2

Boundary Element Methods

The numerical approximation of boundary integral equations leads to boundary element methods in general. Since already the formulation of boundary integral equations is not unique, the choice of an appropriate discretisation scheme gives even more variety. The most common approximation methods are the Collocation scheme and the Galerkin method. In this chapter we first introduce boundary element spaces of piecewise constant piecewise linear basis functions. Then we describe some discretisation methods for different boundary integral formulations and we discuss the corresponding error estimates.

2.1 Boundary Elements

Let $\Gamma = \partial\Omega$ be the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^3$. For $N \in \mathbb{N}$, we consider a sequence of boundary element meshes

$$\Gamma_N = \bigcup_{\ell=1}^{N} \overline{\tau}_\ell \,. \tag{2.1}$$

In the most simple case, we assume that Γ is piecewise polyhedral and that each boundary element mesh (2.1) consists of N plane triangular boundary elements τ_ℓ with mid points x_ℓ^*. Using the reference element

$$\tau \;=\; \left\{\xi \in \mathbb{R}^2 \;:\; 0 < \xi_1 < 1, \quad 0 < \xi_2 < 1 - \xi_1\right\} \,,$$

the boundary element $\tau_\ell = \chi_\ell(\tau)$ with nodes x_{ℓ_i} for $i = 1, 2, 3$ can be described via the parametrisation

$$x(\xi) \;=\; \chi_\ell(\xi) \;=\; x_{\ell_1} + \xi_1(x_{\ell_2} - x_{\ell_1}) + \xi_2(x_{\ell_3} - x_{\ell_1}) \in \tau_\ell \quad \text{for } \xi \in \tau.$$

For the area Δ_ℓ of the boundary element τ_ℓ, we then obtain

$$\Delta_\ell \;=\; \int\limits_{\tau_\ell} ds_x \;=\; \int\limits_{\tau} \sqrt{EG - F^2}\, d\xi \;=\; \frac{1}{2}\sqrt{EG - F^2}\,,$$

where

$$E = \sum_{i=1}^{3} \left(\frac{\partial}{\partial \xi_1} x_i(\xi) \right)^2 = |x_{\ell_2} - x_{\ell_1}|^2,$$

$$G = \sum_{i=1}^{3} \left(\frac{\partial}{\partial \xi_2} x_i(\xi) \right)^2 = |x_{\ell_3} - x_{\ell_1}|^2,$$

$$F = \sum_{i=1}^{3} \frac{\partial}{\partial \xi_1} x_i(\xi) \frac{\partial}{\partial \xi_2} x_i(\xi) = (x_{\ell_2} - x_{\ell_1}, x_{\ell_3} - x_{\ell_1}).$$

Using Δ_ℓ, we define the local mesh size of the boundary element τ_ℓ as

$$h_\ell = \sqrt{\Delta_\ell} \quad \text{for } \ell = 1, \ldots, N$$

implying the global mesh sizes

$$h = h_{\max} = \max_{1 \le \ell \le N} h_\ell, \quad h_{\min} = \min_{1 \le \ell \le N} h_\ell. \tag{2.2}$$

The sequence of boundary element meshes (2.1) is called globally quasi uniform if the mesh ratio

$$\frac{h_{\max}}{h_{\min}} \le c_{\mathrm{G}}$$

is uniformly bounded by a constant c_{G} which is independent of $N \in \mathbb{N}$. Finally, we introduce the element diameter

$$d_\ell = \sup_{x, y \in \tau_\ell} |x - y| .$$

We assume that all boundary elements τ_ℓ are uniformly shape regular, i.e., there exists a global constant c_{B} independent of N such that

$$d_\ell \le c_{\mathrm{B}} h_\ell \quad \text{for all } \ell = 1, \ldots, N.$$

With

$$J_\ell = \begin{pmatrix} x_{\ell_2,1} - x_{\ell_1,1} & x_{\ell_3,1} - x_{\ell_1,1} \\ x_{\ell_2,2} - x_{\ell_1,2} & x_{\ell_3,2} - x_{\ell_1,2} \\ x_{\ell_2,3} - x_{\ell_1,3} & x_{\ell_3,3} - x_{\ell_1,3} \end{pmatrix} \in \mathbb{R}^{3 \times 2}$$

and using the parametrisation $\tau_\ell = \chi_\ell(\tau)$, a function v defined on τ_ℓ can be interpreted as a function $\widetilde{v}_\ell$ with respect to the reference element τ,

$$v(x) = v(x_{\ell_1} + J_\ell \xi) = \widetilde{v}_\ell(\xi) \quad \text{for } \xi \in \tau, \quad x = \chi_\ell(\xi).$$

Vice versa, a function $\widetilde{v}$ defined in the parameter domain τ implies a function v_ℓ on the boundary element τ_ℓ,

$$\widetilde{v}(\xi) = v(x_{\ell_1} + J_\ell \xi) = v_\ell(x) \quad \text{for } \xi \in \tau, \quad x = \chi_\ell(\xi).$$

Hence, we can define boundary element basis functions on τ_ℓ by defining associated shape functions on the reference element τ.

2.2 Basis Functions

Piecewise Constant Basis Functions

The piecewise constant shape function

$$\psi^0(\xi) \;=\; 1 \quad \text{for } \xi \in \tau$$

implies the piecewise constant basis functions on Γ

$$\psi_\ell(x) \;=\; \begin{cases} 1 & \text{for } x \in \tau_\ell, \\ 0 & \text{elsewhere} \end{cases} \tag{2.3}$$

for $\ell = 1, \ldots, N$, and, therefore, the global trial space

$$S_h^0(\Gamma) \;=\; \operatorname{span}\Big\{ \psi_\ell \Big\}_{\ell=1}^{N}, \quad \dim S_h^0(\Gamma) \;=\; N.$$

Note that any $w_h \in S_h^0(\Gamma)$ can be written as

$$w_h \;=\; \sum_{\ell=1}^{N} w_\ell\, \psi_\ell \in S_h^0(\Gamma), \quad w_\ell \in \mathbb{R} \quad \text{for } \ell = 1, \ldots, N.$$

Moreover, a function $w_h \in S_h^0(\Gamma)$ can be identified with the vector $\underline{w} \in \mathbb{R}^N$ defined by the components w_ℓ for $\ell = 1, \ldots, N$.

In what follows, we will consider the approximation property of the trial space $S_h^0(\Gamma) \subset L_2(\Gamma)$. For this, we introduce the L_2 projection of a given function $w \in L_2(\Gamma)$,

$$Q_h w \;=\; \sum_{\ell=1}^{N} w_\ell\, \psi_\ell \in S_h^0(\Gamma),$$

which minimises the error $w - Q_h w$ in the $L_2(\Gamma)$–norm,

$$Q_h w = \arg \min_{w_h \in S_h^0(\Gamma)} \|w - w_h\|^2_{L_2(\Gamma)} \;=\; \arg \min_{w_h \in S_h^0(\Gamma)} \int_\Gamma \Big(w(x) - w_h(x) \Big)^2 ds_x .$$

Note that $Q_h w$ is the unique solution of the variational problem

$$\int_\Gamma (Q_h w)(x) \psi_k(x) ds_x \;=\; \int_\Gamma w(x) \psi_k(x) ds_x \quad \text{for } k = 1, \ldots, N,$$

or,

$$\sum_{\ell=1}^{N} w_\ell \int_\Gamma \psi_\ell(x) \psi_k(x) ds_x \;=\; \int_\Gamma w(x) \psi_k(x) ds_x \quad \text{for } k = 1, \ldots, N.$$

Due to

$$\int_\Gamma \psi_\ell(x)\psi_k(x)ds_x = \begin{cases} \Delta_\ell & \text{for } k = \ell, \\ 0 & \text{for } k \neq \ell \end{cases},$$

we obtain

$$w_\ell = \frac{1}{\Delta_\ell} \int_{\tau_\ell} w(x)ds_x \quad \text{for } \ell = 1, \dots, N.$$

From this explicit representation of w_ℓ, one can prove the error estimate, see Appendix B.2,

$$\|w - Q_h w\|^2_{L_2(\Gamma)} \le c \sum_{\ell=1}^N h_\ell^{2s} \, |w|^2_{H^s(\tau_\ell)} \le c\, h^{2s} \, |w|^2_{H^s_{\text{pw}}(\Gamma)} \tag{2.4}$$

for a sufficiently regular function $w \in H^s_{\text{pw}}(\Gamma)$ and $s \in (0, 1]$. The semi-norm in (2.4) is defined as

$$|w|^2_{H^s(\tau_\ell)} = \int_{\tau_\ell}\int_{\tau_\ell} \frac{|w(x) - w(y)|^2}{|x - y|^{2+2s}} ds_x ds_y \quad \text{for } s \in (0, 1)$$

and

$$|w|^2_{H^1(\tau_\ell)} = \int_\tau |\nabla_\xi w(\chi_\ell(\xi))|^2 d\xi \quad \text{for } s = 1.$$

From the above variational formulation, we conclude the Galerkin orthogonality

$$\int_\Gamma \Big(w(x) - (Q_h w)(x)\Big) v_h(x) ds_x = 0 \quad \text{for all } v_h \in S_h^0(\Gamma),$$

and, therefore, the trivial error estimate

$$\|w - Q_h w\|_{L_2(\Gamma)} \le \|w\|_{L_2(\Gamma)}.$$

Using a duality argument, we further obtain

$$\|w - Q_h w\|_{H^\sigma(\Gamma)} \le c\, h^{s-\sigma} \, |w|_{H^s_{\text{pw}}(\Gamma)}$$

for $\sigma \in [-1, 0]$ and $s \in [0, 1]$.

Summarising the above, we obtain the following approximation property in $S_h^0(\Gamma)$.

Theorem 2.1. *Let $w \in H^s_{pw}(\Gamma)$ for some $s \in [0, 1]$. Then there holds*

$$\inf_{w_h \in S_h^0(\Gamma)} \|w - w_h\|_{H^\sigma(\Gamma)} \le c\, h^{s-\sigma} \, |w|_{H^s_{pw}(\Gamma)} \tag{2.5}$$

for all $\sigma \in [-1, 0]$.

Moreover, the approximation property (2.5) remains valid for all $\sigma \le s \le 1$ with $\sigma < 1/2$.

Piecewise Linear Discontinuous Basis Functions

With respect to the reference element τ, we may also define local polynomial shape functions of higher order. In particular, we introduce the linear shape functions

$$\psi_1^1(\xi) \,=\, 1-\xi_1-\xi_2, \quad \psi_2^1(\xi) \,=\, \xi_1, \quad \psi_3^1(\xi) \,=\, \xi_2 \quad \text{for } \xi\in\tau. \tag{2.6}$$

These shape functions imply globally discontinuous piecewise linear basis functions

$$\psi_{\ell,i}(x) \,=\, \begin{cases} \psi_i^1(\xi) & \text{for } x=\chi_\ell(\xi)\in\tau_\ell, \\ 0 & \text{elsewhere} \end{cases}$$

for $\ell=1,\dots,N$, $i=1,2,3$, and, therefore, the global trial space

$$S_h^{1,-1}(\Gamma) \,=\, \operatorname{span}\Big\{\psi_{\ell,1}(x),\psi_{\ell,2}(x),\psi_{\ell,3}(x)\Big\}_{\ell=1}^N, \quad \dim S_h^{1,-1}(\Gamma) \,=\, 3N.$$

Any function $w_h\in S_h^{1,-1}(\Gamma)$ can be written as

$$w_h \,=\, \sum_{\ell=1}^N\sum_{i=1}^3 w_{\ell,i}\,\psi_{\ell,i} \in S_h^{1,-1}(\Gamma)\,.$$

Moreover, a function $w_h\in S_h^{1,-1}(\Gamma)$ can be identified with the vector $\underline{w}\in\mathbb{R}^{3N}$ which is defined by the coefficients $w_{\ell,i}$ for $i=1,2,3$ and $\ell=1,\dots,N$.

As for piecewise constant basis functions, we may also define the corresponding L_2 projection $Q_hw\in S_h^{1,-1}(\Gamma)\subset L_2(\Gamma)$,

$$Q_hw \,=\, \sum_{\ell=1}^N\sum_{i=1}^3 w_{\ell,i}\,\psi_{\ell,i}\in S_h^{1,-1}(\Gamma),$$

as the unique solution of the variational problem

$$\int_\Gamma (Q_hw)(x)\psi_{k,j}(x)ds_x \,=\, \int_\Gamma w(x)\psi_{k,j}(x)ds_x, \quad j=1,2,3\,,\ k=1,\dots,N$$

satisfying the error estimate

$$\|w-Q_hw\|^2_{L_2(\Gamma)} \,\le\, c\sum_{\ell=1}^N h_\ell^4\,|w|^2_{H^2(\tau_\ell)} \,\le\, c\,h^4\,|w|^2_{H^2_{\mathrm{pw}}(\Gamma)}$$

when assuming $w\in H^2_{\mathrm{pw}}(\Gamma)$. Combining this with the trivial error estimate

$$\|w-Q_hw\|_{L_2(\Gamma)} \,\le\, \|w\|_{L_2(\Gamma)},$$

and using an interpolation argument, the final error estimate

$$\|w - Q_h w\|_{L_2(\Gamma)} \le c\, h^s\, |w|_{H^s_{\mathrm{pw}}(\Gamma)}$$

follows when assuming $w \in H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [0,2]$. Using again a duality argument, we finally obtain

$$\|w - Q_h w\|_{H^\sigma(\Gamma)} \le c\, h^{s-\sigma}\, |w|_{H^s_{\mathrm{pw}}(\Gamma)} \tag{2.7}$$

for $\sigma \in [-2,0]$ and $s \in [0,2]$.

Summarising the above, we obtain the approximation property in $S_h^{1,-1}(\Gamma)$.

Theorem 2.2. *Let $w \in H^s_{pw}(\Gamma)$ for some $s \in [0,2]$. Then there holds*

$$\inf_{w_h \in S_h^{1,-1}(\Gamma)} \|w - w_h\|_{H^\sigma(\Gamma)} \le c\, h^{s-\sigma}\, |w|_{H^s_{pw}(\Gamma)} \tag{2.8}$$

for all $\sigma \in [-2,0]$. Moreover, the approximation property (2.8) *remains valid for all $\sigma \le s \le 2$ with $\sigma < 1/2$.*

Piecewise Linear Continuous Basis Functions

Up to now, we have considered only globally discontinuous basis functions which do not require any admissibility condition of the triangulation (2.1). But such a condition is needed to define globally continuous basis functions. Let $\{x_j\}_{j=1}^M$ be the set of all nodes of the triangulation (2.1). A boundary element mesh consisting of plane triangular elements is called admissible, if the intersection of two neighboured elements $\bar{\tau}_\ell$ and $\bar{\tau}_k$ is just one common edge or one common node. Then $I(j)$ is the index set of all boundary elements τ_ℓ containing the node x_j while $J(\ell)$ is the three–dimensional index set of the nodes defining the triangular element τ_ℓ.

For $j = 1, \ldots, M$, one can define globally continuous piecewise linear basis functions φ_j with

$$\varphi_j(x) = \begin{cases} 1 & \text{for } x = x_j, \\ 0 & \text{for } x = x_i \ne x_j, \\ \text{piecewise linear} & \text{elsewhere.} \end{cases}$$

Note that the restrictions of φ_j onto a boundary element τ_k for $k \in I(j)$ can be represented by the linear shape functions $\psi^1_{j_k}$,

$$\varphi_j(x) = \psi^1_{j_k}(\xi) \quad \text{for } x = \chi_k(\xi) \in \tau_k. \tag{2.9}$$

The basis functions φ_j are used to define the trial space

$$S_h^1(\Gamma) = \mathrm{span}\left\{\varphi_j\right\}_{j=1}^M, \quad \dim S_h^1(\Gamma) = M.$$

The piecewise linear continuous L_2 projection $Q_h w \in S_h^1(\Gamma)$ is then defined as the unique solution of the variational problem

$$\int_\Gamma Q_h w(x)\varphi_j(x)ds_x = \int_\Gamma w(x)\varphi_j(x)ds_x \quad \text{for } j=1,\ldots,M.$$

Due to $S_h^1(\Gamma) \subset S_h^{1,-1}(\Gamma)$ we immediately find the error estimate

$$\|w - Q_h w\|_{H^\sigma(\Gamma)} \leq c\, h^{s-\sigma}\, |w|_{H^s_{\mathrm{pw}}(\Gamma)}$$

when assuming $w \in H^s_{\mathrm{pw}}(\Gamma)$, $\sigma \in [-2,0]$, $s \in [0,2]$.

