2

Fuzzy Variables and Measurement Uncertainty

Chapter 1 has presented a short survey of the basic concepts of the measurement theory. In particular, it has shown that the result of a measurement represents incomplete knowledge of the measurand and this knowledge can be usefully employed only if its ‘incompleteness’ can be somehow estimated and quantified. It has also been shown that this approach requires a suitable mathematics for handling, from the quantitative point of view, incomplete knowledge. Current practice refers mainly to the probability theory, because it is the best known and most assessed mathematical theory that treats incomplete knowledge.

However, the probability theory deals only with that particular kind of incomplete knowledge originated by random effects. As seen in the previous chapter, this implies that all other recognized significant effects, including the systematic ones, are fully compensated.

A systematic effect affects the measurement process always with the same value and sign. If this value was exactly known, it could be treated as an *error* and the measurement result could be corrected by compensating it in the measurement procedure. Of course, this situation does not generally occur. The normal situation is that the presence of a systematic contribution is recognized, but its exact value is not known, even if it is possible to locate it within a closed interval of $\Re$, that is, from a mathematical point of view, an *interval of confidence*. By definition, a systematic contribution always takes the same value, although unknown, within this estimated interval. This means that each value of the interval does not have the same probability to occur, but it has, in absence of further evidence, the same possibility to occur, because no value is preferable to the others.

This situation can be also used to represent a more general case. In many practical situations, there is evidence that some unknown effect is affecting the measurement result. The only available information shows that the contribution related to this unknown effect falls, with a given level of confidence, within a given interval, but it is not known whereabouts. Moreover, nothing else can be assumed, not even if the effect is a systematic or a random one. This situation is called *total ignorance.*

Even if the two kinds of contributions are represented in the same way—an interval of confidence—they should not be confused with each other. In fact, when total ignorance is considered, no other information is available; on the contrary, when a systematic contribution is considered, an important piece of information is available, that is, the systematic behavior of the contribution itself. This additional information should be carefully taken into account when modeling the measurement process.

Let us consider the following simple example. Let m_1 and m_2 be two measurement results for the same quantity, for instance, the weight of an object obtained with the method of the double weighing with a weighbridge. The final measurement result is supposed to be obtained by the arithmetic mean of m_1 and m_2. Let us suppose that no random contributions are present, but only one contribution of a different nature, whose possible values fall within interval $\pm 100\ g$. Two cases can be considered:

1. No additional information is available for this uncertainty contribution, so that it must be classified as total ignorance. In this case the following applies:
$$r = \frac{(m_1 \pm 100\ g) + (m_2 \pm 100\ g)}{2} = \frac{(m_1 + m_2)}{2} \pm 100\ g$$
and the measurement uncertainty associated with the result is the same as the initial one.
2. Additional information is available showing that the uncertainty contribution is due to the different length of the two beams of the weighbridge. In this case, thanks to the available information about this uncertainty contribution, it is known that it affects m_1 and m_2 with the same absolute value, even if not known, and opposite signs (in fact, let us remember that m_1 and m_2 are obtained by placing the object to be weighed on the two different plates of the weighbridge). Therefore, the contribution can be classified as systematic and the following applies:
$$r = \frac{(m_1 \pm 100\ g) + (m_2 \mp 100\ g)}{2} = \frac{(m_1 + m_2)}{2}$$
and the final measurement result has zero associated uncertainty because, in the arithmetic mean, the considered contribution is compensated.

This simple example shows the importance of using all available information while modeling a measurement process. In fact, if the additional information about the reason for the systematic behavior was not taken into account, the measurement uncertainty of the final result would have been overestimated.

Hence, it can be stated, also from a mathematical point of view, that a systematic contribution is a particular case of total ignorance, where additional information is added. If this additional information is given, together with the reasons of the presence of the systematic contribution itself, then it can sometimes be used to suitably and correctly propagate the uncertainty through the measurement process, as in the case of the double weighing with the weighbridge.

The aim of this chapter is to find a mathematical object able to represent total ignorance and its particular cases.

When the probability theory is considered to handle incomplete knowledge, total ignorance is represented by a random variable with a uniform probability distribution over the estimated confidence interval. However, this assumption is inconsistent with the concept of *total* ignorance.

In fact, let us first consider the general case of total ignorance. In this case, by definition, no assumptions can be made about the actual probability distribution, and all probability distributions are, in theory, possible. Hence, assuming a uniform probability distribution means to arbitrarily add information that is not available.

Let us now consider the particular case of a systematic effect. In this case, by definition, only one unknown value has 100% probability to occur, whereas all others have null probability. Hence, assuming a uniform probability, which implies that all values within the given interval have the same probability to occur, leads to a bad interpretation of the available information.

