
Foundations of Solid Mechanics

A major problem in continuum mechanics consists of predicting the deformations and internal forces arising in a material body subjected to a given set of external fields such as the forces of pure mechanical nature, temperature and electromagnetic fields. This Chapter summarizes some of important definitions, relations, and methods commonly employed for the analysis of reformable media although, of course, many aspects of mechanical behavior are left unaccounted for in this approach and referring the unsatisfied reader to monographs in which a level of higher generality is adopted. However, it is hoped that sufficient details are given so that the reader can understand remainder Chapters without constant reference to other books.

As is customary in continuum mechanics studies, material properties and fields are expressed in tensor form in this book. The general technique of tensor analysis is given, e.g. in [99], [1028], [1068] presenting the tensor theory not only as an autonomous mathematical discipline, but also as a preparation for theories of continuum mechanics. The special applications of tensors are described in the books [722], [1114]. The books [21], [364] provide an introduction to the theories of linear elasticity and nonlinear elasticity. More comprehensive treatments of nonlinear and anisotropic elasticity can be found, e.g., in the books [398], [410], [995], [1015], [1082], [832], [722], and [1098]. The coupling effects of mechanical, temperature and electromagnetic fields are considered in [980], [747], [861]. The general reviews of phenomenological plasticity with an extensive list of references are given in [527], [737], [711].

2.1 Elements of Tensor Analysis

This Section provides a remainder of basic notions of tensors of a certain rank which components transforming in a particular way under the transformation of coordinates $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \rightarrow (\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$.

We concern with the rectangular Cartesian coordinate systems and require that the orthonormalized basis is right-handedness $\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = 1$ and $\mathbf{e}_i \cdot \mathbf{e}_i = \delta_{ij}$, where a scalar multiplication $\mathbf{e}_i \cdot \mathbf{e}_j$ denotes the projection of the i th basic vector on the j th one (or vice-versa), and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \quad (2.1)$$

is the Kronecker delta. The vector (cross) product of two basic vectors $\mathbf{e}_i \times \mathbf{e}_j$ can be defined by the permutation tensor ϵ (summation over repeated Latin indices is implied)

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k, \quad \epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123, \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123, \\ 0 & \text{if any two indices are identical.} \end{cases} \quad (2.2)$$

The vector product of the vectors \mathbf{a} , \mathbf{b} and the triple scalar product of the vectors \mathbf{a} , \mathbf{b} , \mathbf{c} can consequently be written as

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_i b_j \mathbf{e}_k, \quad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_i b_j c_k = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad (2.3)$$

The expansion of another orthonormalized basis \mathbf{e}' ($\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$) with the same origin of coordinates into the old one $\mathbf{e}'_{i'} = g_{i'j} \mathbf{e}_j$ is defined by coefficients $g_{i'j}$ called a *cosine matrix*, because each element $g_{i'j}$ is the cosine of the angle between two corresponding axes

$$g_{i'j} = \mathbf{e}'_{i'} \cdot \mathbf{e}_j = \cos \alpha_{i'j}. \quad (2.4)$$

The most frequently used methods of parametric representation of orthogonal transformation of Cartesian coordinates \mathbf{e}' and \mathbf{e} are presented in Appendix A.1. An arbitrary vector \mathbf{x} is completely defined by its magnitude (modulus) and direction in the space

$$\mathbf{x} = \sum x_i \mathbf{e}_i = x_i \mathbf{e}_i, \quad x_i = \mathbf{x} \cdot \mathbf{e}_i = |\mathbf{x}| \cos(\mathbf{x}, \mathbf{e}_i). \quad (2.5)$$

The orthogonal matrix $\mathbf{g} = \|g_{ij}\|$, ($i, j = 1, \dots, 3$) called the rotation matrix links the coordinates \mathbf{x}' of an arbitrary point in the reference orthonormalized frame \mathbf{e}' ($\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$) with the coordinates \mathbf{x} in the crystal orthonormalized coordinate system \mathbf{e} ($\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$) with the same origin of coordinates (see for details Appendix A.1)

$$\mathbf{x}' = \mathbf{g}\mathbf{x}. \quad (2.6)$$

The definitions of tensors of order 0 (scalar) ${}^0\mathbf{T}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \equiv {}^0\mathbf{T}'(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$ and 1 [contravariant vector (2.6)] can be easily generalized to contravariant tensors of order n

$${}^n\mathbf{T} = {}^nT_{i_1 i_2 \dots i_n} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_n} = {}^nT'_{i'_1 i'_2 \dots i'_n} \mathbf{e}'_{i'_1} \otimes \dots \otimes \mathbf{e}'_{i'_n}, \quad (2.7)$$

where the 3^n tensor components in the prime and unprime coordinates are linked by the following transformation

$${}^nT'_{j_1 j_2 \dots j_n} = g_{j_1 i_1} \dots g_{j_n i_n} {}^nT_{i_1 i_2 \dots i_n}. \quad (2.8)$$

A tensor ${}^{n+m}\mathbf{T}$ of rank $n+m$, contravariant of rank n and covariant of rank m , is an object with the induced component transformation

$${}^{n+m}T_{j'_1 j'_2 \dots j'_m}^{j_1 j_2 \dots j_n} = g_{j_1 i_1} \dots g_{j_n i_n} g_{i'_1 j'_1} \dots g_{i'_m j'_m} {}^{n+m}T_{i'_1 i'_2 \dots i'_m}^{i_1 i_2 \dots i_n}. \quad (2.9)$$

It is possible to distinguish four operations with the tensors. The first one joins the summation and multiplication over the number of the tensors T_{i_1, \dots, i_n} and S_{i_1, \dots, i_n} of the same order:

$$\lambda T_{i_1, \dots, i_n} + \beta S_{i_1, \dots, i_n} = R_{i_1, \dots, i_n} \Leftrightarrow \lambda \mathbf{T} + \beta \mathbf{S} = \mathbf{R}, \quad (2.10)$$

where λ and β are scalars. The tensor (or diadic) production of two tensors of orders n and m is a tensor of order $n + m$, defined as

$$\mathbf{T} \otimes \mathbf{S} = (T_{i_1, \dots, i_n} S_{j_1, \dots, j_m}) \mathbf{e}_{i_1} \dots \mathbf{e}_{i_n} \mathbf{e}_{j_1} \dots \mathbf{e}_{j_m}. \quad (2.11)$$

In so doing, in general \mathbf{T} and \mathbf{S} are not commutative, $\mathbf{T} \otimes \mathbf{S} \neq \mathbf{S} \otimes \mathbf{T}$. The third operation called convolution (or contraction) is the action with one tensor reducing his order on two in the framework of the Einstein summation convention, for example

$$T_{i_1, \dots, i_n, k, k, j_1, \dots, j_m} = S_{i_1, \dots, i_n, j_1, \dots, j_m}. \quad (2.12)$$

The fourth operation produces the index permutation leads to a new tensor of the same order but with different index order. For example, one can generate only one new tensor (called transposed one) of the second order: $\mathbf{S} = \mathbf{T}^\top \Leftrightarrow S_{ij} = T_{ji}$. In so doing, a second order tensor \mathbf{T} is called symmetric if $\mathbf{T}^\top = \mathbf{T}$, and it is called antisymmetric if $\mathbf{T} = -\mathbf{T}^\top$.

By straightforward check we are able to ascertain that the new objects formed by mentioned operations are really the tensors. These operations can be arranged in the different combinations. For example, the combination of the diadic production and convolution denoting by a dot leads to the following tensors

$$\mathbf{R} = \mathbf{T} \cdot \mathbf{S} \Leftrightarrow R_{ij} = T_{ik} S_{kj}, \quad (2.13)$$

$$\lambda = \mathbf{R} \cdot \mathbf{T} \Leftrightarrow \lambda = R_{ij} T_{ji}, \quad (2.14)$$

$$\lambda = \mathbf{R} : \mathbf{T} \Leftrightarrow \lambda = R_{ij} T_{ij}, \quad (2.15)$$

where multiple dots are appropriate for complex constructions.

In particular, the simplest tensors ${}^2\mathbf{T}$ of order 2 are the unit tensor $\boldsymbol{\delta}$ (1.1) and the rotation tensor \mathbf{g} (2.4). They hold the property of *orthogonality*: ${}^2\mathbf{T} \cdot {}^2\mathbf{T} = \boldsymbol{\delta}$ and *isotropy*: ${}^2\mathbf{T}' = {}^2\mathbf{T}$. The tensor $\boldsymbol{\delta}$ transform a vector \mathbf{a} into itself while \mathbf{g} transforms the unit base vector \mathbf{e}_i to the unit vector \mathbf{e}'_i of the primed coordinate system: $\mathbf{e}'_i = \mathbf{g} \cdot \mathbf{e}_i$. The simplest tensor of order 4 is the unit tensor \mathbf{I} with the components $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ can be decomposed into the *bulk* and *deviatoric* parts

$$\mathbf{I} = \mathbf{N}_1 + \mathbf{N}_2, \quad \mathbf{N}_1 \equiv \frac{1}{3}\boldsymbol{\delta} \otimes \boldsymbol{\delta}, \quad \mathbf{N}_2 = \mathbf{I} - \mathbf{N}_1 \quad (2.16)$$

which have the property of orthogonality

$$\mathbf{N}_1 : \mathbf{N}_1 = \mathbf{N}_1, \quad \mathbf{N}_2 : \mathbf{N}_2 = \mathbf{N}_2, \quad \mathbf{N}_1 : \mathbf{N}_2 = \mathbf{0}. \quad (2.17)$$

Differentiation denoting by the *del operator* $\partial_i \equiv \partial/\partial x_i$ of a tensor with the suitably smooth components forms another tensors

$$\nabla \cdot \mathbf{T} \equiv \text{Div} \mathbf{T} = (\partial_j T_{j i_1, \dots, i_{n-1}})(\mathbf{x}) \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \dots \mathbf{e}_{i_{n-1}}, \quad (2.18)$$

$$\nabla \otimes \mathbf{T} \equiv \text{Grad} \mathbf{T} = (\partial_{i_1} T_{i_2, \dots, i_{n+1}})(\mathbf{x}) \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \dots \mathbf{e}_{i_{n+1}}, \quad (2.19)$$

$$\nabla \times \mathbf{T} \equiv \text{Curl} \mathbf{T} = [\epsilon_{i_1 j k} (\partial_j T_{k, i_2, \dots, i_n})](\mathbf{x}) \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \dots \mathbf{e}_{i_{n+1}}, \quad (2.20)$$

which are called *divergence*, *gradient*, and *curl* of the tensor field $\mathbf{T} = {}^n \mathbf{T}$ and have the orders $n-1$, $n+1$, and n , respectively. We will also use the other differential operators of the first- and second-order tensors:

$$\text{Def} \mathbf{a} \equiv \frac{1}{2}(\nabla \otimes \mathbf{a} + \mathbf{a} \otimes \nabla) = \frac{1}{2}(a_{i,j} + a_{j,i}) \mathbf{e}_i \otimes \mathbf{e}_j, \quad (2.21)$$

$$\text{Inc} \mathbf{b} \equiv \epsilon_{ijk} \epsilon_{lmn} b_{jn, km} \mathbf{e}_i \otimes \mathbf{e}_j \quad (2.22)$$

called the operator of *deformation*, and *incompatibility*, respectively.

Analogously to differentiation, the integration of a tensor field \mathbf{T} on both the domain w and their boundary ∂w can be defined in terms of the integration of its components. The *Gauss theorem* relates the action of the *del* operator on the tensor field \mathbf{T} in domain \mathcal{E} to its flux across the boundary $\partial \mathcal{E}$, under suitable smoothness of both the tensor \mathbf{T} and the domain boundary ∂w with the unit vector of an outward normal $\mathbf{n}(\mathbf{s}) \perp \partial \mathcal{E}$ ($\mathbf{s} \in \partial \mathcal{E}$):

$$\int_{\mathcal{E}} \nabla * \mathbf{T} \, d\mathbf{x} = \int_{\partial \mathcal{E}} \mathbf{n} * \mathbf{T} \, d\mathbf{s}, \quad (2.23)$$

where $*$ can stand for “ \cdot ”, \otimes , or \times . Let $\partial \mathcal{E}'$ be a portion of an oriented surface with unit outward normal \mathbf{n} and with the surface closed edge C . Then for the tensor field \mathbf{T} continuously differentiable in $\partial \mathcal{E}'$ and continuous in C , the generalized Stokes theorem can be presented in the form

$$\int_{\partial \mathcal{E}'} (\mathbf{n} \times \nabla) * \mathbf{T} \, d\mathbf{s} = \oint_C \mathbf{T} * \, d\mathbf{l}. \quad (2.24)$$

Equation (2.23) is also generalized for the volume \mathcal{E} and surface $\partial \mathcal{E}$, excluding the points of a discontinuity surface τ with unit normal \mathbf{n} which may be sweeping the body. In a similar manner, the surface $\partial \mathcal{E}'$ with edge C in Eq. (2.24) is accompanied by a discontinuity line γ that may be sweeping the surface. Thus (see e.g., [316]),

$$\int_{\mathcal{E} \setminus \tau} \nabla \cdot \mathbf{T} \, d\mathbf{x} = \int_{\partial \mathcal{E} \setminus \tau} \mathbf{T} \cdot \mathbf{n} \, d\mathbf{s} - \int_{\tau} [\mathbf{T}] \cdot \mathbf{n} \, d\mathbf{s}, \quad (2.25)$$

$$\int_{\partial \mathcal{E}' \setminus \gamma} (\mathbf{n} \times \nabla) \cdot \mathbf{T} \, d\mathbf{s} = \oint_{C \setminus \gamma} \mathbf{T} \cdot \, d\mathbf{l} - \oint_{\gamma} [\mathbf{T}] \, d\mathbf{l}, \quad (2.26)$$

where the brackets $[\cdot]$ denote the jump of its enclosure across the discontinuity surface τ or the discontinuity line γ ; $\mathcal{E} \setminus \tau$, $\partial \mathcal{E} \setminus \tau$, and $\partial \mathcal{E}' \setminus \gamma$, $C \setminus \gamma$ exclude points of τ and γ , respectively.

2.2 The Theory of Strains and Stresses

We consider a body occupying the regions \mathcal{E}_0 and \mathcal{E} , with boundaries \mathcal{E}_0 and $\partial \mathcal{E}$, in some fixed reference configuration and in the current one, respectively. Suppose

an arbitrary material point $P^0 \in \mathcal{E}_0$ at time $t = 0$ has the spatial coordinates \mathbf{x} in the orthonormalized coordinate system \mathbf{e}_i ($i = 1, 2, 3$). The point P^0 moves during the time t of deformations to the point $P \in \mathcal{E}$ with coordinates $\boldsymbol{\xi}(t)$ in the same coordinate system. The relations between the coordinates $\boldsymbol{\xi}$ and \mathbf{x} are defined by the smooth one-to-one invertible functions:

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, t), \quad (\xi_i = \xi_i(x_1, x_2, x_3, t)) \quad \text{and} \quad \mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t), \quad (x_i = x_i(\xi_1, \xi_2, \xi_3, t)), \quad (2.27)$$

which are provided by an assumption that the Jacobian

$$J = \det \left\| \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(x_1, x_2, x_3)} \right\| = \begin{vmatrix} \partial\xi_1/\partial x_1 & \partial\xi_1/\partial x_2 & \partial\xi_1/\partial x_3 \\ \partial\xi_2/\partial x_1 & \partial\xi_2/\partial x_2 & \partial\xi_2/\partial x_3 \\ \partial\xi_3/\partial x_1 & \partial\xi_3/\partial x_2 & \partial\xi_3/\partial x_3 \end{vmatrix} \quad (2.28)$$

exists at each point of the configuration \mathcal{E} and that $J > 0$ meaning that the material cannot penetrate itself, and that material element of non-zero volume cannot be compressed to a point or expanded to infinite volume during the motion. The independent variables \mathbf{x} and $\boldsymbol{\xi}$ in Eq. (2.27) are referred to as Lagrange and Euler variables, respectively.