Defining $P_h u \in S_h^1(\Gamma)$ as the unique solution of the variational problem

$$\langle P_h w, v_h\rangle_{H^1(\Gamma)} = \langle w, v_h\rangle_\Gamma \quad \text{for all } v_h \in S_h^1(\Gamma)$$

we can show the error estimate

$$\|w - P_h w\|_{H^\sigma(\Gamma)} \leq c\, h^{s-\sigma}\, |w|_{H^s_{\mathrm{pw}}(\Gamma)}$$

when assuming $w \in H^s_{\mathrm{pw}}(\Gamma)$ and $\sigma \in (0,1]$, $s \in [1,2]$.

Hence, we have the following result.

Theorem 2.3. *Let $v \in H^s_{pw}(\Gamma)$ for some $s \in [1,2]$. Then there holds*

$$\inf_{v_h \in S_h^1(\Gamma)} \|v - v_h\|_{H^\sigma(\Gamma)} \leq c\, h^{s-\sigma}\, |v|_{H^s_{pw}(\Gamma)} \tag{2.10}$$

for all $\sigma \in [-2,1]$. Moreover, the approximation property (2.10) remains valid for all $\sigma \leq s \leq 2$ with $\sigma < 3/2$.

2.3 Laplace Equation

2.3.1 Interior Dirichlet Boundary Value Problem

The solution of the interior Dirichlet boundary value problem (cf. (1.14))

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \quad \gamma_0^{\mathrm{int}} u(x) = g(x) \quad \text{for } x \in \Gamma,$$

is given by the representation formula (cf. (1.6))

$$u(x) = \int_\Gamma u^*(x,y)t(y)ds_y - \int_\Gamma \gamma_{1,y}^{\mathrm{int}} u^*(x,y)g(y)ds_y \quad \text{for } x \in \Omega,$$

where the unknown Neumann datum $t = \gamma_1^{\mathrm{int}} u \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation (cf. (1.15))

$$\frac{1}{4\pi}\int_\Gamma \frac{1}{|x-y|} t(y)ds_y = \frac{1}{2} g(x) + \frac{1}{4\pi}\int_\Gamma \frac{(x-y,\underline{n}(y))}{|x-y|^3} g(y)ds_y \quad \text{for } x \in \Gamma.$$

Replacing $t \in H^{-1/2}(\Gamma)$ by a piecewise constant approximation

$$t_h = \sum_{\ell=1}^N t_\ell\, \psi_\ell \in S_h^0(\Gamma), \tag{2.11}$$

we have to find the unknown coefficient vector $\underline{t} \in \mathbb{R}^N$ from some appropriate system of linear equations.

Collocation Method

Inserting (2.11) into the boundary integral equation (1.15), and choosing the boundary element mid points x_k^* as collocation nodes, we have to solve the collocation equations

$$\frac{1}{4\pi}\int\limits_{\Gamma}\frac{1}{|x_k^*-y|}t_h(y)ds_y \;=\; \frac{1}{2}g(x_k^*)+\frac{1}{4\pi}\int\limits_{\Gamma}\frac{(x_k^*-y,\underline{n}(y))}{|x_k^*-y|^3}g(y)ds_y \tag{2.12}$$

for $k=1,\ldots,N$, or using the definition (2.3) of the piecewise constant basis functions ψ_ℓ,

$$\sum_{\ell=1}^{N} t_\ell\frac{1}{4\pi}\int\limits_{\tau_\ell}\frac{1}{|x_k^*-y|}ds_y \;=\; \frac{1}{2}g(x_k^*)+\frac{1}{4\pi}\int\limits_{\Gamma}\frac{(x_k^*-y,\underline{n}(y))}{|x_k^*-y|^3}g(y)ds_y$$

for $k=1,\ldots,N$. With

$$V_h[k,\ell]=\frac{1}{4\pi}\int\limits_{\tau_\ell}\frac{1}{|x_k^*-y|}ds_y$$

for $k,\ell=1,\ldots,N$, and

$$f_k=\frac{1}{2}g(x_k^*)+\frac{1}{4\pi}\int\limits_{\Gamma}\frac{(x_k^*-y,\underline{n}(y))}{|x_k^*-y|^3}g(y)ds_y$$

for $k=1,\ldots,N$, this results in a linear system of equations,

$$V_h\underline{t} \;=\; \underline{f}.$$

The stiffness matrix V_h of the collocation method is in general non–symmetric and the stability of the collocation scheme (2.12) and therefore the invertibility of the stiffness matrix V_h is still an open problem when Γ is the boundary of a general Lipschitz domain $\Omega\subset\mathbb{R}^3$. When assuming the stability of the collocation scheme (2.12), the quasi optimal error estimate, i.e., Cea's lemma,

$$\|t-t_h\|_{H^{-1/2}(\Gamma)} \;\le\; c \inf_{w_h\in S_h^0(\Gamma)}\|t-w_h\|_{H^{-1/2}(\Gamma)}$$

follows. Combining this with the approximation property (2.5) for $\sigma=-1/2$, we get the error estimate

$$\|t-t_h\|_{H^{-1/2}(\Gamma)} \;\le\; c\,h^{s+1/2}\,|t|_{H^s_{\mathrm{pw}}(\Gamma)}\,,$$

when assuming $t\in H^s_{\mathrm{pw}}(\Gamma)$ for some $s\in[0,1]$. Applying the Aubin–Nitsche trick (for $\sigma<-1/2$) and an inverse inequality argument (for $\sigma\in(-1/2,0]$), we also obtain the error estimate

$$\|t - t_h\|_{H^\sigma(\Gamma)} \leq c\, h^{s-\sigma}\, |t|_{H^s_{\text{pw}}(\Gamma)}\,, \tag{2.13}$$

when assuming $t \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$, and $\sigma \in [-1,0]$. Note that the lower bound $\sigma \geq -1$ is due to the collocation approach, independently of the degree of the used polynomial basis functions.

Inserting the computed solution t_h into the representation formula (1.6), this gives an approximate representation formula

$$\widetilde{u}(x) = \int\limits_\Gamma u^*(x,y)t_h(y)ds_y - \int\limits_\Gamma \gamma^{\text{int}}_{1,y} u^*(x,y)g(y)ds_y$$

for $x \in \Omega$, describing an approximate solution of the Dirichlet boundary value problem (1.14). Note that $\widetilde{u}$ is harmonic, satisfying the Laplace equation, but the Dirichlet boundary conditions are satisfied only approximately. For an arbitrary $x \in \Omega$, the error is given by

$$u(x) - \widetilde{u}(x) = \int\limits_\Gamma u^*(x,y)\Big(t(y) - t_h(y)\Big)\, ds_y\,.$$

Using a duality argument, the error estimate

$$|u(x) - \widetilde{u}(x)| \leq \|u^*(x,\cdot)\|_{H^{-\sigma}(\Gamma)} \|t - t_h\|_{H^\sigma(\Gamma)}$$

for some $\sigma \in \mathbb{R}$ follows. Combining this with the error estimate (2.13) for the minimal possible value $\sigma = -1$, we obtain the pointwise error estimate

$$|u(x) - \widetilde{u}(x)| \leq c\, h^{s+1}\, \|u^*(x,\cdot)\|_{H^1(\Gamma)}\, |t|_{H^s_{\text{pw}}(\Gamma)}\,,$$

when assuming $t \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$. Hence, if t is sufficiently smooth, i.e. $t \in H^1_{\text{pw}}(\Gamma)$, we obtain as the optimal order of convergence for $s = 1$

$$|u(x) - \widetilde{u}(x)| \leq c\, h^2\, \|u^*(x,\cdot)\|_{H^1(\Gamma)} |t|_{H^1_{\text{pw}}(\Gamma)}\,. \tag{2.14}$$

Note that the error estimate (2.14) involves the position of the observation point $x \in \Omega$. In particular, the error estimate (2.14) does not hold in the limiting case $x \in \Gamma$.

Galerkin Method

The boundary integral equation (cf. (1.15))

$$\frac{1}{4\pi}\int\limits_\Gamma \frac{1}{|x-y|} t(y)ds_y = \frac{1}{2}g(x) + \frac{1}{4\pi}\int\limits_\Gamma \frac{(x-y,\underline{n}(y))}{|x-y|^3} g(y)ds_y \quad \text{for } x \in \Gamma$$

is equivalent to the variational problem (1.16),

$$\left\langle Vt, w \right\rangle_\Gamma = \left\langle \left(\frac{1}{2}I + K\right)g, w \right\rangle_\Gamma \quad \text{for all } w \in H^{-1/2}(\Gamma),$$

and to the minimisation problem

$$F(t) = \min_{w \in H^{-1/2}(\Gamma)} F(w)$$

with

$$F(w) = \frac{1}{2}\left\langle Vw, w \right\rangle_\Gamma - \left\langle \left(\frac{1}{2}I + K\right)g, w \right\rangle_\Gamma.$$

Using a sequence of finite dimensional subspaces $S_h^0(\Gamma)$ spanned by piecewise constant basis functions, associated approximate solutions

$$t_h = \sum_{\ell=1}^N t_\ell \psi_\ell \in S_h^0(\Gamma)$$

are obtained from the minimisation problem

$$F(t_h) = \min_{w_h \in S_h^0(\Gamma)} F(w_h).$$

The solution $t_h \in S_h^0(\Gamma)$ of the above minimisation problem is defined via the Galerkin equations

$$\left\langle Vt_h, \psi_k \right\rangle_\Gamma = \left\langle \left(\frac{1}{2}I + K\right)g, \psi_k \right\rangle_\Gamma \quad \text{for } k = 1, \ldots, N. \tag{2.15}$$

With (2.11) and by using the definition (2.3) of the piecewise constant basis functions ψ_ℓ, this is equivalent to

$$\sum_{\ell=1}^N t_\ell \frac{1}{4\pi} \int_{\tau_k} \int_{\tau_\ell} \frac{1}{|x-y|} ds_y ds_x =$$
$$\frac{1}{2} \int_{\tau_k} g(x) ds_x + \frac{1}{4\pi} \int_{\tau_k} \int_{\Gamma} \frac{(x-y, \underline{n}(y))}{|x-y|^3} g(y) ds_y ds_x$$

for $k = 1, \ldots, N$. With

$$V_h[k, \ell] = \frac{1}{4\pi} \int_{\tau_k} \int_{\tau_\ell} \frac{1}{|x-y|} ds_y ds_x$$

for $k, \ell = 1, \ldots, N$, and

$$f_k = \frac{1}{2} \int_{\tau_k} g(x) ds_x + \frac{1}{4\pi} \int_{\tau_k} \int_{\Gamma} \frac{(x-y, \underline{n}(y))}{|x-y|^3} g(y) ds_y ds_x$$

for $k = 1, \ldots, N$, we find the coefficient vector $\underline{t} \in \mathbb{R}^N$ as the unique solution of the linear system

$$V_h \underline{t} = \underline{f} \,. \tag{2.16}$$

The Galerkin stiffness matrix V_h is symmetric and positive definite. Therefore, one may use a conjugate gradient scheme for an iterative solution of the linear system (2.16). Since the spectral condition number of V_h behaves like $\mathcal{O}(h^{-1})$, i.e.,

$$\kappa_2(V_h) \;=\; \|V_h\|_2 \|V_h^{-1}\|_2 \;=\; \frac{\lambda_{\max}(V_h)}{\lambda_{\min}(V_h)} \le c\,\frac{1}{h},$$

an appropriate preconditioning is sometimes needed. Moreover, since the stiffness matrix V_h is dense, fast boundary element methods are required to construct more efficient algorithms, see Chapter 3.

From the $H^{-1/2}(\Gamma)$–ellipticity and the boundedness of the single layer potential

$$V \,:\, H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\,,$$

see Lemma 1.1, we conclude the unique solvability of the Galerkin variational problem (2.15), or, correspondingly, of the linear system (2.16), as well as the quasi optimal error estimate, i.e. Cea's lemma,

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \;\le\; \frac{c_2^V}{c_1^V} \inf_{w_h \in S_h^0(\Gamma)} \|t - w_h\|_{H^{-1/2}(\Gamma)}\,.$$

Combining this with the approximation property (2.5) for $\sigma = -1/2$, we get

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \;\le\; c\,h^{s+\frac{1}{2}}\,|t|_{H^s_{\text{pw}}(\Gamma)}\,,$$

when assuming $t \in H^s_{\text{pw}}(\Gamma)$ and $s \in [0,1]$. Applying the Aubin–Nitsche trick (for $\sigma < -1/2$) and an inverse inequality argument (for $\sigma \in (-1/2, 0]$), we also obtain the error estimate

$$\|t - t_h\|_{H^\sigma(\Gamma)} \;\le\; c\,h^{s-\sigma}\,|t|_{H^s_{\text{pw}}(\Gamma)}\,, \tag{2.17}$$

when assuming $t \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$ and $\sigma \in [-2, 0]$.

Inserting the computed Galerkin solution $t_h \in S_h^0(\Gamma)$ into the representation formula (1.6), this gives an approximate representation formula

$$\widetilde{u}(x) \;=\; \int_\Gamma u^*(x,y) t_h(y) ds_y - \int_\Gamma \gamma_{1,y}^{\text{int}} u^*(x,y) g(y) ds_y \quad \text{for } x \in \Omega\,, \tag{2.18}$$

describing an approximate solution of the Dirichlet boundary value problem (1.14). Note that $\widetilde{u}$ is harmonic satisfying the Laplace equation, but the Dirichlet boundary conditions are satisfied only approximately. For an arbitrary $x \in \Omega$, the error is given by

$$u(x) - \widetilde{u}(x) \;=\; \int_\Gamma u^*(x,y)\Big(t(y) - t_h(y)\Big)\, ds_y\,.$$

Using a duality argument, the error estimate

$$|u(x) - \widetilde{u}(x)| \le \|u^*(x,\cdot)\|_{H^{-\sigma}(\Gamma)} \|t - t_h\|_{H^\sigma(\Gamma)}$$

for some $\sigma \in \mathbb{R}$ follows. Combining this with the error estimate (2.17) for the minimal value $\sigma = -2$, we obtain the pointwise error estimate

$$|u(x) - \widetilde{u}(x)| \le c\, h^{s+2} \|u^*(x,\cdot)\|_{H^2(\Gamma)} \, |t|_{H^s_{\mathrm{pw}}(\Gamma)} \,,$$

when assuming $t \in H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [0,1]$. Hence, if $t \in H^1_{\mathrm{pw}}(\Gamma)$ is sufficiently smooth, we obtain the optimal order of convergence for $s = 1$,

$$|u(x) - \widetilde{u}(x)| \le c\, h^3 \, \|u^*(x,\cdot)\|_{H^2(\Gamma)} \, |t|_{H^1_{\mathrm{pw}}(\Gamma)} \,. \tag{2.19}$$

Note that the error estimate (2.19) involves the position of the observation point $x \in \Omega$ again. In particular, the error estimate (2.19) does not hold in the limiting case $x \in \Gamma$.