Of course, the assumption of whichever other probability distribution brings one to the same considerations. Hence, it can be concluded that the probability theory and the random variables are not able to represent incomplete knowledge, whenever this is due to uncertainty effects that are not explicitly random. Therefore, different mathematical variables should be considered for this aim.

In the second half of the twentieth century, fuzzy variables have been introduced in order to represent incomplete knowledge. This approach is less known than the statistical one, and it is completely lacking in the current standards; therefore, the whole chapter is dedicated to this approach.

2.1 Definition of fuzzy variables

Fuzzy variables and fuzzy sets have been widely used, in the last decades, especially in the field of automatic controls, after Zadeh introduced the basic principles of fuzzy logic and approximate reasoning [Z65]-[Z73]-[Z75]-[Z78].

In the traditional mathematical approach, a variable may only belong or not belong to the set into which it is defined. The function describing the membership of such a crisp variable to its appertaining set can therefore take only the value 1, if the variable belongs to the set, or 0, if the variable does not belong to the set.

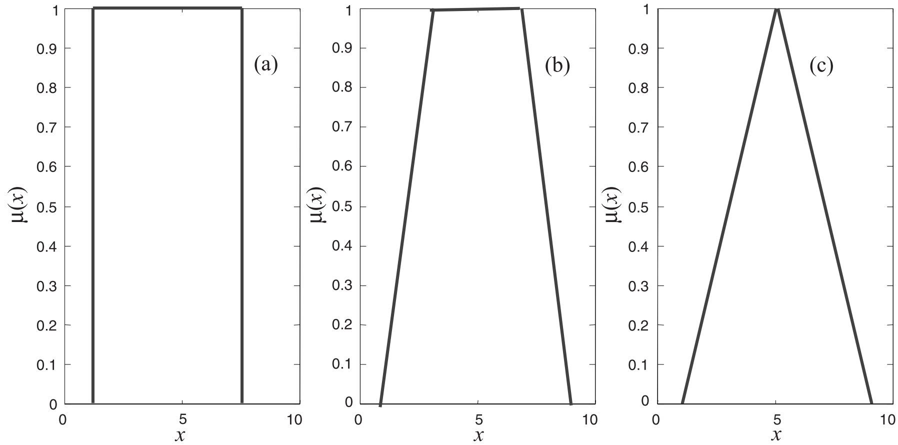

Fig. 2.1. Example of membership functions: (**a**) rectangular, (**b**) trapezoidal, and (**c**) triangular.

When fuzzy variables and fuzzy sets are considered, the function describing the membership of a variable to its appertaining set is allowed to take all values within the 0–1 interval. This means that, given the referential set $\Re$ of the real numbers, a fuzzy variable X is defined by its membership function $\mu_X(x)$, where $x \in \Re$. The membership function of a fuzzy variable satisfies the following properties [KG91]:

- $0 \leq \mu_X(x) \leq 1$;
- $\mu_X(x)$ is convex;
- $\mu_X(x)$ is normal (which means that at least one element x always exists for which $\mu_X(x) = 1$).

Figure 2.1 shows some examples of membership functions: The rectangular one, the trapezoidal one, and the triangular one. Of course, different shapes are allowed, which do not need to be symmetric. Moreover, membership functions that only increase or only decrease qualify as fuzzy variables [KY95] and are used to represent the concept of 'large number' or 'small number' in the context of each particular application. An example of such a membership function is given in Fig. 2.2.

The membership function $\mu_X(x)$ of a fuzzy variable can be also described in terms of α-cuts at different vertical levels α. As the membership function of a fuzzy variable ranges, by definition, between 0 and 1, its α-cuts are defined for values of α between 0 and 1.

Each α-cut, at the generic level α, is defined as

$$X_\alpha = \{x \mid \mu_X(x) \geq \alpha\} \tag{2.1}$$

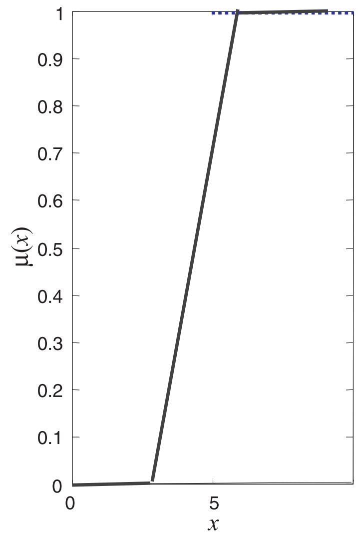

Fig. 2.2. Membership function describing the fuzzy statement 'greater than 5.'

