) deal with forms and operators on *Hilbert spaces* as well as the *Spectral Theorem*. Finally, Appendix F quotes two results from complex *approximation theory*, but giving proofs here would have gone far beyond the scope of this book.

Instead of giving numbers to definitions I decided to incorporate the definitions into the usual text body, with the defined terms printed in boldface letters. All these definitions and some other key-words are collected in the index at the end of the book. There one will find also a list of symbols.

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I dedicate this book to my father, *Hans Albert Haase* (1929–1997).

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