The computation of the right hand side $\underline{f}$ in the linear system (2.16) requires the evaluation of the integrals

$$f_k = \frac{1}{2} \int\limits_{\tau_k} g(x) ds_x + \frac{1}{4\pi} \int\limits_{\tau_k} \int\limits_{\Gamma} \frac{(x - y, \underline{n}(y))}{|x - y|^3} g(y) ds_y ds_x$$

for $k = 1, \ldots, N$. An approximation of the given Dirichlet datum $g \in H^{1/2}(\Gamma)$ by a globally continuous and piecewise linear function

$$g_h = \sum_{j=1}^{M} g_j \, \varphi_j \in S_h^1(\Gamma)$$

can be obtained either by interpolation,

$$g_h = \sum_{j=1}^{M} g(x_j) \, \varphi_j \,, \tag{2.20}$$

or by the L_2 projection,

$$g_h = \sum_{j=1}^{M} g_j \, \varphi_j \,,$$

where the coefficients $g_j \,,\ j = 1, \ldots, M$ satisfy

$$\sum_{j=1}^{M} g_j \langle \varphi_j, \varphi_i \rangle_{L_2(\Gamma)} = \langle g, \varphi_i \rangle_{L_2(\Gamma)} \quad \text{for } i = 1, \ldots, M \,. \tag{2.21}$$

This leads to

$$\widetilde{f}_k = \frac{1}{2}\sum_{j=1}^{M} g_j \int_{\tau_k} \varphi_j(x)ds_x + \sum_{j=1}^{M} g_j \frac{1}{4\pi}\int_{\tau_k}\int_{\Gamma} \frac{(x-y,\underline{n}(y))}{|x-y|^3}\varphi_j(y)ds_y ds_x$$
$$= \sum_{j=1}^{M} g_j \left(\frac{1}{2}M_h[k,j] + K_h[k,j]\right)$$

with the matrix entries

$$M_h[k,j] = \int_{\tau_k} \varphi_j(x)ds_x, \quad K_h[k,j] = \frac{1}{4\pi}\int_{\tau_k}\int_{\Gamma} \frac{(x-y,\underline{n}(y))}{|x-y|^3}\varphi_j(y)ds_y ds_x$$

for $j = 1,\ldots,M$ and $k = 1,\ldots,N$. Instead of the linear system (2.16), we then have to solve a linear system with a perturbed right hand side $\widetilde{\underline{f}}$, yielding a perturbed solution vector $\widetilde{\underline{t}}$, i.e., we have to solve the linear system

$$V_h\widetilde{\underline{t}} = \left(\frac{1}{2}M_h + K_h\right)\underline{g}\,. \tag{2.22}$$

For the perturbed boundary element solution $\widetilde{t}_h \in S_h^0(\Gamma)$, the error estimate

$$\|t - \widetilde{t}_h\|_{H^\sigma(\Gamma)} \le c_1\,\|t - t_h\|_{H^\sigma(\Gamma)} + c_2\,\|g - g_h\|_{H^{\sigma+1}(\Gamma)}$$

follows with $\sigma \in [-2,0]$, when the L_2 projection (2.21) is used to define $g_h \in S_h^1(\Gamma)$. Note that $\sigma \in [-1,0]$ in the case of the interpolation (2.20). Assuming $t \in H^s_{\text{pw}}(\Gamma)$ and $g \in H^{s+1}_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$, we then obtain the error estimate

$$\|t - \widetilde{t}_h\|_{H^\sigma(\Gamma)} \le h^{s-\sigma}\left(c_1\,|t|_{H^s_{\text{pw}}(\Gamma)} + c_2\,|g|_{H^{s+1}_{\text{pw}}(\Gamma)}\right).$$

For the approximate representation formula

$$\widetilde{u}(x) = \int_{\Gamma} u^*(x,y)t_h(y)ds_y - \int_{\Gamma}\gamma_{1,y}^{\text{int}}u^*(x,y)g_h(y)ds_y \quad \text{for } x \in \Omega\,,$$

we then obtain the optimal error estimate

$$|u(x) - \widetilde{u}(x)| \le c(x,t,g)\,h^3\,, \tag{2.23}$$

when using the L_2 projection (2.21) and when assuming $t \in H^1_{\text{pw}}(\Gamma)$ and $g \in H^2_{\text{pw}}(\Gamma)$. When using the interpolation (2.20) instead, the error estimate

$$|u(x) - \widetilde{u}(x)| \le c(x,t,g)\,h^2$$

follows.

2.3.2 Interior Neumann Boundary Value Problem

Let $\Omega \subset \mathbb{R}^3$ be a simply connected domain. The solution of the interior Neumann boundary value problem (cf. (1.21))

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \quad \gamma_1^{\text{int}} u(x) = g(x) \quad \text{for } x \in \Gamma,$$

is given by the representation formula (cf. (1.6))

$$u(x) = \int_\Gamma u^*(x,y) g(y) ds_y - \int_\Gamma \gamma_{1,y}^{\text{int}} u^*(x,y) \bar{u}(y) ds_y \quad \text{for } x \in \Omega,$$

where the unknown Dirichlet datum $\bar{u} = \gamma_0^{\text{int}} u \in H^{1/2}(\Gamma)$ is a solution of the hypersingular boundary integral equation (cf. (1.27))

$$-\gamma_1^{\text{int}} \int_\Gamma \gamma_{1,y}^{\text{int}} u^*(x,y) \bar{u}(y) ds_y = \frac{1}{2} g(x) - \int_\Gamma \gamma_{1,x}^{\text{int}} u^*(x,y) g(y) ds_y$$

for $x \in \Gamma$. Since the hypersingular boundary integral operator D has a non–trivial kernel, we consider the extended variational problem (cf. (1.29)) to find $\bar{u}_\alpha \in H^{1/2}(\Gamma)$ such that

$$\left\langle D\bar{u}_\alpha, v \right\rangle_\Gamma + \left\langle \bar{u}_\alpha, 1 \right\rangle_\Gamma \left\langle v, 1 \right\rangle_\Gamma = \left\langle \left(\frac{1}{2} I - K'\right) g, v \right\rangle_\Gamma + \alpha \left\langle v, 1 \right\rangle_\Gamma$$

is satisfied for all $v \in H^{1/2}(\Gamma)$. Note that from the solvability condition (1.22), we reproduce the scaling condition (1.25). Since the bilinear form of this variational problem is strictly positive, the variational problem is equivalent to the minimisation problem

$$F(\bar{u}_\alpha) = \min_{v \in H^{1/2}(\Gamma)} F(v)$$

with

$$F(v) = \frac{1}{2}\left(\left\langle Dv, v \right\rangle_\Gamma + \left\langle v, 1 \right\rangle_\Gamma^2 \right) - \left\langle \left(\frac{1}{2} I - K'\right) g, v \right\rangle_\Gamma - \alpha \left\langle v, 1 \right\rangle_\Gamma .$$

Using a sequence of finite dimensional subspaces $S_h^1(\Gamma) \subset H^{1/2}(\Gamma)$ spanned by piecewise linear and continuous basis functions, an associated approximate function

$$\bar{u}_{\alpha,h} = \sum_{j=1}^{M} \bar{u}_{\alpha,j} \, \varphi_j \in S_h^1(\Gamma) \tag{2.24}$$

is obtained from the minimisation problem

$$F(\bar{u}_{\alpha,h}) = \min_{v_h \in S_h^1(\Gamma)} F(v_h).$$

The solution $\bar{u}_{\alpha,h} \in S_h^1(\Gamma)$ of the above minimisation problem is then defined via the Galerkin equations

$$\Big\langle D\bar{u}_{\alpha,h}, \varphi_i \Big\rangle_\Gamma + \Big\langle \bar{u}_{\alpha,h}, 1 \Big\rangle_\Gamma \Big\langle \varphi_i, 1 \Big\rangle_\Gamma = \Big\langle \Big(\frac{1}{2}I - K'\Big)g, \varphi_i \Big\rangle_\Gamma + \alpha \Big\langle \varphi_i, 1 \Big\rangle_\Gamma \tag{2.25}$$

for $i = 1, \ldots, M$. Using (2.24), this becomes

$$\sum_{j=1}^{M} \bar{u}_{\alpha,j} \Big(\Big\langle D\varphi_j, \varphi_i \Big\rangle_\Gamma + \Big\langle \varphi_j, 1 \Big\rangle_\Gamma \Big\langle \varphi_i, 1 \Big\rangle_\Gamma \Big) = \Big\langle \Big(\frac{1}{2}I - K'\Big)g, \varphi_i \Big\rangle_\Gamma + \alpha \Big\langle \varphi_i, 1 \Big\rangle_\Gamma$$

for $i = 1, \ldots, M$. With

$$D_h[i,j] = \langle D\varphi_j, \varphi_i \rangle_\Gamma = \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{\big(\underline{\mathrm{curl}}_\Gamma \varphi_j(y)\,,\, \underline{\mathrm{curl}}_\Gamma \varphi_i(x)\big)}{|x-y|} ds_x ds_y,$$

$$a_i = \langle \varphi_i, 1 \rangle_\Gamma = \int_\Gamma \varphi_i(x) ds_x,$$

$$\begin{aligned} f_i &= \Big\langle \Big(\frac{1}{2}I - K'\Big)g, \varphi_i \Big\rangle_\Gamma \\ &= \frac{1}{2} \int_\Gamma g(x)\varphi_i(x) ds_x - \int_\Gamma \varphi_i(x) \int_\Gamma \gamma_{1,x}^{\mathrm{int}} u^*(x,y) g(y) ds_y ds_x \\ &= \frac{1}{2} \int_\Gamma g(x)\varphi_i(x) ds_x - \frac{1}{4\pi} \int_\Gamma \varphi_i(x) \int_\Gamma \frac{(y-x, \underline{n}(x))}{|x-y|^3} g(y) ds_y ds_x \end{aligned}$$

for $i, j = 1, \ldots, M$, we find the coefficient vector $\underline{\bar{u}}_\alpha \in \mathbb{R}^M$ as the unique solution of the linear system

$$\Big(D_h + \underline{a}\,\underline{a}^\top \Big) \underline{\bar{u}}_\alpha = \underline{f} + \alpha\, \underline{a}. \tag{2.26}$$

The extended stiffness matrix $D_h + \underline{a}\,\underline{a}^\top$ is symmetric and positive definite. Therefore, one may use a conjugate gradient scheme for an iterative solution of the linear system (2.26). However, due to the estimate for the spectral condition number

$$\kappa_2(D_h + \underline{a}\,\underline{a}^\top) \le c\,\frac{1}{h},$$

an appropriate preconditioning is sometimes needed.

Note, that instead of a direct evaluation of the hypersingular boundary integral operator D, we apply integration by parts to obtain the representation (1.9) in Lemma 1.4, where

$$\underline{\mathrm{curl}}_\Gamma \varphi_i(x) \,=\, \underline{n}(x) \times \nabla_x \widetilde{\varphi}_i(x) \quad \text{for } x \in \Gamma$$

is the surface curl operator, and $\widetilde{\varphi}_i$ is some locally defined extension of φ_i into the neighbourhood of Γ. Since φ_i is linear on every boundary element τ_k, and defining the extension $\widetilde{\varphi}_i$ to be constant along $\underline{n}(x)$, we obtain $\underline{\mathrm{curl}}_\Gamma \varphi_i$ to be a piecewise constant vector function. Hence, we get

$$D_h[i,j] \,= \tag{2.27}$$
$$\sum_{\tau_k \,\in\, \mathrm{supp}\ \varphi_i} \; \sum_{\tau_\ell \,\in\, \mathrm{supp}\ \varphi_j} \Big(\underline{\mathrm{curl}}_\Gamma \varphi_{i|\tau_k} \,,\, \underline{\mathrm{curl}}_\Gamma \varphi_{j|\tau_\ell}\Big) \frac{1}{4\pi} \int_{\tau_k} \int_{\tau_\ell} \frac{1}{|x-y|} ds_x ds_y \,.$$

Thus, the entries of the stiffness matrix D_h of the hypersingular boundary integral operator D are linear combinations of the entries $V_h[k,\ell]$ of the single layer potential matrix V_h. Hence, we can write

$$D_h \,=\, T^\top \begin{pmatrix} V_h & 0 & 0 \\ 0 & V_h & 0 \\ 0 & 0 & V_h \end{pmatrix} T$$

with some sparse transformation matrix $T \in \mathbb{R}^{M \times 3N}$.

From the $H^{1/2}(\Gamma)$–ellipticity of the extended bilinear form, i.e.,

$$\langle Dv, v\rangle_\Gamma + \langle v, 1\rangle_\Gamma^2 \,\geq\, c_1^D \, \|v\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma),$$

we conclude the unique solvability of the variational problem (2.25), or correspondingly, of the linear system (2.26). Furthermore, the quasi optimal error estimate, i.e., Cea's lemma,

$$\|\bar{u}_\alpha - \bar{u}_{\alpha,h}\|_{H^{1/2}(\Gamma)} \,\leq\, c \inf_{v_h \in S_h^1(\Gamma)} \|\bar{u}_\alpha - v_h\|_{H^{1/2}(\Gamma)}$$

holds. Combining this with the approximation property (2.10) for $\sigma = 1/2$, we get

$$\|\bar{u}_\alpha - \bar{u}_{\alpha,h}\|_{H^{1/2}(\Gamma)} \,\leq\, c\, h^{s-1/2} \, |\bar{u}_\alpha|_{H^s_{\mathrm{pw}}(\Gamma)} \,, \tag{2.28}$$

when assuming $\bar{u}_\alpha \in H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [1/2, 2]$. Applying the Aubin–Nitsche trick, we also obtain the error estimate

$$\|\bar{u}_\alpha - \bar{u}_{\alpha,h}\|_{H^\sigma(\Gamma)} \,\leq\, c\, h^{s-\sigma} \, |\bar{u}_\alpha|_{H^s_{\mathrm{pw}}(\Gamma)} \,,$$

when assuming $\bar{u}_\alpha \in H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [1/2, 2]$ and $\sigma \in [-1, 1/2]$.

Inserting the computed Galerkin solution $\bar{u}_{\alpha,h} \in S_h^1(\Gamma)$ into the representation formula (1.6), this gives an approximate representation formula

$$\widetilde{u}(x) = \int_\Gamma u^*(x,y)g(y)ds_y - \int_\Gamma \gamma_{1,y}^{\text{int}} u^*(x,y)\bar{u}_{\alpha,h}(y)ds_y \quad \text{for } x \in \Omega$$

describing an approximate solution of the Neumann boundary value problem (1.21). For an arbitrary $x \in \Omega$, the error is given by

$$u(x) - \widetilde{u}(x) = \int_\Gamma \gamma_{1,y}^{\text{int}} u^*(x,y)\Big(\bar{u}_{\alpha,h}(y) - \bar{u}_\alpha(y)\Big)ds_y\,.$$

Using a duality argument, the error estimate

$$|u(x) - \widetilde{u}(x)| \le \Big\|\gamma_{1,y}^{\text{int}} u^*(x,\cdot)\Big\|_{H^{-\sigma}(\Gamma)} \|\bar{u}_\alpha - \bar{u}_{\alpha,h}\|_{H^\sigma(\Gamma)}$$

for some $\sigma \in \mathbb{R}$ follows. Combining this with the error estimate (2.28) for the minimal value $\sigma = -1$, we obtain the pointwise error estimate

$$|u(x) - \widetilde{u}(x)| \le c\,h^{s+1} \Big\|\gamma_{1,y}^{\text{int}} u^*(x,\cdot)\Big\|_{H^1(\Gamma)} |\bar{u}_\alpha|_{H^s_{\text{pw}}(\Gamma)}\,,$$

when assuming $\bar{u}_\alpha \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [1/2, 2]$. Hence, if $\bar{u}_\alpha \in H^2_{\text{pw}}(\Gamma)$ is sufficiently smooth, we obtain the optimal order of convergence for $s = 2$,

$$|u(x) - \widetilde{u}(x)| \le c\,h^3\, \|\gamma_{1,y}^{\text{int}} u^*(x,\cdot)\|_{H^1(\Gamma)} |\bar{u}_\alpha|_{H^2_{\text{pw}}(\Gamma)}\,. \tag{2.29}$$

Again, the error estimate (2.29) involves the position of the observation point $x \in \Omega$, and, therefore, it is not valid in the limiting case $x \in \Gamma$.