We define the strain tensor (specifying the change between the points in the deformable body) by introducing the displacement $\mathbf{u} = \boldsymbol{\xi} - \mathbf{x}$, $u_i = \xi_i - x_i$ as well as the tensors of the deformation gradient \mathbf{F} , material displacement gradient \mathbf{H} , and spatial displacement gradient ($J \equiv \det \mathbf{F}(\mathbf{x}) > 0$):

$$\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \frac{\partial \mathbf{u}(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{H}(\mathbf{x}) \equiv \mathbf{F}(\mathbf{x}) - \mathbf{I}, \quad \frac{\partial \mathbf{u}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} = \mathbf{h}(\boldsymbol{\xi}), \quad \mathbf{F}\mathbf{h} = \mathbf{H}, \quad (2.29)$$

respectively. We will reproduce the correspondences between some elements of the regions \mathcal{E}_0 and \mathcal{E} in the reference and current configurations, respectively. The equation $d\boldsymbol{\xi} = \mathbf{F} d\mathbf{x}$ describes how an infinitesimal line element $d\mathbf{x} \in \mathcal{E}_0$ transforms linearly into the element $d\boldsymbol{\xi} \in \mathcal{E}$. The areas ds and dS of the surface elements of $\partial\mathcal{E}$ and $\partial\mathcal{E}_0$ with the (positive) unit normals \mathbf{n} and \mathbf{N} , respectively, are connected by the *Nanson's formula*:

$$\mathbf{n} ds = J \mathbf{B} \mathbf{N} dS, \quad \mathbf{B} = (\mathbf{F}^{-1})^\top. \quad (2.30)$$

The volume dv in the deformed configuration is related with the volume dV in the reference configuration by the equation

$$div = J dV. \quad (2.31)$$

The ratio of current $|\boldsymbol{\xi}|$ to reference $|d\mathbf{x}|$ lengths of a line element that was in the direction \mathbf{M} in the reference configuration defines the *stretch*:

$$\lambda(\mathbf{M}) \equiv \frac{|d\boldsymbol{\xi}|}{|d\mathbf{x}|} = |\mathbf{F}\mathbf{M}| = [\mathbf{M} \cdot (\mathbf{F}^\top \mathbf{F} \mathbf{M})]^{1/2} \quad (2.32)$$

in the direction \mathbf{M} at \mathbf{x} . The quality $\lambda(\mathbf{M}) - 1$ is called the extension ratio in the direction \mathbf{M} , while $|d\boldsymbol{\xi}| - |d\mathbf{x}|$ is the extension.

The deformed states of the body in the vicinity of P^0 are defined the strain tensor specified by the displacement \mathbf{u} . Working with the material \mathbf{x} and spatial $\boldsymbol{\xi}$ variables we can define the strain tensors in Lagrange's (Green's tensor) and Euler's (Almansi's tensor) representations:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{H}^\top + \mathbf{H} + \mathbf{H}^\top \mathbf{H}), \quad \varepsilon_{ij}(\mathbf{x}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right), \quad (2.33)$$

$$\tilde{\boldsymbol{\varepsilon}} = \frac{1}{2}(\mathbf{h}^\top + \mathbf{h} + \mathbf{h}^\top \mathbf{h}), \quad \tilde{\varepsilon}_{ij}(\mathbf{x}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i} - \frac{\partial u_k}{\partial \xi_i} \frac{\partial u_k}{\partial \xi_j} \right), \quad (2.34)$$

Hereafter the values referred to the variables $\boldsymbol{\xi}$ are marked by the symbol $\tilde{\cdot}$. In many practical applications, it is possible to neglect the products of derivatives in Eqs. (2.33) and (2.34). Then the coincidence of the tensors of Green and Almansi yields the well-known infinitesimal strain expression:

$$\boldsymbol{\varepsilon} = \text{Def} \mathbf{u}, \quad \varepsilon_{ij}(\mathbf{x}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (2.35)$$

If a displacements \mathbf{u} are given, the components of the strain tensors (2.33)-(2.35) are easily calculated by proper differentiation of these quantities. However, if six component of a strain tensor are given, they must be interrelated to serve as an integrability conditions ensuring a compatible set of the three independent displacement components. For the linear infinitesimal strain (2.35), these six interrelations, called *Saint-Venant's compatibility equations*, produce only three independent relations between the six components of the strain:

$$\text{Inc} \boldsymbol{\varepsilon} = \mathbf{0}, \quad \varepsilon_{ilm} \varepsilon_{jpk} \frac{\partial^2 \varepsilon_{mn}}{\partial x_l \partial x_q} = 0. \quad (2.36)$$

For an arbitrary symmetric tensor of the second order (including Green and Almansi strain tensors), the characteristic equation $\det \|\varepsilon_{ij} - \varepsilon \delta_{ij}\| = 0$, defining a polynomial of the third degree in ε :

$$\varepsilon^3 - I_\varepsilon \varepsilon^2 + II_\varepsilon \varepsilon - III_\varepsilon = 0 \quad (2.37)$$

has three real roots $\varepsilon_1, \varepsilon_2, \varepsilon_3$ called *principal values*. The coefficients $I_\varepsilon, II_\varepsilon, III_\varepsilon$ of Eq. (2.37) are scalars called *principal basic invariants*, and are given by

$$I_\varepsilon = \text{tr} \boldsymbol{\varepsilon} \equiv \varepsilon_{ii} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad (2.38)$$

$$II_\varepsilon = \frac{1}{2} \varepsilon_{ijk} \varepsilon_{ilm} \varepsilon_{jkl} \varepsilon_{km} \equiv \frac{1}{2} (I_\varepsilon^2 - \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}) = \varepsilon_{11} \varepsilon_{22} + \varepsilon_{22} \varepsilon_{33} + \varepsilon_{33} \varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{13}^2 - \varepsilon_{23}^2 = \varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 + \varepsilon_1 \varepsilon_3, \quad (2.39)$$

$$III_\varepsilon = \det \boldsymbol{\varepsilon} \equiv \frac{1}{6} (2 \varepsilon_{ij} \varepsilon_{jk} \varepsilon_{ki} - 3 I_\varepsilon \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + I_\varepsilon^3) = \varepsilon_1 \varepsilon_2 \varepsilon_3. \quad (2.40)$$

To each principal value ε_i , the corresponding principal direction \mathbf{m}_i is defined by the equation (i not summed) $\boldsymbol{\varepsilon} \cdot \mathbf{m}_i - \varepsilon_i \mathbf{m}_i = \mathbf{0}$. The tensor $\boldsymbol{\varepsilon}$ can be expressed in diagonal form in the principal triad $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$

$$\boldsymbol{\varepsilon} = \sum_{i=1}^3 \varepsilon_i \mathbf{m}_i \otimes \mathbf{m}_i, \quad (2.41)$$

which is orthogonal if $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$: $\mathbf{m}_i \cdot \mathbf{m}_j = \delta_{ij}$. In such a case, the components of the tensor $\boldsymbol{\varepsilon}$ in the principle triad have the form (no sum on i) $\varepsilon_{ij} = \varepsilon_i \delta_{ij}$.

In parallel with the principle basic invariants (2.38)-(2.40) one can introduce other invariants such as, e.g., *basic algebraic invariants*:

$$A_1 = \varepsilon_{ii} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad (2.42)$$

$$A_2 = \varepsilon_{ij} \varepsilon_{ji} = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2, \quad (2.43)$$

$$A_3 = \varepsilon_{ij} \varepsilon_{jk} \varepsilon_{ki} = \varepsilon_1^3 + \varepsilon_2^3 + \varepsilon_3^3 \quad (2.44)$$

as well as the invariant system specified by the intensity e_i of the tensor deviator $\boldsymbol{\varepsilon}'$ and its phase ψ

$$e = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/3, \quad (2.45)$$

$$e_i = \frac{1}{3} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_1 - \varepsilon_3)^2}, \quad (2.46)$$

$$\frac{1}{\sqrt{2}} \cos 3\psi = e_i^{-3} (\varepsilon_1 - e)(\varepsilon_2 - e)(\varepsilon_3 - e). \quad (2.47)$$

The following relationships between the invariants hold true:

$$I_\varepsilon = A_1, \quad 2II_\varepsilon = a_1^2 - A_2, \quad III_\varepsilon = (2A_3 - 3A_2A_1 + A_1^3)/6, \quad (2.48)$$

$$e = \frac{1}{3} A_1, \quad e_i = \frac{1}{3} \sqrt{3A_2 - A_1^2},$$

$$\cos 3\psi = \sqrt{2}(9A_3 - 9A_1A_2 + 2A_1^3)(3A_2 - A_1^2)^{3/2} \quad (2.49)$$

Now we will consider some definitions and notations of stress analysis. Let an element of the internal surface with an area ds have a unit normal vector \mathbf{n} . The total force $d\mathbf{f}$ acting on this element of surface defines the surface traction $\mathbf{t}^{(n)} = d\mathbf{f}/ds$, where the superscript (n) designates the normal to the surface element. The second-order Cauchy (or Eulerian) stress tensor \mathbf{T} is formally related to the stress traction by the symmetric tensor equation:

$$\mathbf{t}^{(n)} = \mathbf{n} \cdot \mathbf{T}, \quad t_j^{(n)} = n_i T_{ij}. \quad (2.50)$$

The tensor \mathbf{T} is symmetric if and only if balance of moment of momentum holds. Then the stress state in the point $\boldsymbol{\xi}$ can be described either by six components T_{ij} of the tensor \mathbf{T} or by three invariants I_T, II_T, III_T called the *principal stresses* [introduced analogously to Eqs. (2.38)-(2.40), $\mathbf{T} = \boldsymbol{\sigma}$]:

$$I_\sigma = \text{tr} \boldsymbol{\sigma} \equiv \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3, \quad (2.51)$$

$$II_\sigma = \frac{1}{2} \epsilon_{ijk} \epsilon_{ilm} \sigma_{jl} \sigma_{km} \equiv \frac{1}{2} (I_\sigma^2 - \boldsymbol{\sigma} : \boldsymbol{\sigma}) = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} \\ - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3, \quad (2.52)$$

$$III_\sigma = \det \boldsymbol{\sigma} \equiv \frac{1}{6} (2\sigma_{ij}\sigma_{jk}\sigma_{ki} - 3I_\sigma \boldsymbol{\sigma} : \boldsymbol{\sigma} + I_\sigma^3) = \sigma_1\sigma_2\sigma_3 \quad (2.53)$$

and by the Eulerian angles specifying the principle triad.

The Cauchy stress tensor defined by Eq. (2.50) refers to the deformed body in the spatial coordinate system. In a similar manner, we can introduce a *nominal stress tensor* (that is transposed of the first Piola-Kirchhoff stress tensor) associated with an original element of undeformed area:

$$\mathbf{T}^0 = J\mathbf{F}^{-1}\mathbf{T}, \quad T_{ij}^0 = J \frac{\partial x_i}{\partial \xi_k} T_{kj}, \quad (2.54)$$

where the Jacobian J is written as a determinant (2.28). The first Piola-Kirchhoff stress tensor is a two-point and, in general, nonsymmetric tensor. For elimination of these two disadvantages, one introduces the symmetric *second Piola-Kirchhoff stress tensor*

$$\boldsymbol{\sigma} = \mathbf{T}^0 \mathbf{B} = J \mathbf{B}^\top \mathbf{T} \mathbf{B}, \quad \sigma_{ij} = T_{im}^0 \frac{\partial x_j}{\partial \xi_m} = J \frac{\partial x_i}{\partial \xi_m} T_{mn} \frac{\partial x_j}{\partial \xi_n}, \quad (2.55)$$

which may be inverted

$$T_{ij} = J^{-1} \frac{\partial \xi_i}{\partial x_m} \sigma_{mn} \frac{\partial \xi_n}{\partial x_j}, \quad T_{im}^0 = \sigma_{ij} \frac{\partial \xi_m}{\partial x_j}. \quad (2.56)$$

2.3 Basic Equations of Solid Mechanics

2.3.1 Conservation Laws, Boundary Conditions, and Constitutive Equation

There are four conservation laws in continuum mechanics: (1) mass conservation, (2) conservation of linear momentum, (3) conservation of angular momentum, and (4) conservation of energy. We will reproduce the first, second, and fourth conservation laws.

The *balance equation of linear momentum* in spatial coordinates in terms of the Cauchy stress tensor \mathbf{T} is

$$\frac{\partial T_{ij}}{\partial \xi_i} + \rho b_i = \rho \left(\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_j}{\partial \xi_i} \right), \quad (2.57)$$

where \mathbf{b} is the body force per unit of mass acting upon the volume element; ρ is the mass density at time t related to the initial mass density ρ_0 by the *equation of continuity* (the first conservation law):

$$\rho = J^{-1} \rho_0 \quad (2.58)$$

according to Eq. (2.31). v_i is a velocity of a particle occupying the coordinate ξ_i at some time t found by taking the time derivative and holding the material coordinates constant: $v_i = \partial \xi_i / \partial t$.

Since the boundary conditions in elastic problems are most easily expressed in material coordinates, we shall reproduce an equivalent equation of motion in

Lagrangian coordinates in terms of the first and second Piola-Kirchhoff stress tensors:

$$\frac{\partial T_{im}^0}{\partial x_i} + \rho_0 b_{0m} = \rho_0 \frac{d^2 u_m(\mathbf{x})}{dt^2}, \quad (2.59)$$

$$\frac{\partial}{\partial x_i} \left[\sigma_{ij} \left(\delta_{jm} + \frac{\partial u_m(\mathbf{x})}{\partial x_j} \right) \right] + \rho_0 b_{0m} = \rho_0 \frac{d^2 u_m(\mathbf{x})}{dt^2}, \quad (2.60)$$

respectively, where the time derivative is actually a simple time derivative.

The basic *energy-conservation equation* (or *first principle of thermodynamic*) equates the rate of change of kinetic K and internal energy E to the rate at which surface and body forces do mechanical work P and the rate at which nonmechanical energy Q is transferred (per unit volume):

$$\dot{K} + \dot{E} = \dot{Q} + P, \quad (2.61)$$

where

$$K = \frac{1}{2\bar{\mathcal{E}}} \int_{\mathcal{E}} \rho v_i v_i \, dv, \quad E = \frac{1}{\bar{\mathcal{E}}} \int_{\mathcal{E}} \rho e \, dv, \quad (2.62)$$

$$Q = -\frac{1}{\bar{\mathcal{E}}} \int_{\partial\mathcal{E}} q_k n_k \, ds + \frac{1}{\bar{\mathcal{E}}} \int_{\mathcal{E}} h \, dv = \frac{1}{\bar{\mathcal{E}}} \int_{\mathcal{E}} (-\nabla \cdot \mathbf{q} + h) \, dv, \quad (2.63)$$

$$P = \frac{1}{\bar{\mathcal{E}}} \int_{\mathcal{E}} b_i v_i \, dv + \frac{1}{\bar{\mathcal{E}}} \int_{\partial\mathcal{E}} T_{ij} n_j v_i \, ds, \quad (2.64)$$

where e is a specific internal energy per unit mass, \mathbf{q} is a nonmechanical energy flux tensor defining the rate $q_k n_k ds$ of any nonmechanical energy transmitted outward through the surface element ds with outward directed normal \mathbf{n} , and h is heat created per unit volume in the body. In Eq. (2.61) K , E , Q , and P are global quantities represented as integrals over the volume and its surface (2.62)–(2.64), but they also have local analogies (for which we will use the same notations) defined pointwise within the body. Equation (2.61) can be transformed into the differential form by the use of the equation of continuity (2.58) and the equations of motion:

$$\rho \frac{de}{dt} = T_{ij} d_{ij} - \frac{\partial q_j}{\partial \xi_j} + \rho h, \quad d_{ij} = \frac{1}{2} \left(\frac{\partial v_j}{\partial \xi_i} + \frac{\partial v_i}{\partial \xi_j} \right), \quad (2.65)$$

where, \mathbf{d} called the *rate strain tensor*, is the symmetric part of the tensor $v_{i,j}$, and $d/dt = \partial/\partial t + v_i \partial/\partial \xi_i$ is a *material derivative*.

The *second law of thermodynamics* states the existence of the total differential for reversible processes:

$$dS = \frac{dQ}{T}, \quad (2.66)$$

where S is the entropy regarded as a measure of energy dissipation with respect to the absolute temperature T , and

$$\dot{S} + \int_{\partial\mathcal{E}} \left(\frac{\mathbf{q}}{T} \right) \cdot \mathbf{n} \, ds - \int_{\mathcal{E}} \frac{h}{T} \, dv \geq 0. \quad (2.67)$$

The last expression is referred to as the second law of thermodynamics, known as the Clausius-Duhem inequality, which can be rewritten as

$$\int_{\mathcal{E}} \left[\dot{s} + \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) - \frac{h}{T} \right] \text{div} \geq 0, \quad (2.68)$$

or

$$T\dot{s} + \nabla \cdot \mathbf{q} - \frac{1}{T} \mathbf{q} \cdot \nabla T - h \geq 0, \quad (2.69)$$

which is the local form of the Clausius-Duhem inequality. Here and henceforth s is the specific entropy density, and the term “specific” will mean “per unit volume”. Without loss of generality, it is assumed $\text{mes } \mathcal{E} = 1$.

