Introduction

In this book we are concerned with the study of a certain class of infinite matrices and two important properties of them: their *Fredholmness* and the *stability* of the approximation by their finite truncations. Let us take these two properties as a starting point for the big picture that shall be presented in what follows.

$\boxed{\text{Fredholmness}}$ $\boxed{\text{Stability}}$

We think of our infinite matrices as bounded linear operators on a Banach space E of two-sided infinite sequences. Probably the simplest case to start with is the space $E = \ell^2$ of all complex-valued sequences $u = (u_m)_{m=-\infty}^{+\infty}$ for which $|u_m|^2$ is summable over $m \in \mathbb{Z}$.

The class of operators we are interested in consists of those bounded and linear operators on E which can be approximated in the operator norm by band matrices. We refer to them as *band-dominated operators*. Of course, these considerations are not limited to the space $E = \ell^2$. We will widen the selection of the underlying space E in three directions:

- We pass to the classical sequence spaces ℓ^p with $1 \le p \le \infty$.
- Our elements $u = (u_m) \in E$ have indices $m \in \mathbb{Z}^n$ rather than just $m \in \mathbb{Z}$.
- We allow values u_m in an arbitrary fixed Banach space $\mathbf{X}$ rather than $\mathbb{C}$.

So the space E we are dealing with is characterized by the parameters $p \in [1, \infty]$, $n \in \mathbb{N}$ and the Banach space $\mathbf{X}$; it will be denoted by $\ell^p(\mathbb{Z}^n, \mathbf{X})$. Note that this variety of spaces E includes all classical Lebesgue spaces $L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$ if we identify a function $f \in L^p(\mathbb{R}^n)$ with the sequence of its restrictions to the cubes $m + [0, 1)^n$ for $m \in \mathbb{Z}^n$, understood as elements of $\mathbf{X} = L^p(\,[0, 1)^n\,)$.

For our infinite matrices $[a_{ij}]$ acting on $E = \ell^p(\mathbb{Z}^n, \mathbf{X})$, the indices i, j are now in $\mathbb{Z}^n$, and the entries a_{ij} are linear operators on $\mathbf{X}$. Clearly, such band-dominated operators can be found in countless fields of mathematics and physics. Just to mention a few examples, we find them in wave scattering and propagation problems [21], quantum mechanics [38], signal processing [65], small-world networks [52], and biophysical neural networks [11]. Prominent examples are convolution-type operators, Schrödinger (for example, Almost Mathieu) operators, Jacobi operators and other discretizations of partial differential and pseudo-differential equations.

Stability of the Approximation by Finite Truncations

If a bounded and linear operator on E, generated by an infinite matrix A, is invertible, then, for every right-hand side $b \in E$, the equation

$$Au = b \tag{1}$$

has a unique solution $u \in E$. To find this solution, one often replaces equation (1) by the sequence of finite matrix-vector equations

$$A_m u_m = b_m, \qquad m = 1, 2, \ldots \tag{2}$$

where $A_m = [a_{ij}]_{|i|,|j| \le m}$ is the so-called mth finite section of the infinite matrix A and b_m is the respective finite subvector of the right-hand side b.

The naive but often successful idea behind this procedure is to "keep fingers crossed" that (2) is uniquely solvable – at least for all sufficiently large m – and that the solutions u_m of (2) componentwise tend to the solution u of (1) as m goes to infinity. One can show that this is the case for every right-hand side $b \in E$ if and only if A is invertible and (A_m) is *stable*, the latter meaning that all matrices A_m with a sufficiently large index m are invertible and their inverses are uniformly bounded.

Fredholm Operators

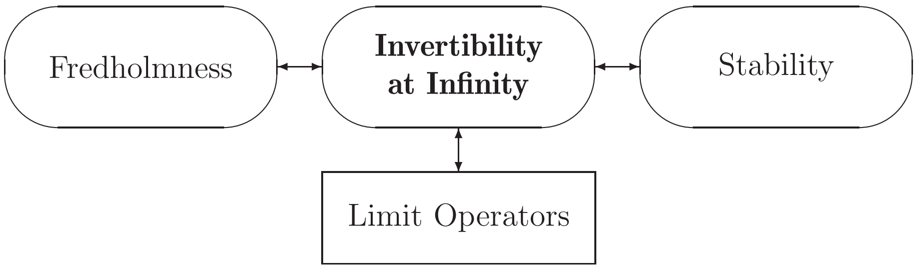
If a bounded and linear operator A on E is not invertible, then it is not injective or not surjective; that is, either