As in the boundary element method for the Dirichlet boundary value problem, we may also approximate the given Neumann datum $g \in H^{-1/2}(\Gamma)$ first. If $g_h \in S_h^0(\Gamma)$ is defined by the L_2 projection, i.e. if it is the unique solution of the variational problem

$$\int_\Gamma g_h(x)\psi_k(x)\,ds_x = \int_\Gamma g(x)\psi_k(x)\,ds_x \quad \text{for } k = 1,\ldots,N\,,$$

then the error estimate

$$\|g - g_h\|_{H^\sigma(\Gamma)} \le c\,h^{s-\sigma}\,|g|_{H^s_{\text{pw}}(\Gamma)}$$

holds, when assuming $g \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$ and $\sigma \in [-1,0]$. Hence, if g is sufficiently smooth, i.e., $g \in H^1_{\text{pw}}(\Gamma)$, we get the optimal error estimate

$$\|g - g_h\|_{H^{-1}(\Gamma)} \le c\,h^2\,|g|_{H^1_{\text{pw}}(\Gamma)}. \tag{2.30}$$

Then,

$$
\begin{aligned}
\widetilde{f}_i &= \Big\langle \Big(\frac{1}{2}I - K'\Big) g_h, \varphi_i \Big\rangle_\Gamma \\
&= \frac{1}{2}\sum_{\ell=1}^N g_\ell \int_{\tau_\ell} \varphi_i(x) ds_x - \sum_{\ell=1}^N g_\ell \int_\Gamma \varphi_i(x) \int_{\tau_\ell} \gamma_{1,x}^{\text{int}} u^*(x,y) ds_y ds_x \\
&= \frac{1}{2}\sum_{\ell=1}^N g_\ell \int_{\tau_\ell} \varphi_i(x) ds_x - \sum_{\ell=1}^N g_\ell \frac{1}{4\pi} \int_\Gamma \varphi_i(x) \int_{\tau_\ell} \frac{(y-x, \underline{n}(x))}{|x-y|^3} ds_y ds_x \\
&= \sum_{\ell=1}^N g_\ell \left(\frac{1}{2} M_h[\ell, i] - K_h[\ell, i] \right) .
\end{aligned}
$$

Instead of the linear system (2.26), we now have to solve a linear system with a perturbed right hand side $\underline{\widetilde{f}}$ yielding a perturbed solution vector $\underline{\widetilde{u}}_\alpha$, i.e., we have to solve the linear system

$$
\Big(D_h + \underline{a}\,\underline{a}^\top\Big)\underline{\widetilde{u}}_\alpha \;=\; \Big(\frac{1}{2}M_h^\top - K_h^\top\Big)\underline{g} + \alpha \underline{a} . \tag{2.31}
$$

For the associated boundary element solution $\widetilde{u}_{\alpha,h} \in S_h^1(\Gamma)$, the error estimate

$$
\begin{aligned}
\|\bar{u}_\alpha - \widetilde{u}_{\alpha,h}\|_{H^{1/2}(\Gamma)} &\le \|\bar{u}_\alpha - \bar{u}_{\alpha,h}\|_{H^{1/2}(\Gamma)} + c\,\|g - g_h\|_{H^{-1/2}(\Gamma)} \\
&\le c\, h^{3/2} \Big(|\bar{u}_\alpha|_{H^2_{\text{pw}}(\Gamma)} + |g|_{H^1_{\text{pw}}(\Gamma)} \Big) ,
\end{aligned}
$$

holds, when assuming $\bar{u} \in H^2_{\text{pw}}(\Gamma)$ and $g \in H^1_{\text{pw}}(\Gamma)$. Applying the Aubin–Nitsche trick to obtain an error estimate in lower order Sobolev spaces, the restriction due to the error estimate (2.30) has to be considered. Hence, we obtain the error estimate

$$
\begin{aligned}
\|\bar{u} - \widetilde{u}_h\|_{H^\sigma(\Gamma)} &\le c_1\,\|\bar{u} - \bar{u}_h\|_{H^\sigma(\Gamma)} + c_2\,\|g - g_h\|_{H^{\sigma-1}(\Gamma)} \\
&\le c\, h^{2-\sigma} \Big(|\bar{u}|_{H^2_{\text{pw}}(\Gamma)} + |g|_{H^1_{\text{pw}}(\Gamma)} \Big) ,
\end{aligned}
$$

when assuming $\bar{u}_\alpha \in H^2_{\text{pw}}(\Gamma)$, $g \in H^1_{\text{pw}}(\Gamma)$, and $\sigma \ge 0$. Therefore, the optimal error estimate reads

$$
\|\bar{u}_\alpha - \widetilde{u}_{\alpha,h}\|_{L_2(\Gamma)} \;\le\; c\, h^2 \Big(|\bar{u}_\alpha|_{H^2_{\text{pw}}(\Gamma)} + |g|_{H^1_{\text{pw}}(\Gamma)} \Big) . \tag{2.32}
$$

For the approximate representation formula

$$
\widetilde{u}(x) \;=\; \int_\Gamma \gamma_{0,y}^{\text{int}} u^*(x,y) g_h(y) ds_y - \int_\Gamma \gamma_{1,y}^{\text{int}} u^*(x,y) \widetilde{u}_{\alpha,h}(y) ds_y \tag{2.33}
$$

for $x \in \Omega$, we then obtain the best possible error estimate

$$
|u(x) - \widetilde{u}(x)| \;\le\; c(x, g, \bar{u}_\alpha)\, h^2 , \tag{2.34}
$$

when assuming $\bar{u}_\alpha \in H^2_{\text{pw}}(\Gamma)$ and $g \in H^1_{\text{pw}}(\Gamma)$.

2.3.3 Mixed Boundary Value Problem

The solution of the mixed boundary value problem (cf. (1.34))

$$\begin{aligned}
-\Delta u(x) &= 0 && \text{for } x \in \Omega, \\
\gamma_0^{\text{int}} u(x) &= g(x) && \text{for } x \in \Gamma_D, \\
\gamma_1^{\text{int}} u(x) &= f(x) && \text{for } x \in \Gamma_N
\end{aligned}$$

is given by the representation formula

$$\begin{aligned}
u(x) = &\int_{\Gamma_D} u^*(x,y)\gamma_1^{\text{int}} u(y) ds_y + \int_{\Gamma_N} u^*(x,y) f(y) ds_y \\
&- \int_{\Gamma_D} \gamma_{1,y}^{\text{int}} u^*(x,y) g(y) ds_y - \int_{\Gamma_N} \gamma_{1,y}^{\text{int}} u^*(x,y) \gamma_0^{\text{int}} u(y) ds_y
\end{aligned}$$

for $x \in \Omega$, where we have to find the yet unknown Cauchy data $\gamma_0^{\text{int}} u$ on Γ_N and $\gamma_1^{\text{int}} u$ on Γ_D. Let $\widetilde{g} \in H^{1/2}(\Gamma)$ and $\widetilde{f} \in H^{-1/2}(\Gamma)$ be some arbitrary, but fixed extensions of the given boundary data $g \in H^{1/2}(\Gamma_D)$ and $f \in H^{-1/2}(\Gamma_N)$, respectively.

The new Cauchy data

$$\widetilde{u} = \gamma_0^{\text{int}} u - \widetilde{g} \in \widetilde{H}^{1/2}(\Gamma_N)$$

and

$$\widetilde{t} = \gamma_1^{\text{int}} u - \widetilde{f} \in \widetilde{H}^{-1/2}(\Gamma)$$

are the unique solutions of the variational problem (cf. (1.35))

$$a(\widetilde{t}, \widetilde{u}\,;\, w, v) \;=\; F(w,v)$$

for all $v \in \widetilde{H}^{1/2}(\Gamma_N)$ and $w \in \widetilde{H}^{-1/2}(\Gamma_D)$ with the bilinear form

$$\begin{aligned}
a(\widetilde{t}, \widetilde{u}\,;\, w, v) = &\frac{1}{4\pi} \int_{\Gamma_D} w(x) \int_{\Gamma_D} \frac{1}{|x-y|} \widetilde{t}(y) ds_y ds_x \\
&- \frac{1}{4\pi} \int_{\Gamma_D} w(x) \int_{\Gamma_N} \frac{(x-y, \underline{n}(y))}{|x-y|^3} \widetilde{u}(y) ds_y ds_x \\
&+ \frac{1}{4\pi} \int_{\Gamma_N} v(x) \int_{\Gamma_D} \frac{(y-x, \underline{n}(x))}{|x-y|^3} \widetilde{t}(y) ds_y ds_x \\
&+ \frac{1}{4\pi} \int_{\Gamma} \int_{\Gamma} \frac{\left(\underline{\text{curl}}_\Gamma v(x)\,,\, \underline{\text{curl}}_\Gamma \widetilde{u}(y)\right)}{|x-y|} ds_y ds_x
\end{aligned}$$

and with the linear form

$$F(w,v) = \frac{1}{2}\int\limits_{\Gamma_D} w(x)g(x)ds_x + \frac{1}{4\pi}\int\limits_{\Gamma_D} w(x)\int\limits_{\Gamma} \frac{(x-y,\underline{n}(y))}{|x-y|^3}\widetilde{g}(y)ds_y ds_x$$
$$- \frac{1}{4\pi}\int\limits_{\Gamma_D} w(x)\int\limits_{\Gamma} \frac{1}{|x-y|}\widetilde{f}(y)ds_y ds_x + \frac{1}{2}\int\limits_{\Gamma_N} v(x)f(x)ds_x$$
$$- \frac{1}{4\pi}\int\limits_{\Gamma_N} v(x)\int\limits_{\Gamma} \frac{(y-x,\underline{n}(x))}{|x-y|}\widetilde{f}(y)ds_y ds_x$$
$$- \frac{1}{4\pi}\int\limits_{\Gamma_N}\int\limits_{\Gamma} \frac{\big(\underline{\mathrm{curl}}_\Gamma v(x)\,,\ \underline{\mathrm{curl}}_\Gamma \widetilde{g}(y)\big)}{|x-y|}ds_y ds_x.$$

To be able to define approximate solutions of the above variational problem, we first define suitable trial spaces,

$$S_h^0(\Gamma_D) = S_h^0(\Gamma)\cap \widetilde{H}^{-1/2}(\Gamma_D) \;=\; \mathrm{span}\Big\{\psi_\ell\Big\}_{\ell=1}^{N_D},$$
$$S_h^1(\Gamma_N) = S_h^1(\Gamma)\cap \widetilde{H}^{1/2}(\Gamma_N) \;=\; \mathrm{span}\Big\{\varphi_j\Big\}_{j=1}^{M_N}.$$

The Galerkin formulation of the variational problem (1.35) is to find

$$\widetilde{t}_h \in S_h^0(\Gamma_D)$$

and

$$\widetilde{u}_h \in S_h^1(\Gamma_N)$$

such that

$$a(\widetilde{t}_h,\widetilde{u}_h\,;\,w_h,v_h) \;=\; F(w_h,v_h) \tag{2.35}$$

is satisfied for all $w_h \in S_h^0(\Gamma_D)$ and $v_h \in S_h^1(\Gamma_N)$. This formulation is equivalent to a linear system of equations

$$\begin{pmatrix} V_h & -K_h \\ K_h^\top & D_h \end{pmatrix}\begin{pmatrix} \underline{\widetilde{t}} \\ \underline{\widetilde{u}} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix} \tag{2.36}$$

with the following blocks:

$$V_h \in \mathbb{R}^{N_D\times N_D}, \quad K_h \in \mathbb{R}^{N_D\times M_N}, \quad D_h \in \mathbb{R}^{M_N\times M_N}.$$

The matrix entries of these blocks are defined by

$$V_h[k,\ell] = \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\tau_\ell} \frac{1}{|x-y|}ds_y ds_x,$$
$$K_h[k,j] = \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\Gamma} \frac{(x-y,\underline{n}(y))}{|x-y|^3}\varphi_j(y)ds_y ds_x,$$
$$D_h[i,j] = \frac{1}{4\pi}\int\limits_{\Gamma}\int\limits_{\Gamma} \frac{(\underline{\mathrm{curl}}_\Gamma\varphi_j(y),\underline{\mathrm{curl}}_\Gamma\varphi_i(x))}{|x-y|}ds_y ds_x$$

for all $k, \ell = 1, \ldots, N_D$ and $i, j = 1, \ldots, M_N$. The components of the right hand side, $\underline{g} \in \mathbb{R}^{N_D}$ and $\underline{f} \in \mathbb{R}^{M_N}$, are given by

$$\begin{aligned} g_k &= \frac{1}{2} \int\limits_{\tau_k} g(x) ds_x + \frac{1}{4\pi} \int\limits_{\tau_k} \int\limits_{\Gamma} \frac{(x-y, \underline{n}(y))}{|x-y|^3} \widetilde{g}(y) ds_y ds_x \\ &\quad - \frac{1}{4\pi} \int\limits_{\tau_k} \int\limits_{\Gamma} \frac{1}{|x-y|} \widetilde{f}(y) ds_y ds_x \,, \\ f_i &= \frac{1}{2} \int\limits_{\Gamma_N} \varphi_i(x) f(x) ds_x - \frac{1}{4\pi} \int\limits_{\Gamma_N} \varphi_i(x) \int\limits_{\Gamma} \frac{(y-x, \underline{n}(x))}{|x-y|} \widetilde{f}(y) ds_y ds_x \\ &\quad - \frac{1}{4\pi} \int\limits_{\Gamma_N} \int\limits_{\Gamma} \frac{(\underline{\mathrm{curl}}_\Gamma \widetilde{g}(y), \underline{\mathrm{curl}}_\Gamma \varphi_i(x))}{|x-y|} ds_y ds_x \end{aligned}$$

for all $k = 1, \ldots, N_D$ and $i = 1, \ldots, M_N$.

Since the trial spaces $S_h^0(\Gamma_D) \subset S_h^0(\Gamma)$ and $S_h^1(\Gamma_N) \subset S_h^1(\Gamma)$ are subspaces of the trial spaces already used for the Dirichlet and for the Neumann boundary value problems, the blocks of the matrix in (2.36) are submatrices of the stiffness matrices already used in (2.22) and in (2.31), respectively. In particular, the evaluation of the discrete hypersingular integral operator D_h can be reduced to the evaluation of some linear combinations of the matrix entries of the discrete single layer potential V_h.

Since the stiffness matrix in (2.36) is positive definite but block skew symmetric, we have to apply a generalised Krylov subspace method such as the Generalised Minimal Residual Method (GMRES) (see Appendix C.3) to solve (2.36) by an iterative method. Here we will describe two alternative approaches to apply the conjugate gradient scheme to solve (2.36).

Since the discrete single layer potential V_h is symmetric and positive definite and hence invertible, we can solve the first equation in (2.36) to find

$$\widetilde{\underline{t}} = V_h^{-1} \left(\underline{g} + K_h \widetilde{\underline{u}} \right).$$

Inserting this into the second equation in (2.36), this gives the Schur complement system

$$S_h \widetilde{\underline{u}} = \underline{f} - K_h^\top V_h^{-1} \underline{g} \tag{2.37}$$

with the symmetric and positive definite Schur complement matrix

$$S_h = D_h + K_h^\top V_h^{-1} K_h \,.$$

Therefore, we can apply a conjugate gradient scheme to solve (2.37), where we eventually need a suitable preconditioning matrix for S_h. Note that the matrix by vector multiplication with the Schur complement matrix S_h involves one application of the inverse single layer potential matrix V_h. This can be realised either by a direct inversion, if the dimension N_D is small, or by the application of an inner conjugate gradient scheme. Again, a suitable preconditioning matrix is eventually needed, which is spectrally equivalent to V_h.