If the right-hand side of motion equations (2.57), (2.59), (2.60) are zero, the resulting equations are called the equilibrium equation. One way to maintain a body in equilibrium is to apply suitable boundary conditions. The typical boundary conditions involves the prescribed displacements \mathbf{u} specifying on part of the boundary $\partial\mathcal{E}_{\mathbf{u}}^0 \subset \partial\mathcal{E}^0$, and the stress vector on the remained, $\partial\mathcal{E}_{\sigma}^0$, so that

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^{\partial}(\mathbf{x}), \quad \text{on } \partial\mathcal{E}_{\mathbf{u}}^0, \quad (2.70)$$

$$\mathbf{T}^{0\top} \cdot \mathbf{N} = \mathbf{t}^{0(\mathbf{N})}(\mathbf{F}, \mathbf{x}) \quad \text{on } \partial\mathcal{E}_{\sigma}^0, \quad (2.71)$$

where $\partial\mathcal{E}_{\mathbf{u}}^0 \cap \partial\mathcal{E}_{\sigma}^0 = \partial\mathcal{E}^0$, $\partial\mathcal{E}_{\mathbf{u}}^0 \cup \partial\mathcal{E}_{\sigma}^0 = \partial\mathcal{E}^0$. A general *configuration dependent loading* (2.70) if reduced to a *dead-load traction* in the surface traction introduced by (2.70) is independent of \mathbf{F} .

The aforementioned equations of conservation laws are in general insufficient to determine the body motion produced by given boundary conditions and body forces. They need to be accompanied by a *constitutive equation* characterizing the constitution of the body. It is assumed that there are no stresses and no strains in the initial (virgin) strain of the body. For *Cauchy elastic materials*, it is described by a symmetric single-valued *response function* \mathbf{G} : $\mathbf{T} = \mathbf{G}(\mathbf{F})$, which does not depend on the path of deformation. In a special case of Cauchy elasticity called hyperelasticity, there exists a *specific strain-energy function* $w = w(\mathbf{F})$ defined on the space of deformation gradients:

$$\mathbf{T}^0 = \frac{\partial w}{\partial \mathbf{F}}, \quad \mathbf{T} = J^{-1} \mathbf{F} \frac{\partial w}{\partial \mathbf{F}}. \quad (2.72)$$

Equations (2.72) are the constitutive equations for finite-deformation elasticity. The function w defined per unit volume in \mathcal{E}^0 represents the work done per unit volume at $\mathbf{x} \in \mathcal{E}^0$ in changing the deformation gradient from \mathbf{I} to \mathbf{F} . The function w can be considered as depending on either $\boldsymbol{\varepsilon}$ (2.33) or H_{ij} (2.29). Then we get the representations

$$\sigma_{ij} = \frac{\partial w(\boldsymbol{\varepsilon})}{\varepsilon_{ij}}, \quad T_{ij}^0 = \frac{\partial w(\mathbf{H})}{H_{ij}} \quad (2.73)$$

relating the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}$ to the Green deformation tensor $\boldsymbol{\varepsilon}$, and the nominal stress tensor \mathbf{T}^0 to the material displacement gradient \mathbf{H} , respectively.

A material is said to be *isotropic* if there is no orientation effect in the material. In such a case, the strain energy can be presented either as a symmetric function of the principal stretches or as a function of three independent invariants, such as (2.38)–(2.40) or (2.42)–(2.44). For an incompressible material

$J = III_\varepsilon \equiv 1$ holds, and the function w depends on only two independent invariants. In particular, the *Mooney-Rivlin* form of strain energy

$$w = C_1(I_\varepsilon - 3) + C_2(II_\varepsilon - 3), \quad (C_1, C_2 = \text{const.}) \quad (2.74)$$

is reduced to the *neo-Hookean* form

$$w = C_1(I_\varepsilon - 3). \quad (2.75)$$

Using Eq. (2.73₁) and the forms of the strain-energy functions (2.74) and (2.75), the stress tensor take the forms $\boldsymbol{\sigma} = -p\mathbf{I} + 2C_1\mathbf{F}\mathbf{F}^\top$ and $\boldsymbol{\sigma} = -p\mathbf{I} + 2C_1\mathbf{F}\mathbf{F}^\top - 2C_2(\mathbf{F}\mathbf{F}^\top)^{-1}$ for Mooney-Rivlin and neo-Hookean materials, respectively; here p is an arbitrary scalar. Many other forms of the strain energy have been analyzed for both incompressible and compressible nonlinear rubberlike solids in [26], [830].

Now we will consider the strain-energy functions of physically nonlinear elasticity defined on the space of infinitesimal strain tensors (2.35). A general quadratic form of the function w composed by the algebraic invariants (2.42) and (2.43) is

$$w = a_1 A_1^2 + a_2 A_2, \quad (2.76)$$

leading to the known constitutive equation (2.73₁) for the isotropic medium:

$$\boldsymbol{\sigma} = \mathbf{L}\boldsymbol{\varepsilon}, \quad L = 3\lambda\mathbf{N}_1 + 2\mu\mathbf{I} \quad (2.77)$$

by the use of two, λ and μ , Lamé's elastic constants. A more general special form of specific strain energy depending on three basic algebraic invariants (2.42)–(2.44) is the *Murnaghan* form, defined by

$$w = \frac{1}{2}\lambda A_1^2 + \mu A_2 + \frac{a}{3}A_1^3 + bA_1A_2 + \frac{c}{3}A_3, \quad (2.78)$$

where λ and μ are the Lamé' elastic constants of the second order, and a, b, c are the elastic constants of the third order. Equations (2.73₁) and (2.78) yield the following tensor constitutive equations:

$$\boldsymbol{\sigma} = \mathbf{L}\boldsymbol{\varepsilon} + \mathbf{L}^{(3)}\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}, \quad (2.79)$$

$$L_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad (2.80)$$

$$L_{ijklmn} = a\delta_{ij}\delta_{mn}\delta_{kl} + b(\delta_{ij}I_{mnkl} + \delta_{mn}I_{ijkl} + \delta_{kl}I_{mnij}) + cJ_{ijmnkl}, \quad (2.81)$$

$$J_{ijmnkl} = \frac{1}{2}(I_{ipkl}I_{pqmn} + I_{ipmn}I_{pjkl}). \quad (2.82)$$

The Kauderer potential depending on two basic algebraic invariants (2.42)–(2.43) and three constant (λ, μ, γ) is known:

$$w = \frac{1}{2}\lambda A_1^2 + \mu A_2 + \frac{1}{3}\gamma(A_2 - \frac{1}{3}A_1)^2. \quad (2.83)$$

The strain energy (2.83) is associated with constitutive equations

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} + 2\gamma \varepsilon_{eq} (\varepsilon_{ij} - \frac{1}{3} \delta_{ij} \varepsilon_{kk}), \quad (2.84)$$

$$\varepsilon_{ij} = \frac{1}{9k} \sigma_{kk} \delta_{ij} + \frac{1}{2\mu} (1 - \frac{\gamma}{9\mu^3} \tau^2) s_{ij}, \quad (2.85)$$

where $k \equiv \lambda + 2\mu/3$ is the bulk modulus, and τ and ε_{eq} are the stress and strain intensities

$$\tau = \left(\frac{3}{2} \mathbf{s} : \mathbf{s} \right)^{1/2}, \quad \varepsilon_{eq} = \left(\frac{2}{3} \mathbf{e} : \mathbf{e} \right)^{1/2}, \quad (2.86)$$

respectively, where $\mathbf{s} = \mathbf{N}_2 \boldsymbol{\sigma}$ and $\mathbf{e} = \mathbf{N}_2 \boldsymbol{\varepsilon}$. The constitutive equation (2.84) can be presented in more compact form:

$$\boldsymbol{\sigma} = \mathbf{L} \boldsymbol{\varepsilon} + \boldsymbol{\Psi}(\boldsymbol{\varepsilon}), \quad \mathbf{L} = 3k \mathbf{N}_1 + 2\mu \mathbf{N}_2, \quad \boldsymbol{\Psi} = 2\gamma \varepsilon_{eq}^2 \mathbf{N}_2, \quad (2.87)$$

where the tensor $\boldsymbol{\Psi}$ depending on the effective strains ε_{eq} describes the nonlinear material properties.

Also used is an asymptotic expansion of the specific strain energy in a Taylor series about the state of zero strain and stress as

$$w = \frac{1}{2} L_{ijkl}^{(2)} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{6} L_{ijklmn}^{(3)} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{mn} + \dots \quad (2.88)$$

assuming $w(\mathbf{0}) = 0$. One can get from Eq. (2.88) that

$$\mathbf{L}^{(n)} = \frac{\partial^{(n)} w}{\partial \boldsymbol{\varepsilon}^{(n)}} \quad (2.89)$$

where $n = 2, 3, \dots$ and $\boldsymbol{\varepsilon}^{(n)} = \boldsymbol{\varepsilon} \otimes \dots \otimes \boldsymbol{\varepsilon}$ is n -multiple tensor production of the tensor $\boldsymbol{\varepsilon}$. The equality (2.89) yields the following symmetry properties of the elasticity tensors $\mathbf{L}^{(n)}$ of the $2n$ order with respect to the index pairs

$$L_{ijkl}^{(2)} = L_{jikl}^{(2)} = L_{ijlk}^{(2)} = L_{lkij}^{(2)}, \quad (2.90)$$

$$L_{ijklpq}^{(3)} = L_{jiklpq}^{(3)} = L_{ijlkpq}^{(3)} = L_{ijklqp}^{(3)} = L_{klijpq}^{(3)} = L_{pqklij}^{(3)} = L_{ijpqkl}^{(3)}, \quad (2.91)$$

and so forth. In so doing, the “pair symmetry” described by two first equalities of Eq. (2.90) and corresponding to an interchange of the indices of the first pair and interchange of the indices of the last pair are followed from the symmetry of both the $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ tensors. The “diagonal symmetry” corresponding to an interchange of the first pair of indices with the second one is described by the third equality of Eq. (2.90) and defined by Eq. (2.88). Fourth-order tensors with pair and diagonal-symmetry are referred as “full-symmetric” tensors.

2.3.2 The Equations of Linear Elasticity

We will reproduce the reduction of the elasticity theory of finite deformation to a linear theory, which is a special case of small deformations superposed on a finite deformations with the special values $\mathbf{b}_0 = \mathbf{0}$, $\mathbf{F}_0 = \mathbf{I}$, $\mathbf{T}_0^0 = \mathbf{0}$. Then all stress tensors coincide and can be recognized as a Cauchy stress. All strain tensors likewise reduce to the infinitesimal strain tensor (2.35). Then the equalities

$$\boldsymbol{\sigma} = \mathbf{T} = \mathbf{T}^0, \quad \boldsymbol{\varepsilon} = \tilde{\boldsymbol{\varepsilon}}, \quad J = 1 + \varepsilon_{ii} \quad (2.92)$$

simplify the basic equations of elasticity theory

$$\nabla \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}}, \quad \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (2.93)$$

$$\boldsymbol{\sigma} = \mathbf{L} \boldsymbol{\varepsilon}, \quad \sigma_{ij} = L_{ijkl} \varepsilon_{kl}, \quad (2.94)$$

$$\boldsymbol{\varepsilon} = \text{Def} \mathbf{u}, \quad \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (2.95)$$

The mixed boundary conditions on $\partial \mathcal{E}$ with the unit outward normal $\mathbf{n}^{\partial \mathcal{E}}$ will be considered

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^{\partial \mathcal{E}}(\mathbf{x}), \quad \mathbf{x} \in \partial \mathcal{E}_{\mathbf{u}}, \quad (2.96)$$

$$\boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n}^{\partial \mathcal{E}}(\mathbf{x}) = \mathbf{t}^{\partial \mathcal{E}}(\mathbf{x}), \quad \mathbf{x} \in \partial \mathcal{E}_{\boldsymbol{\sigma}}, \quad (2.97)$$

where $\partial \mathcal{E}_{\mathbf{u}}$ and $\partial \mathcal{E}_{\boldsymbol{\sigma}}$ are prescribed displacement and traction non-intersected boundary conditions such that $\partial \mathcal{E}_{\mathbf{u}} \cup \partial \mathcal{E}_{\boldsymbol{\sigma}} = \partial \mathcal{E}$; $\mathbf{u}^{\partial \mathcal{E}}(\mathbf{x})$ and $\mathbf{t}^{\partial \mathcal{E}}(\mathbf{x})$ are prescribed the displacement on $\partial \mathcal{E}_{\mathbf{u}}$ and traction on $\partial \mathcal{E}_{\boldsymbol{\sigma}}$, respectively; mixed boundary conditions, such as in the case of elastic supports are also possible. When $\partial \mathcal{E}_{\boldsymbol{\sigma}}$ is empty, the mixed boundary conditions (2.96) and (2.97) reduce to the *displacement problem* or the *first* boundary value problem. If $\partial \mathcal{E}_{\mathbf{u}}$ is empty, the boundary conditions (2.96), (2.97) becomes the *traction problem* or the *second* boundary value problem. As usual we shall distinguish the interior from the exterior problem according to whether the body occupies the interior or the exterior domain with respect to \mathcal{E} .

Of special practical interest are the homogeneous boundary conditions:

$$\mathbf{u}^{\partial \mathcal{E}}(\mathbf{x}) = \boldsymbol{\varepsilon}^{\partial \mathcal{E}} \mathbf{x}, \quad \boldsymbol{\varepsilon}^{\partial \mathcal{E}}(\mathbf{x}) \equiv \text{const.}, \quad \mathbf{x} \in \partial \mathcal{E}, \quad (2.98)$$

$$\boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n}^{\partial \mathcal{E}}(\mathbf{x}) = \boldsymbol{\sigma}^{\partial \mathcal{E}}(\mathbf{x}) \cdot \mathbf{n}^{\partial \mathcal{E}}(\mathbf{x}), \quad \boldsymbol{\sigma}^{\partial \mathcal{E}}(\mathbf{x}) = \text{const.}, \quad \mathbf{x} \in \partial \mathcal{E}, \quad (2.99)$$

where $\boldsymbol{\varepsilon}^{\partial \mathcal{E}}(\mathbf{x}) = \frac{1}{2} [\nabla \otimes \mathbf{u}^{\partial \mathcal{E}}(\mathbf{x}) + (\nabla \otimes \mathbf{u}^{\partial \mathcal{E}}(\mathbf{x}))^\top]$, $\mathbf{x} \in \partial \mathcal{E}_{\mathbf{u}}$.

2.3.3 Extremum Principles of Elastostatic

The principle of minimum potential energy and the principle of minimum complementary energy completely characterize the solution of the mixed boundary problem (2.93)–(2.97) of elastostatic ($\ddot{\mathbf{u}} \equiv \mathbf{0}$ in Eq. (2.93)). For definition of these energy functions we introduce the notion of admissible fields. A sufficiently smooth (of class C^1) displacement field \mathbf{u} with the displacement boundary conditions:

$$\mathcal{A}^k(\mathbf{u}) = \{\mathbf{u} | \mathbf{u}(\mathbf{x}) \in C^1(\mathcal{E}), \quad \mathbf{u}(\mathbf{y}) = \mathbf{u}^{\partial \mathcal{E}}(\mathbf{y}), \quad \mathbf{y} \in \partial \mathcal{E}_{\mathbf{u}}\} \quad (2.100)$$

is called a *kinematically admissible displacement field*. A kinematically admissible state is an ordered array $s = \{\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}\}$ (forming the set $\mathcal{A}^k(s)$) of kinematically admissible displacement field $\mathbf{u} \in \mathcal{A}^k(\mathbf{u})$ (2.100) generating the symmetric fields

ε and σ according to Eqs. (2.95) and (2.94), respectively. While $\varepsilon(\mathbf{u})$ is compatible, $\sigma(\mathbf{u})$ may not necessarily satisfy the equilibrium equations (2.93).