According to Eq. (2.1), each α-cut defines an interval $[x_1^\alpha, x_2^\alpha]$, where it is always $x_1^\alpha \leq x_2^\alpha$. The equality of x_1^α and x_2^α can be reached only for $\alpha = 1$ and only if the membership function has a single peak value, for instance, in the case of the triangular membership function reported in Fig. 2.1c. Generally, x_1^α and x_2^α take finite values, but in some cases, as the one in Fig. 2.2, it could be $x_1^\alpha = -\infty$ and/or $x_2^\alpha = +\infty$. If, for any value α, it is $x_1^\alpha = x_2^\alpha$, then the fuzzy variable degenerates into a crisp variable. An example of α-cut, for level $\alpha = 0.3$, is given in Fig. 2.3.

The importance of representing a fuzzy variable in terms of its α-cuts is that the α-cuts of a fuzzy variable and the corresponding levels α can be considered as a set of intervals of confidence and associated levels of certitude. The level of certitude contains information about how certain a person is about its knowledge. If a person, for instance, remembers exactly the birthday of a friend, his knowledge is certain; but if he only remembers the month, but not the day, the certainty of his knowledge is lower. In the first case, the available knowledge can be represented by a crisp value, that is, the exact day of the year (April, 15^{th}); in the second case, the available knowledge can be represented by a set, which contains the 30 days of April. Hence, as the level of certitude increases, the width of the corresponding confidence interval decreases.

As simply shown by the previous example, the link between the level α of certitude and the confidence interval at the same level corresponds to the natural, often implicit, mechanism of human thinking in the subjective estimation of a value for a measurement.

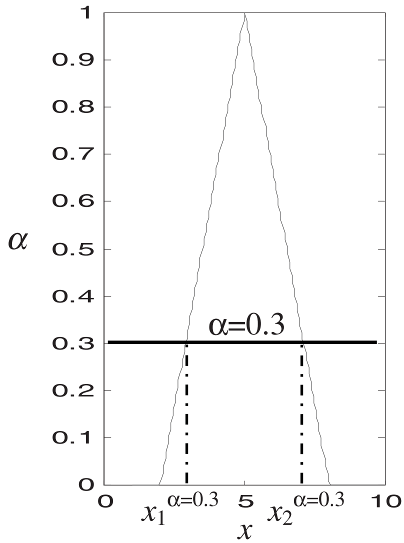

Fig. 2.3. α-cut of a triangular membership function for level $\alpha = 0.3$.

The following example shows how a fuzzy variable can be built, starting from the information, or personal idea, of one, or more human beings. Let us consider again the birthday of a friend, whose name is Sam. Tom is sure that Sam's birthday is April 15$^{\text{th}}$; John is not so sure about the date, even if he remembers well that the birthday falls in April. From a mathematical point of view, the interval of confidence estimated by Tom is [April 15, April 15], whereas the interval of confidence estimated by John is [April 1, April 30]. The levels of certitude associated with the two intervals are 1 and 0, respectively. If now a fuzzy variable must be associated with the available information, the following applies.

Let us suppose the only available information is from Tom. In this case, the available information can be represented by a crisp value. In fact, in this case, full certainty is given.

Let us now suppose the only available information is from John. In this case, the available information can be represented by a rectangular fuzzy variable, like the one in Fig. 2.1a.

However, it is also possible to combine the information given by Tom and John. As the level of certitude associated with John's interval is zero and those associated with Tom's interval is one, these two intervals can be considered as the α-cuts of the fuzzy variable, which represents Sam's birthday, at levels of α zero and one, respectively. Since, as the level of certitude increases, the interval of confidence becomes narrower, the fuzzy variable could be, for instance, triangular, like the one in Fig. 2.1c.

This simple example also shows, in an intuitive way, a relationship between the level of certitude and the level of confidence, which, according to the Theory of Uncertainty, should always be associated with an interval of confidence. The level of certitude indicates how much a person is sure about a certain event. If, for instance, the example of Sam's birthday is considered again, this means that the surer a person is about the birthday's date, the smaller is the extimated range of days. On the other hand, the level of confidence indicates the probability of a certain event. Therefore, considering the same example of Sam's birthday, the smaller the given range of days, the smaller the probability that Sam's birthday falls within those dates. If only one date is given, the probability that this is exactly Sam's birthday is zero, as also shown by Eq. (1.4). In other words, although the level of certitude increases as the width of the confidence interval decreases, the opposite applies to the levels of confidence. Hence, the levels of confidence equal to one and zero are assigned to intervals [April 1, April 30] and [April 15, April 15], respectively. Moreover, intuitively, it can also be assessed that, given the confidence interval at level α, the associated level of confidence is $1 - \alpha$[1] [FS03, FGS04, FS04, FS05a, FS05b, FS05c].