$$\ker A := \{u \in E : Au = 0\} \neq \{0\} \qquad \text{or} \qquad \operatorname{im} A := \{Au : u \in E\} \neq E,$$

or both. As an indication of how badly injectivity and surjectivity are violated, one looks at the dimension of $\ker A$ and the co-dimension of $\operatorname{im} A$ by defining the integers $\alpha := \dim \ker A$ and $\beta := \dim(E/\operatorname{im} A)$, provided $\operatorname{im} A$ is closed. The operator A is called a *Fredholm operator* if its image $\operatorname{im} A$ is closed and both α and β are finite. In this case, the integer $\alpha - \beta$ is called the *Fredholm index* of A.

So, if A is a Fredholm operator, then A, while not necessarily being invertible, is still reasonably well-behaved in the sense that the equation (1) is solvable for all right-hand sides b in a closed subspace of finite co-dimension, and the solution u is unique up to perturbations in a finite-dimensional space, namely $\ker A$.

As stated earlier, we are going to study these two properties, the stability of the finite section approximation and the Fredholm property, for our infinite matrices alias band-dominated operators. To do this we shall introduce a third property, called *invertibility at infinity*, that is closely related to both Fredholmness and stability, and we present a tool for its study: the method of *limit operators*.



Invertibility at Infinity

A band-dominated operator A is said to be *invertible at infinity* if there are two band-dominated operators B and C and an integer m such that

$$Q_m AB \;=\; Q_m \;=\; CAQ_m \tag{3}$$

holds, where Q_m is the operator of multiplication by the function that is 0 in and 1 outside the discrete cube $\{-m, m\}^n$.

This property is intimately related with Fredholmness on $E = \ell^p(\mathbb{Z}^n, \mathbf{X})$. Indeed, if $1 < p < \infty$, then Fredholmness implies invertibility at infinity whereas the implication holds the other way round if $\mathbf{X}$ is a finite-dimensional space. Thus both properties coincide if $E = \ell^2$, for example.

Our other main issue is that concerning the stability of the sequence (A_m) in (2). One easily reduces this problem to an associated invertibility at infinity problem. Instead of the sequence of finite matrices $A_1, A_2, \ldots$ we look at the infinite block diagonal matrix

$$A' := \operatorname{diag}(A_1, A_2, \ldots).$$

The sequence (A_m) turns out to be stable if and only if A' is invertible at infinity.

We will also present a slightly more involved method of assembling a sequence (A_m) of operators to one operator A' by increasing the dimension of the problem from n to $n + 1$. Roughly speaking, we stack infinitely many copies of the space E, together with the operators A_m acting on them, into the $(n + 1)$th dimension. With this stacking idea we can also study the stability of approximations by infinite matrices A_m.

Limit Operators

To get an idea of how to study invertibility at infinity, we think of a band-dominated operator A on E as an infinite matrix $[a_{ij}]$ again. For every $m \in \mathbb{N}$, the operators $Q_m A$ and AQ_m in (3), and hence the invertibility at infinity of A, are independent of the matrix entries a_{ij} with $i, j \in \{-m, m\}^n$. Consequently, all information about invertibility at infinity is hidden in the asymptotics of the entries a_{ij} towards infinity.

For band-dominated operators, the only interesting directions for the study of these asymptotics are the parallels to the main diagonal since the matrix entries decay to zero in all other directions.

For our journey along the diagonal, we choose a sequence $h = (h_m) \subset \mathbb{Z}^n$ tending to infinity and observe the sequence of matrices $[a_{i+h_m,\, j+h_m}]$ as $m \to \infty$. If this sequence of matrices, alias operators on E, converges in a certain sense, then we denote its limit by A_h and call it the *limit operator* of A with respect to the sequence h.

We call A a *rich* operator if it has sufficiently many limit operators in the sense that every sequence h tending to infinity has a subsequence g such that A_g exists. In this case, all behaviour of A at infinity is accurately stored in the collection of all limit operators of A. We denote this set by $\sigma^{\mathsf{op}}(A)$ and refer to it as the *operator spectrum* of A. For operators $A \in \mathrm{BDO}^p_{\mathcal{S},\$}$; that is the set of all rich band-dominated operators with the additional technical requirement that A is the adjoint of another operator if $p = \infty$, we prove the following nice theorem.