A sufficiently smooth (of class C^1) stress field satisfying the equilibrium equation (2.93) and the traction boundary condition (2.97):

$$\mathcal{A}^s(\sigma) = \{\sigma | \sigma(\mathbf{x}) \in C^1(\mathcal{E}), \nabla \cdot \sigma + \rho \mathbf{b} = \mathbf{0}, \sigma(\mathbf{y}) \cdot \mathbf{n}^{\partial \mathcal{E}}(\mathbf{y}) = \mathbf{t}^{\partial \mathcal{E}}(\mathbf{y}), \mathbf{y} \in \partial \mathcal{E}_\sigma\} \quad (2.101)$$

is referred to as a *statically admissible stress field*. The ordered array $s^s = \{\mathbf{u}, \varepsilon, \sigma\}$ (forming the set $\mathcal{A}^s(s)$) is called a statically admissible state if $\sigma \in \mathcal{A}^s(\sigma)$ (2.101) generates the symmetric tensor ε according to the constitutive equation (2.94) and the displacement field \mathbf{u} is related to ε by the use of Eq. (2.95). In so doing, $\varepsilon(\sigma)$ is not necessarily compatible.

We also define the total strain energy $W(\varepsilon)$ and the total stress energy $W^c(\sigma)$ by

$$W(\varepsilon) = \frac{1}{2} \int_{\mathcal{E}} \varepsilon : \mathbf{L} : \varepsilon \, d\mathbf{x}, \quad W^c(\sigma) = \frac{1}{2} \int_{\mathcal{E}} \sigma : \mathbf{M} : \sigma \, d\mathbf{x}, \quad (2.102)$$

respectively. Let $\delta \mathbf{u} = \delta \mathbf{u}(\mathbf{x})$ be a virtual, or imaginative, infinitesimal displacement field associated with a virtual strain field from the current state $\delta \varepsilon = \text{Def}(\delta \mathbf{u})$. The components of the virtual displacement vector $\delta \mathbf{u}$ vanish at $\partial \mathcal{E}_{\mathbf{u}}$: $\delta \mathbf{u}(\mathbf{y}) \equiv \mathbf{0}$ at $\mathbf{y} \in \partial \mathcal{E}_{\mathbf{u}}$. It can be shown from the equations of linear and angular momentum, that it holds the *principle of virtual work* (PVW):

$$\int_{\mathcal{E}} \sigma : \delta \varepsilon \, d\mathbf{x} = \int_{\partial \mathcal{E}} \mathbf{t}^{\partial \mathcal{E}_\sigma} \cdot \delta \mathbf{u} \, ds + \int_{\mathcal{E}} \rho \mathbf{b} \cdot \delta \mathbf{u} \, d\mathbf{x}, \quad (2.103)$$

which is a starting point in developing minimum principles.

As a consequence, one can obtain a work–energy relation (theorem of work expended) involving the admissible stress field σ and the admissible displacement field \mathbf{u} [$\varepsilon(\mathbf{u})$ is the related strain field] corresponding to the given external forces $\rho \mathbf{b}(\mathbf{x})$ and $\mathbf{t}^{\partial \mathcal{E}_\sigma}(\mathbf{y})$ ($\mathbf{x} \in \mathcal{E}$, $\mathbf{y} \in \partial \mathcal{E}_\sigma$):

$$\int_{\mathcal{E}} \sigma : \varepsilon \, d\mathbf{x} = \int_{\partial \mathcal{E}} \mathbf{t}^{\partial \mathcal{E}_\sigma} \cdot \mathbf{u} \, ds + \int_{\mathcal{E}} \rho \mathbf{b} \cdot \mathbf{u} \, d\mathbf{x}, \quad (2.104)$$

which looks similar to the first law of thermodynamics (2.61), although there is no thermodynamic expression in (2.104). We note that σ and ε need not be connected by a specific stress-strain relation.

We will present the classical extremum principles.

Principle of minimum of potential energy. Let Π , called the *potential energy* (corresponding to the given external forces $\rho \mathbf{b}(\mathbf{x})$ and $\mathbf{t}^{\partial \mathcal{E}_\sigma}(\mathbf{y})$, $\mathbf{x} \in \mathcal{E}$, $\mathbf{y} \in \partial \mathcal{E}_\sigma$), be the function defined on the set of kinematically admissible states $\tilde{s}^k = \{\tilde{\mathbf{u}}, \tilde{\varepsilon}, \tilde{\sigma}\} \in \mathcal{A}^k(\tilde{s})$ by the equation

$$\Pi(\tilde{s}^k) = W(\tilde{\varepsilon}) - \int_{\mathcal{E}} \rho \mathbf{b} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} - \int_{\partial \mathcal{E}_\sigma} \mathbf{t}^{\partial \mathcal{E}_\sigma} \cdot \tilde{\mathbf{u}} \, ds. \quad (2.105)$$

Then the actual displacement field \mathbf{u} with corresponding admissible state s^k renders the potential energy Π an absolute minimum

$$\Pi(s^k) \leq \Pi(\tilde{s}^k). \quad (2.106)$$

Principle of minimum of complementary energy. Let Π^c called the *complementary energy* (corresponding to the given boundary displacement $\mathbf{u}^{\partial\mathcal{E}_u}(\mathbf{y}) \in \partial\mathcal{E}_u$) be the function defined on the set of statically admissible states $\tilde{s}^s = \{\tilde{\mathbf{u}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\sigma}}\} \in \mathcal{A}^s(\tilde{s})$ by the equation

$$\Pi^c(\tilde{s}^k) = W^c(\tilde{\boldsymbol{\sigma}}) - \int_{\partial\mathcal{E}_u} \tilde{\mathbf{t}}_{\mathbf{n}} \cdot \mathbf{u}^{\partial\mathcal{E}_u} d\mathbf{s}, \quad \tilde{\mathbf{t}}_{\mathbf{n}} \equiv \tilde{\boldsymbol{\sigma}} \cdot \mathbf{n}^{\partial\mathcal{E}}. \quad (2.107)$$

Then the actual stress field $\boldsymbol{\sigma}$ with corresponding admissible state s^s renders the complementary energy Π^c an absolute minimum

$$\Pi^c(s^s) \leq \Pi^c(\tilde{s}^s). \quad (2.108)$$

2.4 Basic Equations of Thermoelasticity and Electroelasticity

2.4.1 Thermoelasticity Equations

In classical thermodynamics we are concerned with the small neighborhood of thermodynamic equilibrium. We also consider the infinitesimal deformations (2.92). In such a case the first law of thermodynamics (2.65) can be recast in the form (hereafter a unit volume is considered)

$$dE = dQ + \sigma_{ij} d\varepsilon_{ij}, \quad (2.109)$$

To derive the constitutive relations of thermoelasticity theory, we transform (2.109) for the reversible processes (2.66) to the thermodynamic relation:

$$dE = \sigma_{ij} d\varepsilon_{ij} + T dS \quad (2.110)$$

equating the total differential of the internal energy (2.62₂) and the sum of the increment of the deformation work and the amount of the heat introduced into the considered volume; here T is the absolute temperature and S is the entropy density (2.68). We see that E is a state function of $\boldsymbol{\varepsilon}$ and S for the reversible adiabatic (isentropic, $S = \text{const.}$) processes, and according to the ordinary rules of differentiation, we get

$$\sigma_{ij} = \left. \frac{\partial E}{\partial \varepsilon_{ij}} \right|_S, \quad T = \left. \frac{\partial E}{\partial S} \right|_{\boldsymbol{\varepsilon}}. \quad (2.111)$$

On the other hand, if the process is isothermal ($T = \text{const.}$), the dependence of the stresses $\boldsymbol{\sigma}$ on the strains $\boldsymbol{\varepsilon}$ is found by introducing the *Helmholtz free-energy function*:

$$F = E - TS, \quad dF = \sigma_{ij} d\varepsilon_{ij} - S dT \quad (2.112)$$

with the strains and temperature as its independent variables. The value $-TS$ is the irreversible heat energy due to entropy as related to temperature, with

the negative sign indicating that the compressive reaction results from thermal expansion in a restrained body. Then, from Eqs. (2.110) and (2.112), we get

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}} \Big|_T, \quad S = - \frac{\partial F}{\partial T} \Big|_\varepsilon. \quad (2.113)$$

Thus, the strain energy density w can be identified with the internal energy E in an isentropic process ($S \equiv \text{const.}$) and the free energy F in an isothermal process ($T \equiv \text{const.}$). The temperature is determined by Eq. (4.3₂) when using the potential E and the entropy is determined by Eq. (2.113) when using the potential F .

For small deformations and small temperature changes $\theta = T - T^0$ ($\theta/T_0 \ll 1$), $F(\varepsilon, T)$ can be expanded in a power series of its arguments in the neighborhood of the virgin state ($\varepsilon = \mathbf{0}$, $\theta = 0$):

$$F(\varepsilon, T) = \frac{1}{2} L_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \alpha_{ij}^T \varepsilon_{ij} \theta - \frac{C_\varepsilon}{2T_0} \theta^2, \quad (2.114)$$

where one introduces the notations

$$L_{ijkl} = \frac{\partial^2 F(\mathbf{0}, T_0)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}, \quad \alpha_{ij}^T = \frac{\partial^2 F(\mathbf{0}, T_0)}{\partial \varepsilon_{ij} \partial T}, \quad \frac{C_\varepsilon}{T_0} = - \frac{\partial^2 F(\mathbf{0}, T_0)}{\partial T^2} \quad (2.115)$$

and C_ε denotes the specific heat at constant strain, which can be also determined as

$$C_\varepsilon = \frac{dQ}{dT} \Big|_\varepsilon = \frac{\partial E}{\partial T} \Big|_\varepsilon = T \frac{\partial S}{\partial T} \Big|_\varepsilon. \quad (2.116)$$

The negative sign for the last term on the right-hand side of Eq. (2.114) indicates that a temperature rise leads to a compressive reaction on the restricted body. This term signifies the thermal energy due to temperature while the second term $\alpha^T \varepsilon \theta$ describes the coupling effect between temperature and mechanical deformation. The difference between the isentropic moduli $\tilde{\mathbf{L}}$ and isothermic moduli \mathbf{L} :

$$\tilde{\mathbf{L}} - \mathbf{L} = \frac{T_0}{\rho C_\varepsilon} \boldsymbol{\alpha}^T \otimes \boldsymbol{\alpha}^T, \quad \tilde{\mathbf{L}} = \frac{\partial^2 E}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} \Big|_S, \quad \mathbf{L} = \frac{\partial^2 F}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} \Big|_T$$

is of as order of 1% or less for metals and ceramics.

To determine the entropy as a function of ε and T , consider the total differential of the function $S(\varepsilon, T)$:

$$dS = \frac{\partial S}{\partial \boldsymbol{\varepsilon}} \Big|_T : d\boldsymbol{\varepsilon} + \frac{\partial S}{\partial T} \Big|_\varepsilon dT. \quad (2.117)$$

Taking Eqs. (2.113) and (2.116) into account and that

$$\frac{\partial \boldsymbol{\sigma}}{\partial T} \Big|_\varepsilon = \frac{\partial^2 F}{\partial T \partial \boldsymbol{\varepsilon}} \Big|_\varepsilon = \boldsymbol{\alpha}^T, \quad (2.118)$$

equation (2.117) becomes

$$dS = -\boldsymbol{\alpha}^T : d\boldsymbol{\varepsilon} + \frac{C_\varepsilon}{T} dT, \quad (2.119)$$

which after integration under the virgin state conditions $\boldsymbol{\varepsilon} = \mathbf{0}$ and $T = T_0$ gives, for small temperature change ($\theta/T_0 \ll 1$),

$$S = -\boldsymbol{\alpha}^T : \boldsymbol{\varepsilon} + \frac{C_\varepsilon}{T_0} \theta. \quad (2.120)$$

The Gibbs thermodynamic potential per unit volume is defined for the case where the stress and temperature are chosen as the independent state variables

$$G_e = F - TS, \quad dG_e = -\varepsilon_{ij} d\sigma_{ij} - S dT. \quad (2.121)$$

Since dG_e is a total differential (as dE and dF), it follows that

$$\left. \frac{\partial G_e}{\partial T} \right|_\sigma = -S, \quad \left. \frac{\partial G_e}{\partial \sigma_{ij}} \right|_T = -\varepsilon_{ij}. \quad (2.122)$$

Thus, $-G_e$ can be identified with the complementary energy function W^c , which has the property that $\partial W^c / \partial \sigma_{ij} = \varepsilon_{ij}$. In the neighborhood of the virgin state ($\theta/T_0 \ll 1$, $\mathbf{M} :: (\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}) \ll 1$; \mathbf{M} is the compliance tensor) we have a power series expansion

$$G_e(\boldsymbol{\sigma}, T) = -\frac{1}{2} M_{ijkl} \sigma_{ij} \sigma_{kl} - \beta_{ij}^T \sigma_{ij} \theta + \frac{1}{2} C_\sigma \frac{\theta^2}{T_0} \quad (2.123)$$

depending on the coefficient of thermal expansion $\boldsymbol{\beta}^T$ and the specific heat at constant stress C_σ

$$\boldsymbol{\beta}^T \equiv \left. \frac{\partial \boldsymbol{\varepsilon}}{\partial T} \right|_\sigma = -\mathbf{M} \boldsymbol{\alpha}^T, \quad C_\sigma = \left. \frac{dQ}{dT} \right|_\sigma, \quad (2.124)$$

where C_σ is related to C_ε by the equation

$$C_\sigma = C_\varepsilon + L_{ijkl} \beta_{ij}^T \beta_{kl}^T T_0. \quad (2.125)$$

Analogously to the tensor \mathbf{L} (2.90), the compliance tensor \mathbf{M} is also full-symmetric.

Substitution of the representation (2.114) into Eq. (2.113₁) yields the general *Duhamel-Newmann form of Hook's law* for an anisotropic body in the form of invertible relations:

$$\boldsymbol{\sigma} = \mathbf{L} \boldsymbol{\varepsilon} + \boldsymbol{\alpha}, \quad \boldsymbol{\varepsilon} = \mathbf{M} \boldsymbol{\sigma} + \boldsymbol{\beta} \quad (2.126)$$

where $\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}^T \theta$, $\boldsymbol{\beta}^\top = \boldsymbol{\beta} / \theta$ is the coefficient of thermal expansion related to $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha} = -\mathbf{L} \boldsymbol{\beta}$, $\alpha_{ij} = -L_{ijkl} \beta_{kl}$; see Eq. (2.124). $\boldsymbol{\beta}(\mathbf{x})$ and $\boldsymbol{\alpha}(\mathbf{x}) \equiv -\mathbf{L}(\mathbf{x}) \boldsymbol{\beta}(\mathbf{x})$ are second-order tensors of local eigenstrains and eigenstresses, respectively (frequently called transformation fields) which may arise by thermal expansion, phase transformation, twinning, and other changes of shape or volume of the material. Substitution of the constitutive equation (2.126₁) into the motion equation (2.60) at small deformations (2.92) leads to the Navier equation:

$$\nabla(\mathbf{L}\nabla\mathbf{u}) + \rho\mathbf{b} = \rho\ddot{\mathbf{u}} - \boldsymbol{\alpha}^T \nabla\theta, \quad \frac{\partial}{\partial x_j} \left(L_{ijkl} \frac{\partial u_k}{\partial x_l} \right) + \rho b_i = \rho \frac{\partial^2 u_j}{\partial t^2} - \alpha_{ij}^T \frac{\partial \theta}{\partial x_j}, \quad (2.127)$$

which should be considered simultaneously with the generalized heat condition equation:

$$\nabla(\boldsymbol{\lambda}\nabla\theta) - C_\varepsilon \dot{\boldsymbol{\varepsilon}} + T_0 \boldsymbol{\alpha}^T : \dot{\boldsymbol{\varepsilon}} = -h, \quad \frac{\partial}{\partial x_i} \left(\lambda_{ij} \frac{\partial \theta}{\partial x_j} \right) - C_\varepsilon \frac{\partial \theta}{\partial t} + T_0 \alpha_{ij}^T \frac{\partial \varepsilon_{ij}}{\partial t} = -h, \quad (2.128)$$

where h is the heat source function. The coefficient $\boldsymbol{\alpha}^T$ couples the equations of the dynamic thermoelastic problem (2.127) and (2.128). Dropping the term $T_0 \boldsymbol{\alpha}^T : \dot{\boldsymbol{\varepsilon}}$ in the left-hand side of Eq. (2.128), we obtain the uncoupled equations of dynamic thermoelasticity. Further neglecting the inertial term $\rho\ddot{\mathbf{u}}$ in Eq. (2.127) leads to a system of equations of the quasistatic thermoelastic problem.