The use of fuzzy variables in the context of measurement uncertainty is particularly interesting, if we remember that the most useful way to express the result of a measurement is in terms of confidence intervals and '*the ideal method for evaluating and expressing measurement uncertainty should be capable of readily providing such a confidence interval*' [ISO93]. Therefore, it can be stated that a fuzzy variable can be effectively employed to represent the result of a measurement, because it provides all available information about the result itself: the confidence intervals and the associated levels of confidence.

The good measurement practice also requires that the uncertainty of a measurement result is directly usable as a component in evaluating the uncertainty of another measurement in which the first result is used [ISO93]; that is, measurement uncertainties have to be composed among each other. Thus, arithmetic operations among fuzzy variables must be defined, in order to ensure that fuzzy variables are able to propagate measurement results and related uncertainties.

2.2 Mathematics of fuzzy variables

As shown in the previous section, a fuzzy variable A can be described in two ways: by means of its membership function $\mu_A(x)$, for $x \in \Re$; or by means of its α-cuts A_α, for $0 \leq \alpha \leq 1$. These two ways to represent a fuzzy variable are, of course, equivalently valid and contain the same information, because the α-cuts can be determined starting from the membership function and vice versa. Similarly, the mathematics of fuzzy variables can also be defined in two different ways, which refer to membership functions and α-cuts, respectively.

[1] This statement will be proven, in a more general context, in the next chapter. For the moment, let us rely on the given intuitive example.

As the α-cuts of a fuzzy variable are confidence intervals and the aim of a measurement process is indeed to find confidence intervals, the second approach is preferable and hence reported in this section.

Let us consider fuzzy variables with a finite support, like, for instance, the ones in Fig. 2.1. Even if these variables are only a part of the whole kind of possible fuzzy variables, as shown in the previous section, this assumption is coherent with the measurement practice, where the possible values that can be assumed by a measurand are almost always confined into a closed interval. Moreover, even when the probability distribution of the measurement results is supposed to be greater than zero in every point of $\Re$, like, for instance, a Gaussian distribution, the probability that the measurand takes values outside a suitable confidence interval is very small and it is possible to consider this last interval as the support of the fuzzy variable, which correspond to the Gaussian distribution.[2]

Under the assumption of finite support, fuzzy arithmetic can be then defined on the basis of the two following properties:

- Each fuzzy variable can fully and uniquely be represented by its α-cuts.
- The α-cuts of each fuzzy variable are closed intervals of real numbers.

These properties enable one to define arithmetic operations on fuzzy numbers in terms of arithmetic operations on their α-cuts or, in other words, arithmetic operations on closed intervals [KG91, KY95]. These operations are a topic of the *interval analysis*, a well-established area of classic mathematics. Therefore, an overview of the arithmetic operations on closed intervals is previously given.

Let $*$ denote any of the four arithmetic operations on closed intervals: addition $+$, subtraction $-$, multiplication $\times$, and division $/$. Then, a general property of all arithmetic operations on closed intervals is given by

$$[a,b] * [d,e] = \{f * g \,|\, a \le f \le b, d \le g \le e\} \tag{2.2}$$

except that the division is not defined when $0 \in [d,e]$. The meaning of Eq. (2.2) is that the result of an arithmetic operation on closed intervals is again a closed interval.

This last interval is given by the values assumed by the proper operation $f*g$ between numbers f and g, taken from the original intervals. In particular, the four arithmetic operations on closed intervals are defined as follows:

$$[a,b] + [d,e] = [a+d, b+e] \tag{2.3}$$

$$[a,b] - [d,e] = [a-e, b-d] \tag{2.4}$$

$$[a,b] \times [d,e] = [\min(ad, ae, bd, be), \max(ad, ae, bd, be)] \tag{2.5}$$

[2] The information contained in a probability distribution can also always be represented in terms of a fuzzy variable. This is possible thanks to suitable probability–possibility transformations, as will be shown in Chapter 5.

and, provided that $0 \notin [d, e]$

$$\begin{aligned} [a, b] \,/\, [d, e] &= [a, b] \times [1/e, 1/d] \\ &= [\min(a/d, a/e, b/d, b/e), \max(a/d, a/e, b/d, b/e)] \end{aligned} \tag{2.6}$$

It can be noted that a real number r may also be regarded to as a special (degenerated) interval $[r, r]$. In this respect, Eqs. (2.3) to (2.6) also describe operations that involve real numbers and closed intervals. Of course, when both intervals degenerate, the standard arithmetic on real numbers is obtained.