Following [14], we can also apply a suitable transformation to (2.36) to obtain a linear system with a symmetric, positive definite matrix. In particular, the transformed matrix

$$\begin{pmatrix} V_h C_V^{-1} - I & 0 \\ -K_h^\top C_V^{-1} & I \end{pmatrix} \begin{pmatrix} V_h & -K_h \\ K_h^\top & D_h \end{pmatrix} = \begin{pmatrix} V_h C_V^{-1} V_h - V_h & (I - V_h C_V^{-1}) K_h \\ K_h^\top (I - C_V^{-1} V_h) & D_h + K_h^\top C_V^{-1} K_h \end{pmatrix}$$

is symmetric and positive definite. Hence, instead of (2.36), we now solve the transformed linear system

$$\begin{pmatrix} V_h C_V^{-1} V_h - V_h & (I - V_h C_V^{-1}) K_h \\ K_h^\top (I - C_V^{-1} V_h) & D_h + K_h^\top C_V^{-1} K_h \end{pmatrix} \begin{pmatrix} \underline{\widetilde{t}} \\ \underline{\widetilde{u}} \end{pmatrix} = \begin{pmatrix} V_h C_V^{-1} - I & 0 \\ -K_h^\top C_V^{-1} & I \end{pmatrix} \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix} \tag{2.38}$$

by a preconditioned conjugate gradient scheme. In the above, C_V is a suitable preconditioning matrix, which is spectrally equivalent to the discrete single layer potential V_h, i.e.,

$$c_1\,(C_V \underline{w}, \underline{w}) \le (V_h \underline{w}, \underline{w}) \le c_2\,(C_V \underline{w}, \underline{w}) \quad \text{for all } \underline{w} \in \mathbb{R}^{N_D}.$$

To ensure that (2.38) is equivalent to (2.36), we have to require the invertibility of

$$V_h C_V^{-1} - I = (V_h - C_V) C_V^{-1}.$$

Due to

$$((V_h - C_V)\underline{w}, \underline{w}) \ge (c_1 - 1)(C_V \underline{w}, \underline{w}) \quad \text{for all } \underline{w} \in \mathbb{R}^{N_D},$$

a sufficient condition is $c_1 > 1$, which ensures the positive definiteness of $V_h - C_V$, and, therefore, its invertibility. A suitable preconditioning matrix for (2.38) is

$$C_M = \begin{pmatrix} V_h - C_V & 0 \\ 0 & C_S \end{pmatrix},$$

where C_S is a preconditioning matrix for the Schur complement S_h.

From the $\widetilde{H}^{-1/2}(\varGamma_D) \times \widetilde{H}^{1/2}(\varGamma_N)$–ellipticity of the underlying bilinear form $a(\cdot, \cdot; \cdot, \cdot)$, we conclude the unique solvability of the Galerkin variational problem (2.35), and, therefore, of the linear system (2.36). In particular, we obtain the quasi optimal error estimate

$$\|\widetilde{t}-\widetilde{t}_h\|^2_{H^{-1/2}(\Gamma)}+\|\widetilde{u}-\widetilde{u}_h\|^2_{H^{1/2}(\Gamma)}$$
$$\leq c\left(\inf_{w_h\in S_h^0(\Gamma_D)}\|\widetilde{t}-w_h\|^2_{H^{-1/2}(\Gamma)}+\inf_{v_h\in S_h^1(\Gamma_N)}\|\widetilde{u}-v_h\|^2_{H^{1/2}(\Gamma)}\right)$$

from Cea's lemma. Using the approximation property (2.5) for $\sigma=-1/2$ as well as the approximation property (2.10) for $\sigma=1/2$, this gives

$$\|\widetilde{t}-\widetilde{t}_h\|^2_{H^{-1/2}(\Gamma)}+\|\widetilde{u}-\widetilde{u}_h\|^2_{H^{1/2}(\Gamma)}\leq c_1h^{2s_1+1}|\widetilde{t}|^2_{H^{s_1}_{\mathrm{pw}}(\Gamma)}+c_2h^{2s_2-1}|\widetilde{u}|^2_{H^{s_2}_{\mathrm{pw}}(\Gamma)}\,,$$

when assuming $\widetilde{t}\in H^{s_1}_{\mathrm{pw}}(\Gamma)$ for some $s_1\in[-1/2,1]$, and $\widetilde{u}\in H^{s_2}_{\mathrm{pw}}(\Gamma)$ for some $s_2\in[1/2,2]$. Since, in general, those regularity estimates result from a regularity estimate for the solution $u\in H^s(\Omega)$ of the mixed boundary value problem (1.34), we obtain $\gamma_0^{\mathrm{int}}u\in H^{s-1/2}_{\mathrm{pw}}(\Gamma)$ and $\gamma_1^{\mathrm{int}}u\in H^{s-3/2}_{\mathrm{pw}}(\Gamma)$ by applying the trace theorems, and, therefore, $s_1=s-3/2$ and $s_2=s-1/2$. Thus, if $u\in H^s(\Omega)$ is the solution of the mixed boundary value problem (1.34) for some $s\in[1,5/2]$, we then obtain the error estimate

$$\|\widetilde{t}-\widetilde{t}_h\|^2_{H^{-1/2}(\Gamma)}+\|\widetilde{u}-\widetilde{u}_h\|^2_{H^{1/2}(\Gamma)}\;\leq\;c\,h^{2(s-1)}\,|u|^2_{H^s(\Omega)}.$$

As for the Dirichlet and for the Neumann boundary value problem, applying the Aubin–Nitsche trick (for $\sigma\in[-2,1/2)$) and an inverse inequality argument (for $\sigma\in(-1/2,0]$), we obtain the error estimate

$$\|\widetilde{t}-\widetilde{t}_h\|^2_{H^{\sigma}(\Gamma)}+\|\widetilde{u}-\widetilde{u}_h\|^2_{H^{\sigma+1}(\Gamma)}\;\leq\;c\,h^{2(s-\sigma)-3}\,|u|^2_{H^s(\Omega)}\,,\tag{2.39}$$

when assuming $u\in H^s(\Omega)$ for some $s\in[1,5/2]$ and $\sigma\in[-2,0]$.

Inserting the computed Galerkin solutions $\widetilde{t}_h\in S_h^0(\Gamma_D)$ and $\widetilde{u}_h\in S_h^1(\Gamma_N)$ into the representation formula (1.6), this gives an approximate representation formula

$$\widetilde{u}(x)=\int_\Gamma u^*(x,y)\Big(\widetilde{t}_h(y)+\widetilde{f}(y)\Big)ds_y-\int_\Gamma\gamma^{\mathrm{int}}_{1,y}u^*(x,y)\Big(\widetilde{u}_h(y)+\widetilde{g}(y)\Big)ds_y$$

for $x\in\Omega$. The above formula describes an approximate solution of the mixed boundary value problem (1.34). For an arbitrary $x\in\Omega$, the error is given by

$$u(x)-\widetilde{u}(x)=$$
$$\int_{\Gamma_N}u^*(x,y)\Big(\widetilde{t}(y)-\widetilde{t}_h(y)\Big)ds_y-\int_{\Gamma_D}\gamma^{\mathrm{int}}_{1,y}u^*(x,y)\Big(\widetilde{u}(y)-\widetilde{u}_h(y)\Big)ds_y\,.$$

Using a duality argument, the error estimate

$$|u(x)-\widetilde{u}(x)|\leq$$
$$\|u^*(x,\cdot)\|_{H^{-\sigma_1}(\Gamma)}\|\widetilde{t}-\widetilde{t}_h\|_{H^{\sigma_1}(\Gamma)}+\Big\|\gamma_1^{\mathrm{int}}u^*(x,\cdot)\Big\|_{H^{-\sigma_2}(\Gamma)}\|\widetilde{u}-\widetilde{u}_h\|_{H^{\sigma_2}(\Gamma)}$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}$ follows. Combining this with the error estimate (2.39) for the minimal values $\sigma_1 = -2$ and $\sigma_2 = -1$, we obtain the pointwise error estimate

$$|u(x) - \widetilde{u}(x)| \leq c\, h^{2s+1} \Big(\|u^*(x,\cdot)\|_{H^2(\Gamma)} + \|\gamma_1^{\text{int}} u^*(x,\cdot)\|_{H^1(\Gamma)} \Big)\, |u|_{H^s(\Omega)}\,,$$

when assuming $u \in H^s(\Omega)$ for some $s \in [1, 5/2]$. In particular, for $s = 5/2$, we obtain the optimal order of convergence,

$$|u(x) - \widetilde{u}(x)| \leq c(x)\, h^3\, |u|_{H^{5/2}(\Omega)}\,. \tag{2.40}$$

Note that the error estimate (2.40) is based on the exact use of the given boundary data $g \in H^{1/2}(\Gamma_D)$ and $f \in H^{-1/2}(\Gamma_N)$, and their extensions $\widetilde{g} \in H^{1/2}(\Gamma)$ and $\widetilde{f} \in H^{-1/2}(\Gamma)$.

Starting from an approximation $u_h \in S_h^1(\Gamma)$ of the complete Dirichlet datum $\gamma_0^{\text{int}} u$,

$$u_h = \sum_{j=1}^{M} u_j \varphi_j = \sum_{j=1}^{M_N} u_j \varphi_j + \sum_{j=M_N+1}^{M} u_j \varphi_j = \widetilde{u}_h + g_h\,,$$

we first have to find the coefficients u_j for $j = M_N + 1, \ldots, M$ of the approximate Dirichlet datum $g_h \in S_h^1(\Gamma) \cap H^{1/2}(\Gamma_N)$. This can be done, e.g., by applying the L_2 projection,

$$\sum_{j=M_N+1}^{M} u_j \int\limits_{\Gamma_D} \varphi_j(x) \varphi_i(x) dx = \int\limits_{\Gamma_D} g(x) \varphi_i(x) ds_x \quad \text{for } i = M_N + 1, \ldots, M.$$

In a similar way, we obtain an approximation $f_h \in S_h^0(\Gamma_N)$ of the given Neumann datum $f \in H^{-1/2}(\Gamma_N)$,

$$\sum_{\ell=N_D+1}^{N} t_\ell \int\limits_{\Gamma_N} \psi_\ell(x) \psi_k(x) dx = \int\limits_{\Gamma_N} f(x) \psi_k(x) ds_x \quad \text{for } k = N_D + 1, \ldots, N.$$

Hence, we have to find the remaining Cauchy data

$$\widetilde{t}_h \in S_h^0(\Gamma_D) \quad \text{and} \quad \widetilde{u}_h \in S_h^1(\Gamma_N)$$

from the variational problem

$$a(\widetilde{t}_h, \widetilde{u}_h\,; \psi_k, \varphi_i) = \widetilde{F}(\psi_k, \varphi_i)$$

for $k = 1, \ldots, N_D$ and $i = 1, \ldots, M_N$, where the perturbed linear form is now given by

$$
\begin{aligned}
\widetilde{F}(\psi_k,\varphi_i) &= \frac{1}{2}\int\limits_{\tau_k} g_h(x)ds_x + \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\Gamma}\frac{(x-y,\underline{n}(y))}{|x-y|^3}g_h(y)ds_y ds_x \\
&\quad - \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\Gamma_N}\frac{1}{|x-y|}f_h(y)ds_y ds_x + \frac{1}{2}\int\limits_{\Gamma_N} f_h(x)\varphi_i(x)ds_x \\
&\quad - \frac{1}{4\pi}\int\limits_{\Gamma_N}\varphi_i(x)\int\limits_{\Gamma_N}\frac{(y-x,\underline{n}(x))}{|x-y|^3}f_h(y)ds_y ds_x \\
&\quad - \frac{1}{4\pi}\int\limits_{\Gamma_N}\int\limits_{\Gamma}\frac{\big(\underline{\mathrm{curl}}_\Gamma\varphi_i(x)\,,\,\underline{\mathrm{curl}}_\Gamma g_h(y)\big)}{|x-y|}ds_y ds_x.
\end{aligned}
$$

The above perturbed variational problem is now equivalent to a linear system of equations

$$
\begin{pmatrix} V_h & -K_h \\ K_h^\top & D_h \end{pmatrix}\begin{pmatrix} \underline{\widetilde{t}} \\ \underline{\widetilde{u}} \end{pmatrix} = \begin{pmatrix} -\bar{V}_h & \frac{1}{2}\bar{M}_h + \bar{K}_h \\ \frac{1}{2}\bar{M}_h^\top - \bar{K}_h^\top & -\bar{D}_h \end{pmatrix}\begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}. \tag{2.41}
$$

Note that the right hand side of this system differs from the one in (2.36). The blocks on the right have the following dimensions:

$$
\bar{V}_h \in \mathbb{R}^{N_D\times(N-N_D)}, \quad \bar{M}_h \in \mathbb{R}^{N_D\times(M-M_N)}, \quad \bar{K}_h \in \mathbb{R}^{N_D\times(M-M_N)}
$$

and the following entries

$$
\begin{aligned}
\bar{V}_h[k,\ell] &= \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\tau_\ell}\frac{1}{|x-y|}ds_y ds_x, \\
\bar{M}_h[k,j] &= \int\limits_{\tau_k}\varphi_j(x)ds_x, \\
\bar{K}_h[k,j] &= \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\Gamma}\frac{(x-y,\underline{n}(y))}{|x-y|^3}\varphi_j(y)ds_y ds_x, \\
\bar{D}_h[i,j] &= \frac{1}{4\pi}\int\limits_{\Gamma}\int\limits_{\Gamma}\frac{\big(\underline{\mathrm{curl}}_\Gamma\varphi_j(y)\,,\,\underline{\mathrm{curl}}_\Gamma\varphi_i(x)\big)}{|x-y|}ds_y ds_x
\end{aligned}
$$

for

$$
\ell = N_D+1,\ldots,N\,,\ k=1,\ldots,N_D\,,\ j=M_N+1,\ldots,M\,,\ i=1,\ldots,M_N\,.
$$

Note that the matrices $\bar{V}_h$, $\bar{M}_h$, $\bar{K}_h$, and $\bar{D}_h$ are also submatrices of the stiffness matrices already used in (2.16) and (2.26) to handle the Dirichlet and Neumann boundary value problem, respectively.

The solution of the perturbed linear system (2.41) can be realised as for the linear system (2.36). The error estimates for the resulting approximations can be obtained as in the previous cases, however, the approximations of the given boundary data have to be recognised accordingly. This can be done as for the Dirichlet boundary value problem and as for the Neumann boundary value problem. In particular, the error estimate (2.39) holds for $\sigma \in [-1,0]$, and instead of (2.40), we obtain only the pointwise error estimate

$$|u(x) - \widetilde{u}(x)| \le c(x)\, h^2\, |u|_{H^{5/2}(\Omega)} \tag{2.42}$$

for $x \in \Omega$, when assuming $u \in H^{5/2}(\Omega)$.

2.3.4 Interface Problem

We consider the interface problem (1.56)–(1.58), i.e., the system of partial differential equations (1.56),

$$-\alpha_i \Delta u_i(x) = f(x) \quad \text{for } x \in \Omega, \quad -\alpha_e \Delta u_e(x) = 0 \quad \text{for } x \in \Omega^e,$$

the transmission conditions (1.57),

$$\gamma_0^{\text{int}} u_i(x) = \gamma_0^{\text{ext}} u_e(x), \quad \alpha_i \gamma_1^{\text{int}} u_i(x) = \alpha_e \gamma_1^{\text{ext}} u_e(x) \quad \text{for } x \in \Gamma,$$

and the radiation condition (1.58) with $u_0 = 0$,

$$|u_e(x)| = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty\,.$$

Introducing $\bar{u} = \gamma_0^{\text{int}} u_i = \gamma_0^{\text{ext}} u_e \in H^{1/2}(\Gamma)$, we have to solve the resulting variational problem (1.59),

$$\langle (\alpha_i S^{\text{int}} + \alpha_e S^{\text{ext}})\bar{u}, v\rangle_\Gamma = \langle S^{\text{int}} \gamma_0^{\text{int}} u_p - \gamma_1^{\text{int}} u_p, v\rangle_\Gamma$$

for all $v \in H^{1/2}(\Gamma)$, where u_p is a particular solution satisfying $-\Delta u_p = f$ in Ω.