The system (2.127) and (2.128) should be accompanied by appropriate boundary and initial conditions. The initial conditions specify the temperature, displacements, and velocities at all points $\mathbf{x} \in \mathcal{E}$ within \mathcal{E} : $\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x})$, $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$, $\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$ with the prescribed functions $\theta_0, \mathbf{u}_0, \mathbf{v}_0$ on \mathcal{E} . The most widely used boundary condition for the temperature function θ is expressed by

$$\boldsymbol{\lambda} : (\nabla\theta(\mathbf{x}, t) \otimes \mathbf{n}) + \alpha_{\partial\mathcal{E}}[\theta(\mathbf{x}, t) - \theta_{\partial\mathcal{E}}(\mathbf{x}, t)] = q_0(\mathbf{x}, t) \quad (2.129)$$

with $\theta_{\partial\mathcal{E}}, q_0$ as known functions of the time and point $\mathbf{x} \in \partial\mathcal{E}$. If $q_0 \equiv 0$, Eq. (2.129) describes heat exchange of the third kind called Newton's law. The conditions $\alpha_{\partial\mathcal{E}} = \infty$ and $q_0 \equiv 0$ reduce Eq. (2.129) to the boundary conditions of the first kind. We obtain the second kind of boundary conditions if $\alpha_{\partial\mathcal{E}} \equiv 0$. The mechanical boundary conditions (2.96) and (2.97) should be added to the thermal boundary conditions (2.129). In so doing, the traction $\mathbf{t}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})\mathbf{n}(\mathbf{x})$ acting on any surface with the normal $\mathbf{n}(\mathbf{x})$ through the point \mathbf{x} can be represented in terms of displacements:

$$\mathbf{t}(\mathbf{x}) = \hat{\mathbf{t}}(\mathbf{n}, \nabla)\mathbf{u}(\mathbf{x}) + \boldsymbol{\alpha}\mathbf{n}, \quad \hat{t}_{ik}(\mathbf{n}, \nabla) \equiv L_{ijkl} n_j(\mathbf{x}) \partial/\partial x_l, \quad (2.130)$$

where $\hat{t}_{ik}(\mathbf{n}, \nabla)$ is the so-called "stress operator."

We will consider the uncoupled quasistatic thermoelasticity theory [$\rho\ddot{\mathbf{u}} \equiv \mathbf{0}$ in Eq. (2.127) and $T_0 \boldsymbol{\alpha}^T : \dot{\boldsymbol{\varepsilon}} \equiv \mathbf{0}$ in Eq. (2.128)], where the temperature field is determined by Eq. (2.128) with no influence of the latent heat due to the change of strain. It takes place according to so-called *body force analogy* [197] asserting that $\{\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}\}$ is a solution of the mixed problem of thermoelastostatics corresponding to the external loading conditions $(\mathbf{b}, \mathbf{t}^{\partial\mathcal{E}_\sigma}, \mathbf{u}^{\partial\mathcal{E}_u}, \theta)$ on $\mathcal{E} \cup \partial\mathcal{E}_\sigma \cup \partial\mathcal{E}_u$ if and only if $\{\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma} - \boldsymbol{\alpha}\}$ is a solution of the mixed problem of elastostatic corresponding to the external loading $(\mathbf{b} + 1/\rho \nabla \boldsymbol{\alpha}, \mathbf{t}^{\partial\mathcal{E}_\sigma} - \boldsymbol{\alpha} \cdot \mathbf{n}, \mathbf{u}^{\partial\mathcal{E}_u})$ on $\mathcal{E} \cup \partial\mathcal{E}_\sigma \cup \partial\mathcal{E}_u$.

To present the extremum principles of linear thermoelastostatic, determine the total strain energy $W(\boldsymbol{\varepsilon})$ and total stress energy $W^c(\boldsymbol{\sigma})$:

$$\begin{aligned} W(\boldsymbol{\varepsilon}) &= \int_{\mathcal{E}} w(\boldsymbol{\varepsilon}) \, d\mathbf{x}, \quad w(\boldsymbol{\varepsilon}) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\beta}) : \mathbf{L} : (\boldsymbol{\varepsilon} - \boldsymbol{\beta}), \\ W^c(\boldsymbol{\sigma}) &= \int_{\mathcal{E}} w^c(\boldsymbol{\sigma}) \, d\mathbf{x}, \quad w^c(\boldsymbol{\sigma}) = \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\alpha}) : \mathbf{M} : (\boldsymbol{\sigma} - \boldsymbol{\alpha}) \end{aligned} \quad (2.131)$$

yielding Eqs. (2.102) at $\boldsymbol{\alpha} = \boldsymbol{\beta} \equiv \mathbf{0}$. $w(\boldsymbol{\varepsilon})$ and $w^c(\boldsymbol{\sigma})$ are potential functions for determining the stress and strain according to the equations

$$\sigma_{ij} = \frac{\partial w(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}}, \quad \varepsilon_{ij} = \frac{\partial w^c(\boldsymbol{\sigma})}{\partial \sigma_{ij}}, \quad (2.132)$$

respectively, which are valid for both elasticity and thermoelasticity; in linear theory $w(\boldsymbol{\varepsilon})$ and $w^c(\boldsymbol{\sigma})$ are equal: $w = w^c$. Positiveness of w and w^c (at $\boldsymbol{\varepsilon}, \boldsymbol{\sigma} \neq \mathbf{0}$) leads to the positive definiteness of \mathbf{L} and \mathbf{M} , respectively (i.e. $\boldsymbol{\varepsilon} : \mathbf{L} : \boldsymbol{\varepsilon} > 0$ for $\forall \boldsymbol{\varepsilon} \neq \mathbf{0}$).

The classical extremum principles (2.106) and (2.108) for pure mechanical loading can be generalized for the thermostatic case in the following manner. We determine the potential Π and complementary Π^c energies:

$$\begin{aligned} \Pi(\tilde{s}^k) &= W(\tilde{\boldsymbol{\varepsilon}}) - \int_{\mathcal{E}} \rho \mathbf{b} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} - \int_{\partial \mathcal{E}_\sigma} \mathbf{t}^{\partial \mathcal{E}_\sigma} \cdot \tilde{\mathbf{u}} \, d\mathbf{s}, \\ \Pi^c(\tilde{s}^s) &= W^c(\tilde{\boldsymbol{\sigma}}) - \int_{\partial \mathcal{E}_u} \tilde{\mathbf{t}}_{\mathbf{n}} \cdot \mathbf{u}^{\partial \mathcal{E}_u} \, d\mathbf{s}, \quad \tilde{\mathbf{t}}_{\mathbf{n}} \equiv \tilde{\boldsymbol{\sigma}} \cdot \mathbf{n}^{\partial \mathcal{E}}. \end{aligned} \quad (2.133)$$

defined on the sets of kinematically $\tilde{s}^k = \{\tilde{\mathbf{u}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\sigma}}\} \in \mathcal{A}^k(\tilde{s})$ (2.100) and statically $\tilde{s}^s = \{\tilde{\mathbf{u}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\sigma}}\} \in \mathcal{A}^s(\tilde{s})$ (2.101) admissible states, respectively. In so doing one assumes that Eqs. (2.94) is replaced by Eq. (2.126₁) to include the $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\varepsilon}}$ generated by $\tilde{\mathbf{u}} \in \mathcal{A}^k(\tilde{\mathbf{u}})$ and by $\tilde{\boldsymbol{\sigma}} \in \mathcal{A}^s(\tilde{\boldsymbol{\sigma}})$, respectively. Then the actual state $s = \{\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}\}$ renders the potential energy Π and complementary energy Π^c an absolute minimum

$$\Pi(s) \leq \Pi(\tilde{s}^k), \quad \Pi^c(s) \leq \Pi^c(\tilde{s}^s). \quad (2.134)$$

2.4.2 Electroelastic Equations

Another example of the theory of coupled fields is associated with electroelasticity and deals with phenomena caused by interactions between the elastic, electric, and thermal fields. The relations between the mechanical, electrical, and thermal properties can be demonstrated from equilibrium thermodynamics described by the potential *Gibbs function*:

$$G_e = E - \mathbf{E} \cdot \mathbf{D} - ST, \quad (2.135)$$

whose differential form

$$dG_e = \sigma_{ij} d\varepsilon_{ij} - D_i dE_i - S dT \quad (2.136)$$

yields the relations

$$\sigma_{ij} = \left. \frac{\partial G_e}{\partial \varepsilon_{ij}} \right|_{\mathbf{E}, T}, \quad D_m = - \left. \frac{\partial G_e}{\partial E_m} \right|_{\varepsilon, T}, \quad S = - \left. \frac{\partial G_e}{\partial T} \right|_{\varepsilon, \mathbf{E}}. \quad (2.137)$$

Differentiating Eqs. (2.137) gives

$$\left. \frac{\partial \sigma_{ij}}{\partial E_m} \right|_{\varepsilon, T} = - \left. \frac{\partial D_m}{\partial \varepsilon} \right|_{\mathbf{E}, T}, \quad \left. \frac{\partial \sigma_{ij}}{\partial T} \right|_{\varepsilon, \mathbf{E}} = - \left. \frac{\partial S}{\partial \varepsilon_{ij}} \right|_{\mathbf{E}, T}, \quad \left. \frac{\partial D_m}{\partial T} \right|_{\varepsilon, \mathbf{E}} = \left. \frac{\partial S}{\partial E_m} \right|_{\varepsilon, T}. \quad (2.138)$$

It was assumed that the strains ε , the electric field intensity \mathbf{E} , and the temperature $T = T_0 + \theta$ are considered the independent variables, and the dependent variables will be the stresses $\boldsymbol{\sigma}$, the vector of induction \mathbf{D} , and the entropy S . G_e is a state function of ε , \mathbf{E} , and T , and the ordinary rules of differentiation yield the explicite representation of $d\boldsymbol{\sigma}$, $d\mathbf{D}$, and dS as the functions of $d\varepsilon$, $d\mathbf{E}$, and dT :

$$\begin{aligned} d\sigma_{ij} &= \left. \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \right|_{\mathbf{E}, T} d\varepsilon_{kl} + \left. \frac{\partial \sigma_{ij}}{\partial E_m} \right|_{\varepsilon, T} dE_m + \left. \frac{\partial \sigma_{ij}}{\partial T} \right|_{\mathbf{E}, \varepsilon} dT, \\ dD_m &= \left. \frac{\partial D_m}{\partial \varepsilon_{kl}} \right|_{\mathbf{E}, T} d\varepsilon_{kl} + \left. \frac{\partial D_m}{\partial E_k} \right|_{\varepsilon, T} dE_k + \left. \frac{\partial D_m}{\partial T} \right|_{\varepsilon, \mathbf{E}} dT, \\ dS &= \left. \frac{\partial S}{\partial \varepsilon_{kl}} \right|_{\mathbf{E}, T} d\varepsilon_{kl} + \left. \frac{\partial S}{\partial E_m} \right|_{\varepsilon, T} dE_m + \left. \frac{\partial S}{\partial T} \right|_{\varepsilon, \mathbf{E}} dT, \end{aligned} \quad (2.139)$$

For small variations of ε , \mathbf{E} , and T in the neighborhood of virgin state ($\varepsilon, \mathbf{E}, \theta = \mathbf{0}$), the partial derivatives can be considered as constants and Eqs. (2.139) becomes integrable. This allows us to obtain the constitutive relations of a deformable piezoelectric medium (see, e.g., [747], [861]):

$$\begin{aligned} \sigma_{ij} &= L_{ijkl}\varepsilon_{kl} + e_{kij}E_k + \alpha_{ij}^T\theta, \\ D_i &= e_{ijk}\varepsilon_{jk} - k_{ij}E_j + p_i\theta, \quad dS = \alpha_{ij}\varepsilon_{ij} + p_iE_i + c_\varepsilon\theta/T_0, \end{aligned} \quad (2.140)$$

where the coefficients of the system (2.140) denote the particular derivatives:

$$\begin{aligned} L_{ijkl} &= \left. \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \right|_{\mathbf{E}, T}, \quad e_{ijk} = \left. \frac{\partial \sigma_{ij}}{\partial E_k} \right|_{\varepsilon, T}, \\ p_i &= \left. \frac{\partial D_i}{\partial T} \right|_{\varepsilon, \mathbf{E}}, \quad k_{ij} = \left. \frac{\partial D_i}{\partial E_j} \right|_{\varepsilon, T}, \\ \alpha_{ij}^T &= \left. \frac{\partial \sigma_{ij}}{\partial T} \right|_{\varepsilon, \mathbf{E}}, \quad \frac{c_\varepsilon}{T_0} = \left. \frac{\partial S}{\partial T} \right|_{\varepsilon, \mathbf{E}} \end{aligned} \quad (2.141)$$

The constitutive equations (2.140₁) and (2.140₂) can be inverted

$$\varepsilon_{ij} = M_{ijkl}\sigma_{kl} + d_{kij}D_k + \beta_{ij}^T\theta, \quad (2.142)$$

$$E_i = d_{ijk}\sigma_{jk} - b_{ij}D_j + q_i\theta. \quad (2.143)$$

The coefficients of Eqs. (2.140) and (2.142), (2.143) are denoted as follows: \mathbf{L} and \mathbf{M} are the elastic and compliance tensors, \mathbf{k} and \mathbf{b} are the tensors of dielectric permeability and impermeability, \mathbf{e} and \mathbf{d} are the piezoelectric moduli, $\boldsymbol{\alpha}^T$ and $\boldsymbol{\beta}^T$ are the coefficients of thermoelastic stress and expansion, \mathbf{p} and \mathbf{q} are the pyroelectric coefficients; and c_ε is a specific heat per unit volume at constant strain. To obtain a symmetric matrix of coefficients we replace the electric field \mathbf{E} by $-\mathbf{E}$. Hereinafter in Subsection 2.4.3, the Latin indexes range from 1 to 3,

and the Greek ones range from 1 to 4. We will assume that the electric and elastic fields are fully coupled, but temperature enters the problem only as a parameter through the constitutive equation. The tensor coefficients in the above equations are related as $e_{kij} = d_{kmn}L_{mnij}$, $d_{kij} = e_{kmn}M_{mnij}$, $M_{ijmn}L_{mnkl} = I_{ijkl}$. We will use the matrix notation instead of the tensor one, as it is accepted in the theory of elasticity. For notational convenience the elastic and electric variable will be treated on equal footing, and with this in mind we recast the local linear constitutive relations of thermoelectroelasticity for this material in the notation introduced in [38]. For this purpose, we introduce the matrices of generalized stresses Σ and strains \mathcal{E} ($i, j = 1, 2, 3$):

$$\Sigma = \begin{pmatrix} \sigma \\ \mathbf{D} \end{pmatrix}, \quad \Sigma_{ij} = \sigma_{ij}, \quad \Sigma_{i4} = D_i, \quad (2.144)$$

$$\mathcal{E} = \begin{pmatrix} \varepsilon \\ \mathbf{E} \end{pmatrix}, \quad \mathcal{E}_{ij} = \varepsilon_{ij}, \quad \mathcal{E}_{4i} = E_i, \quad (2.145)$$

which are interrelated through the matrices of generalized elastic coefficients $\mathbb{L}_{\alpha\beta\gamma\delta}$, compliance $\mathbb{M}_{\alpha\beta\gamma\delta}$, and generalized coefficients of thermal expansion $\Lambda_{\alpha\beta}$:

$$\Sigma = \mathbb{L}(\mathcal{E} - \Lambda), \quad \mathcal{E} = \mathbb{M}\Sigma + \Lambda, \quad (2.146)$$

where the generalized coefficients can be presented in matrix form:

$$\mathbb{L} = \begin{pmatrix} \mathbf{L} & \mathbf{e}^\top \\ \mathbf{e} & -\mathbf{k} \end{pmatrix}, \quad \mathbb{M} = \begin{pmatrix} \mathbf{M} & \mathbf{d}^\top \\ \mathbf{d} & -\mathbf{b} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \beta^\top \theta \\ \mathbf{q} \theta \end{pmatrix} \quad (2.147)$$

as well as in component form:

$$\mathbb{L}_{ijkl} = L_{ijkl}, \quad \mathbb{L}_{4ikl} = \mathbb{L}_{i4kl} = e_{ikl}, \quad (2.148)$$

$$\mathbb{L}_{ij4l} = \mathbb{L}_{ijl4} = e_{lij}, \quad \mathbb{L}_{44kl} = \mathbb{L}_{kl44} = -k_{kl}, \quad (2.149)$$

$$\mathbb{M}_{ijkl} = M_{ijkl}, \quad \mathbb{M}_{4ikl} = \mathbb{M}_{i4kl} = d_{ikl}, \quad (2.150)$$

$$\mathbb{M}_{ij4l} = \mathbb{M}_{ijl4} = d_{lij}, \quad \mathbb{M}_{44kl} = \mathbb{M}_{kl44} = -b_{kl}, \quad (2.151)$$

$$\Lambda_{ij} = \beta_{ij}\theta, \quad \Lambda_{4i} = \Lambda_{i4} = q_i\theta \quad (2.152)$$

The generalized static equation and the Cauchy conditions of small generalized deformations are expressed in the form:

$$\nabla_\alpha \Sigma_{\alpha\beta} = f_\beta, \quad \nabla_4 \equiv 0, \quad (2.153)$$

$$\mathcal{E}_{\alpha\beta} = (\nabla_\alpha U_\beta + \nabla_\beta U_\alpha)/2, \quad (2.154)$$

where $U = (u_1, u_2, u_3, -\phi)^\top$, ϕ is the electrostatic potential, and f_β is a density of generalized body forces. Equations (2.153) and (2.154) should be complemented by the boundary conditions of the first or the second type. Except for notations, these equations coincide with the equations of linear thermoelasticity (2.126). Because of this the theory of piezoelectric composite materials (PCM) retraces at a particular instant the path of development of the theory of microinhomogeneous elastic media, exhibiting substantial progress. In light of the analogy mentioned in our brief survey we will not consider in detail the PCM; one may refer instead to the appropriate scheme of thermoelastic composites.