The following examples illustrate the arithmetic operations over closed intervals, as defined by Eqs. (2.3)–(2.6):

$$[2, 5] + [1, 3] = [3, 8] \quad [0, 1] + [-6, 5] = [-6, 6]$$

$$[0, 1] - [-6, 5] = [-5, 7] \quad [2, 5] - [1, 3] = [-1, 4]$$

$$[3, 4] \times [2, 2] = [6, 8] \quad [-1, 1] \times [-2, -0.5] = [-2, 2]$$

$$[-1, 1] \,/\, [-2, -0.5] = [-2, 2] \quad [4, 10] \,/\, [1, 2] = [2, 10]$$

Arithmetic operations on closed intervals satisfy some useful properties. Let us take $A = [a_1, a_2]$, $B = [b_1, b_2]$, $C = [c_1, c_2]$, $D = [d_1, d_2]$, $\mathbf{0} = [0, 0]$, and $\mathbf{1} = [1, 1]$. Then, the properties can be formulated as follows:

1. Commutativity: $A + B = B + A$; $A \times B = B \times A$.
2. Associativity: $(A + B) + C = A + (B + C)$; $(A \times B) \times C = A \times (B \times C)$.
3. Identity: $A = 0 + A = A + 0$; $A = 1 \times A = A \times 1$.
4. Subdistributivity: $A \times (B + C) \subseteq A \times B + A \times C$.
5. Distributivity:
   a. If $b \times c \geq 0$ for every $b \in B$ and $c \in C$, then $A \times (B + C) = A \times B + A \times C$.
   b. If $A = [a, a]$, then $a \times (B + C) = a \times B + a \times C$.
6. $0 \in A - A$ and $1 \in A/A$.
7. Inclusion monotonicity: If $A \subseteq C$ and $B \subseteq D$, then $A * B \subseteq C * D$.

These properties can be readily proven from Eqs. (2.3)–(2.6).

These same equations can be also used to define the arithmetic of fuzzy variables. In fact, as stated, a fuzzy variable can be fully and uniquely represented by its α-cuts, which are indeed closed intervals of real numbers. Therefore, it is possible to apply the interval analysis over each α-cut of the fuzzy variable.

Let A and B denote two fuzzy variables, whose generic α-cuts are A_α and B_α. Let $*$ be any of the four basic arithmetic operations. As A and B are fuzzy variables, $A * B$ is also a fuzzy variable. The α-cuts of the result $A * B$, denoted by $(A * B)_\alpha$, can be easily evaluated from A_α and B_α as

$$(A * B)_\alpha = A_\alpha * B_\alpha \tag{2.7}$$

for any α between 0 and 1. Let us remember that, when the division is considered, it is required that $0 \notin B_\alpha$ for every α between 0 and 1.

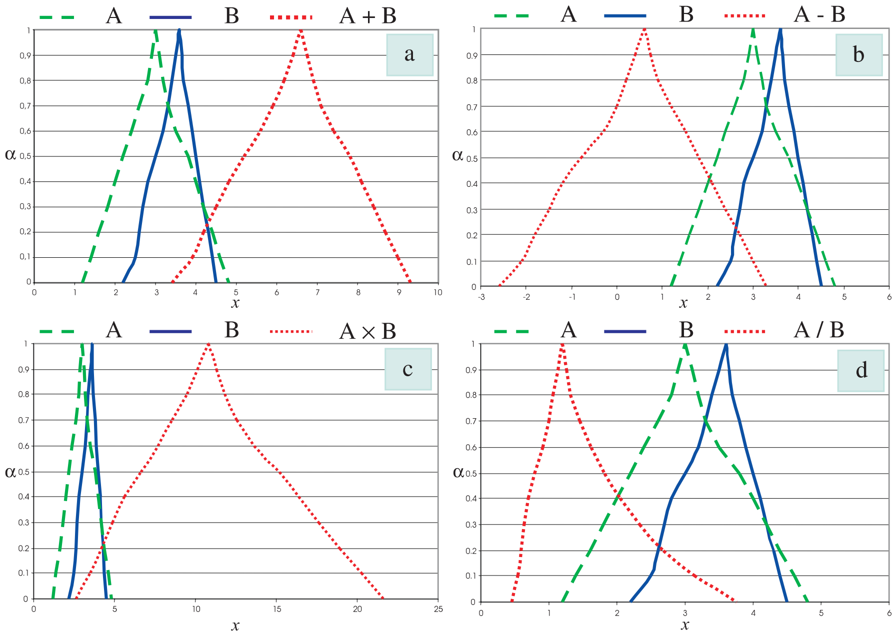

Fig. 2.4. Arithmetic operations on fuzzy variables: (**a**) addition, (**b**) subtraction, (**c**) multiplication, and (**d**) division.