Using a sequence of finite dimensional subspaces $S_h^1(\Gamma) \subset H^{1/2}(\Gamma)$ spanned by piecewise linear and continuous basis functions, an associated approximate solution

$$\bar{u}_h = \sum_{j=1}^{M} \bar{u}_j \varphi_j \in S_h^1(\Gamma)$$

can be found as the unique solution of the Galerkin equations

$$\langle (\alpha_i S^{\text{int}} + \alpha_e S^{\text{ext}})\bar{u}_h, \varphi_i\rangle_\Gamma = \langle S^{\text{int}} \gamma_0^{\text{int}} u_p - \gamma_1^{\text{int}} u_p, \varphi_i\rangle_\Gamma \tag{2.43}$$

for $i = 1, \ldots, M$. This is equivalent to a system of linear equations,

$$S_h \underline{\bar{u}} = \underline{f},$$

with $S_h \in \mathbb{R}^{M\times M}$ and $\underline{f} \in \mathbb{R}^M$ with the entries

$$S_h[i,j] = \langle(\alpha_i S^{\text{int}} + \alpha_e S^{\text{ext}})\varphi_j, \varphi_i\rangle_\Gamma,$$

$$f_i = \langle S^{\text{int}}\gamma_0^{\text{int}} u_p - \gamma_1^{\text{int}} u_p, \varphi_i\rangle_\Gamma$$

for $i,j = 1,\ldots,M$. Since the Steklov–Poincaré operators

$$\begin{aligned}
(S^{\text{int}}\bar{u})(x) &= V^{-1}\Big(\frac{1}{2}I + K\Big)\bar{u}(x)\\
&= \left(D + \Big(\frac{1}{2}I + K'\Big)V^{-1}\Big(\frac{1}{2}I + K\Big)\right)\bar{u}(x),\\
(S^{\text{ext}}\bar{u})(x) &= V^{-1}\Big(-\frac{1}{2}I + K\Big)\bar{u}(x)\\
&= \left(D + \Big(-\frac{1}{2}I + K'\Big)V^{-1}\Big(-\frac{1}{2}I + K\Big)\right)\bar{u}(x)
\end{aligned}$$

do not allow a direct evaluation of both, the stiffness matrix and the right hand side, additional approximations are required. The application of the Steklov–Poincaré operator S^{int} related to the interior Dirichlet boundary value problem can be written as

$$\begin{aligned}
(S^{\text{int}}\bar{u})(x) &= \left(D + \Big(\frac{1}{2}I + K'\Big)V^{-1}\Big(\frac{1}{2}I + K\Big)\right)\bar{u}(x)\\
&= (D\bar{u})(x) + \Big(\frac{1}{2}I + K'\Big)t_i(x)\,,
\end{aligned}$$

where

$$t_i = V^{-1}\Big(\frac{1}{2}I + K\Big)\bar{u} \in H^{-1/2}(\Gamma)$$

is the unique solution of the variational problem

$$\langle Vt_i, w\rangle_\Gamma \;=\; \Big\langle\Big(\frac{1}{2}I + K\Big)\bar{u}, w\Big\rangle_\Gamma \quad \text{for all } w \in H^{-1/2}(\Gamma).$$

Let $t_{i,h} \in S_h^0(\Gamma)$ be the unique solution of the Galerkin variational problem

$$\Big\langle Vt_{i,h}, w_h\Big\rangle_\Gamma \;=\; \Big\langle\Big(\frac{1}{2}I + K\Big)\bar{u}, w_h\Big\rangle_\Gamma \quad \text{for all } w_h \in S_h^0(\Gamma).$$

Then,

$$(\widetilde{S}^{\text{int}}\bar{u})(x) \;=\; (D\bar{u})(x) + \Big(\frac{1}{2}I + K'\Big)t_{i,h}(x)$$

defines an approximate Steklov–Poincaré operator associated to the interior Dirichlet boundary value problem. In the same way, we define an approximate Steklov–Poincaré operator

$$(\widetilde{S}^{\text{ext}}\bar{u})(x) \,=\, (D\bar{u})(x) + \Big(-\frac{1}{2}I + K'\Big)t_{e,h}(x)\,,$$

which is associated to the exterior Dirichlet boundary value problem, and where $t_{e,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin equations

$$\langle Vt_{e,h}, w_h\rangle_\Gamma \,=\, \Big\langle \Big(-\frac{1}{2}I + K\Big)\bar{u}, w_h\Big\rangle_\Gamma \quad \text{for all } w_h \in S_h^0(\Gamma).$$

Now, instead of the variational problem (2.43), we consider the perturbed problem

$$\langle(\alpha_i\widetilde{S}^{\text{int}} + \alpha_e\widetilde{S}^{\text{ext}})\widetilde{u}_h, \varphi_i\rangle_\Gamma \,=\, \langle\widetilde{S}^{\text{int}}u_{p,h} - t_{p,h}, \varphi_i\rangle_\Gamma \tag{2.44}$$

for $i = 1, \ldots, M$. In (2.44), $t_{p,h} \in S_h^0(\Gamma)$ and $u_{p,h} \in S_h^1(\Gamma)$ are suitable approximations (L_2 projections) of the Cauchy data of the particular solution u_p, i.e.,

$$\langle t_{p,h}, \psi_k\rangle_{L_2(\Gamma)} \,=\, \langle\gamma_1^{\text{int}}u_p, \psi_k\rangle_{L_2(\Gamma)}$$

for $k = 1, \ldots, N$ and

$$\langle u_{p,h}, \varphi_i\rangle_{L_2(\Gamma)} \,=\, \langle\gamma_0^{\text{int}}u_p, \varphi_i\rangle_{L_2(\Gamma)}$$

for $i = 1, \ldots, M$. From (2.44), we then obtain the linear system

$$\begin{aligned}
&\Bigg(\alpha_i\left(D_h + \Big(\frac{1}{2}M_h^\top + K_h^\top\Big)V_h^{-1}\Big(\frac{1}{2}M_h + K_h\Big)\right) \\
&\quad +\alpha_e\left(D_h + \Big(-\frac{1}{2}M_h^\top + K_h^\top\Big)V_h^{-1}\Big(-\frac{1}{2}M_h + K_h\Big)\right)\Bigg)\,\underline{\widetilde{u}} = \\
&\left(D_h + \Big(\frac{1}{2}M_h^\top + K_h^\top\Big)V_h^{-1}\Big(\frac{1}{2}M_h + K_h\Big)\right)\underline{u}_p - M_h^\top\underline{t}_p\,,
\end{aligned} \tag{2.45}$$

where

$$V_h \in \mathbb{R}^{N\times N}, \quad M_h \in \mathbb{R}^{N\times M}, \quad K_h \in \mathbb{R}^{N\times M}, \quad D_h \in \mathbb{R}^{M\times M}$$

are the Galerkin stiffness matrices, which have already been used for the Dirichlet and for the Neumann boundary value problems. The entries of these matrices are defined as

$$\begin{aligned}
V_h[k,\ell] &= \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\tau_\ell}\frac{1}{|x-y|}ds_y ds_x, \\
M_h[k,j] &= \int\limits_{\tau_k}\varphi_j(x)ds_x, \\
K_h[k,j] &= \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\Gamma}\frac{(x-y,\underline{n}(y))}{|x-y|^3}\varphi_j(y)ds_y ds_x, \\
D_h[i,j] &= \frac{1}{4\pi}\int\limits_{\Gamma}\int\limits_{\Gamma}\frac{(\underline{\text{curl}}_\Gamma\varphi_j(y), \underline{\text{curl}}_\Gamma\varphi_i(x))}{|x-y|}ds_y ds_x
\end{aligned}$$

for $k, \ell = 1, \ldots, N$ and $i, j = 1, \ldots, M$.

Instead of the linear system (2.45) we may also solve the equivalent coupled system

$$\begin{pmatrix} \alpha_i V_h & 0 & -\alpha_i(\frac{1}{2}M_h + K_h) \\ 0 & \alpha_e V_h & -\alpha_e(-\frac{1}{2}M_h + K_h) \\ \alpha_i(\frac{1}{2}M_h^\top + K_h^\top) & \alpha_e(-\frac{1}{2}M_h^\top + K_h^\top) & (\alpha_i + \alpha_e)D_h \end{pmatrix} \begin{pmatrix} \underline{t}_i \\ \underline{t}_e \\ \underline{\widetilde{u}} \end{pmatrix}$$

$$= \begin{pmatrix} -(\frac{1}{2}M_h + K_h)\underline{u}_p \\ \underline{0} \\ D_h \underline{u}_p - M_h^\top \underline{t}_p \end{pmatrix}, \tag{2.46}$$

which is of the same structure as the linear system (2.36), i.e. block skew symmetric but positive definite. Note that (2.45) is the Schur complement system of (2.46).

As for the Neumann boundary value problem, we conclude the error estimate

$$\|\bar{u} - \widetilde{u}_h\|_{H^{1/2}(\Gamma)} \leq c_1 \inf_{v_h \in S_h^1(\Gamma)} \|\bar{u} - v_h\|_{H^{1/2}(\Gamma)}$$
$$+ c_2 \inf_{w_h \in S_h^0(\Gamma)} \|S^{\mathrm{int}}\bar{u} - w_h\|_{H^{-1/2}(\Gamma)} + c_3 \inf_{w_h \in S_h^0(\Gamma)} \|S^{\mathrm{ext}}\bar{u} - w_h\|_{H^{-1/2}(\Gamma)} .$$

Hence, assuming $\bar{u} \in H^2_{\mathrm{pw}}(\Gamma)$ and $S^{\mathrm{int/ext}}\bar{u} \in H^1_{\mathrm{pw}},(\Gamma)$, we obtain the error estimate

$$\|\bar{u} - \widetilde{u}_h\|_{H^{1/2}(\Gamma)} \leq c\, h^{3/2} \left(\|\bar{u}\|_{H^2_{\mathrm{pw}}(\Gamma)} + \|S^{\mathrm{int}}\bar{u}\|_{H^1_{\mathrm{pw}}(\Gamma)} + \|S^{\mathrm{ext}}\bar{u}\|_{H^1_{\mathrm{pw}}(\Gamma)} \right),$$

and by applying the Aubin–Nitsche trick, we get

$$\|\bar{u} - \widetilde{u}_h\|_{L_2(\Gamma)} \leq c(\bar{u})\, h^2 .$$

When the Dirichlet datum $\bar{u}_h$ is known, one can compute the remaining Neumann datum by solving both, the interior and exterior Dirichlet boundary value problems. Since those boundary value problems are Dirichlet boundary value problems with approximated boundary data, the corresponding error estimates are still valid.

2.4 Lame Equations

For a simply connected domain $\Omega \subset \mathbb{R}^3$, we consider the mixed boundary value problem (1.79)

$$
\begin{aligned}
-\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\underline{u}, x) = 0 & \qquad \text{for } x \in \Omega\,, \\
\gamma_0^{\text{int}} u_i(x) = g_i(x) & \quad \text{for } x \in \Gamma_{D,i}\,, \\
(\gamma_1^{\text{int}} \underline{u})_i(x) \;=\; \sum_{j=1}^{3} \sigma_{ij}(\underline{u}, x) n_j(x) = f_i(x) & \quad \text{for } x \in \Gamma_{N,i}\,,
\end{aligned}
$$

for $i = 1, 2, 3$. Note that we assume

$$
\Gamma = \overline{\Gamma}_{N,i} \cup \overline{\Gamma}_{D,i}\,, \quad \Gamma_{N,i} \cap \Gamma_{D,i} = \varnothing\,, \quad \text{meas}\, \Gamma_{D,i} > 0
$$

for $i = 1, 2, 3$. To find the yet unknown Cauchy data $(\gamma_1^{\text{int}} \underline{u})_i$ on $\Gamma_{D,i}$ and $\gamma_0^{\text{int}} u_i$ on $\Gamma_{N,i}$, we consider the variational problem (1.80), which is related to the symmetric formulation of boundary integral equations. Hence, we have to find

$$
\widetilde{t}_i = (\gamma_1^{\text{int}} \underline{u})_i - \widetilde{f}_i \in \widetilde{H}^{-1/2}(\Gamma_{D,i})
$$

and

$$
\widetilde{u}_i = \gamma_0^{\text{int}} u_i - \widetilde{g}_i \in \widetilde{H}^{1/2}(\Gamma_{N,i})
$$

such that

$$
a(\widetilde{\underline{t}}, \widetilde{\underline{u}}\,;\, \underline{w}, \underline{v}) \;=\; F(\underline{w}, \underline{v})
$$

is satisfied for all $w_i \in \widetilde{H}^{-1/2}(\Gamma_{D,i})$ and $v_i \in \widetilde{H}^{1/2}(\Gamma_{N,i})$ for $i = 1, 2, 3$. Note that the bilinear form is given by

$$
\begin{aligned}
a(\widetilde{\underline{t}}, \widetilde{\underline{u}}\,;\, \underline{w}, \underline{v}) = & \sum_{i=1}^{3} \Big\langle (V^{\text{Lame}} \widetilde{\underline{t}})_i, w_i \Big\rangle_{\Gamma_{D,i}} - \sum_{i=1}^{3} \Big\langle (K^{\text{Lame}} \widetilde{\underline{u}}\,)_i, w_i \Big\rangle_{\Gamma_{D,i}} \\
& + \sum_{i=1}^{3} \Big\langle \widetilde{t}_i, (K^{\text{Lame}} \underline{v}\,)_i \Big\rangle_{\Gamma_{N,i}} + \sum_{i=1}^{3} \Big\langle (D^{\text{Lame}} \widetilde{\underline{u}}\,)_i, v_i \Big\rangle_{\Gamma_{N,i}}\,,
\end{aligned}
$$

while the linear form is

$$
\begin{aligned}
& F(\underline{w}, \underline{v}) = \\
& \sum_{i=1}^{3} \left(\frac{1}{2} \Big\langle g_i, w_i \Big\rangle_{\Gamma_{D,i}} + \Big\langle (K^{\text{Lame}} \widetilde{\underline{g}}\,)_i, w_i \Big\rangle_{\Gamma_{D,i}} - \Big\langle (V^{\text{Lame}} \widetilde{\underline{f}}\,)_i, w_i \Big\rangle_{\Gamma_{D,i}} \right) + \\
& \sum_{i=1}^{3} \left(\frac{1}{2} \Big\langle f_i, v_i \Big\rangle_{\Gamma_{N,i}} - \Big\langle \widetilde{f}_i, (K^{\text{Lame}} \underline{v}\,)_i \Big\rangle_{\Gamma_{N,i}} - \Big\langle (D^{\text{Lame}} \widetilde{\underline{g}}\,)_i, v_i \Big\rangle_{\Gamma_{N,i}} \right).
\end{aligned}
$$