2.4.3 Matrix Representation of Some Symmetric Tensors

Many material (e.g., elastic moduli (2.90)) and field (e.g., the stresses (2.50)) tensors have internal symmetry with respect to pairs of indices. Although all calculations with these tensors can be done by common methods of tensor analysis (see Section 2.2), it is sometimes convenient to introduce a contracted notation significantly simplifying the calculations. It is possible to form six different pairs ij for permuted indices i, j ($i, j = 1, 2, 3$). Then the pair of these Latin indices ij can be substituted by a single Greek index α ranging from 1 to 6 according to the Voight notation:

$$\begin{array}{l} \text{tensor notation } ij = 11 \quad 22 \quad 33 \quad 23 \quad 32 \quad 13 \quad 31 \quad 12 \quad 21 \\ \text{matrix notation } \alpha = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \end{array}, \quad (2.155)$$

which can be presented in a formula form (no sum on i, j):

$$\alpha = i\delta_{ij} + (9 - i - j)(1 - \delta_{ij}). \quad (2.156)$$

To avoid confusion, it should be mentioned that the prescribed reduction rule $ij \leftrightarrow \alpha$ between the Latin and Greek indices is also used for Latin indices $ij \leftrightarrow k$ ($i, j = 1, 2, 3, k = 1, \dots, 6$) where the correspondence between the pair indices and one index is provided by the scheme (2.155). We will also use this traditional form of an accordance.

We accept the convention that for the tensors with the prescribed aforementioned symmetry, Greek indices α, β, γ range from 1 to 6 and are connected with the pairs of Latin indices ij, kl and others by relation (2.155). In such a case, the tensor representations of the thermoelastic constitutive equations (2.126) can be presented in matrix notations:

$$\sigma_\gamma = L_{\gamma\delta} \varepsilon_\delta + \alpha_\gamma, \quad \varepsilon_\gamma = M_{\gamma\delta} \sigma_\delta + \beta_\gamma. \quad (2.157)$$

It is also required that the tensor representations of widely used scalar productions $\mathbf{L} : \mathbf{M}$ and $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ have an appropriate vector form $L_{\lambda\mu} M_{\mu\nu} = \delta_{\lambda\nu}$, $\sigma_{ij} \varepsilon_{ij} = \sigma_\lambda \varepsilon_\lambda$ with the standard assumption of summation over the repeated indices. However, it is conceivable to show that if the quantities corresponding to the different tensors are formed by one and the same rule (such as e.g. $\sigma_{ij} = \sigma_\lambda$), the equalities either (2.156) or (2.157) cannot be fulfilled. The mentioned inconsistency can be avoided if the following matrix representations of tensors are used ($i, j, k, l = 1, 2, 3; \lambda, \mu = 1, \dots, 6$)

$$\varepsilon_{ij} = \frac{\varepsilon_\lambda}{2 - \delta_{ij}}, \quad \sigma_{ij} = \sigma_\lambda, \quad (2.158)$$

$$L_{ijkl} = L_{\lambda\mu}, \quad M_{ijkl} = \frac{M_{\lambda\mu}}{(2 - \delta_{ij})(2 - \delta_{kl})}. \quad (2.159)$$

In so doing, the transition to Greek indices for all symmetric second-rank tensors could be done in the unique manner $\alpha_{ij} = (2 - \delta_{ij})^{-1/2} \alpha_\lambda$.

Since the components of the stress and strain tensors are functions of the orientation of the reference system, the elastic coefficients in Eqs. (2.126₁) (and,

analogously, in Eq. (2.126₂)) are also tensor functions of this orientation and they can be presented in a new coordinate system:

$$\mathbf{L}' = \mathbf{g}\mathbf{g}\mathbf{L}\mathbf{g}^\top \mathbf{g}^\top, \quad L'_{ijkl} = q_{im}g_{jn}g_{ko}g_{lp}L_{mnop} \quad (2.160)$$

with the relevant transformation for the strain tensors:

$$\boldsymbol{\varepsilon}' = \mathbf{g}\mathbf{g}\boldsymbol{\varepsilon}, \quad \varepsilon'_{ij} = g_{ik}g_{jl}\varepsilon_{kl}. \quad (2.161)$$

These tensors as well as the thermal expansion and thermoelasticity coefficients are transformed using Voight reduced components as

$$\mathbf{L}' = \mathbf{q}^L \mathbf{L} (\mathbf{q}^L)^\top, \quad \mathbf{M}' = \mathbf{q}^R \mathbf{M} (\mathbf{q}^R)^\top, \quad (2.162)$$

$$\boldsymbol{\sigma}' = \mathbf{q}^L \boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}' = \mathbf{q}^R \boldsymbol{\varepsilon}, \quad \boldsymbol{\alpha}^{T'} = \mathbf{q}^L \boldsymbol{\alpha}^T, \quad \boldsymbol{\beta}^{T'} = \mathbf{q}^R \boldsymbol{\beta}^T. \quad (2.163)$$

where the fourth-order tensors \mathbf{q}^L and $\mathbf{q}^R = [(\mathbf{q}^L)^{-1}]^\top$ presented in the form of (6×6) transformation matrices:

$$\mathbf{q}^L = \begin{pmatrix} \mathbf{K}_1 & 2\mathbf{K}_2 \\ \mathbf{K}_3 & \mathbf{K}_4 \end{pmatrix}, \quad \mathbf{q}^R = \begin{pmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ 2\mathbf{K}_3 & \mathbf{K}_4 \end{pmatrix}, \quad (2.164)$$

are combined by four matrices (3×3) :

$$\begin{aligned} \mathbf{K}_1 &= \begin{pmatrix} g_{11}^2 & g_{12}^2 & g_{13}^2 \\ g_{21}^2 & g_{22}^2 & g_{23}^2 \\ g_{31}^2 & g_{32}^2 & g_{33}^2 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} g_{12}g_{13} & g_{13}g_{11} & g_{11}g_{12} \\ g_{22}g_{23} & g_{23}g_{21} & g_{21}g_{22} \\ g_{32}g_{33} & g_{33}g_{31} & g_{31}g_{32} \end{pmatrix}, \\ \mathbf{K}_3 &= \begin{pmatrix} g_{21}g_{31} & g_{22}g_{32} & g_{23}g_{33} \\ g_{31}g_{11} & g_{32}g_{12} & g_{33}g_{13} \\ g_{11}g_{21} & g_{12}g_{22} & g_{13}g_{23} \end{pmatrix}, \\ \mathbf{K}_4 &= \begin{pmatrix} g_{22}g_{33} + g_{23}g_{32} & g_{23}g_{31} + g_{21}g_{33} & g_{21}g_{32} + g_{22}g_{31} \\ g_{32}g_{13} + g_{33}g_{12} & g_{33}g_{11} + g_{31}g_{13} & g_{31}g_{12} + g_{32}g_{11} \\ g_{12}g_{23} + g_{13}g_{22} & g_{13}g_{21} + g_{11}g_{23} & g_{11}g_{22} + g_{12}g_{21} \end{pmatrix}. \end{aligned} \quad (2.165)$$

The elements of the matrices \mathbf{q}^L and \mathbf{q}^R can be represented by means of the formulae

$$\frac{1}{2}(g_{i'k}g_{j'l} + g_{i'l}g_{j'k}) = \frac{q_{\lambda'\mu}^R}{2 - \delta_{i'j'}} = \frac{q_{\lambda'\mu}^L}{2 - \delta_{kl}}, \quad (2.166)$$

where the index λ' corresponds to the pair $i'j'$, and μ to the pair kl .

The equations describing the piezoelectric effect (2.142) and (2.143) can be presented in the matrix form:

$$\varepsilon_\gamma = M_{\gamma\delta}\sigma_\delta + d_{\gamma i}E_i + \lambda_\gamma, \quad D_i = e_{i\gamma}\varepsilon_\gamma - k_{ij}E_j + p_i\theta, \quad (2.167)$$

which leads to the following recalculation rules for the coefficients ($jk \leftrightarrow \gamma$): $d_{ijk} = (2 - \delta_{jk})^{-1}d_{i\gamma}$, $e_{ijk} = e_{i\gamma}$ with Latin indices ranging from 1 to 3. The transformation formulae for quantities $d_{i\gamma}$ and $e_{i\gamma}$ in the old \mathbf{e}_k and new \mathbf{e}'_k ($k = 1, 2, 3$) coordinate systems with the cosine matrix g_{ij} (2.4) are

$$d'_{i\gamma} = g_{ij}q_{\gamma\delta}^R d_{j\delta}, \quad d_{j\delta} = g_{ji}q_{\delta\gamma}^L d'_{i\gamma}, \quad e'_{i\gamma} = g_{ij}q_{\gamma\delta}^L e_{j\delta}, \quad e_{j\delta} = g_{ji}q_{\delta\gamma}^R e'_{i\gamma}. \quad (2.168)$$

The recalculation rules and the transformation formulae can be obtained by the same method for other coefficients characterizing piezoelectric properties of crystals (for details see [1015]).

2.5 Symmetry of Elastic Properties

Under an orthogonal transformation (2.6) the elastic stiffnesses L'_{ijkl} referred to the \mathbf{e}'_i coordinate system are described by Eq. (2.160). When $\mathbf{L}' = \mathbf{L}$, i.e.,

$$L_{ijkl} = g_{ip}g_{jq}g_{kr}g_{st}L_{pqrt}, \quad (2.169)$$

the material is said to possess a *symmetry* with respect to \mathbf{g} , which is called a symmetry transformation. If \mathbf{g} is a symmetry transformation, $\mathbf{g}^{-1} = \mathbf{g}^\top$ is also a symmetry transformation, and if \mathbf{g} and \mathbf{q} are two symmetry transformations then \mathbf{gq} is also a symmetry transformation. The set of all symmetry transformations at $\mathbf{x} \in \mathcal{E}$ forms a symmetry group $S_{\mathbf{x}}$ with the unit transformation δ . The group $S_{\mathbf{x}}$ always contains the two-element subgroup $\{\delta, -\delta\}$. We describe by $\mathbf{g}(\omega\mathbf{n})$ (A.1.2) a right-handed rotation by the angle ω , $0 < \omega < \pi$, about an axis oriented in the direction of unit vector \mathbf{n} . The transformation $\mathcal{R}(\mathbf{n}) = -\mathbf{g}(\pi\mathbf{n})$ is called a reflection in the plane $P(\mathbf{n})$ with the normal unit vector \mathbf{n} . A unit vector \mathbf{n} is called an axis of symmetry at \mathbf{x} if $\mathbf{gn} = \mathbf{n}$ for some $\mathbf{g} \in S_{\mathbf{x}}$, with $\mathbf{g} \neq \delta$. A plane $P(\mathbf{n}_1, \mathbf{n}_2)$, spanned by two mutually orthogonal unit vectors \mathbf{n}_1 and \mathbf{n}_2 , is called a plane of symmetry at \mathbf{x} if $\mathbf{gn}_1 = \mathbf{n}_1$ and $\mathbf{gn}_2 = \mathbf{n}_2$, for some $\mathbf{g} \in S_{\mathbf{x}}$, with $\mathbf{g} \neq \delta$. The condition symmetry (2.169) introduces a restriction on the elastic moduli \mathbf{L} . A *triclinic* material has a minimum symmetry group $S_{\mathbf{x}}$ formed by the two-element group $\{\delta, -\delta\}$. According to Eq. (2.90), the number of independent elastic coefficients for the general anisotropic linearly elastic triclinic materials is reduced from $3^4 = 81$ to 21, which is the number of terms on the main diagonal of the matrix \mathbf{L} and the five subdiagonals above it: $6+5+4+3+2+1 = 21$. This number is reduced when the symmetry of the crystal class of medium is accounted for. A material is called *monoclinic* if for any $\mathbf{x} \in \mathcal{E}$ the symmetry group $S_{\mathbf{x}}$ is formed by the transformations $\pm\delta, \pm\mathbf{g}(\pi, \mathbf{n})$ for any $\mathbf{x} \in \mathcal{E}$. An *orthotropic* has the symmetry group $S_{\mathbf{x}}$ formed for any $\mathbf{x} \in \mathcal{E}$ by the transformations $\pm\delta, \pm\mathbf{g}(\pi, \mathbf{n}_1), \pm\mathbf{g}(\pi, \mathbf{n}_2), \pm\mathbf{g}(\pi, \mathbf{n}_3)$, where $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are three mutually orthogonal unit vectors. A material is called *transversally isotropic* with respect to the direction \mathbf{n} if its symmetry group $S_{\mathbf{x}}$ ($\forall \mathbf{x} \in \mathcal{E}$) consists of the transformations $\pm\delta$ and $\pm\mathbf{g}(\omega, \mathbf{n})$ with $0 < \omega < \pi$.

Depending on the number of rotations and/or reflection symmetry a crystal possesses, the 6×6 matrix \mathbf{L} can be represented by one of the eight groups summarized in [821]. If the matrix \mathbf{L}' referred to a different coordinate system \mathbf{e}'_i is desired, the transformation (2.160) is used.