For the sake of clearness, let us consider the three fuzzy variables A, B, and C, where $C = A * B$, and let us denote the generic α-cuts of A, B, and C as $A_\alpha = [a_1^\alpha, a_2^\alpha]\,, B = [b_1^\alpha, b_2^\alpha]$, and $C = [c_1^\alpha, c_2^\alpha]$, respectively. Then:

- ADDITION: $C = A + B$
  $[c_1^\alpha, c_2^\alpha] = [a_1^\alpha + b_1^\alpha,\ a_2^\alpha + b_2^\alpha]$
- SUBTRACTION: $C = A - B$
  $[\ c_1^\alpha, c_2^\alpha] = [a_1^\alpha - b_2^\alpha,\ a_2^\alpha - b_1^\alpha]$
- MULTIPLICATION: $C = A \times B$
  $[c_1^\alpha, c_2^\alpha] = [\min\,(a_1^\alpha b_2^\alpha,\ a_2^\alpha b_1^\alpha,\ a_1^\alpha b_1^\alpha,\ a_2^\alpha b_2^\alpha), \max\,(a_1^\alpha b_2^\alpha,\ a_2^\alpha b_1^\alpha,\ a_1^\alpha b_1^\alpha,\ a_2^\alpha b_2^\alpha)]$
- DIVISION: $C = A/B$
  Provided that $0 \notin [b_1^\alpha, b_2^\alpha]$:
  $[c_1^\alpha, c_2^\alpha] =$
  $[\min\,(a_1^\alpha/b_2^\alpha,\ a_2^\alpha/b_1^\alpha,\ a_1^\alpha/b_1^\alpha,\ a_2^\alpha/b_2^\alpha), \max\,(a_1^\alpha/b_2^\alpha,\ a_2^\alpha/b_1^\alpha,\ a_1^\alpha/b_1^\alpha,\ a_2^\alpha/b_2^\alpha)]$
  for every α between 0 and 1.

Figure 2.4 shows an example of the four arithmetic operations on the two fuzzy variables A and B, represented by the dashed and solid lines, respectively. Starting from the arithmetic operations between fuzzy variables, it is

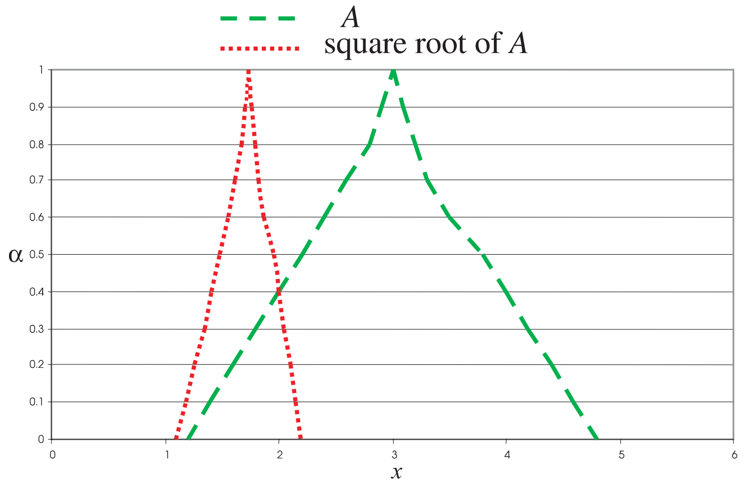

Fig. 2.5. Square root of a fuzzy variable.

also possible to define other mathematical operations. As an example, the square root of a fuzzy variable can be defined as follows.

Let A be a fuzzy variable, defined by its α-cuts $A_\alpha = [a_1^\alpha, a_2^\alpha]$, and let us consider the square root $C = \sqrt{A}$. Provided that the fuzzy number A is positive, that is, $0 \leq a_1^\alpha \leq a_2^\alpha$ for every α, the generic α-cut of C is

$$[c_1^\alpha, c_2^\alpha] = [\sqrt{a_1^\alpha}, \sqrt{a_2^\alpha}] \tag{2.8}$$

Figure 2.5 shows an example.

It is important to underline that, in some particular applications, it could be also necessary to perform a square root of a fuzzy variable that falls across the zero value. In fact, when the measurement uncertainty of a value near zero is considered, the correspondent confidence interval (and the correspondent fuzzy variable too) contains the zero value. Hence, when the measurement uncertainty of the square root of such a value is considered, the correspondent confidence interval (and the correspondent fuzzy variable too) still contains the zero value. In this case, Eq. (2.8) modifies as follows. If the fuzzy variable is like the one in Fig. 2.6a, that is, the zero value is crossed by the left side of its membership function, it is

$$c_1^\alpha = \begin{cases} -\sqrt{-a_1^\alpha} & \alpha < k^* \\ \sqrt{a_1^\alpha} & \alpha \geq k^* \end{cases}$$

$$c_2^\alpha = \sqrt{a_2^\alpha}$$

where $k^* \; = \alpha|_{a_1^\alpha = 0}$.