As for the Laplace equation, we first define suitable trial spaces,

$$
\begin{aligned}
S_h^0(\Gamma_{D,i}) = S_h^0(\Gamma) \cap \widetilde{H}^{-1/2}(\Gamma_{D,i}) \;=\; \text{span}\Big\{ \psi_\ell^i \Big\}_{\ell=1}^{N_{D,i}}, \\
S_h^1(\Gamma_{N,i}) = S_h^1(\Gamma) \cap \widetilde{H}^{1/2}(\Gamma_{N,i}) \;=\; \text{span}\Big\{ \varphi_j^i \Big\}_{j=1}^{M_{N,i}}
\end{aligned}
$$

for $i = 1, 2, 3$. The Galerkin formulation of the variational problem (1.80) is to find $\widetilde{t}_{i,h} \in S_h^0(\Gamma_{D,i})$ and $\widetilde{u}_{i,h} \in S_h^1(\Gamma_{N,i})$ such that

$$a(\underline{\widetilde{t}}_h, \underline{\widetilde{u}}_h\,;\, \underline{w}_h, \underline{v}_h) \;=\; F(\underline{w}_h, \underline{v}_h)$$

is satisfied for all $w_i \in S_h^0(\Gamma_{D,i})$ and $v_i \in S_h^1(\Gamma_{M,i})$ for $i = 1, 2, 3$. This formulation is equivalent to a linear system of equations

$$\begin{pmatrix} \bar{V}_h^{\mathrm{Lame}} & -\bar{K}_h^{\mathrm{Lame}} \\ \left(\bar{K}_h^{\mathrm{Lame}}\right)^\top & \bar{D}_h^{\mathrm{Lame}} \end{pmatrix} \begin{pmatrix} \underline{\widetilde{t}} \\ \underline{\widetilde{u}} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix}, \tag{2.47}$$

having the blocks

$$\bar{V}_h^{\mathrm{Lame}} \in \mathbb{R}^{N_D \times N_D}, \quad \bar{K}_h^{\mathrm{Lame}} \in \mathbb{R}^{N_D \times M_N}, \quad \bar{D}_h^{\mathrm{Lame}} \in \mathbb{R}^{M_N \times M_N},$$

where

$$N_D \;=\; \sum_{i=1}^{3} N_{D,i}, \quad M_N \;=\; \sum_{i=1}^{3} M_{N,i}.$$

While the blocks in the linear system (2.47) recover only the unknown coefficients $\widetilde{t}_{i,\ell}$ and $\widetilde{u}_{i,j}$, an implementation based on the complete stiffness matrices may be advantageous. Let

$$S_h^0(\Gamma) \;=\; \mathrm{span}\Big\{\psi_\ell\Big\}_{\ell=1}^{N}, \quad S_h^1(\Gamma) \;=\; \mathrm{span}\Big\{\varphi_j\Big\}_{j=1}^{M}$$

be the boundary element spaces spanned by piecewise constant and piecewise linear continuous basis functions, respectively. Note that both $S_h^0(\Gamma)$ and $S_h^1(\Gamma)$ are defined with respect to a boundary element mesh of the complete surface Γ. By $P_i : \mathbb{R}^N \to \mathbb{R}^{N_{D,i}}$ and $Q_i : \mathbb{R}^M \to \mathbb{R}^{M_{N,i}}$, we denote some nodal projection operators describing the imbedding $\underline{w}^i = P_i \underline{w} \in \mathbb{R}^{N_{D,i}}$ for $\underline{w} \in \mathbb{R}^N$ with

$$w_h^i(x) \;=\; \sum_{\ell=1}^{N_{D,i}} w_\ell^i \psi_\ell^i(x) \in S_h^0(\Gamma_{D,i}), \quad w_h(x) \;=\; \sum_{\ell=1}^{N} w_\ell \psi_\ell(x) \in S_h^0(\Gamma)$$

as well as the imbedding $\underline{v}^i = Q_i \underline{v} \in \mathbb{R}^{M_{N,i}}$ for $\underline{v} \in \mathbb{R}^M$ with

$$v_h^i(x) \;=\; \sum_{j=1}^{M_{N,i}} v_j^i \varphi_j^i(x) \in S_h^1(\Gamma_{N,i}), \quad v_h(x) \;=\; \sum_{j=1}^{N} v_j \varphi_j(x) \in S_h^1(\Gamma).$$

From this we obtain the representations

$$\bar{V}_h^{\mathrm{Lame}} \;=\; P V_h^{\mathrm{Lame}} P^\top, \; \bar{K}_h^{\mathrm{Lame}} \;=\; P K_h^{\mathrm{Lame}} Q^\top, \quad \bar{D}_h^{\mathrm{Lame}} \;=\; Q D_h^{\mathrm{Lame}} Q^\top,$$

where the stiffness matrices V_h^{Lame}, K_h^{Lame}, and D_h^{Lame} correspond to the Galerkin discretisation of the associated boundary integral operators V^{Lame},

$K^{\rm Lame}$ and $D^{\rm Lame}$ with respect to the boundary element spaces $[S_h^0(\Gamma)]^3$ and $[S_h^1(\Gamma)]^3$. In particular, for the discrete single layer potential $\bar{V}_h$ we have the representation

$$V_h^{\rm Lame} = \tag{2.48}$$
$$\frac{1}{2}\frac{1}{E}\frac{1+\nu}{1-\nu}\left((3-4\nu)\begin{pmatrix} V_h & 0 & 0 \\ 0 & V_h & 0 \\ 0 & 0 & V_h \end{pmatrix} + \begin{pmatrix} V_{11,h} & V_{21,h} & V_{13,h} \\ V_{21,h} & V_{22,h} & V_{23,h} \\ V_{31,h} & V_{32,h} & V_{33,h} \end{pmatrix}\right)$$

with the matrix $V_h \in \mathbb{R}^{N\times N}$ having the entries

$$V_h[k,\ell] = \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\tau_\ell}\frac{1}{|x-y|}ds_y ds_x\,, \tag{2.49}$$

and six further matrices $V_{ij,h} \in \mathbb{R}^{N\times N}$ defined by

$$\begin{aligned} V_{ij,h}[k,\ell] &= \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\tau_\ell}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}ds_y ds_x \\ &= \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\tau_\ell}(x_i-y_i)\frac{\partial}{\partial y_j}\frac{1}{|x-y|}ds_y ds_x \end{aligned} \tag{2.50}$$

for $k,\ell = 1,\ldots,N$ and $i,j=1,2,3$. Note that V_h is just the Galerkin stiffness matrix of the single layer potential for the Laplace operator, while the matrix entries $V_{ij,h}[\ell,k]$ are similar to the Galerkin discretisation of the double layer potential for the Laplace operator.

From Lemma 1.16, we find the representation for the double layer potential $K^{\rm Lame}$

$$(K^{\rm Lame}\underline{v})(x) \;=\; (K\underline{v})(x) - \Big(VM(\partial,\underline{n})\underline{v}\Big)(x) + \frac{E}{1+\nu}\Big(V^{\rm Lame}M(\partial,\underline{n})\underline{v}\Big)(x)$$

for $x\in\Gamma$, and, therefore, the matrix representation

$$K_h^{\rm Lame} = \tag{2.51}$$
$$\begin{pmatrix} K_h & 0 & 0 \\ 0 & K_h & 0 \\ 0 & 0 & K_h \end{pmatrix} - \begin{pmatrix} V_h & 0 & 0 \\ 0 & V_h 0 & \\ 0 & 0 & V_h \end{pmatrix}\widetilde{T} + \frac{E}{1+\nu}V_h^{\rm Lame}\widetilde{T}\,,$$

where V_h and K_h are the Galerkin matrices related to the single and double layer potential of the Laplace operator. Furthermore, $\widetilde{T}$ is a transformation matrix related to the matrix surface curl operator $M(\partial,\underline{n})$.

Using the representation of the bilinear form of the hypersingular boundary integral operator $D^{\rm Lame}$ as given in Lemma 1.18, one can derive a similar representation for the Galerkin matrix $D_h^{\rm Lame}$, which is based on the transformation matrix $\widetilde{T}$ and on the Galerkin matrices related to the single layer potential of both, the Laplace operator and the system of linear elastostatics.

2.5 Helmholtz Equation

2.5.1 Interior Dirichlet Problem

The solution of the interior Dirichlet boundary value problem (cf. (1.104)),

$$-\Delta u(x) - \kappa^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega, \quad \gamma_0^{\text{int}} u(x) \;=\; g(x) \quad \text{for } x \in \Gamma\,,$$

is given by the representation formula (cf. (1.95))

$$u(x) \;=\; \int\limits_\Gamma u_\kappa^*(x,y)t(y)ds_y - \int\limits_\Gamma \gamma_{1,y}^{\text{int}} u_\kappa^*(x,y)g(y)ds_y \quad \text{for } x \in \Omega,$$

where the unknown Neumann datum $t = \gamma_1^{\text{int}} u \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation (cf. (1.105))

$$(V_\kappa t)(x) \;=\; \frac{1}{2} g(x) + (K_\kappa g)(x) \quad \text{for } x \in \Gamma.$$

Note that for the unique solvability, we have to assume that κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.108). Then, $t \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem (cf. (1.106))

$$\Big\langle V_\kappa t, w \Big\rangle_\Gamma \;=\; \Big\langle \Big(\frac{1}{2} I + K_\kappa\Big) g, w \Big\rangle_\Gamma \quad \text{for all } w \in H^{-1/2}(\Gamma).$$

Using a sequence of finite dimensional subspaces $S_h^0(\Gamma)$ spanned by piecewise constant basis functions, associated approximate solutions

$$t_h \;=\; \sum_{\ell=1}^N t_\ell \psi_\ell \in S_h^0(\Gamma)$$

are obtained from the Galerkin equations

$$\Big\langle V_\kappa t_h, \psi_k \Big\rangle_\Gamma \;=\; \Big\langle \Big(\frac{1}{2} I + K_\kappa\Big) g, \psi_k \Big\rangle_\Gamma \quad \text{for } k = 1, \ldots, N. \tag{2.52}$$

Hence, we find the coefficient vector $\underline{t} \in \mathbb{C}^N$ as the unique solution of the linear system

$$V_{\kappa,h} \underline{t} \;=\; \underline{f}$$

with

$$V_{\kappa,h}[k,\ell] = \frac{1}{4\pi} \int\limits_{\tau_k} \int\limits_{\tau_\ell} \frac{e^{\imath\kappa|x-y|}}{|x-y|} ds_y ds_x\,, \tag{2.53}$$

for $k, \ell = 1, \ldots, N$, and

$$f_k = \frac{1}{2}\int\limits_{\tau_k} g(x)ds_x + \frac{1}{4\pi}\int\limits_{\tau_k}\int\limits_{\Gamma} \big(1 - \imath\,\kappa\,|x-y|\big)e^{\imath\,\kappa|x-y|}\frac{(x-y,\underline{n}(y))}{|x-y|^3} g(y)ds_y ds_x$$

for $k = 1, \ldots, N$.

Since the single layer potential $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is coercive, i.e. V_κ satisfies (1.97), and since V_κ is injective when κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.108), we conclude the unique solvability of the Galerkin variational problem (2.52), as well as the quasi optimal error estimate, i.e. Cea's lemma,

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \le c \inf_{w_h \in S_h^0(\Gamma)} \|t - w_h\|_{H^{-1/2}(\Gamma)}.$$

Combining this with the approximation property (2.5) for $\sigma = -1/2$, we get

$$\|t - t_h\|_{H^{-1/2}(\Gamma)} \le c\, h^{s+\frac{1}{2}} |t|_{H^s_{\mathrm{pw}}(\Gamma)},$$

when assuming $t \in H^s_{\mathrm{pw}}(\Gamma)$ and $s \in [0,1]$. Applying the Aubin–Nitsche trick (for $\sigma < -1/2$) and the inverse inequality argument (for $\sigma \in (-1/2, 0]$), we also obtain the error estimate

$$\|t - t_h\|_{H^\sigma(\Gamma)} \le c\, h^{s-\sigma} |t|_{H^s_{\mathrm{pw}}(\Gamma)}, \tag{2.54}$$

when assuming $t \in H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [0,1]$ and $\sigma \in [-2, 0]$.

Inserting the computed Galerkin solution $t_h \in S_h^0(\Gamma)$ into the representation formula (1.95), this gives an approximate representation formula

$$\widetilde{u}(x) = \int\limits_{\Gamma} \gamma_0^{\mathrm{int}} u^*_\kappa(x,y) t_h(y) ds_y - \int\limits_{\Gamma} \gamma_1^{\mathrm{int}} u^*_\kappa(x,y) g(y) ds_y, \tag{2.55}$$

for $x \in \Omega$, describing an approximate solution of the Dirichlet boundary value problem (1.104). Note that $\widetilde{u}$ satisfies the Helmholtz equation, but the Dirichlet boundary conditions are satisfied only approximately. For an arbitrary $x \in \Omega$, the error is given by

$$u(x) - \widetilde{u}(x) = \int\limits_{\Gamma} u^*_\kappa(x,y)\Big(t(y) - t_h(y)\Big) ds_y.$$

Using a duality argument, the error estimate

$$|u(x) - \widetilde{u}(x)| \le \|u^*_\kappa(x,\cdot)\|_{H^{-\sigma}(\Gamma)} \|t - t_h\|_{H^\sigma(\Gamma)}$$

for some $\sigma \in \mathbb{R}$ follows. Combining this with the error estimate (2.54) for the minimal value $\sigma = -2$, we obtain the pointwise error estimate

$$|u(x) - \widetilde{u}(x)| \le c\, h^{s+2}\, \|u^*_\kappa(x,\cdot)\|_{H^2(\Gamma)}\, |t|_{H^s_{\mathrm{pw}}(\Gamma)}.$$

Hence, if $t \in H^1_{\rm pw}(\Gamma)$ is sufficiently smooth, we obtain the optimal order of convergence for $s = 1$,

$$|u(x) - \widetilde{u}(x)| \,\leq\, c\, h^3 \, \|u^*_\kappa(x,\cdot)\|_{H^2(\Gamma)} \, |t|_{H^1_{\rm pw}(\Gamma)}. \tag{2.56}$$

Again, the error estimate (2.56) involves the position of the observation point $x \in \Omega$, and, therefore, it is not valid in the limiting case $x \in \Gamma$.

As for the Dirichlet problem for the Laplace equation, the computation of f_k requires the evaluation of the integrals

$$f_k \;=\; \frac{1}{2} \int\limits_{\tau_k} g(x) ds_x + \frac{1}{4\pi} \int\limits_{\tau_k} \int\limits_{\Gamma} \big(1 - \imath\, \kappa\, |x-y|\big) e^{\imath\,\kappa |x-y|} \frac{(x-y, \underline{n}(y))}{|x-y|^3} g(y) ds_y ds_x.$$

When using a piecewise linear approximation $g_h \in S^1_h(\Gamma)$ of the given Dirichlet datum $g \in H^{1/2}(\Gamma)$, we find a perturbed solution vector $\underline{\widetilde{t}} \in \mathbb{C}^N$ from the linear system

$$V_{\kappa,h} \underline{\widetilde{t}} \;=\; \Big(\frac{1}{2} M_h + K_{\kappa,h}\Big) \underline{g} \tag{2.57}$$

with additional matrices defined by the entries

$$M_h[k,j] = \int\limits_{\tau_k} \varphi_j(x) ds_x \,,$$

and

$$K_{\kappa,h}[k,j] = \frac{1}{4\pi} \int\limits_{\tau_k} \int\limits_{\Gamma} \big(1 - \imath\, \kappa\, |x-y|\big) e^{\imath\,\kappa |x-y|} \frac{(x-y, \underline{n}(y))}{|x-y|^3} \varphi_j(y) ds_y ds_x \tag{2.58}$$

for $k = 1, \ldots, N$ and $j = 1, \ldots, M$. Then, the exact Galerkin solution t_h has to be replaced by the perturbed solution $\widetilde{t}_h$ to obtain an approximate solution of the Dirichlet problem (1.104) for $x \in \Omega$,

$$\widetilde{u}(x) \;=\; \int\limits_{\Gamma} u^*_\kappa(x,y) \widetilde{t}_h(y) ds_y - \int\limits_{\Gamma} \gamma_1^{\rm int} u^*_\kappa(x,y) g_h(y) ds_y \,.$$

Thus, we obtain the optimal error estimate

$$|u(x) - \widetilde{u}(x)| \,\leq\, c(x,t,g)\, h^3 \,, \tag{2.59}$$

when using a L_2 projection to approximate the boundary conditions, and when assuming $t \in H^1_{\rm pw}(\Gamma)$ and $g \in H^2_{\rm pw}(\Gamma)$.