Let us consider polycrystalline aggregates of an orthorhombic system. Then for the classes of symmetry with nine independent elastic constants, the tensors of the elastic moduli and compliance of a crystallite in the crystallographic system of coordinates can be represented in the form (see, e.g., [995]):

$$\begin{aligned} L_{ijkl} = \sum_n & \left[\lambda_n \delta_{in} \delta_{jn} \delta_{kn} \delta_{ln} + \mu_n (\delta_{in} \delta_{jn} \delta_{kl} + \delta_{ij} \delta_{kn} \delta_{ln}) \right. \\ & \left. + \nu_n (\delta_{in} \delta_{jk} \delta_{ln} + \delta_{jn} \delta_{ik} \delta_{ln} + \delta_{in} \delta_{jl} \delta_{kn} + \delta_{jn} \delta_{il} \delta_{kn}) \right], \quad (2.170) \\ M_{ijkl} = \sum_n & \left[p_n \delta_{in} \delta_{jn} \delta_{kn} \delta_{ln} + q_n (\delta_{in} \delta_{jn} \delta_{kl} + \delta_{ij} \delta_{kn} \delta_{ln}) \right] \end{aligned}$$

$$+ r_n(\delta_{in}\delta_{jk}\delta_{ln} + \delta_{jn}\delta_{ik}\delta_{ln} + \delta_{in}\delta_{jl}\delta_{kn} + \delta_{jn}\delta_{il}\delta_{kn})\Big], \quad (2.171)$$

where the elastic constants λ_n, μ_n, ν_n and p_n, q_n, r_n ($n = 1, 2, 3$) are defined by the matrix representation of the elastic moduli L_{mn} and stiffness M_{ij} in the crystallographic system:

$$\begin{aligned} \lambda_1 &= L_{11} + L_{23} + 2L_{44} - (L_{12} + L_{13} + 2L_{55} + 2L_{66}), \\ \lambda_2 &= L_{22} + L_{13} + 2L_{55} - (L_{12} + L_{23} + 2L_{44} + 2L_{66}), \\ \lambda_3 &= L_{33} + L_{12} + 2L_{66} - (L_{13} + L_{23} + 2L_{44} + 2L_{55}), \\ 2\mu_1 &= L_{12} + L_{13} - L_{23}, \quad 2\nu_1 = L_{55} + L_{66} - L_{44}, \\ 2\mu_2 &= L_{12} + L_{23} - L_{13}, \quad 2\nu_2 = L_{44} + L_{66} - L_{55}, \\ 2\mu_3 &= L_{13} + L_{23} - L_{12}, \quad 2\nu_3 = L_{44} + L_{55} - L_{66}, \\ p_1 &= M_{11} + M_{23} + M_{44}/2 - (2M_{12} + 2M_{13} + M_{55} + M_{66})/2, \\ p_2 &= M_{22} + M_{13} + M_{55}/2 - (2M_{12} + 2M_{23} + M_{44} + M_{66})/2, \\ p_3 &= M_{33} + M_{12} + M_{66}/2 - (2M_{13} + 2M_{23} + M_{44} + M_{55})/2, \\ 2q_1 &= M_{12} + M_{13} - M_{23}, \quad 8r_1 = M_{55} + M_{66} - M_{44}, \\ 2q_2 &= M_{12} + M_{23} - M_{13}, \quad 8r_2 = M_{44} + M_{66} - M_{55}, \\ 2q_3 &= M_{13} + M_{23} - M_{12}, \quad 8r_3 = M_{44} + M_{55} - M_{66}, \end{aligned} \quad (2.172)$$

In particular, for tetragonal crystals, the relations $L_{11} = L_{22}$, $L_{13} = L_{23}$, $L_{44} = L_{55}$ reduce the number of unknown elastic constants from nine to six:

$$\begin{aligned} \lambda_1 &= \lambda_2 = L_{11} - (L_{12} + L_{66}), \\ \lambda_3 &= L_{33} + L_{12} + 2L_{66} - 2(L_{13} + 2L_{44}), \\ 2\mu_1 &= 2\mu_2 = L_{12}, \quad 2\mu_3 = 2L_{13} - L_{12}, \\ 2\nu_1 &= 2\nu_2 = L_{66}, \quad 2\nu_3 = 2L_{44} - L_{66}. \end{aligned} \quad (2.173)$$

For hexagonal crystals, the addition condition $2L_{66} = L_{11} - L_{12}$ takes place substitution of which into Eq. (2.173) leads to

$$\begin{aligned} \lambda_1 &= \lambda_2 = 0, \quad \lambda_3 = L_{11} + L_{33} - 2(L_{13} + 2L_{44}), \\ 2\mu_1 &= 2\mu_2 = L_{12}, \quad 2\mu_3 = 2L_{13} - L_{12}, \\ 4\nu_1 &= 4\nu_2 = L_{11} - L_{12}, \quad 4\nu_3 = 4L_{44} + L_{12} - L_{11}. \end{aligned} \quad (2.174)$$

For cubic symmetry, substitution of the conditions $L_{22} = L_{33}$, $L_{12} = L_{13}$, $L_{44} = L_{66}$ into Eq. (2.173) gives

$$\begin{aligned} \lambda_1 &= \lambda_2 = \lambda_3 = L_{11} - L_{12} - 2L_{44}, \\ \mu_1 &= \mu_2 = \mu_3 = L_{12}/2, \quad \nu_1 = \nu_2 = \nu_3 = L_{44}/2. \end{aligned} \quad (2.175)$$

At last, a material is called elastically isotropic if its elastic properties are independent of direction. In the case of cubic crystal symmetry this is realized when $L_{44} = (L_{11} - L_{12})/2$. The deviation of the cubic lattice from an isotropic one can be quantified by the so-called *Zener anisotropy ratio*

$$Z = \frac{2L_{44}}{L_{11} - L_{12}}. \quad (2.176)$$

Substitution of an additional condition $2L_{44} = L_{11} - L_{12}$ for the isotropic medium into Eq. (2.175) yields

$$\begin{aligned} \lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \mu_1 = \mu_2 = \mu_3 = L_{12}/2, \\ \nu_1 = \nu_2 = \nu_3 = (L_{11} - L_{12})/2. \end{aligned} \quad (2.177)$$

The representation of elastic moduli in the reference coordinate system (Ox', Oy', Oz') can be obtained by the use of rotation $\mathbf{g} \in O^3$ from the tensors \mathbf{L} (2.170) and \mathbf{M} (2.171) according to Eqs. (2.160) and (2.161), respectively. The final representation is simplified in exploiting of the identities $g_{ij}\delta_{jn} = g_{in}$ and $g_{ij}g_{kj} = g_{ik}$:

$$\begin{aligned} L'_{ijkl} = \sum_n \left[\lambda_n g_{in} g_{jn} g_{kn} g_{ln} + \mu_n (g_{in} g_{jn} \delta_{kl} + \delta_{ij} g_{kn} g_{ln}) \right. \\ \left. + \nu_n (g_{in} \delta_{jk} g_{ln} + g_{jn} \delta_{ik} g_{ln} + g_{in} \delta_{jl} g_{kn} + g_{jn} \delta_{il} g_{kn}) \right], \end{aligned} \quad (2.178)$$

$$\begin{aligned} M'_{ijkl} = \sum_n \left[p_n g_{in} g_{jn} g_{kn} g_{ln} + q_n (g_{in} g_{jn} \delta_{kl} + \delta_{ij} g_{kn} g_{ln}) \right. \\ \left. + r_n (g_{in} \delta_{jk} g_{ln} + g_{jn} \delta_{ik} g_{ln} + g_{in} \delta_{jl} g_{kn} + g_{jn} \delta_{il} g_{kn}) \right]. \end{aligned} \quad (2.179)$$

where the elastic constants λ_n, μ_n, ν_n and p_n, q_n, r_n ($n = 1, 2, 3$) are defined by the matrix representations L_{ij} and M_{ij} in the crystallographic coordinate system (2.172).

In particularly, for the *tetragonal* crystals (2.173),

$$\begin{aligned} L'_{ijkl} = L_{12} \delta_{ij} \delta_{kl} + L_{66} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + (L_{11} - L_{12} - 2L_{66}) \sum_{ijkl} \\ + (L_{33} - L_{11} + 2L_{12} - 2L_{13} + 4L_{66} - 4L_{44}) T_{ijkl} \\ + 2(L_{44} - L_{66}) (T_{(ik} \delta_{jl)} + T_{(il} \delta_{jk)}) + 2(L_{13} - L_{12}) T_{(ij} \delta_{kl)}, \end{aligned} \quad (2.180)$$

for *hexagonal* crystals (2.174),

$$\begin{aligned} L'_{ijkl} = L_{12} \delta_{ij} \delta_{kl} + (L_{11} - L_{12}) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ + (L_{33} + L_{11} - 2L_{13} - 4L_{44}) T_{ijkl} + 2(L_{13} - L_{12}) T_{(ij} \delta_{kl)} \\ + (L_{12} - L_{11} + 2L_{44}) (T_{(ik} \delta_{jl)} + T_{(il} \delta_{jk)}), \end{aligned} \quad (2.181)$$

for *cubic* crystals (2.175),

$$L'_{ijkl} = L_{12} \delta_{ij} \delta_{kl} + L_{44} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + (L_{11} - L_{12} - 2L_{44}) \sum_{ijkl}, \quad (2.182)$$

and for the *isotropic* material (2.177),

$$L'_{ijkl} = L_{12} \delta_{ij} \delta_{kl} + L_{44} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (2.183)$$

Here one introduces the notations

$$\sum_{ijkl} = \sum_{n=1}^3 g_{in}g_{jn}g_{kn}g_{ln}, \quad T_{ij} = g_{i3}g_{j3}, \quad T_{ijkl} = g_{i3}g_{j3}g_{k3}g_{l3}, \quad (2.184)$$

where the tensors T_{ijkl} and \sum_{ijkl} show the properties

$$T_{iikl} = T_{kl}, \quad T_{ijkl}T_{klpq} = T_{ijpq}, \quad \sum_{iikl} = \delta_{kl}, \quad \sum_{ijkl} \sum_{klpq} = \sum_{ijpq}. \quad (2.185)$$

The representation (2.182) for the *cubic* crystals can be also presented in more detailed form:

$$\begin{aligned} L'_{11} &= L_{11} - 2L(g_{11}^2g_{12}^2 + g_{12}^2g_{13}^2 + g_{13}^2g_{11}^2), \\ L'_{12} &= L_{12} + L(g_{11}^2g_{21}^2 + g_{12}^2g_{22}^2 + g_{13}^2g_{23}^2), \\ L'_{44} &= L_{44} + L(g_{12}^2g_{13}^2 + g_{22}^2g_{32}^2 + g_{23}^2g_{33}^2), \quad L \equiv L_{11} - L_{12} - 2L_{44}. \end{aligned} \quad (2.186)$$

The inverse transformations from the coefficients $L_{\gamma\nu}$ to $M_{\gamma\nu}$ can be carried out by direct inversion of the matrix of known coefficients because $L_{\gamma\delta}M_{\delta\nu} = \delta_{\gamma\nu}$. However, in some cases the simple relations between the coefficients $L_{\gamma\nu}$ and $M_{\gamma\nu}$ are useful. They are presented below for some symmetry groups described, at most, by seven independent parameters:

Trigonal materials:

$$\begin{aligned} M_{11} + M_{12} &= L_{33}/l, \quad M_{11} - M_{12} = L_{44}/l', \quad M_{13} = -L_{13}/l, \\ M_{14} &= -L_{14}/l', \quad M_{33} = (L_{11} + L_{12})/l, \quad M_{44} = (L_{11} - L_{12})/l', \\ l &\equiv L_{33}(L_{11} + L_{12}) - 2L_{13}^2, \quad l' \equiv L_{44}(L_{11} - L_{12}) - 2L_{14}^2. \end{aligned} \quad (2.187)$$

Hexagonal materials:

$$\begin{aligned} M_{11} + M_{12} &= L_{33}/l, \quad M_{11} - M_{12} = (L_{11} - L_{12})^{-1}, \quad M_{13} = -L_{13}/l, \\ M_{33} &= (L_{11} + L_{12})/l, \quad M_{44} = L_{44}^{-1}, \\ l &\equiv L_{33}(L_{11} + L_{12}) - 2L_{13}^2. \end{aligned} \quad (2.188)$$

Cubic materials:

$$\begin{aligned} M_{11} &= (L_{11} + L_{12})(L_{11} - L_{12})^{-1}(L_{11} + 2L_{12})^{-1}, \\ M_{12} &= -L_{12}(L_{11} - L_{12})^{-1}(L_{11} + 2L_{12})^{-1}, \\ M_{44} &= L_{44}^{-1}. \end{aligned} \quad (2.189)$$

In particular, the elastic constants in the isotropic matrix ($\mathbf{A} = \mathbf{L}, \mathbf{M}$; $A_{ij} = L_{ij}, M_{ij}$):

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{12} & 0 & 0 & 0 \\ A_{12} & A_{11} & A_{12} & 0 & 0 & 0 \\ A_{12} & A_{12} & A_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{44} \end{pmatrix}, \quad (2.190)$$

(where $L_{44} = (L_{11} - L_{12})/2$, $M_{44} = 2(M_{11} - M_{12})$) are usually written in the engineering notations:

$$L_{11} = \frac{E}{1+\nu} \frac{1-\nu}{1-2\nu}, \quad L_{12} = \frac{E}{1+\nu} \frac{\nu}{1-2\nu}, \quad L_{44} = \frac{E}{2(1+\nu)}, \quad (2.191)$$

$$M_{11} = \frac{1}{E}, \quad M_{12} = -\frac{\nu}{E}, \quad M_{44} = \frac{2(1+\nu)}{E}, \quad (2.192)$$

where two independent constants *Young's modulus* E and *Poisson's ratio* ν can be also used in the tensor representations ($\mathbf{N}_1 = \boldsymbol{\delta} \otimes \boldsymbol{\delta}/3$)

$$\mathbf{L} = \frac{3\nu E}{(1+\nu)(1-2\nu)} \mathbf{N}_1 + \frac{E}{1+\nu} \mathbf{I}, \quad \mathbf{M} = -\frac{3\nu}{E} \mathbf{N}_1 + \frac{1+\nu}{E} \mathbf{I}. \quad (2.193)$$

Another popular representations of the elastic isotropic properties using *Lame's constants* λ and μ can be formed in both the matrix

$$L_{11} = \lambda + \mu, \quad L_{12} = \lambda, \quad L_{66} = \mu, \quad (2.194)$$

and tensor

$$\mathbf{L} = 3\lambda \mathbf{N}_1 + 2\mu \mathbf{I}, \quad \mathbf{M} = 3s \mathbf{N}_1 + 2q \mathbf{I}, \quad (2.195)$$

notations where the compliances s and q are expressed in terms of elastic constants:

$$s = \frac{1}{3(3\lambda + 2\mu)}, \quad q = \frac{1}{4\mu}. \quad (2.196)$$

Due to the decomposition of the unit tensor \mathbf{I} into the orthogonal bulk \mathbf{N}_1 and deviatoric \mathbf{N}_2 tensors (2.16), the isotropic tensors \mathbf{L} and \mathbf{M} can also be decomposed:

$$\mathbf{L} = (3k, 2\mu) \equiv 3k \mathbf{N}_1 + 2\mu \mathbf{N}_2, \quad \mathbf{M} = (3p, 2q). \quad (2.197)$$

where k and μ , called the bulk and shear moduli, are related to the appropriate compliance coefficients p , q by means of the “multiplication table” for the products between the elementary idempotent tensors \mathbf{N}_1 and \mathbf{N}_2 (2.17). Then the inverse tensor $\mathbf{L}_1^{-1} \equiv \mathbf{M}$, and the product $\mathbf{L}_1 : \mathbf{L}_2$ of such tensors, $\mathbf{L}_1 = (3k_1, 2\mu_1)$ and $\mathbf{L}_2 = (3k_2, 2\mu_2)$, can be estimated by the compact formulae:

$$\mathbf{L}_1^{-1} = (3p, 2q) = \left(\frac{1}{3k}, \frac{1}{2\mu} \right), \quad \mathbf{L}_1 : \mathbf{L}_2 = (9k_1 k_2, 4\mu_1 \mu_2). \quad (2.198)$$

Table 2.1 shows the relationships between the various elastic constants of isotropic materials.

Table 2.1. Relationships between elastic moduli

Constant	λ, μ	K, μ	μ, ν	E, ν	E, μ
λ	λ	$K - \frac{2}{3}\mu$	$\frac{2\mu\nu}{1-2\nu}$	$\frac{\nu E}{(1+\nu)(1-3\nu)}$	$\frac{\mu(E-2\mu)}{3\mu-E}$
μ	μ	μ	μ	$\frac{E}{2(1+\nu)}$	μ
K	$\lambda + \frac{2}{3}\mu$	K	$\frac{2\mu(1-\nu)}{3(1-2\nu)}$	$\frac{E}{3(1-2\nu)}$	$\frac{E\mu}{3(3\mu-E)}$
E	$\frac{(3\lambda+2\mu)\mu}{\lambda+\mu}$	$\frac{9K\mu}{3K+\mu}$	$2(1+\nu)\mu$	E	E
ν	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3K-2\mu}{6K+2\mu}$	ν	ν	$\frac{E}{2\mu}-1$

The bulk and shear components of the isotropic tensors \mathbf{L} and \mathbf{M} are found through a suitable contraction of the indices of \mathbf{L} and \mathbf{M} in the following relations

$$3k = \frac{1}{3}L_{iikk}, \quad 2\mu = \frac{1}{5}(L_{ikik} - \frac{1}{3}L_{iikk}), \quad (2.199)$$

$$3p = \frac{1}{3}M_{iikk}, \quad 2q = \frac{1}{5}(M_{ikik} - \frac{1}{3}M_{iikk}), \quad (2.200)$$

which can be used at the estimation of average moduli of isotropic polycrystals.