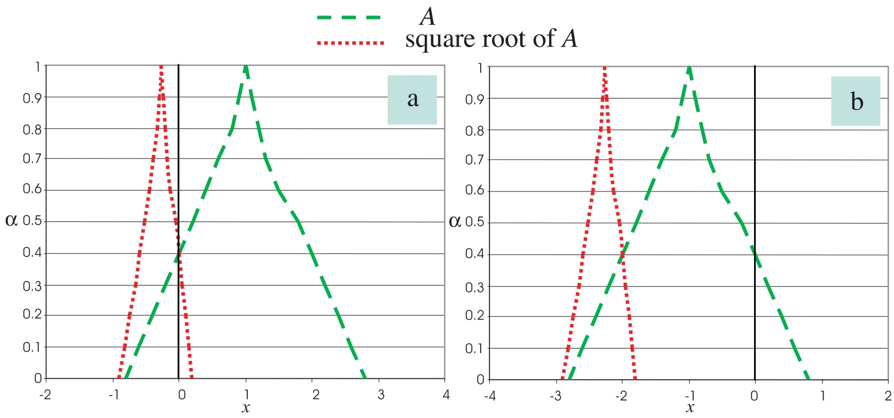

Fig. 2.6. Square root of a fuzzy variable across the zero value. In this example, $k^* = 0.4$.

If the fuzzy variable is like the one in Fig. 2.6b, that is, the zero value is crossed by the right side of its membership function, it is

$$c_1^\alpha = -\sqrt{-a_1^\alpha}$$

$$c_2^\alpha = \begin{cases} -\sqrt{-a_2^\alpha} & \alpha > k^* \\ \sqrt{a_2^\alpha} & \alpha \leq k^* \end{cases}$$

where $k^* = \alpha|_{a_2^\alpha=0}$.

2.3 A simple example of application of the fuzzy variables to represent measurement results

It has already been stated that a fuzzy variable can be suitably employed to represent a measurement result together with its associated uncertainty [MBFH00, MLF01, UW03, FS03]; in fact, a fuzzy variable can be represented by a set of confidence intervals (the α-cuts) and associated levels of confidence (strictly related to levels α).

Moreover, when an indirect measurement is considered, the mathematics of fuzzy variables allows one to directly obtain the final measurement result in terms of a fuzzy variable. This means that the final measurement result and associated uncertainty can be obtained together, in a single step, which involves fuzzy variables.

In fact, if $y = f(x_1, x_2, \ldots, x_n)$ is the measurement algorithm, the fuzzy variable Y associated with the measurement result and its associated uncertainty is readily given by $Y = f(X_1, X_2, \ldots, X_n)$, where $X_1, X_2, \ldots, X_n$ are the fuzzy variables associated with the input quantities. The operations are of course performed according to the mathematics of fuzzy variables.

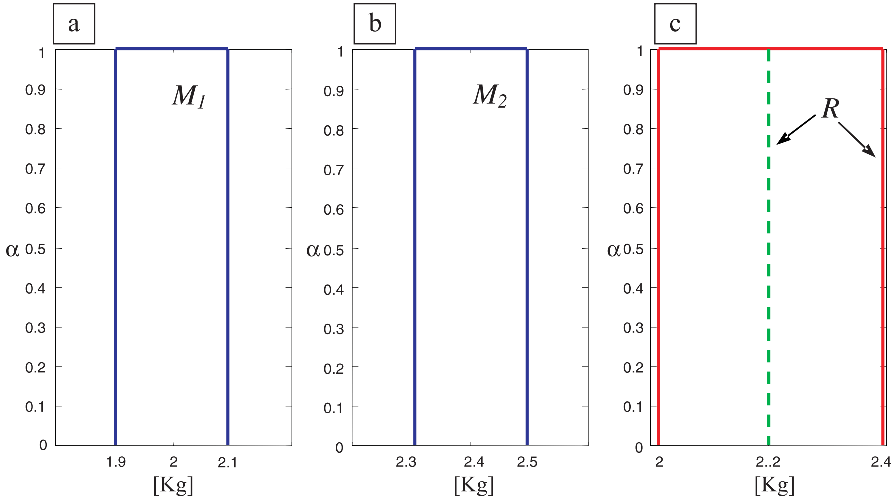

Fig. 2.7. Method of the double weighing. M_1 and M_2 are the two direct measurements, and R is the final result in the two different cases.

Let us consider again the example of the double weighing, and let m_1 and m_2 be 2 kg and 2.4 kg, respectively. The two measurement results can be represented, together with their uncertainty, by two fuzzy variables M_1 and M_2. Let us suppose again that the contributions to uncertainty fall in the interval $\pm 100\ g$ and are not random.