2.5.2 Interior Neumann Problem

Next we consider the interior Neumann boundary value problem (1.109),

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega, \quad \gamma_1^{\text{int}} u(x) = g(x) \quad \text{for } x \in \Gamma.$$

The solution is given by the representation formula for $x \in \Omega$ (cf. (1.95))

$$u(x) = \int_\Gamma u_\kappa^*(x,y) g(y) ds_y - \int_\Gamma \gamma_{1,y}^{\text{int}} u_\kappa^*(x,y) \gamma_0^{\text{int}} u(y) ds_y.$$

We assume that κ^2 is not an eigenvalue of the Neumann eigenvalue problem (1.113). In this case, the unknown Dirichlet datum $\hat{u} = \gamma_0^{\text{int}} u \in H^{1/2}(\Gamma)$ is the unique solution of the boundary integral equation (1.111),

$$(D_\kappa \hat{u})(x) = \frac{1}{2} g(x) - (K_\kappa' g)(x) \quad \text{for } x \in \Gamma\,,$$

or of the equivalent variational problem (1.112),

$$\Big\langle D_\kappa \hat{u}, v \Big\rangle_\Gamma = \Big\langle \Big(\frac{1}{2} I - K_\kappa'\Big) g, v \Big\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma)\,.$$

Using a sequence of finite dimensional subspaces $S_h^1(\Gamma)$ spanned by piecewise linear continuous basis functions, associated approximate solutions

$$\hat{u}_h = \sum_{j=1}^M \hat{u}_j \varphi_j \in S_h^1(\Gamma)$$

are obtained from the Galerkin equations

$$\Big\langle D_\kappa \hat{u}_h, \varphi_i \Big\rangle_\Gamma = \Big\langle \Big(\frac{1}{2} I - K_\kappa'\Big) g, \varphi_i \Big\rangle_\Gamma \quad \text{for } i = 1, \ldots, M. \tag{2.60}$$

Hence, we find the coefficient vector $\underline{\hat{u}} \in \mathbb{C}^M$ as the unique solution of the linear system

$$D_{\kappa,h} \underline{\hat{u}} = \underline{f} \tag{2.61}$$

with

$$\begin{aligned} D_{\kappa,h}[i,j] &= \langle D_\kappa \varphi_j, \varphi_i \rangle_\Gamma \qquad (2.62) \\ &= \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{e^{\imath\kappa|x-y|}}{|x-y|} (\underline{\text{curl}}_\Gamma \varphi_j(y), \underline{\text{curl}}_\Gamma \varphi_i(x)) ds_y ds_x \\ &\quad - \frac{\kappa^2}{4\pi} \int_\Gamma \int_\Gamma \frac{e^{\imath\kappa|x-y|}}{|x-y|} \varphi_j(y) \varphi_i(x) (\underline{n}(x), \underline{n}(y)) ds_y ds_x\,, \end{aligned}$$

for $i, j = 1, \ldots, M$, and

$$\begin{aligned} f_i &= \frac{1}{2} \int_\Gamma g(x) \varphi_i(x) ds_x \\ &\quad - \frac{1}{4\pi} \int_\Gamma \varphi_i(x) \int_\Gamma \big(1 - \imath\,\kappa\,|x-y|\big) e^{\imath\,\kappa|x-y|} \frac{(x-y, \underline{n}(y))}{|x-y|^3} g(y) ds_y ds_x \end{aligned}$$

for $i = 1, \ldots, M$. Note that for the computation of the matrix entries $D_{\kappa,h}[i,j]$, we can reuse the discrete single layer potential $V_{\kappa,h}$ for picewise constant basis functions, but we also need to have the Galerkin discretisation with piecewise linear continuous basis functions of the operator

$$(C_\kappa u)(x) = \int\limits_\Gamma \frac{e^{\imath\kappa|x-y|}}{|x-y|} \left(\underline{n}(x), \underline{n}(y)\right) u(y) ds_y \,, \tag{2.63}$$

which is similar to the single layer potential operator.

Since the hypersingular integral operator

$$D_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is coercive, i.e. D_κ satisfies (1.98), and since D_κ is injective when κ^2 is not an eigenvalue of the Neumann eigenvalue problem (1.113), we conclude the unique solvability of the Galerkin variational problem (2.60), as well as the quasi optimal error estimate, i.e. Cea's lemma,

$$\|\bar{u} - \bar{u}_h\|_{H^{1/2}(\Gamma)} \leq c \inf_{v_h \in S_h^1(\Gamma)} \|\bar{u} - v_h\|_{H^{1/2}(\Gamma)}.$$

Combining this with the approximation property (2.10) for $\sigma = 1/2$, we get

$$\|\bar{u} - \bar{u}_h\|_{H^{1/2}(\Gamma)} \leq c\, h^{s-\frac{1}{2}} \|\bar{u}\|_{H^s_{\rm pw}(\Gamma)} \,,$$

when assuming $\bar{u} \in H^s_{\rm pw}(\Gamma)$ and $s \in [1,2]$. Applying the Aubin–Nitsche trick we also obtain the error estimate

$$\|\bar{u} - \bar{u}_h\|_{H^\sigma(\Gamma)} \leq c\, h^{s-\sigma} \|\bar{u}\|_{H^s_{\rm pw}(\Gamma)} \,, \tag{2.64}$$

when assuming $\bar{u} \in H^s_{\rm pw}(\Gamma)$ for some $s \in [1,2]$ and $\sigma \in [-1, 1/2]$.

Inserting the computed Galerkin solution $\hat{u}_h \in S_h^1(\Gamma)$ into the representation formula (1.95), this gives an approximate representation formula for $x \in \Omega$,

$$\widetilde{u}(x) = \int\limits_\Gamma u_\kappa^*(x,y) g(y) ds_y - \int\limits_\Gamma \gamma_{1,y}^{\rm int} u_\kappa^*(x,y) \hat{u}_h(y) ds_y, \tag{2.65}$$

describing an approximate solution of the Neumann boundary value problem (1.109). Note that $\widetilde{u}$ satisfies the Helmholtz equation, but the Neumann boundary conditions are satisfied only approximately. For an arbitrary $x \in \Omega$, the error is given by

$$u(x) - \widetilde{u}(x) = \int\limits_\Gamma \gamma_{1,y}^{\rm int} u_\kappa^*(x,y) \Big(\hat{u}_h(y) - \hat{u}(y)\Big) ds_y.$$

Using a duality argument, the error estimate

$$|u(x) - \widetilde{u}(x)| \leq \|u_\kappa^*(x,\cdot)\|_{H^{-\sigma}(\Gamma)} \|\bar{u} - \bar{u}_h\|_{H^\sigma(\Gamma)}$$

for some $\sigma \in \mathbb{R}$ follows. Combining this with the error estimate (2.64) for the minimal value $\sigma = -1$, we obtain the pointwise error estimate

$$|u(x) - \widetilde{u}(x)| \leq c\, h^{s+1} \|u_\kappa^*(x,\cdot)\|_{H^1(\Gamma)} |\hat{u}|_{H^s_{\mathrm{pw}}(\Gamma)}.$$

Hence, if $\hat{u} \in H^2_{\mathrm{pw}}(\Gamma)$ is sufficiently smooth, we get the optimal order of convergence for $s = 2$,

$$|u(x) - \widetilde{u}(x)| \leq c\, h^3 \|u_\kappa^*(x,\cdot)\|_{H^1(\Gamma)} |\bar{u}|_{H^2_{\mathrm{pw}}(\Gamma)}. \tag{2.66}$$

Again, the error estimate (2.66) involves the position of the observation point $x \in \Omega$, and, therefore, is not valid in the limiting case $x \in \Gamma$.

When using a piecewise constant approximation $g_h \in S_h^0(\Gamma)$ of the given Neumann datum $g \in H^{-1/2}(\Gamma)$, we can compute a perturbed piecewise linear approximation $\widetilde{u}_h \in S_h^1(\Gamma)$ from the Galerkin equations

$$\Big\langle D_\kappa \hat{u}_h, \varphi_i \Big\rangle_\Gamma = \Big\langle \Big(\frac{1}{2} I - K_\kappa'\Big) g_h, \varphi_i \Big\rangle_\Gamma \quad \text{for } i = 1, \ldots, M$$

or from the equivalent linear system

$$D_{\kappa,h} \underline{\widetilde{u}} = \Big(\frac{1}{2} M_h^\top - K_{\kappa,h}'\Big) \underline{g}$$

with

$$M_h^\top[i,\ell] = \int_{\tau_\ell} \varphi_i(x) ds_x = M_h[\ell, i],$$

$$K_{\kappa,h}'[i,\ell] = \frac{1}{4\pi} \int_\Gamma \varphi_i(x) \int_{\tau_\ell} \big(1 - \imath\, \kappa\, |x-y|\big) e^{\imath\, \kappa |x-y|} \frac{(x-y, \underline{n}(y))}{|x-y|^3} ds_y ds_x\,.$$

An approximate solution of the interior Neumann boundary value problem is then given for $x \in \Omega$,

$$\widetilde{u}(x) = \int_\Gamma u_\kappa^*(x,y) g_h(y) ds_y - \int_\Gamma \gamma_{1,y}^{\mathrm{int}} u_\kappa^*(x,y) \widetilde{u}_h(y) ds_y.$$

As for the perturbed linear system (2.31) for the Neumann boundary value problem of the Laplace equation, we obtain the error estimate

$$|u(x) - \widetilde{u}(x)| \leq c(x,t,g)\, h^2\,, \tag{2.67}$$

when using a L_2 projection to approximate the boundary conditions, when assuming $g \in H^1_{\mathrm{pw}}(\Gamma)$ and $\bar{u} \in H^2_{\mathrm{pw}}(\Gamma)$.

2.5.3 Exterior Dirichlet Problem

The solution of the exterior Dirichlet boundary value problem (cf. (1.114))

$$-\Delta u(x) - \kappa^2 u(x) \,=\, 0 \quad \text{for } x \in \Omega^e, \quad \gamma_0^{\text{ext}} u(x) \,=\, g(x) \quad \text{for } x \in \Gamma,$$

where, in addition, we have to require the Sommerfeld radiation condition (1.101), is given by the representation formula for $x \in \Omega^e$ (cf. 1.103)

$$u(x) \,=\, -\int\limits_\Gamma u_\kappa^*(x,y)t(y)ds_y + \int\limits_\Gamma \gamma_{1,y}^{\text{ext}} u_\kappa^*(x,y)g(y)ds_y.$$

Again we assume that κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.108). The unknown Neumann datum $t = \gamma_1^{\text{ext}} \in H^{-1/2}(\Gamma)$ is then the unique solution of the boundary integral equation (cf. (1.115))

$$(V_\kappa t)(x) \,=\, -\frac{1}{2} g(x) + (K_\kappa g)(x) \quad \text{for } x \in \Gamma.$$

To compute an approximate solution of this boundary integral equation, and, therefore, of the exterior Dirichlet problem, we can proceed as in the case of the interior Dirichlet problem. In particular, when using a piecewise linear approximation $g_h \in S_h^1(\Gamma)$, we find a perturbed piecewise constant approximation $\widetilde{t}_h \in S_h^0(\Gamma)$ from the Galerkin equations

$$\Big\langle V_\kappa \widetilde{t}_h, \psi_k \Big\rangle_\Gamma \,=\, \Big\langle \Big(-\frac{1}{2} I + K_\kappa \Big) g_h, \psi_k \Big\rangle_\Gamma \quad \text{for } k = 1, \ldots, N\,.$$

Hence, we obtain the coefficient vector $\underline{\widetilde{t}} \in \mathbb{C}^N$ as the unique solution of the linear system

$$V_{\kappa,h} \underline{\widetilde{t}} \,=\, \Big(-\frac{1}{2} M_h + K_{\kappa,h} \Big) \underline{g},$$

and an approximate solution of the exterior Dirichlet problem for $x \in \Omega$,

$$\widetilde{u}(x) \,=\, -\int\limits_\Gamma u_\kappa^*(x,y)\widetilde{t}_h(y)ds_y + \int\limits_\Gamma \gamma_{1,y}^{\text{ext}} u_\kappa^*(x,y)g_h(y)ds_y\,. \tag{2.68}$$

Moreover, as for the interior Dirichlet problem, there holds the optimal error estimate

$$|u(x) - \widetilde{u}(x)| \,\leq\, c(x,t,g)\, h^3\,, \tag{2.69}$$

when using a L_2 projection to approximate the boundary conditions, and when assuming $t \in H^1_{\text{pw}}(\Gamma)$ and $g \in H^2_{\text{pw}}(\Gamma)$.

2.5.4 Exterior Neumann Problem

The solution of the exterior Neumann boundary value problem (cf. (1.120))

$$-\Delta u(x) - \kappa^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega^e, \quad \gamma_1^{\text{ext}} u(x) \;=\; g(x) \quad \text{for } x \in \Gamma,$$

where, in addition, we have to require the Sommerfeld radiation condition (1.101), is given by the representation formula for $x \in \Omega^c$ (cf. (1.95))

$$u(x) \;=\; -\int_\Gamma u_\kappa^*(x,y) g(y) ds_y + \int_\Gamma \gamma_{1,y}^{\text{ext}} u_\kappa^*(x,y) \gamma_0^{\text{ext}} u(y) ds_y.$$

Again, we assume that κ^2 is not eigenvalue of the Neumann eigenvalue problem (1.113). The unknown Dirichlet datum $\bar{u} = \gamma_0^{\text{ext}} u \in H^{1/2}(\Gamma)$ is then the unique solution of the boundary integral equation (cf. (1.121))

$$(D_\kappa \bar{u})(x) \;=\; -\frac{1}{2} g(x) - (K'_\kappa g)(x) \quad \text{for } x \in \Gamma.$$

To compute an approximate solution of this boundary integral equation, and, therefore, of the exterior Neumann problem, we can proceed as in the case of the interior Neumann problem. In particular, when using a piecewise constant approximation $g_h \in S_h^0(\Gamma)$ of the given Neumann datum g, we find a perturbed piecewise linear approximation $\widetilde{u}_h \in S_h^1(\Gamma)$ from the Galerkin equations

$$\Big\langle D_\kappa \widetilde{u}_h, \varphi_i \Big\rangle_\Gamma \;=\; \Big\langle \Big(-\frac{1}{2} I - K'_\kappa \Big) g_h, \varphi_i \Big\rangle_\Gamma \quad \text{for } i = 1, \ldots, M.$$

Hence, we obtain the coefficient vector $\underline{\widetilde{u}} \in \mathbb{C}^M$ as the unique solution of the linear system

$$D_{\kappa,h} \underline{\widetilde{u}} \;=\; \Big(-\frac{1}{2} M_h^\top - K'_{\kappa,h} \Big) \underline{g},$$

and an approximate solution of the exterior Neumann problem for $x \in \Omega$,

$$\widetilde{u}(x) \;=\; -\int_\Gamma u_\kappa^*(x,y) g_h(y) ds_y + \int_\Gamma \gamma_{1,y}^{\text{ext}} u_\kappa^*(x,y) \widetilde{u}_h(y) ds_y.$$

Moreover, we obtain the error estimate

$$|u(x) - \widetilde{u}(x)| \;\le\; c(x,t,g)\, h^2\,, \tag{2.70}$$

when using the L_2 projection to approximate the boundary conditions, and when assuming $g \in H^1_{\text{pw}}(\Gamma)$ and $\hat{u} \in H^2_{\text{pw}}(\Gamma)$.

2.6 Bibliographic Remarks

The numerical analysis of boundary element methods was introduced independently by J.–C. Nédélec and J. Planchard [79] and by G. C. Hsiao and W. L. Wendland [57]. While the stability and error analysis of the Galerkin boundary element methods follow as in the case of the finite element methods, the stability of the collocation boundary element methods for general Lipschitz boundaries is still open, see [4, 5, 100, 101] for some special cases. The Aubin–Nitsche trick to obtain higher order error estimates for boundary element methods was first given in [58].

Since the implementation of boundary element methods often requires numerical integration techniques, an appropriate numerical analysis is mandatory. Galerkin collocation schemes were first discussed in [54, 68]. Further investigations on the use of numerical integration schemes were made in [45, 97, 98, 102]. In [76], the influence on an additional boundary approximation was considered.

Further references on boundary element methods are, for example, [12, 15, 21, 40, 50, 104, 117] and [99, 105].

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