Theorem 1. *An operator $A \in \mathrm{BDO}^p_{\mathcal{S},\$}$ is invertible at infinity if and only if its operator spectrum $\sigma^{\mathsf{op}}(A)$ is uniformly invertible.*

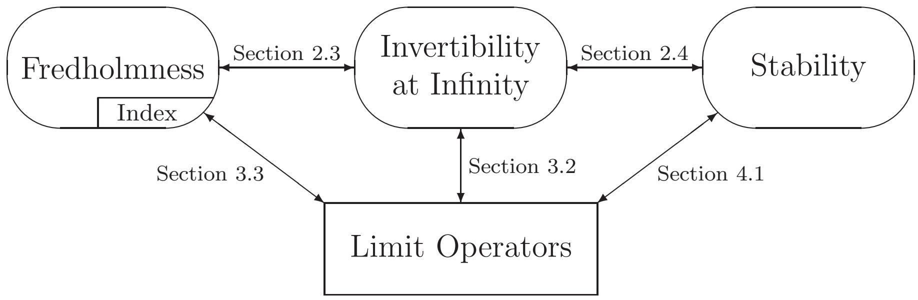
The term “uniformly invertible” means that

① all elements A_h of $\sigma^{\mathsf{op}}(A)$ are invertible, and
② their inverses are uniformly bounded, $\sup \|A_h^{-1}\| < \infty$.

This theorem yields the vertical arrow in our picture on page ix, and, in a sense, it is the heart of the whole theory and the justification of the study of limit operators.

A big question, that is as old as the first versions of Theorem 1 itself, is whether or not condition ② is redundant. On the one hand, the presence of condition ② often makes the application of Theorem 1 technically difficult. In his review of the article [61], Albrecht Böttcher justifiably points out that “Condition ② is nasty to work with.” There is nothing to add to this. On the other hand, we do not know of a single example where ② is not redundant, which is why we ask this question. We will address this issue, and we will single out at least some subclasses of $\mathrm{BDO}^p_{\mathcal{S},\$}$ for which the “nasty condition” is indeed known to be redundant.

Equipped with this tool, the limit operator concept, we can now study Fredholmness and stability. The following picture should be seen as a rough guide to this book.



We relate Fredholmness, including the computation of the Fredholm index $\alpha - \beta$, directly to the operator spectrum of A. Moreover, we formulate sufficient and necessary criteria on the applicability of the finite section method (2) in terms of limit operators of the operator A under consideration.

For a brief history of the whole subject, see Sections 1.8, 2.5, 3.10 and 4.4, at the end of each chapter.

About this Book

The original intention of this book was to enrich my PhD thesis by a number of remarks, examples and explanations to increase its readability and to make it accessible to a larger audience. During the actual process of writing this book I found myself slightly deviating from this goal. The present book did not only grow around my thesis, it also became an introductory text to the subject with several branches reaching up to the current frontier of research. It includes a number of new contributions to both theory and applications of band-dominated operators and their limit operators.

There is a noticeable focus on readability in this book. It contains many examples, figures, and remarks, coupled with a healthy amount of very human language. The main ideas and the main actors, *band-dominated operators*, *invertibility at infinity* and *limit operators*, are introduced and illustrated by looking at them from different angles, which might be helpful for readers with various backgrounds.

There is naturally a lot of overlap with the book [70], "Limit Operators and their Applications in Operator Theory" by Vladimir Rabinovich, Steffen Roch and Bernd Silbermann, published at Birkhäuser in 2004, which is and will be the 'bible' of the limit operator business. However, the non-specialist might appreciate the introductory character and the considerable effort expended on the presentation and readability of the present book. Moreover, it should be mentioned that this book covers a number of topics not included in [70]. Most notably, and this was the main thrust of my PhD thesis, this book treats the spaces ℓ^p with $1 \leq p \leq \infty$ rather than just $1 < p < \infty$. The case $p = 1$ is interesting in, for example, stochastic theory, while the treatment of $p = \infty$ opens the door to the study of operators on the space BC of bounded and continuous functions, to mention only one example. We demonstrate the latter in Sections 4.2.3 and 4.3, and we discuss an application of the developed techniques to boundary integral equations on unbounded rough surfaces.

I experienced the work on this book as equally breathtaking and delightful, and I would be very pleased if the reader will occasionally sense that, too.

Marko Lindner
Reading in May 2006

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