The available information is that the measurement result belongs to the given interval and that each point of the interval is as plausible as the others. Therefore, the fuzzy variables M_1 and M_2 have a rectangular membership function (Fig. 2.7a and b): The confidence interval is given with a level of confidence equal to one, and no other assumptions can be done.

The final measurement result, that is, the arithmetic mean of M_1 and M_2, obtained with the proposed approach, is the fuzzy variable:

$$R = \frac{M_1 + M_2}{2}$$

shown with a solid line in Fig. 2.7c.

Obviously, as stated at the beginning of this chapter, all available information should be used in modeling the measurement process. Therefore, if it was known that the uncertainty contribution behaves systematically and it is due to a difference in length between the two beams of the weighbridge, and thus always takes the (unknown) value a, belonging to interval $\pm 100\ g$, then the double weighing naturally compensates it, because it affects measurement m_1 with positive sign ($+a$) and measurement m_2 with negative sign ($-a$). Hence, the following applies:

$$R = \frac{(m_1 + a) + (m_2 - a)}{2} = \frac{m_1 + m_2}{2}$$

and the final measurement result, shown with a dashed line in Fig. 2.7c, is not affected by this uncertainty contribution. The method of the double weighing is indeed used for the compensation of the systematic error due to the different length of the beams.

This particular example has been suitably chosen in order to underline the importance of the measurement model associated with the available information. Of course, this is a really particular case and the fact that the systematic contribution to uncertainty compensates depends on the kind of considered contribution and the considered measurement procedure. The readers should not erroneously think that compensation is a particular characteristic of systematic contributions! In fact, if the two measurements m_1 and m_2 were performed on the same plate (instead of one on each opposite plate), the knowledge that the uncertainty contribution behaves systematically would not be useful in any way and a similar result as in the case of total ignorance would be obtained.

2.4 Conclusions

In this chapter, fuzzy variables and their mathematics have been defined. Furthermore, it has been shown how they can be used to represent and propagate measurement results each time the uncertainty affecting the results themselves is due to totally unknown contributions, including the systematic ones, when their presence can be only supposed but nothing else can be said about their behavior.

The simple example given in the previous section has shown how immediate is the use of fuzzy variables. When this approach is followed, many advantages can be drawn. Operations are performed directly on fuzzy variables, and no further elaborations are needed in order to process measurement uncertainties.

The measurement result is directly provided in terms of a fuzzy variable, which contains all available information about the measurement result. For instance, if the confidence interval at level of confidence 1 is needed, it simply corresponds to the α-cut at level $\alpha = 0$.

The computational burden is low.

According to their mathematics, the approach based on fuzzy variables can be seen as an attempt to modernize the old theory of errors. In fact, the mathematics are similar (both are based on the interval analysis), but the approach based on fuzzy variables does not refer to the unknowable true value of the measurand.

On the other hand, this approach can be also seen as a complement of the modern Theory of Uncertainty. In fact, it encompasses all concepts of confidence intervals and levels of confidence, while providing a different way to represent incomplete knowledge.

However, the approach based on fuzzy variables must not be considered as an alternative to the statistical approach. In fact, fuzzy variables and random variables are defined in a different way and they obey different mathematics.

Fuzzy variables compose each other according to the mathematics of the intervals; this means, for instance, that the sum of the fuzzy variables A and B (see Fig. 2.4a) is a fuzzy variable whose α-cuts show a width that is the sum of the widths of the corresponding α-cuts of A and B. Therefore, it can be assessed that fuzzy variables are not subject to compensation phenomena. On the other hand, random variables compose each other according to statistics; this means, for instance, that the sum of the random variables A and B is a random variable whose standard deviation is smaller than the sum of the standard deviations of A and B. Hence, random variables are subject to a natural compensation phenomena.

As a result, fuzzy and random variables can be considered to represent different kind of physical phenomena: Fuzzy variables may represent totally unknown phenomena, which physically do not compensate each other, including the systematic ones; random variable may represent, as widely known, random phenomena, which physically compensate each other. Hence, with respect to the Theory of Uncertainty, fuzzy variables and random variables cannot be considered competitive but, rather, complementary. In fact, uncertainty arises in the measurement processes because of the presence of all kinds of uncertainty contributions (random, systematic, unknown). Thus, both approaches are needed.

However, it is not practical and not effective, of course, to follow the two approaches separately. Therefore, a unifying mathematical theory, as well as a unifying mathematical variable, should be found.

This theory is the Theory of Evidence, which encompasses both the probability theory, the mother of random variables, and the possibility theory, the mother of fuzzy variables. The theory of evidence is the topic of Chapter 3.

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