Transversely isotropic materials have five independent elastic parameters. If the plane of isotropy coincides with the x_1, x_2 -plane, the elastic and compliance matrices ($\mathbf{A} = \mathbf{L}, \mathbf{M}$; $A_{ij} = L_{ij}, M_{ij}$):

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{12} & A_{11} & A_{13} & 0 & 0 & 0 \\ A_{13} & A_{13} & A_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{55} \end{pmatrix}, \quad (2.201)$$

(where $L_{55} = (L_{11} - L_{12})/2$, $M_{55} = 2(M_{11} - M_{12})$) can be presented in engineering notations:

$$L_{11} = l \left(\frac{1}{EE_3} - \frac{\nu_3^2}{E_3^2} \right), \quad L_{12} = l \left(\frac{\nu}{EE_3} + \frac{\nu_3^2}{E_3^2} \right), \quad L_{13} = l \frac{(1 + \nu)\nu_3}{EE_3},$$

$$L_{33} = l \frac{1 - \nu^2}{E^2}, \quad L_{44} = \mu_3, \quad l = \frac{E^2 E_3^2}{(1 + \nu)[(1 - \nu)E_3 - 2\nu_3^2 E]}, \quad (2.202)$$

$$M_{11} = \frac{1}{E}, \quad M_{12} = -\frac{\nu}{E}, \quad M_{13} = -\frac{\nu_3}{E_3}, \quad M_{33} = \frac{1}{E_3}, \quad M_{44} = \frac{1}{\mu_3}. \quad (2.203)$$

Here the Young modulus and the Poisson ratios in the x_1, x_2 -plane are the same, say, $E_1 = E_2 = E$ and $\nu_{12} = \nu_{21} = \nu$, respectively. The corresponding shear modulus $\mu_{12} = \mu_{21} = \mu$ is formed as $\mu = E/2(1 + \nu)$. Young modulus, Poisson ratio, and the shear modulus associated with the x_3 -direction and a direction in the $x_1 x_2$ -plane are denoted by, say, E_3 , $\nu_{13} = \nu_{23} = \nu_3$, and $\mu_{13} = \mu_{23} = \mu_3$, respectively.

For the *orthotropic* case, the elastic and compliance matrices are defined by nine independent elastic parameters and can be presented in a coordinate system coincident with the material symmetry directions as ($\mathbf{A} = \mathbf{L}, \mathbf{M}$; $A_{ij} = L_{ij}, M_{ij}$):

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{12} & A_{11} & A_{23} & 0 & 0 & 0 \\ A_{13} & A_{23} & A_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{pmatrix}, \quad (2.204)$$

where, for example, the compliance matrix in engineering terminology becomes

$$\mathbf{M} = \begin{pmatrix} 1/E_1 - \nu_{21}/E_2 - \nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 - \nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 - \nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\mu_{12} \end{pmatrix}. \quad (2.205)$$

We will now consider a unique representation of the constitutive laws and elastic moduli for a d -dimensional linear isotropic material with bulk modulus $k_{[d]}$, shear modulus $\mu_{[d]}$, Young's modulus $E_{[d]}$, and Poisson's ratio $\nu_{[d]}$ in the space dimensionality d ($i, j, k, l = 1, \dots, d$; $d = 2, 3$) (see, e.g., [1106]):

$$\sigma_{ij} = (k_{[d]} - \frac{2}{d}\mu_{[d]})\varepsilon_{kk}\delta_{ij} + 2\mu_{[d]}\varepsilon_{ij}, \quad \varepsilon_{ij} = -\frac{\nu_{[d]}}{E_{[d]}}\sigma_{kk}\delta_{ij} + \frac{(1 + \nu_{[d]})}{E_{[d]}}\sigma_{ij} \quad (2.206)$$

and

$$\mathbf{L} = dk_{[d]}\mathbf{N}_1 + 2\mu_{[d]}\mathbf{N}_2, \quad \mathbf{M} = \frac{1 + \nu_{[d]}(1 - d)}{E_{[d]}}\mathbf{N}_1 + \frac{1 + \nu_{[d]}}{E_{[d]}}\mathbf{N}_2, \quad (2.207)$$

where the projection tensors

$$\mathbf{N}_{1|ijkl} = \frac{1}{d}\delta_{ij}\delta_{kl}, \quad \mathbf{N}_{2|ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{d}\delta_{ij}\delta_{kl} \quad (2.208)$$

satisfy the normality conditions (2.17) allowing the inverse of relation (2.207₁)

$$\mathbf{M} = \frac{1}{dk_{[d]}}\mathbf{N}_1 + \frac{1}{2\mu_{[d]}}\mathbf{N}_2. \quad (2.209)$$

Comparison of representations (2.207₂) and (2.209) leads to the interrelations

$$\begin{aligned} k_{[d]} &= \frac{E_{[d]}}{d[1 + \nu_{[d]}(1 - d)]}, \quad \mu_{[d]} = \frac{E_{[d]}}{2(1 + \nu_{[d]})}, \\ \frac{1}{E_{[d]}} &= \frac{1}{d^2k_{[d]}} + \frac{d - 1}{2d\mu_{[d]}}, \quad \nu_{[d]} = \frac{dk_{[d]} - 2\mu_{[d]}}{d(d - 1)k_{[d]} + 2\mu_{[d]}}. \end{aligned} \quad (2.210)$$

Passages to the limits of the positive moduli $k_{[d]}/\mu_{[d]} \rightarrow \infty$ and $\mu_{[d]}/k_{[d]} \rightarrow \infty$ in the expression (2.210₄) yield the upper and lower limits of $\nu_{[d]}$: $-1 \leq \nu_{[d]} \leq (d - 1)^{-1}$, respectively.

Finally, we will consider either *plane-strain* ($\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{13}$ in Eq. (2.206) with $d = 3$) or *plane-stress* ($\sigma_{11} = \sigma_{12} = \sigma_{13}$ in Eq. (2.206) with $d = 3$) elasticity. If we compare these simplified three-dimensional expressions to relations (2.206) with $d = 2$, we obtained the interrelations:

$$E_{[2]} = \frac{E_{[3]}}{(1 + \nu_{[3]})(1 + \nu_{[3]})}, \quad \nu_{[2]} = \frac{\nu_{[3]}}{1 - \nu_{[3]}}, \quad k_{[2]} = k_{[3]} + \frac{1}{3}\mu_{[3]}, \quad \mu_{[2]} = \mu_{[3]}, \quad (2.211)$$

for plane-strain elasticity and

$$E_{[2]} = E_{[3]}, \quad \nu_{[2]} = \nu_{[3]}, \quad k_{[2]} = \frac{9k_{[3]}\mu_{[3]}}{3k_{[3]} + 4\mu_{[3]}}, \quad \mu_{[2]} = \mu_{[3]} \quad (2.212)$$

for plane-stress elasticity, respectively.

2.6 Basic Equations of Thermoelastoplastic Deformations

2.6.1 Incremental Theory of Plasticity

We assume that the thermomechanical properties of the composite medium with, generally speaking, anisotropic components are described by the theory of small elastic-plastic strains under arbitrarily varying external loading. Additive decomposition of the increments of total strain tensor ε is assumed:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^t + d\varepsilon^p, \quad (2.213)$$

with the increments of the elastic strains $d\varepsilon^e$, the so-called “transformation strains” $d\varepsilon^t$ including thermal strains and plastic strains $d\varepsilon^p$.

The stress increment $d\sigma$ relates to the elastic part of the strain increment $d\varepsilon$ by the elasticity relation given in the form

$$d\sigma(\mathbf{x}) = \mathbf{L}(\mathbf{x}) d\varepsilon^e(\mathbf{x}), \quad (2.214)$$

where $\mathbf{L}(\mathbf{x})$ is the fourth-order elasticity tensor (2.88).

The transformation strain increment $d\varepsilon^t$ (2.213) may consist of contributions of different physical origins. For example, if only thermal effects are considered, $\varepsilon^t = \beta^T \theta$, where β^T is the tensor of linear thermal expansion coefficients and θ is the temperature change from the reference value to the current temperature.

In the six-dimensional stress space, consider a yield surface $f(\mathbf{x}, \dots) = 0$ bonding the region of plastic deformation, in which $f(\sigma, \dots) > 0$; here dots stand for temperature and for internal variables characterizing material hardening. The behavior is elastic if $f < 0$, or if $f = 0$ and $[(\partial f / \partial \sigma) : d\sigma + (\partial f / \partial \theta) d\theta] \leq 0$ (for the elastic unloading and neutral loading); elastic-plastic deformations take place under active loading, when $f = 0$ and $[(\partial f / \partial \sigma) : d\sigma + (\partial f / \partial \theta) d\theta] > 0$. In the general case, the yield surface can depend on a variety of tensor \mathbf{a}^p and scalar hardening parameters. The hardening tensor parameter, in particular, may simply coincide with the plastic-strain tensor. Regarding the constitutive equations for the elastic-plastic materials we use the so-called J_2 -flow theory with combined isotropic-kinematic hardening. The von Mises form of the yield surface is given by

$$f \equiv \tau - F(\gamma, \theta) = 0, \quad F(0, \theta) = \tau_0(\theta) \quad (2.215)$$

in terms of von Mises effective stress, τ , and the accumulated effective plastic strain increment, γ , respectively, as defined by

$$\tau = \left(\frac{3}{2} s_{ij}^a s_{ij}^a \right)^{1/2}, \quad d\gamma = \left(\frac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p \right)^{1/2}, \quad \mathbf{s}^a = \mathbf{N}_2(\sigma - \mathbf{a}^p). \quad (2.216)$$

Here τ_0 is the initial yield stress, F is a nonlinear function describing the hardening effect and temperature dependence, for example as in the modified Ludwik equation:

$$F(\gamma, \theta) = \tau_0(\theta) + h(\theta) \gamma^{n(\theta)}, \quad (2.217)$$

where h and n are hardening parameters. \mathbf{s}^a is the active stress deviator; \mathbf{a}^p is a symmetric second order tensor corresponding to the “back-stresses” defining the location of the center of the yield surface in the deviatoric stress space. For evaluation of the back stress tensor \mathbf{a}^p we use Ziegler’s assumption:

$$d\mathbf{a}^p = d\gamma A \mathbf{s}^a, \quad A \equiv A(\gamma), \quad (2.218)$$

or Prager’s rule

$$d\mathbf{a}^p = B d\varepsilon^p, \quad B \equiv B(\gamma), \quad (2.219)$$

respectively, with $\mathbf{a}^p = 0$ if $\gamma = 0$. In the case $\mathbf{a}^p \equiv \mathbf{0}$, $h \neq 0$, Eqs. (2.215) and (2.216) are reduced to the isotropic hardening; the case $h \equiv 0$, $\mathbf{a}^p \neq \mathbf{0}$ corresponds to the kinematic hardening. Though the von Mises yield criterion (2.215) is assumed in this study, modifications of the present method for introducing other yield criteria and hardening laws are possible.

Drucker’s postulate states that the work of additional stresses is positive over the whole cycle. Then for any stress state $\boldsymbol{\sigma}^0$ within or on the convex yield surface the local maximum principle $(\boldsymbol{\sigma} - \boldsymbol{\sigma}^0) : d\boldsymbol{\varepsilon}^p \geq 0$. Drucker’s postulate leads to the necessity of the associated law of plastic flow when the smooth yield function f is taken as plastic potential function from which the incremental plastic strain can be derived as

$$\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \text{for } f = 0, \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} > 0, \quad (2.220)$$

where the plastic flow is fixed in the direction along the normal to the yield surface, while its magnitude $d\lambda$, called a proportionality factor, is undetermined. For consistency during plastic deformation:

$$df \equiv \frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial f}{\partial \mathbf{a}^p} : d\mathbf{a}^p + \frac{\partial f}{\partial \gamma} d\gamma + \frac{\partial f}{\partial \theta} d\theta = 0. \quad (2.221)$$

Substitution of (2.216), (2.218), (2.219) in (2.221) leads to the determination of $d\lambda$ by

$$d\lambda = G \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial f}{\partial \theta} d\theta \right), \quad (2.222)$$

where the proportionality parameter G is defined by the explicit form of hardening rule (2.217)–(2.219). For example, for isotropic-kinematic hardening, (2.217), (2.218), we have

$$G = - \left(\frac{\partial f}{\partial \gamma} + A(\gamma) \frac{\partial f}{\partial \mathbf{a}^p} : \mathbf{s}^a \right)^{-1} \left(\frac{2}{3} \frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}} \right)^{-1/2}. \quad (2.223)$$

The associated flow law (2.220) requires the smoothness of the yield surface with a defined normal to the surface. However, the plastic flow corresponding to edges or vertexes of the yield surface $f = 0$ originated by the intersections of smooth surfaces $f_\alpha = 0$ ($\alpha = 1, \dots$) can be estimated from superposition principle by Koiter [581]:

$$d\boldsymbol{\varepsilon}^p = \sum_{\alpha=1} d\lambda_{\alpha} \frac{\partial f_{\alpha}}{\partial \boldsymbol{\sigma}}, \quad (2.224)$$

$$f_{\alpha} = 0, \quad \frac{\partial f_{\alpha}}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} \geq 0. \quad (2.225)$$

A state of stress on the yield surface is described by a value of zero of one or more yield functions, all other yield functions being negative. In so doing, the nonzero proportionality factors $d\lambda_{\alpha}$ correspond only to the yield surfaces satisfying the conditions (2.225).

2.6.2 Deformation Theory of Plasticity

It is assumed that the rheological properties of isotropic media are described by the theory of small elastoplastic strains under monotonic, proportional loading when the ratio of the stress components $\sigma_{11} : \sigma_{22} : \sigma_{33} : \sigma_{13} : \sigma_{23} : \sigma_{12}$ is held constant at all time. Specifically, the total strain $\boldsymbol{\varepsilon}$ is written as the sum of elastic $\boldsymbol{\varepsilon}^e$ and plastic $\boldsymbol{\varepsilon}^p$ contributions:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p, \quad (2.226)$$

where the mean stress and mean strain are linearly related: $\sigma_{ii} = 3k\varepsilon_{ii}$. The local equation for the elastic material state, which relates the stress tensor $\boldsymbol{\sigma}(\mathbf{x})$ and the elastic strain tensor $\boldsymbol{\varepsilon}^e(\mathbf{x})$, is given in the form (2.214) where $\mathbf{L}(\mathbf{x})$ is an isotropic fourth-order elasticity tensor described by the Young's modulus E and the Poisson's ratio ν (2.193).

The relation between flow stress and plastic strains is represented by the expression

$$\tau = \tau_0 + f(\varepsilon_{\text{eq}}^p), \quad f(0) = 0 \quad (2.227)$$

in terms of the von Mises effective stress τ and the effective plastic strain $\varepsilon_{\text{eq}}^p$, defined as

$$\tau = \left(\frac{3}{2} s_{ij} s_{ij} \right)^{1/2}, \quad \varepsilon_{\text{eq}}^p = \left(\frac{2}{3} \varepsilon_{ij}^p \varepsilon_{ij}^p \right)^{1/2}, \quad \mathbf{s} \equiv \mathbf{N}_2 \boldsymbol{\sigma}. \quad (2.228)$$

In Eq. (2.227), τ_0 is the initial yield stress and f is a nonlinear function describing the material's hardening behavior, for example:

$$f(\varepsilon_{\text{eq}}^p) = h(\varepsilon_{\text{eq}}^p)^n, \quad (2.229)$$

where h and n are the strength coefficient and the work-hardening exponent, respectively. The material remains elastic, i.e., $\boldsymbol{\varepsilon}^p \equiv \mathbf{0}$, when $\tau < \tau_0$. In addition, Hencky's flow rule

$$\varepsilon_{ij}^p = \frac{3\varepsilon_{\text{eq}}^p}{2\tau} \sigma_{ij} \quad (2.230)$$

is adopted. A more general relation given in Prager's form is

$$\varepsilon_{ij}^p = \alpha s_{ij} + \beta t_{ij}, \quad (2.231)$$

where $t_{ij} = s_{ik}s_{kj} - 2/3 J_2 \delta_{ij}$ is the deviation of the square of the stress deviation, and for an isotropic materials α and β are the functions of the invariants $J_2 = s_{ij}s_{ij}/2$ and $J_3 = s_{ij}s_{jk}s_{ki}/3$.

Based on relation (2.226) one may determine the secant modulus as

$$E^s = \left[\frac{1}{E} + \frac{\epsilon_{eq}^p}{\tau_0 + f(\epsilon_{eq}^p)} \right]^{-1}. \quad (2.232)$$

Due to the plastic incompressibility the secant bulk modulus k^s is equal to k , and we may accordingly define the secant Poisson's ratio and shear modulus:

$$\nu^s = \frac{1}{2} - \left(\frac{1}{2} - \nu \right) \frac{E^s}{E}, \quad \mu^s = \frac{E^s}{2(1 + \nu^s)}. \quad (2.233)$$

Both here and below the superscript s indicates the calculation of the parameter under consideration with the help of the secant modulus E^s . Therefore, under a monotonic, proportional loading, the plastic state can be described by a single secant modulus, say E^s , and the other two elastic constants. For these loading the incremental flow rule can be integrated to yield the deformation rule.



<http://www.springer.com/978-0-387-36827-6>

Micromechanics of Heterogeneous Materials

Buryachenko, V.

2007, XX, 687 p. 180 illus., Hardcover

ISBN: 978-0-387-36827-6